Problems and Solutions Albert Natian, Section Editor

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This section of the Journal offers readers an opportunity to exchange interesting mathematical problems and solutions. Please email them to Prof. Albert Natian at Department of Mathematics, Los Angeles Valley College. Please present all proposed solutions and proposed problems according to formatting requirements delineated near the end of this document. Also, please make sure every proposed problem or proposed solution is provided in both ***LaTeX*** and pdf documents. *Thank you!*

To propose problems, email them to: problems4ssma@gmail.com

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Solutions to previously published problems can be seen at <www.ssma.org/publications>.

Solutions to the problems published in this issue should be submitted *before* December 1, 2024.

• **5775** *Proposed by Ovidiu Furdui and Alina Sîntămărian, Technical University of Cluj-Napoca, Cluj-Napoca, Romania.*

Calculate

$$\sum_{n=1}^{\infty}\left[\frac{1}{2n-1}-\left(\frac{1}{n}-\frac{1}{n+1}+\frac{1}{n+2}-\cdots\right)\right].$$

• **5776** *Proposed by Paolo Perfetti, dipartimento di matematica Universitą di "Tor Vergata", Rome, Italy .*

Let p a positive real number and let $\{a_n\}_{n\geqslant 1}$ be a sequence defined by $a_1 = 1$, $a_{n+1} = \dfrac{a_n}{1+a_n^p}$. Find those real values $q > 0$, such that the following series converges

$$\sum_{n=1}^{\infty}\left|a_n-(pn)^{-\frac{1}{p}}+\frac{p-1}{2p^2}\frac{\ln n}{n^{1+1/p}}\right|^q$$

• **5777** *Proposed by Seán M. Stewart, Physical Science and Engineering Division, King Abdullah University of Science and Technology, Thuwal 23955-6900, Saudi Arabia.*

Consider the sequence of polynomials $\left\{P_n(x)\right\}_{n=0}^{\infty}$ defined by the exponential generating function

$$\frac{1}{(1-x)e^t+x}=\sum_{n=0}^{\infty}P_n(x)\frac{t^n}{n!}.$$

Show that

$$\sum_{k=0}^{n}\binom{n}{k}\int_0^1 P_k(x)P_{n-k}(x)\,dx=(-1)^n.$$

• **5778** *Proposed by Narendra Bhandari, Bajura, Nepal.*

Prove

$$\sum_{m=1}^{\infty}\sum_{n=0}^{\infty}\frac{(-1)^m}{(2m+n)^2}\left(1-(-1)^n+(-1)^{n-m}\left(3+\frac{m-3n}{n+m}+\frac{n^2}{(m+n)^2}\right)\right)=G,$$

where G is the Catalan's constant, which is defined as $\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{(2n-1)^2}$.

• **5779** *Proposed by Daniel Sitaru, National Economic College "Theodor Costescu," Drobeta Turnu - Severin, Romania..*

If $0<a\leqslant b$ then:

$$e^{ab}+e^{\left(\frac{a+b}{2}\right)^2}\leqslant e^{\left(\frac{2ab}{a+b}\right)^2}+e^{\left(\sqrt{ab}+\frac{a+b}{2}-\frac{2ab}{a+b}\right)^2}$$

• **5780** *Proposed by Goran Conar, Varaždin, Croatia.*

Let α,β,γ be angles of an arbitrary triangle. Prove that the following inequality holds

$$\frac{\alpha\cos\alpha+\beta\cos\beta+\gamma\cos\gamma}{\alpha\sin\alpha+\beta\sin\beta+\gamma\sin\gamma}\leqslant\cot\left(\frac{\alpha\sin\alpha+\beta\sin\beta+\gamma\sin\gamma}{\sin\alpha+\sin\beta+\sin\gamma}\right).$$

When does equality occur?

Solutions

To Formerly Published Problems

• **5751** *Proposed by Daniel Sitaru, National Economic College "Theodor Costescu," Drobeta Turnu - Severin, Romania.*

Show that if $0<a\leqslant b<\frac{\pi}{2}$, then

$$6\log\left|\frac{\cos(b)}{\cos(a)}\right|+3(b^2-a^2)+2\int_a^b x^2\tan(x)\,dx\leqslant 0.$$

Solution 1 by Albert Stadler, Herrliberg, Switzerland.

It is sufficient to prove that

$$\tan x\ \geqslant\frac{3x}{3-x^2},\ 0\leqslant x<\frac{\pi}{2},$$

for then

$$0 \geqslant 2\int_a^b \left(\left(x^2-3\right)\tan\left(x\right)\ +3x\right)dx = \ 6\log\left(\cos\left(x\right)\ \right)\Big|_{x=a}^{x=b} +\ 3x^2\Big|_{x=a}^{x=b} + 2\int_a^b x^2 \tan\left(x\right)\,dx =$$

$$= 6\log\left|\frac{\cos\left(b\right)}{\cos\left(a\right)}\right|\ + 3\left(b^2-a^2\right) + 2\int_a^b x^2\tan\left(x\right)\,dx.$$

To prove the initially stated inequality we start from the product representation of the cosine function:

$$\cos x = \prod_{n=1}^{\infty}\left(1-\frac{4x^2}{\pi^2(2n-1)^2}\right).$$

Logarithmic differentiation then gives

$$\tan x\ = -\frac{d}{dx}\log\left(\cos\left(x\right)\ \right)\ = 2\sum_{n=1}^{\infty}\frac{\frac{4x}{\pi^2(2n-1)^2}}{1-\frac{4x^2}{\pi^2(2n-1)^2}} = 2\sum_{k=1}^{\infty}\sum_{n=1}^{\infty}\frac{4^k x^{2k-1}}{\pi^{2k}(2n-1)^{2k}} =$$

$$= 2\sum_{k=1}^{\infty}\frac{4^k x^{2k-1}}{\pi^{2k}}\left(\sum_{n=1}^{\infty}\frac{1}{n^{2k}}-\sum_{n=1}^{\infty}\frac{1}{(2n)^{2k}}\right) = 2\sum_{k=1}^{\infty}\frac{4^k\left(4^k-1\right)(2k)}{(2\pi)^{2k}}x^{2k-1}.$$

Thus, if $0 \leqslant x < \frac{\pi}{2}$,

$$\tan x\ -\frac{3x}{3-x^2} = 2\sum_{k=1}^{\infty}\frac{4^k\left(4^k-1\right)(2k)}{(2\pi)^{2k}}x^{2k-1} - \sum_{k=1}^{\infty}\frac{1}{3^{k-1}}x^{2k-1} =$$

$$= \sum_{k=3}^{\infty}\left(2\frac{4^k\left(4^k-1\right)(2k)}{(2\pi)^{2k}} - \frac{1}{3^{k-1}}\right)x^{2k-1} \geqslant 0,$$

taking into account that $\frac{\pi}{2} < \sqrt{3},\ \ (2) = \frac{\pi^2}{6},\ \ (4) = \frac{\pi^4}{90},\ \ (2k) > 1,\ \ k \geqslant 1,\ \ (2\pi)^2 < 40$ so that

$$2\frac{4^k\left(4^k-1\right)(2k)}{(2\pi)^{2k}} - \frac{1}{3^{k-1}} > 2\frac{4^k-1}{10^k} - \frac{1}{3^{k-1}} > 0,\ k \geqslant 3.$$

Solution 2 by Michel Bataille, Rouen, France.

The inequality is equivalent to

$$-3\int_a^b \tan(x)\,dx + 3\int_a^b x\,dx + \int_a^b x^2\tan(x)\,dx \leqslant 0,$$

that is, to $\int_a^b f(x)\,dx \geqslant 0$ where

$$f(x) = 3\tan(x) - 3x - x^2\tan(x).$$

Thus, it suffices to prove that $f(x) \geqslant 0$ for $x \in [0, \frac{\pi}{2})$. Since $f(0) = 0$, it is even sufficient to prove that $f'(x) \geqslant 0$.
A simple calculation gives $f'(x) = \dfrac{1}{\cos^2(x)} \cdot g(x)$ where $g(x) = 3\sin^2(x) - x\sin(2x) - x^2$.
Now, for $x \in [0, \frac{\pi}{2})$, we obtain

$$g'(x) = 6\sin(x)\cos(x) - \sin(2x) - 2x\cos(2x) - 2x = 2\sin(2x) - 2x(1 + \cos(2x)) = 4\cos^2(x)(\tan(x) - x);$$

since $\tan(x) \geqslant x$, we have $g'(x) \geqslant 0$, hence $g(x) \geqslant g(0) = 0$ and consequently $f'(x) \geqslant 0$, as desired.

Solution 3 by Moti Levy, Rehovot, Israel.

We rewrite the problem statement as follows:

$$\int_a^b x^2 \tan(x)\, dx \leqslant -3\ln\big(\cos(b)\big) - \frac{3}{2}b^2 + 3\ln\big(\cos(a)\big) + \frac{3}{2}a^2. \tag{1}$$

Let

$$F(x) := -\left(3\ln\big(\cos(x)\big) + \frac{3}{2}x^2\right), \tag{2}$$

The inequality is equivalent to

$$\int_a^b x^2 \tan(x)\, dx \leqslant F(b) - F(a), \tag{3}$$

but

$$F(b) - F(a) = \int_a^b 3\big(\tan(x) - x\big)\, dx.$$

Hence the original inequality is equivalent to

$$\int_a^b x^2 \tan(x)\, dx \leqslant \int_a^b 3\big(\tan(x) - x\big)\, dx,$$

or to

$$\int_a^b \left(\left(x^2 - 3\right)\tan(x) + x\right) dx \leqslant 0. \tag{4}$$

We now prove (4) by showing that the integrand is negative in (a, b) where $0 < a \leqslant b < \dfrac{\pi}{2}$

$$\left(x^2 - 3\right)\tan(x) + x \leqslant 0. \tag{5}$$

Inequality (5) is equivalent to

$$\frac{\tan(x)}{x} \geqslant \frac{1}{3 - x^2}. \tag{6}$$

The series expansion of $\dfrac{\tan(x)}{x}$ implies that

$$\frac{\tan(x)}{x} \geqslant 1 + \frac{1}{3}x^2 + \frac{2}{15}x^4. \tag{7}$$

One can check that

$$1 + \frac{1}{3}x^2 + \frac{1}{15}x^4 - \frac{1}{3-x^2} \geqslant 0,$$

since the function $30 - 2x^4 - x^6$ is concave in $0 < x < \dfrac{\pi}{2}$ then

$$1 + \frac{1}{3}x^2 + \frac{1}{15}x^4 - \frac{1}{3-x^2} = \frac{30 - 2x^4 - x^6}{15(3-x^2)} \geqslant 0 \text{ for } 0 < x < \frac{\pi}{2}. \tag{8}$$

It follows from (7) and (8) that the inequality $\dfrac{\tan(x)}{x} \geqslant \dfrac{1}{3-x^2}$ holds for $0 < x < \dfrac{\pi}{2}$.

Solution 4 by Perfetti Paolo, dipartimento di matematica Universitą di "Tor Vergata", Roma, Italy.

$$\frac{d}{da}\left(6\log\left|\frac{\cos(b)}{\cos(a)}\right| + 3(b^2-a^2) + 2\int_a^b x^2\tan(x)dx\right) = 2\Big((3-a^2)\tan a - 3a\Big)$$

$3 - a^2 \geqslant 3 - \pi^2/4 > 0$ and $\tan a \geqslant a + \dfrac{a^3}{3} + \dfrac{2a^5}{15} + \dfrac{17a^5}{315}$ thus

$$\begin{aligned}(3-a^2)\tan a - 3a &\geqslant (3-a^2)\Big(a + \frac{a^3}{3} + \frac{2a^5}{15} + \frac{17a^5}{315}\Big) - 3a \geqslant \\ &= \frac{a^5}{315}(21 + 9a^2 - 17a^4) \geqslant 0 \quad \text{for} \quad a \leqslant \Big(\frac{9+\sqrt{1509}}{34}\Big)^{\frac{1}{2}} \sim 1.186\end{aligned}$$

Thus for $a \leqslant 1.18$ the inequality is proved.

Now let's define $b = \pi/2 - a$. The inequality $(3-a^2)\tan a - 3a$ becomes

$$\Big(3 - (\frac{\pi}{2} - b)^2\Big)\frac{\cos b}{\sin b} - 3(\frac{\pi}{2} - b) \geqslant \Big(3 - (\frac{\pi}{2} - b)^2\Big)\frac{1 - \frac{b^2}{2}}{b} - 3(\frac{\pi}{2} - b) \tag{1}$$

for $0 \leqslant b \leqslant \dfrac{\pi}{2} - 1.18 \sim 0.3907$ and $\cos b \geqslant 1 - b^2/2$, and $\sin b \leqslant b$. The r.h.s. of (1)

$$\frac{24 + 4b^2 - 2\pi^2 + \pi^2 b^2 - 4\pi b - 4\pi b^3 + 4b^4}{8b} \geqslant 0, \quad 0 \leqslant b \leqslant 2/5$$

$f(b) = 24 + 4b^2 - 2\pi^2 + \pi^2 b^2 - 4\pi b - 4\pi b^3 + 4b^4$ and

$$f'(b) = 16b^3 + 8b + 2\pi^2 b - 4\pi - 32\pi b^2 \leqslant 16b^2 + 8b + 2\pi^2 b - 4\pi - 32\pi b^2 \leqslant 0$$

if and only if

$$2\pi - b(4+\pi^2) + (16\pi - 8)b^2 \geqslant 0 \text{ (true by) } (4+\pi^2)^2 - 8\pi(16\pi - 8) \sim -1011 < 0$$

This implies that $f(b)$ decreases and since

$$f(2/5) = \frac{15464}{625} - \frac{46\pi^2}{25} - \frac{232\pi}{125} \sim 0.75 \Longrightarrow f(b) > 0$$

and this in turn implies that through (1) the inequality $(3-a^2)\tan a - 3a > 0$ also for $1.18 \leqslant a \leqslant \pi/2$. This implies that

$$6\log\left|\frac{\cos(b)}{\cos(a)}\right| + 3(b^2-a^2) + 2\int_a^b x^2\tan(x)dx$$

increases with a and then the maximum value is attained when $a = b$ thus proving the inequality

Solution 5 by proposed by G. C. Greubel, Newport News, VA.

Using the series

$$\ln(\cos(x)) = -\frac{1}{2}\sum_{k=1}^{\infty}\frac{\alpha_{2k}\,x^{2k}}{k}$$

$$\tan(x) = \sum_{k=1}^{\infty}\alpha_{2k}\,x^{2k-1},$$

and integral

$$\int x^2\,\tan(x)\,dx = \sum_{k=2}^{\infty}\frac{\alpha_{2k-2}\,x^{2k}}{2k},$$

where

$$\alpha_{2k} = \frac{4^k\,(4^k-1)\,|B_{2k}|}{(2k)!}$$

with B_n being the Bernoulli numbers, then

$$\begin{aligned} S &= 6\,\ln\left|\frac{\cos(b)}{\cos(a)}\right| + 3\,(b^2-a^2) + 2\int_a^b x^2\,\tan(x)\,dx \\ &= -3\sum_{k=1}^{\infty}\frac{\alpha_{2k}}{k}\,(b^{2k}-a^{2k}) + 3\,(b^2-a^2) + 2\sum_{k=2}^{\infty}\frac{\alpha_{2k-2}}{2k}\,(b^{2k}-a^{2k}) \\ &= -3\sum_{k=2}^{\infty}\frac{\alpha_{2k}}{k}\,(b^{2k}-a^{2k}) + \sum_{k=2}^{\infty}\frac{\alpha_{2k-2}}{k}\,(b^{2k}-a^{2k}) \\ &= -\sum_{k=2}^{\infty}\frac{3\,\alpha_{2k}-\alpha_{2k-2}}{k}\,(b^{2k}-a^{2k}). \end{aligned}$$

Since $\alpha_2 = 1$ and $3\,\alpha_4 = 1$ then

$$S = -\sum_{k=3}^{\infty} \frac{3\,\alpha_{2k} - \alpha_{2k-2}}{k}\,(b^{2k} - a^{2k}).$$

It is evident that $3\,\alpha_{2n} > \alpha_{2n-2}$ for $n \geqslant 3$ and leads to

$$6\,\ln\left|\frac{\cos(b)}{\cos(a)}\right| + 3\,(b^2 - a^2) + 2\int_a^b x^2\,\tan(x)\,dx \leqslant 0$$

for $b \geqslant a$. Equality occurs when $b = a$.

Also solved by the problem proposer.

• **5752** *Proposed by Raluca Maria Caraion, Călăraşi, Romania and Florică Anastase, Lehliu-Gară, Romania.*

Show that if $x, y, z > 0$, then

$$\sum_{cyc} \frac{(kx^3 + y^3)z}{x^3y^3(1 + nz)} \geqslant \frac{3(k+1)}{8}\left(\frac{15}{x^2 + y^2 + z^2} - n^2\right).$$

Solution 1 by proposed by Moti Levy, Rehovot, Israel.

We rewrite the left hand side as follows,

$$\sum_{cyc} \frac{\left(kx^3 + y^3\right)z}{x^3y^3\,(1 + nz)} = \sum_{cyc} \frac{kx^3z}{x^3y^3\,(1 + nz)} + \sum_{cyc} \frac{y^3z}{x^3y^3\,(1 + nz)}$$
$$= \sum_{cyc} \frac{kz}{y^3\,(1 + nz)} + \sum_{cyc} \frac{z}{x^3\,(1 + nz)}$$

Without loss of generality, we may assume that $x \geqslant y \geqslant z$.
Then $\frac{x}{1 + nx} \geqslant \frac{y}{1 + ny} \geqslant \frac{z}{1 + nz}$ and $\frac{1}{z^3} \geqslant \frac{1}{y^3} \geqslant \frac{1}{x^3}$.

By the rearrangement inequality

$$\sum_{cyc} \frac{1}{x^3}\frac{z}{(1 + nz)} \geqslant \sum_{cyc} \frac{1}{y^3}\frac{z}{(1 + nz)}.$$

It follows that

$$\sum_{cyc} \frac{1}{y^3}\frac{kz}{1 + nz} + \sum_{cyc} \frac{1}{x^3}\frac{z}{1 + nz} \geqslant (k+1)\sum_{cyc} \frac{z}{y^3\,(1 + nz)}. \qquad (9)$$

Assuming $k+1>0$ then (9) implies that it is enough to prove,

$$\sum_{cyc} \frac{z}{y^3(1+nz)} \geqslant \frac{3}{8}\left(\frac{15}{x^2+y^2+z^2}-n^2\right).$$

Again, by the rearrangement inequality

$$\sum_{cyc} \frac{z}{y^3(1+nz)} \geqslant \sum_{cyc} \frac{x}{x^3(1+nx)},$$

then it is enough to prove that

$$\sum_{cyc} \frac{1}{x^2(1+nx)} \geqslant \frac{3}{8}\left(\frac{15}{x^2+y^2+z^2}-n^2\right). \tag{10}$$

Let

$$f(y) := \frac{1}{y\left(1+n\sqrt{y}\right)},$$

$$\frac{d^2 f}{dy^2} = \frac{15n^2 y + 21n\sqrt{y}+8}{4y^3\left(n\sqrt{y}+1\right)^3} > 0 \quad for \quad y>0.$$

Hence, $f(y)$ is convex function at $y>0$. Now we apply Jensen's inequality to obtain,

$$\sum_{cyc} \frac{1}{x^2(1+nx)} = \sum_{cyc} f\left(x^2\right) = \sum_{cyc} \frac{1}{x^2\left(1+n\sqrt{x^2}\right)} \geqslant \frac{9}{\left(\sum_{cyc} x^2\right)\left(1+\frac{n}{\sqrt{3}}\sqrt{\sum_{cyc} x^2}\right)} \tag{11}$$

Let $t := \sum_{cyc} x^2$, then we rewrite (11) as follows,

$$\frac{9}{t\left(1+\frac{n}{\sqrt{3}}\sqrt{t}\right)} \geqslant \frac{3}{8}\left(\frac{15}{t}-n^2\right). \tag{12}$$

Simplifying (12), we get the equivalent inequality,

$$9-\frac{3}{8}\left(\frac{15}{t}-n^2\right)t\left(1+\frac{n}{\sqrt{3}}\sqrt{t}\right) \geqslant 0.$$

After factoring,

$$9-\frac{3}{8}\left(\frac{15}{t}-n^2\right)t\left(1+\frac{n}{\sqrt{3}}\sqrt{t}\right) = \frac{\sqrt{3}}{24}\left(n\sqrt{t}+3\sqrt{3}\right)\left(\sqrt{3}n\sqrt{t}-3\right)^2.$$

Now we further assume that $n>0$, to conclude that

$$\frac{\sqrt{3}}{24}\left(n\sqrt{t}+3\sqrt{3}\right)\left(\sqrt{3}n\sqrt{t}-3\right)^2 \geqslant 0.$$

Also solved by the problem proposer.

• **5753** *Proposed by Goran Conar, Varaždin, Croatia.*

Let $x_1, \ldots, x_n > 0$ and set $s = \sum_{i=1}^{n} x_i$. Prove

$$\prod_{i=1}^{n} x_i^{x_i} \geqslant \left(\frac{s}{n+s}\right)^s \prod_{i=1}^{n} (1+x_i)^{x_i}.$$

When does equality occur?

Solution 1 by proposed by Albert Stadler, Herrliberg, Switzerland.

We take logarithms on both sides and obtain the equivalent inequality

$$\sum_{i=1}^{n} x_i \ln\left(\frac{x_i}{1+x_i}\right) \geqslant s \ln\left(\frac{s}{n+s}\right) .$$

The function $x \to x\ln\left(\frac{x}{1+x}\right)$ is convex, since $\frac{d^2}{dx^2} x\ln\left(\frac{x}{1+x}\right) = \frac{1}{x(1+x)^2} > 0$. Hence, by Jensen's inequality,

$$\sum_{i=1}^{n} x_i \ln\left(\frac{x_i}{1+x_i}\right) \geqslant n\frac{s}{n}\ln\left(\frac{\frac{s}{n}}{1+\frac{s}{n}}\right) = s\ln\left(\frac{s}{n+s}\right),$$

as required. Equality in Jensen's inequality occurs if and only if x_i=s/n for all i, 1⩽i⩽n.

Solution 2 by proposed by Perfetti Paolo, dipartimento di matematica Universitą di "Tor Vergata," Roma, Italy.

The inequality is

$$\sum_{i=1}^{n} x_i \ln \frac{x_i}{1+x_i} \geqslant s \ln \frac{s}{n+s}$$

$$\left(\frac{x}{1+x}\right)'' = \frac{1}{x(1+x)^2} > 0$$

hence is convex and

$$\sum_{i=1}^{n} x_i \ln \frac{x_i}{1+x_i} = s \sum_{i=1}^{n} \frac{x_i}{s} \ln \frac{x_i}{1+x_i} \geqslant s \ln \left(\frac{\frac{x_1^2+\ldots+x_n^2}{s}}{1+\frac{x_1^2+\ldots+x_n^2}{s}}\right)$$

$x_1^2 + \ldots + x_n^2 \geqslant \dfrac{(x_1 + \ldots + x_n)^2}{n} = \dfrac{s^2}{n}$ and $\ln(x/(1+x))$ is monotonic increasing.

$x_1^2 + \ldots + x_n^2 \geqslant \dfrac{(x_1 + \ldots + x_n)^2}{n}$ may be proved by the convexity of $f(x) = x^2$

$$n\frac{x_1^2}{n} + \ldots + n\frac{x_n^2}{n} \geqslant n\left(\frac{x_1}{n} + \ldots + \frac{x_n}{n}\right)^2$$

The monotonicity of $\ln(x/(1+x))$ follows by $\left(\ln \dfrac{x}{1+x}\right)' = \dfrac{1}{x(1+x)}$

It follows

$$s\ln\left(\frac{\frac{x_1^2+\ldots+x_n^2}{s}}{1+\frac{x_1^2+\ldots+x_n^2}{s}}\right) \geqslant s\ln\frac{\frac{s}{n}}{1+\frac{s}{n}} = s\ln\frac{s}{n+s}$$

The inequality occurs when

$$x_1^2 + \ldots + x_n^2 = \frac{(x_1 + \ldots + x_n)^2}{n} \iff x_1 = \ldots = x_n = +s/n$$

The proof is complete.

Solution 3 by proposed by Prakash Pant, Mathematics Initiatives in Nepal, Bardiya, Nepal.

We first rearrange the problem as follows:

$$\prod_{i=1}^{n}(\frac{x_i}{1+x_i})^{x_i} \geqslant (\frac{s}{n+s})^s$$

We further modify the problem taking log on both sides,

$$\sum_{i=1}^{n} x_i \ln(\frac{x_i}{1+x_i}) \geqslant s\ln(\frac{s}{n+s}) \qquad \text{..........(1)}$$

Now, we will focus on proving this statement.

Then, consider a function $f(x) = x\ln(\dfrac{x}{1+x})$. Then f"(x) = $\dfrac{1}{x(1+x)^2} > 0\ \forall x > 0$. So, the function is convex in our required interval.
Now, using Jensen's inequality,

$$\frac{\sum_{i=1}^{n} x_i \ln(\frac{x_i}{1+x_i})}{n} \geqslant \frac{\sum_{i=1}^{n} x_i}{n}\ln(\frac{\frac{\sum_{i=1}^{n} x_i}{n}}{1+\frac{\sum_{i=1}^{n} x_i}{n}})$$

which on simplification gives

$$\sum_{i=1}^{n} x_i \ln(\frac{x_i}{1+x_i}) \geqslant s\ln(\frac{s}{n+s})$$

which is what was to be proved from (1). And the equality holds when $x_1 = x_2 = ... = x_n$.

Solution 4 by proposed by Toyesh Prakash Sharma, Agra college, Agra, India.

Consider a function $f(x) = x\ln\left(\dfrac{x}{1+x}\right) = x\ln x - x\ln(1+x)$ then differentiating it with respect to x we have $f'(x) = (1+\ln x) - \left(\ln(1+x) + \dfrac{x}{1+x}\right)$ again differentiation gives $f''(x) = \dfrac{1}{x} - \left(\dfrac{1}{1+x} + \dfrac{1}{(1+x)^2}\right) = \dfrac{1}{x(1+x)} - \dfrac{1}{(1+x)^2} = \dfrac{1}{x(1+x)^2} > 0$ then we can say that considered function is an example of convex function then therefore using Jensen's Inequality for convex function we can obtain

$$\frac{1}{n}\sum_{i=1}^{n} f(x_i) \geqslant f\left(\frac{1}{n}\sum_{i=1}^{n} x_i\right) \Leftrightarrow \frac{1}{n}\sum_{i=1}^{n} f(x_i) \geqslant f\left(\frac{s}{n}\right)$$

$$\Longrightarrow \frac{1}{n}\sum_{i=1}^{n} x_i \ln\left(\frac{x_i}{1+x_i}\right) \geqslant \left(\frac{s}{n}\right)\ln\left(\frac{\frac{s}{n}}{1+\frac{s}{n}}\right) \Longrightarrow \sum_{i=1}^{n}\ln\left(\frac{x_i}{1+x_i}\right)^{x_i} \geqslant \ln\left(\frac{s}{n+s}\right)^{s}$$

$$\Longrightarrow \prod_{i=1}^{n}\left(\frac{x_i}{1+x_i}\right)^{x_i} \geqslant \left(\frac{s}{n+s}\right)^{s} \Longleftrightarrow \prod_{i=1}^{n} x_i^{x_i} \geqslant \left(\frac{s}{n+s}\right)^{s}\prod_{i=1}^{n}(1+x_i)^{x_i}$$

Equality occurs at $x_1 = x_2 = \cdots = x_n$.

Solution 5 by proposed by Henry Ricardo, Westchester Area Math Circle, Purchase, NY.

The *log sum inequality*, useful for proving results in information theory, states that

$$\sum_{i=1}^{n} a_i \log\frac{a_i}{b_i} \geqslant a\log\frac{a}{b},$$

where the a_i and b_i are nonnegative numbers, $a = \sum a_i$, and $b = \sum b_i$. If we set $a_i = x_i$ and $b_i = 1 + x_i$, then $a = s$ and $b = n + s$. Thus we have

$$\sum_{i=1}^{n} x_i \log\left(\frac{x_i}{1+x_i}\right) \geqslant s\log\left(\frac{s}{n+s}\right), \text{ or } \sum_{i=1}^{n} x_i \log x_i \geqslant s\log\left(\frac{s}{n+s}\right) + \sum_{i=1}^{n} x_i \log(1+x_i).$$

Exponeniating this last expression yields

$$\prod_{i=1}^{n} x_i^{x_i} \geqslant \left(\frac{s}{n+s}\right)^{s}\prod_{i=1}^{n}(1+x_i)^{x_i}.$$

Equality holds if and only if $x_1 = x_2 = \cdots = x_n$.

Another Solution: Let $f(x) = x \log x$, a convex function for $x > 0$. Jensen's inequality yields

$$\begin{aligned}
\sum_{i=1}^{n} x_i \log\left(\frac{x_i}{1+x_i}\right) &= \sum_{i=1}^{n}(1+x_i)\cdot\frac{x_i}{1+x_i}\log\left(\frac{x_i}{1+x_i}\right)\\
&= (n+s)\sum_{i=1}^{n}\frac{1+x_i}{n+s}\cdot\frac{x_i}{1+x_i}\log\left(\frac{x_i}{1+x_i}\right)\\
&\geqslant (n+s)\sum_{i=1}^{n}\frac{1+x_i}{n+s}\cdot\frac{x_i}{1+x_i}\log\left(\sum_{i=1}^{n}\frac{1+x_i}{n+s}\cdot\frac{x_i}{1+x_i}\right)\\
&= (n+s)\left(\frac{1}{n+s}\sum_{i=1}^{n}x_i\right)\log\left(\frac{1}{n+s}\sum_{i=1}^{n}x_i\right)\\
&= (n+s)\cdot\frac{s}{n+s}\log\left(\frac{s}{n+s}\right)\\
&= s\log\left(\frac{s}{n+s}\right).
\end{aligned}$$

Exponentiating the inequality yields the desired result. Equality holds if and only if $x_1 = x_2 = \cdots = x_n$.

Also solved by the problem proposer.

Note: Michel Bataille (Rouen, France) states that "This [#5753] is a repetition of problem 5738, proposed in April 2023. 8 solutions were featured in the December 2023 issue."

Editor's note: It is not easy to characterize the 'dynamics' responsible for this repetition. It is possible that the proposer of this problem has sent the same problem twice under two different filenames. It is also possible that the proposer of the problem has sent the problem only once but that the editor has mistakenly puslished the problem twice without realizing having done so at the time of publication. It is important that the problem proposers create a personal system for maintaining their store of different problems under respectively different filenames.

• **5754** *Proposed by Ovidiu Furdui and Alina Sîntămărian, Technical University of Cluj-Napoca, Cluj-Napoca, Romania.*

Calculate $S = \sum_{n=1}^{\infty}(2n-1)\left(\sum_{k=0}^{\infty}\frac{(-1)^k}{(n+k)^2}\right)^2.$

Solution 1 by proposed by Yunyong Zhang, Chinaunicom, Yunnan, China.

$$S = \sum_{n=1}^{\infty}(2n-1)\left[\frac{1}{n^2} - \frac{1}{(n+1)^2} + \frac{1}{(n+2)^2} - \cdots\right]^2$$

Let $a_k = 2k-1,\ A_n = \sum_{k=1}^{n} a_k = \sum_{k=1}^{n}(2k-1) = n^2$

Let $b_k = \left[(\frac{1}{k^2} - \frac{1}{(k+1)^2}) + \frac{1}{(k+2)^2} - \cdots\right]^2$

$$b_k - b_{n+1} = \left[(\frac{1}{k^2} - \frac{1}{(k+1)^2}) + \frac{1}{(k+2)^2} - \cdots\right]^2 - \left[(\frac{1}{(k+1)^2} - \frac{1}{(k+2)^2}) + \frac{1}{(k+3)^2} - \cdots\right]^2$$

$$= \frac{1}{k^2}\left[(\frac{1}{k^2} - \frac{2}{(k+1)^2}) + \frac{2}{(k+2)^2} - \cdots\right]$$

$$= \frac{1}{k^2}\left[(-\frac{1}{k^2} + \frac{2}{k^2} - \frac{2}{(k+1)^2}) + \frac{2}{(k+2)^2} - \cdots\right]$$

$$= \frac{2}{k^2}\left[\frac{1}{k^2} - \frac{1}{(k+1)^2} + \frac{1}{(k+2)^2} - \cdots\right] - \frac{1}{k^4}$$

According to Abel Theorem $S = \sum_{k=1}^{\infty} a_k b_k = \lim_{n\to\infty} A_n b_{n+1} + \sum_{k=1}^{\infty} A_k(b_k - b_{k+1})$

$$= \lim_{n\to\infty} n^2\left[\frac{1}{(n+1)^2} - \frac{1}{(n+2)^2} + \frac{1}{(n+3)^2} - \cdots\right]^2 + \sum_{k=1}^{\infty} k^2\left\{\frac{2}{k^2}\left[\frac{1}{k^2} - \frac{1}{(k+1)^2} + \frac{1}{(k+2)^2} - \cdots\right] - \frac{1}{k^4}\right\}$$

$$= \lim_{n\to\infty}\left[\frac{n}{(n+1)^2} - \frac{n}{(n+2)^2} + \frac{n}{(n+3)^2} - \cdots\right]^2 + 2\sum_{k=1}^{\infty}\left[\frac{1}{k^2} - \frac{1}{(k+1)^2} + \frac{1}{(k+2)^2} - \cdots\right] - \sum_{k=1}^{\infty}\frac{1}{k^2}$$

$$= 2\left(\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \cdots\right) - \frac{\pi^2}{6}$$

$$= 2\times\frac{\pi^2}{8} - \frac{\pi^2}{6}$$

$$= \frac{\pi^2}{12}.$$

Solution 2 by proposed by Perfetti Paolo, dipartimento di matematica Universitą di "Tor Vergata," Roma, Italy.

$$\sum_{k=0}^{\infty}\frac{(-1)^k}{(n+k)^2} = \sum_{k=n}^{\infty}\frac{(-1)^{n+k}}{k^2} = \sum_{k=n}^{\infty}(-1)^{n+k}\int_0^{\infty} e^{-kt}t\,dt = \int_0^{\infty}\sum_{k=n}^{\infty}(-1)^{k+n}e^{-kt}t\,dt =$$

$$= (-1)^n\int_0^{\infty}\frac{(-1)^n te^{-nt}}{1+e^{-t}}dt = \int_0^{\infty}\frac{te^{t(1-n)}dt}{1+e^t} \qquad (n-1=p)$$

Now let's integrate by parts

$$\frac{te^{-pt}}{-p(1+e^t)}\Big|_0^\infty + \frac{1}{p}\int_0^\infty e^{-pt}\frac{1+e^t-te^t}{(1+e^t)^2}dt =$$
$$= e^{-pt}\frac{1+e^t-te^t}{-p^2(1+e^t)^2}\Big|_0^\infty + \int_0^\infty \frac{e^{-pt}(e^t(-t+te^t-2-2e^t))}{p^2(1+e^t)^3}dt =$$
$$= \frac{1}{2p^2} - \frac{e^{-pt}(e^t(-t+te^t-2-2e^t))}{p^3(1+e^t)^3}\Big|_0^\infty - \int_0^\infty \frac{e^{-pt}e^t(t+te^{2t}+3-3e^{2t}-4te^t)}{p^3(1+e^t)^4}dt =$$
$$= \frac{1}{2p^2} - \frac{1}{2p^3} + \frac{e^{-pt}e^t(t+te^{2t}+3-3e^{2t}-4te^t)}{p^4(1+e^t)^4}\Big|_0^\infty +$$
$$+ \int_0^\infty e^{-pt}\frac{e^t(t-11te^t+11te^{2t}-te^{3t}+4-12e^t-12e^{2t}+4e^{3t})}{p^4(1+e^t)^5}dt =$$
$$= \frac{1}{2p^2} - \frac{1}{2p^3} + O(\frac{1}{p^5})$$

Now let's apply Abel "Abel's "summation by parts"

$$\sum_{n=1}^N a_n b_n = A_N b_N - A_0 b_0 + \sum_{n=1}^N A_{n-1}(b_{n-1}-b_n), \qquad A_N = \sum_{n=1}^N a_n,\ A_0 = 0$$
$$\sum_{n=1}^N a_n b_n = A_N b_N + \sum_{n=2}^N A_{n-1}(b_{n-1}-b_n), \tag{1}$$

$$A_n = \sum_{k=1}^n (2k-1) = n(n+1) - n = n^2, \quad b_n = \left(\sum_{k=n}^\infty \frac{(-1)^{n+k}}{k^2}\right)^2 = O(\frac{1}{n^4})$$

$$b_n - b_{n-1} = \left(\sum_{k=n}^\infty \frac{(-1)^{n+k}}{k^2} - \sum_{k=n-1}^\infty \frac{(-1)^{n-1+k}}{k^2}\right)\left(\sum_{k=n}^\infty \frac{(-1)^{n+k}}{k^2} + \sum_{k=n-1}^\infty \frac{(-1)^{n-1+k}}{k^2}\right) =$$
$$= \left(\sum_{k=n}^\infty \frac{(-1)^{n+k}}{k^2} + \sum_{k=n-1}^\infty \frac{(-1)^{n+k}}{k^2}\right)\left(\sum_{k=n}^\infty \frac{(-1)^{n+k}}{k^2} - \sum_{k=n-1}^\infty \frac{(-1)^{n+k}}{k^2}\right) =$$
$$= \left(\frac{-1}{(n-1)^2} + \sum_{k=n}^\infty \frac{2(-1)^{n+k}}{k^2}\right)\frac{1}{(n-1)^2}$$

(1) becomes

$$A_N b_N + \sum_{n=2}^N A_{n-1}(b_{n-1}-b_n) = N^2 O(\frac{1}{N^4}) + \sum_{n=2}^N \frac{(n-1)^2}{(n-1)^2}\left(\frac{1}{(n-1)^2} - \sum_{k=n}^\infty \frac{2(-1)^{n+k}}{k^2}\right) =$$
$$= o(1) + \frac{\pi^2}{6} - 2\sum_{n=2}^N (-1)^n \sum_{k=n}^\infty \frac{(-1)^k}{k^2} \doteq o(1) + \frac{\pi^2}{6} - 2\sum_{n=2}^N c_n d_n \tag{2}$$

"Abel's "summation by parts" again yields

$$\sum_{n=2}^N c_n d_n = C_N d_N - C_1 d_1 + \sum_{n=2}^N C_{n-1}(d_{n-1}-d_n)$$

$C_n = \sum_{k=1}^{n}(-1)^k$ equals zero if n is even and -1 if n is odd.

$$d_n = \sum_{k=n}^{\infty}\frac{(-1)^k}{k^2} = O(\frac{1}{n^2}) \implies C_N d_N \to 0, \qquad C_1 d_1 = (-1)\cdot\frac{-\pi^2}{12} = \frac{\pi^2}{12}$$

$$\sum_{n=2}^{\infty} C_{n-1}(d_{n-1}-d_n) = \sum_{n=2}^{\infty} C_{n-1}\frac{(-1)^{n-1}}{(n-1)^2} = \sum_{n=1}^{\infty} C_n\frac{(-1)^n}{n^2} = \sum_{p=0}^{N}\frac{1}{(2p+1)^2} = \frac{\pi^2}{8}$$

The result is

$$S = \frac{\pi^2}{6} - 2\left(-\frac{\pi^2}{12}+\frac{\pi^2}{8}\right) = \frac{\pi^2}{12}.$$

Solution 3 by proposed by Moti Levy, Rehovot, Israel.

Denote the tail by

$$\overline{\tau}_n := \sum_{k=0}^{\infty}\frac{(-1)^k}{(n+k)^2} = (-1)^n\sum_{k=n}^{\infty}(-1)^k\frac{1}{k^2}.$$

We immediately observe that $\sum_{n=1}^{\infty}\overline{\tau}_n$ converges,

$$\sum_{n=1}^{\infty}\overline{\tau}_n = \sum_{n=1}^{\infty}(-1)^n\sum_{k=n}^{\infty}(-1)^k\frac{1}{k^2} = \sum_{m=1}^{\infty}\frac{1}{(2m-1)^2} = \frac{1}{8}\pi^2. \tag{13}$$

The required sum is

$$S = \sum_{n=1}^{\infty}(2n-1)\,\overline{\tau}_n^2 \tag{14}$$

We apply summation by parts,

$$\sum_{n=1}^{\infty} a_n b_n = \lim_{n\to\infty} a_n B_n - \sum_{n=1}^{\infty} B_n\left(a_{n+1}-a_n\right), \qquad B_n = \sum_{k=1}^{n} b_k, \tag{15}$$

on the sum S.

Set $a_n = \overline{\tau}_n^2$ and $b_n = 2n-1$

$$B_n = \sum_{k=1}^{n}(2k-1) = n^2.$$

By (15),

$$S = \lim_{n\to\infty}\overline{\tau}_n^2 n^2 - \sum_{n=1}^{\infty} n^2\left(\overline{\tau}_{n+1}^2-\overline{\tau}_n^2\right) = -\sum_{n=1}^{\infty} n^2\left(\overline{\tau}_{n+1}-\overline{\tau}_n\right)\left(\overline{\tau}_{n+1}+\overline{\tau}_n\right). \tag{16}$$

Since $\overline{\tau}_n > 0$ for $n \geqslant 1$, then the convergence of the $\sum_{n=1}^{\infty} \overline{\tau}_n$ (as shown in (13)) implies that $\overline{\tau}_n \sim \frac{1}{n^s}$, $s \geqslant 2$. Hence $\lim_{n\to\infty} \overline{\tau}_n^2 n^2 = 0$.

It is straightforward to see that

$$\overline{\tau}_{n+1} - \overline{\tau}_n = \frac{1}{n^2} - 2\overline{\tau}_n. \tag{17}$$

It follows that

$$\overline{\tau}_{n+1} + \overline{\tau}_n = \frac{1}{n^2}. \tag{18}$$

By plugging (17) and (18) into (16), we obtain the required sum,

$$S = -\sum_{n=1}^{\infty} n^2 \left(\frac{1}{n^2} - 2\overline{\tau}_n\right)\left(\frac{1}{n^2}\right) = -\sum_{n=1}^{\infty}\left(\frac{1}{n^2} - 2\overline{\tau}_n\right) = 2\left(\sum_{n=1}^{\infty}\overline{\tau}_n\right) - \frac{\pi^2}{6}$$

$$S = 2 * \frac{\pi^2}{8} - \frac{\pi^2}{6} = \frac{1}{12}\pi^2 \cong 0.822467.$$

Solution 4 by proposed by Michel Bataille, Rouen, France.

Let $R_n = \sum_{k=n}^{\infty} \frac{(-1)^k}{k^2}$ and $s_n = \sum_{k=1}^{n}(2k-1) = n^2$. Summing by parts, we obtain that for any integer $N > 1$

$$\begin{aligned}
\sum_{n=1}^{N}(2n-1)\left(\sum_{k=0}^{\infty}\frac{(-1)^k}{(n+k)^2}\right)^2 &= \sum_{n=1}^{N}(2n-1)R_n^2 = s_N R_N^2 + \sum_{n=1}^{N-1} s_n(R_n^2 - R_{n+1}^2)\\
&= N^2R_N^2 + \sum_{n=1}^{N-1} n^2\left(\frac{1}{n^4} + 2R_{n+1}\frac{(-1)^n}{n^2}\right)\\
&= N^2R_N^2 + \sum_{n=1}^{N-1}\frac{1}{n^2} + 2\sum_{n=1}^{N-1}(-1)^n R_{n+1} \qquad (1)
\end{aligned}$$

where we used $R_n^2 - R_{n+1}^2 = (R_n - R_{n+1})(R_n + R_{n+1}) = \frac{(-1)^n}{n^2}\left(\frac{(-1)^n}{n^2} + 2R_{n+1}\right)$.

Now, $R_N = \frac{(-1)^N}{N^2} + R'_N$ with $|R'_N| \leqslant \frac{1}{(N+1)^2}$ (since the sequence $(1/k^2)_{k\geqslant n}$ is decreasing towards 0), hence $\lim_{N\to\infty} NR_N = 0$.

Also, it is readily checked that

$$\sum_{n=1}^{N-1}(-1)^n R_{n+1} = -R_2 + R_3 - R_4 + R_5 - \cdots + (-1)^{N-1}R_N = -\sum_{n=1}^{M-1}\frac{1}{(2n)^2} - \alpha R_{2M}$$

where $M = \lfloor\frac{N+1}{2}\rfloor$ and $\alpha = 1$ if $N = 2M$, $\alpha = 0$ if $N = 2M-1$.

It follows that $\lim_{N\to\infty} \sum_{n=1}^{N-1} (-1)^n R_{n+1} = -\frac{1}{4} \cdot \frac{\pi^2}{6} = -\frac{\pi^2}{24}$.
Returning to (1) and letting $N \to \infty$, we obtain

$$S = 0 + \frac{\pi^2}{6} - 2 \cdot \frac{\pi^2}{24} = \frac{\pi^2}{12}.$$

Solution 5 by proposed by G. C. Greubel Newport News, VA.

Using

$$\frac{1}{(n+k)^2} = \int_0^\infty e^{-(n+k)t}\, t\, dt$$

then

$$\begin{aligned}
\sum_{k=0}^{\infty} \frac{(-1)^k}{(n+k)^2} &= \sum_{k=0}^{\infty} (-1)^k \int_0^\infty e^{-(n+k)t}\, t\, dt \\
&= \int_0^\infty e^{-nt}\, t \left(\sum_{k=0}^{\infty} \left(-e^{-t}\right)^k \right) dt \\
&= \int_0^\infty \frac{e^{-nt}\, t}{1+e^{-t}}\, dt
\end{aligned}$$

With this then the series in question becomes

$$\begin{aligned}
S &= \sum_{n=1}^{\infty} (2n-1) \left(\sum_{k=0}^{\infty} \frac{(-1)^k}{(n+k)^2} \right)^2 \\
&= \sum_{n=1}^{\infty} (2n-1) \int_0^\infty \int_0^\infty \frac{u\, t\, e^{-n(t+u)}\, dt\, du}{(1+e^{-u})(1+e^{-t})} \\
&= \int_0^\infty \int_0^\infty \frac{ut\, e^{-u-t}\,(1+e^{-u-t})\, du\, dt}{(1+e^{-t})(1+e^{-u})(1-e^{-t-u})^2} \\
&= \int_0^\infty \frac{t\, e^{-t}\, dt}{1+e^{-t}} \int_0^\infty \frac{u\, e^{-u}\,(1+e^{-t-u})\, du}{(1+e^{-u})(1-e^{-t-u})^2} \\
&= \int_0^\infty \frac{t\, e^{-t}}{1+e^{-t}}\, J(t)\, dt.
\end{aligned}$$

The integral $J(t)$ can be evaluated by using the expansion

$$\begin{aligned}
\frac{1+e^{-t-u}}{(1+e^{-u})(1-e^{-t-u})^2} = \frac{1-e^{-t}}{(1+e^{-t})^2} &\left(\frac{1}{1+e^{-u}} + \frac{e^{-t}}{1-e^{-t-u}} \right) \\
&+ \frac{2\, e^{-t}}{1+e^{-t}}\, \frac{1}{(1-e^{-t-u})^2}
\end{aligned}$$

which leads to

$$J(t) = \frac{1-e^{-t}}{(1+e^{-t})^2}\,(I_1 + e^{-t}\,I_2) + \frac{2\,e^{-t}}{1+e^{-t}}\,I_3,$$

where, with $\mathrm{Li}_2(x)$ being the dilogarithm function,

$$\begin{aligned} I_1 &= \int_0^\infty \frac{u\,e^{-u}\,du}{1+e^{-u}} = \frac{\zeta(2)}{2} \\ I_2 &= \int_0^\infty \frac{u\,e^{-u}\,du}{1-e^{-t-u}} = e^t\,\mathrm{Li}_2(e^{-t}) \\ I_3 &= \int_0^\infty \frac{u\,e^{-u}\,du}{(1-e^{-t-u})^2} = -e^t\,\ln(1-e^{-t}) \end{aligned}$$

for which

$$J(t) = \frac{1-e^{-t}}{(1+e^{-t})^2}\left(\frac{\zeta(2)}{2} + \mathrm{Li}_2\left(e^{-t}\right)\right) + \frac{2\,\ln(1-e^{-t})}{1+e^{-t}}.$$

Now,

$$S = \int_0^\infty \frac{t\,e^{-t}\,(1-e^{-t})}{(1+e^{-t})^3}\left(\frac{\zeta(2)}{2} + \mathrm{Li}_2(e^{-t})\right)\,dt - 2\int_0^\infty \frac{t\,e^{-t}\,\ln(1-e^{-t})}{(1+e^{-t})^2}.$$

Since $1-e^{-t} = 2-(1+e^{-t})$ then

$$S = \frac{\zeta(2)}{2}\,(2\,I_4 - I_5) + 2\,I_6 - I_7 - 2\,I_8,$$

where

$$\begin{aligned} I_4 &= \int_0^\infty \frac{t\,e^{-t}\,dt}{(1+e^{-t})^3} = \frac{1}{4} + \frac{\ln(2)}{2} \\ I_5 &= \int_0^\infty \frac{t\,e^{-t}\,dt}{(1+e^{-t})^2} = \ln(2) \\ I_6 &= \int_0^\infty \frac{t\,e^{-t}\,\mathrm{Li}_2(e^{-t})\,dt}{(1+e^{-t})^3} = \frac{\ln^2(2)}{2} - \frac{\zeta(2)}{8} + \frac{5\,\zeta(2)\,\ln(2)}{4} - \frac{9\,\zeta(3)}{8} \\ I_7 &= \int_0^\infty \frac{t\,e^{-t}\,\mathrm{Li}_2(e^{-t})\,dt}{(1+e^{-t})^2} = \frac{5\,\zeta(2)\,\ln(2)}{2} - \frac{9\,\zeta(3)}{4} \\ I_8 &= \int_0^\infty \frac{t\,e^{-t}\,\ln(1-e^{-t})\,dt}{(1+e^{-t})^2} = -\frac{\zeta(2)}{4} + \frac{\ln^2(2)}{2}, \end{aligned}$$

leads S into the final result of $2\,S = \zeta(2)$. In another form it can be stated that

$$S = \sum_{n=1}^\infty (2n-1)\left(\sum_{k=0}^\infty \frac{(-1)^k}{(n+k)^2}\right)^2 = \frac{\zeta(2)}{2}.$$

Solution 6 by proposed by Albert Stadler, Herrliberg, Switzerland.

Put $\sigma_n := \sum_{k=n}^{\infty} \frac{(-1)^k}{k^2}$. Clearly, $\sigma_n = O\left(\frac{1}{n^2}\right)$, $n \to \infty$, and

$$\sum_{k=0}^{\infty} \frac{(-1)^k}{(n+k)^2} = (-1)^n \sum_{k=n}^{\infty} \frac{(-1)^k}{k^2} = (-1)^n \sigma_n.$$

Then

$$\sum_{n=1}^{N} (2n-1) \left(\sum_{k=0}^{\infty} \frac{(-1)^k}{(n+k)^2} \right)^2 = \sum_{n=1}^{N} \left(n^2 - (n-1)^2 \right) \sigma_n^2 =$$

$$= \sum_{n=1}^{N} n^2 \sigma_n^2 - \sum_{n=1}^{N-1} n^2 \sigma_{n+1}^2 = \sum_{n=1}^{N} n^2 \sigma_n^2 - \sum_{n=1}^{N} n^2 \sigma_{n+1}^2 + N^2 \sigma_{N+1}^2 =$$

$$= \sum_{n=1}^{N} n^2 \left(\sigma_n - \sigma_{n+1} \right) \left(\sigma_n + \sigma_{n+1} \right) + N^2 \sigma_{N+1}^2 = \sum_{n=1}^{N} (-1)^n \left(\sigma_n + \sigma_{n+1} \right) + N^2 \sigma_{N+1}^2 =$$

$$= \sum_{n=1}^{N} (-1)^n \sigma_n - \sum_{n=2}^{N+1} (-1)^n \sigma_n + N^2 \sigma_{N+1}^2 = -\sigma_1 + (-1)^N \sigma_{N+1} + N^2 \sigma_{N+1}^2.$$

We let N tend to infinity and find

$$S = \sum_{n=1}^{\infty} (2n-1) \left(\sum_{k=0}^{\infty} \frac{(-1)^k}{(n+k)^2} \right)^2 = -\sigma_1 = \sum_{k=1}^{\infty} \frac{1}{k^2} - 2 \sum_{k=1}^{\infty} \frac{1}{(2k)^2} = \frac{1}{2} (2) = \frac{\pi^2}{12}.$$

Also solved by the problem proposer.

• **5755** *Proposed by Paolo Perfetti, dipartimento di matematica Universitą di "Tor Vergata", Rome, Italy .*

Formally assuming that $(\sin 0)/0 = 1$, prove

$$\forall x \in \left[0, \pi/2\right]: \quad \frac{\sin x}{x} + \frac{(\sin x)^4}{x^4} \geqslant 2 \cos x.$$

Solution 1 by proposed by Henry Ricardo, Westchester Area Math Circle, Purchase, NY.

We will prove the generalization that for every $x \in [0, \pi/2]$ and $r \leqslant 5$

$$\frac{\sin x}{x} + \left(\frac{\sin x}{x} \right)^r \geqslant 2 \cos x.$$

Clearly this holds if $x = 0$ or $x = \pi/2$. It is sufficient to prove this result for $r = 5$.

First we apply the AGM inequality to see that

$$\frac{\sin x}{x} + \left(\frac{\sin x}{x}\right)^r \geqslant 2\left(\frac{\sin x}{x}\right)^{(r+1)/2},$$

and now we show that $(\sin x/x)^{(r+1)/2} \geqslant \cos x$. Let $f(x) = x - \sin x(\cos x)^{-2/(r+1)}$. Then

$$f'(x) = 1 - (\cos x)^{(r-1)/(r+1)} - \frac{2}{r+1}\sin^2 x(\cos x)^{-(r+3)/(r+1)},$$

$$f''(x) = \left(\frac{r-1}{r+1}\right)^2 \sin x(\cos x)^{-2\left(\frac{r+2}{r+1}\right)}\left[\cos^2 x - \frac{2(r+3)}{(r-1)^2}\right]$$

If $r = 5$ and $0 < x < \pi/2$, we have $f''(x) < 0$, $f'(x) < f'(0) = 0$, and $f(x) < f(0) = 0$, which proves our result.

Solution 2 by proposed by Michel Bataille, Rouen, France.

We first show the following lemma: $\tan x(\sin x)^2 \geqslant x^3$ whenever $0 \leqslant x < \pi/2$.
Let $f(x) = \tan x(\sin x)^2 - x^3$, $\quad g(x) = \tan x \sin x - x^2$. The derivatives of f and g are given by

$$f'(x) = 3\sin^2 x + (\sin x \tan x)^2 - 3x^2, \quad g'(x) = \sin x + \frac{\sin x}{\cos^2 x} - 2x.$$

Thus, $g(0) = 0$ and $g'(x) = \tan x\left(\cos x + \frac{1}{\cos x}\right) - 2x \geqslant 2(\tan x - x) \geqslant 0$, hence $g(x) \geqslant 0$.
It follows that

$$f'(x) \geqslant 3\sin^2 x + x^4 - 3x^2 = \frac{3}{2}\left(1 - \frac{(2x)^2}{2!} + \frac{(2x)^4}{4!} - \cos 2x\right) \geqslant 0$$

and therefore $f(x) \geqslant 0$ (since $f(0) = 0$).
Now, the required inequality clearly holds for $x = 0$ (by assumption) and for $x = \pi/2$; for $0 < x < \pi/2$, the lemma yields $(\tan x)^2(\sin x)^4 \geqslant x^6$, hence $\frac{(\sin x)^4}{x^4} \geqslant \frac{x^2}{(\tan x)^2}$ and therefore

$$\frac{\sin x}{x} + \frac{(\sin x)^4}{x^4} \geqslant \frac{\sin x}{x} + \frac{x^2}{(\tan x)^2} \geqslant 2\sqrt{\frac{\sin x}{x} \cdot \frac{x^2}{(\tan x)^2}} = 2\cos x\sqrt{\frac{x}{\sin x}} \geqslant 2\cos x$$

(using $\sin x \leqslant x$ for the last inequality).

Note. The lemma above also shows that $(\tan x)^r(\sin x)^s \geqslant x^{r+s}$ for $0 \leqslant x < \pi/2$ if $r > 0$ and $s \leqslant 2r$ (see problem 823, *The College Math. Journal*, Vol. 38, No 2, March 2007, p. 153).

Solution 3 by proposed by Moti Levy, Rehovot, Israel.

The famous Cusa - Huygens inequality is

$$\frac{\sin(x)}{x} \geqslant \frac{1+\cos(x)}{2}, \quad \forall x \in \left[0, \frac{\pi}{2}\right] \tag{19}$$

Bagul-Chesneau inequality is

$$\frac{\sin(x)}{x}\cos(x) + 2\frac{\sin(x)}{x} - 1 \geqslant 2\cos(x), \quad \forall x \in \left[0, \frac{\pi}{2}\right]. \tag{20}$$

see Yogesh J. Bagul, Christophe Chesneau, "*Two double sided inequalities involving sinc and hyperbolic sinc functions*", Int. J. Open Problems Compt. Math., Vol. 12, No. 4, December 2019.

By (20) it is enough to prove

$$\frac{\sin(x)}{x} + \left(\frac{\sin(x)}{x}\right)^4 \geqslant \left(\frac{\sin(x)}{x}\cos(x) + 2\frac{\sin(x)}{x} - 1\right),$$

or

$$\left(\frac{\sin(x)}{x}\right)^4 - \frac{\sin(x)}{x} + 1 \geqslant \frac{\sin(x)}{x}\cos(x). \tag{21}$$

By (19), we have

$$\cos(x) \leqslant 2\frac{\sin(x)}{x} - 1. \tag{22}$$

Plugging (22) into the right hand side of (21), we get

$$\left(2\frac{\sin(x)}{x} - 1\right)\frac{\sin(x)}{x} \geqslant \frac{\sin(x)}{x}\cos(x). \tag{23}$$

Hence it is enough to prove, the following inequality

$$\left(\frac{\sin(x)}{x}\right)^4 - \frac{\sin(x)}{x} + 1 - \left(2\frac{\sin(x)}{x} - 1\right)\frac{\sin(x)}{x} \geqslant 0. \tag{24}$$

Simplifying (24) we get

$$\begin{aligned}&\left(\frac{\sin(x)}{x}\right)^4 - \frac{\sin(x)}{x} + 1 - \left(2\frac{\sin(x)}{x} - 1\right)\frac{\sin(x)}{x} \\ &= \left(\frac{\sin(x)}{x}\right)^4 - 2\left(\frac{\sin(x)}{x}\right)^2 + 1 = \left(\left(\frac{\sin(x)}{x}\right)^2 - 1\right)^2 \geqslant 0.\end{aligned}$$

Solution 4 by proposed by Albert Stadler, Herrliberg, Switzerland.

We prove more precisely that

$$x \in \left[0, \frac{\pi}{2}\right]: \quad \frac{\sin x}{x} + \left(\frac{\sin x}{x}\right)^5 \geqslant 2\cos x,$$

and that if there is a nonnegative a such that $\frac{\sin x}{x}+\left(\frac{\sin x}{x}\right)^a \geqslant 2\cos x$ for all $x\in\left[0,\frac{\pi}{2}\right]$ then a⩽5.

The Taylor expansion of $\frac{\sin x}{x}+\left(\frac{\sin x}{x}\right)^a - 2\cos x$ around x=0 is

$$\frac{\sin x}{x}+\left(\frac{\sin x}{x}\right)^a - 2\cos x = \frac{5-a}{6}x^2+\frac{1}{360}\left(5a^2-2a-27\right)x^4+O\left(x^6\right)$$

so that necessarily a⩽5.

Suppose that a⩽5. Then, by the AM-GM inequality,

$$\frac{\sin x}{x}+\left(\frac{\sin x}{x}\right)^a \geqslant \frac{\sin x}{x}+\left(\frac{\sin x}{x}\right)^5 \geqslant 2\left(\frac{\sin x}{x}\right)^3.$$

It remains to prove that $\left(\frac{\sin x}{x}\right)^3 \geqslant \cos x$ for 0⩽x<π/2. (The inequality is trivially true for x=π/2.)

By the product representation of the sine and cosine function,

$$\frac{\sin x}{x}=\prod_{n=1}^{\infty}\left(1-\frac{x^2}{\pi^2 n^2}\right),$$

$$\cos x=\prod_{n=1}^{\infty}\left(1-\frac{4x^2}{\pi^2(2n-1)^2}\right).$$

We need to prove that

$$\prod_{n=1}^{\infty}\left(1-\frac{x^2}{\pi^2 n^2}\right)^3 \geqslant \prod_{n=1}^{\infty}\left(1-\frac{4x^2}{\pi^2(2n-1)^2}\right),\quad 0\leqslant x<\frac{\pi}{2}.$$

This inequality is equivalent to each of the following lines:

$$3\sum_{n=1}^{\infty}\log\left(1-\frac{x^2}{\pi^2 n^2}\right) \geqslant \sum_{n=1}^{\infty}\log\left(1-\frac{4x^2}{\pi^2(2n-1)^2}\right),$$

$$\sum_{k=1}^{\infty}\frac{1}{k}\sum_{n=1}^{\infty}\frac{4^k x^{2k}}{\pi^{2k}(2n-1)^{2k}} \geqslant 3\sum_{k=1}^{\infty}\frac{1}{k}\sum_{n=1}^{\infty}\frac{x^{2k}}{\pi^{2k}n^{2k}},$$

$$\sum_{k=1}^{\infty}\frac{1}{k}\frac{x^{2k}}{\pi^{2k}}\sum_{n=1}^{\infty}\left(\frac{4^k}{(2n-1)^{2k}}-\frac{3}{n^{2k}}\right)\geqslant 0,$$

$$\sum_{k=2}^{\infty}\frac{1}{k}\frac{x^{2k}}{\pi^{2k}}(2k)\left(4^k-4\right)\geqslant 0,$$

and the last inequality holds true. We have used that

$$\sum_{n=1}^{\infty}\frac{1}{(2n-1)^{2k}}=\sum_{n=1}^{\infty}\frac{1}{n^{2k}}-\sum_{n=1}^{\infty}\frac{1}{(2n)^{2k}}=(2k)\left(1-\frac{1}{2^{2k}}\right).$$

Also solved by the problem proposer.

• **5756** *Proposed by Toyesh Prakash Sharma (Undergraduate Student) Agra College, India.*

Calculate $T = \int_0^\infty \frac{dx}{x^2\left(\tan^2 x + \cot^2 x\right)}$.

Solution 1 by proposed by Albert Stadler, Herrliberg, Switzerland.

The function $x \to \tan^2 x + \cot^2 x$ is periodic with period $\pi/2$. Thus

$$T = \frac{1}{2}\int_{-\infty}^{\infty} \frac{dx}{x^2\left(\tan^2 x + \cot^2 x\right)} = \frac{1}{2}\int_0^{\frac{\pi}{2}} \left(\sum_{k=-\infty}^{\infty} \frac{1}{\left(x + k\frac{\pi}{2}\right)^2}\right) \frac{dx}{\left(\tan^2 x + \cot^2 x\right)}.$$

It is well known that

$$\frac{1}{\sin^2 x} = \sum_{k=-\infty}^{\infty} \frac{1}{(x + k\pi)^2},$$

and the series is absolutely and uniformly convergent in any compact set contained in $\mathbb{R}\backslash\mathbb{Z}$. The formula is easily derived from the partial fraction expansion of the cotangent function (https://proofwiki.org/wiki/Parti by termwise differentiation. So

$$\sum_{k=-\infty}^{\infty} \frac{1}{\left(x + k\frac{\pi}{2}\right)^2} = \frac{4}{\sin^2(2x)} = \frac{1}{\sin^2 x \cos^2 x}$$

and

$$T = \frac{1}{2}\int_0^{\frac{\pi}{2}} \frac{1}{\sin^2 x \cos^2 x} \frac{1}{\left(\tan^2 x + \cot^2 x\right)} dx.$$

We perform the change of variable $y = \tan x$, $dy = \frac{1}{\cos^2 x} dx$. Then

$$T = \frac{1}{2}\int_0^\infty \frac{1 + \frac{1}{y^2}}{\left(y^2 + \frac{1}{y^2}\right)} dy = \frac{1}{2}\int_0^\infty \frac{d\left(y - \frac{1}{y}\right)}{\left(y - \frac{1}{y}\right)^2 + 2} = \frac{1}{2}\int_{-\infty}^{\infty} \frac{dz}{z^2 + 2} = \frac{1}{2\sqrt{2}} \arctan\left(\frac{z}{\sqrt{2}}\right)\Bigg|_{z=-\infty}^{z=\infty} = \frac{\pi}{2\sqrt{2}}.$$

Solution 2 proposed by Yunyong Zhang, Chinaunicom, Yunnan, China.

According to Mittag-Leffler

$$\cot(x) = \frac{1}{x} + 2\sum_{k=1}^{\infty} \frac{x}{x^2 - (k\pi)^2}$$
$$= \frac{1}{x} + \sum_{k=-\infty}^{\infty} \frac{x}{x^2 - (k\pi)^2} - \frac{1}{x}$$
$$= \sum_{k=-\infty}^{\infty} \frac{x}{x^2 - (k\pi)^2}$$

Taking the derivative of the above equation

$$-\csc^2 x = \sum_{k=-\infty}^{\infty} -\frac{x^2 + \pi^2 k^2}{[x^2 - (k\pi)^2]^2}$$
$$\text{i.e.} \quad \csc^2 x = \sum_{k=-\infty}^{\infty} \frac{x^2 + \pi^2 k^2}{(x - k\pi)^2 (x + k\pi)^2}$$
$$= \frac{1}{2}\sum_{k=-\infty}^{\infty} \frac{1}{(x + k\pi)^2} + \frac{1}{(x - k\pi)^2} = \sum_{k=-\infty}^{\infty} \frac{1}{(x + k\pi)^2}$$
$$\therefore \quad T = \frac{1}{2}\int_{-\infty}^{\infty} \frac{\mathrm{d}x}{x^2(\tan^2 x + \cot^2 x)}$$
$$= \frac{1}{2}\sum_{k=-\infty}^{\infty} \int_{k\pi}^{(k+1)\pi} \frac{\mathrm{d}x}{x^2(\tan^2 x + \cot^2 x)}$$
$$= \frac{1}{2}\sum_{k=-\infty}^{\infty} \int_0^{\pi} \frac{1}{(x + k\pi)^2} \frac{\mathrm{d}x}{\tan^2 x + \cot^2 x}$$
$$= \frac{1}{2}\int_0^{\pi} \frac{1}{\sin^2 x} \frac{\mathrm{d}x}{\tan^2 x + \cot^2 x}$$
$$= \frac{1}{2}\int_0^{\pi} \frac{\cos^2 x \mathrm{d}x}{\sin^4 x + \cos^4 x}$$
$$= \frac{\sqrt{2}}{4}\pi.$$

Solution 3 proposed by Prakash Pant, Mathematics Initiatives in Nepal, Bardiya, Nepal.

$$\text{Using } \tan(x) = \frac{\sin(x)}{\cos(x)} \text{ and } \cot(x) = \frac{\cos(x)}{\sin(x)}, \text{ we get}$$
$$= \int_0^{\infty} \frac{\sin^2(x)\cos^2(x)}{x^2} . \frac{dx}{(\sin^4(x) + \cos^4(x))}$$
$$\text{Using } \sin(2x) = 2\sin(x)\cos(x) \text{ and } a^2 + b^2 = (a+b)^2 - 2ab, \text{ we get}$$
$$= \int_0^{\infty} \frac{\sin^2(2x)}{(2x)^2} . \frac{dx}{(\sin^2(x) + \cos^2(x))^2 - 2\sin^2(x)\cos^2(x)}$$
$$\text{Using } \sin^2(x) + \cos^2(x) = 1 \text{ and } \sin(2x) = 2\sin(x)\cos(x), weget$$

$$= \int_0^\infty \frac{\sin^2(2x)}{(2x)^2} . \frac{dx}{\left(1 - \frac{1}{2}\sin^2(2x)\right)}$$

Now, we make a u-substitution such that u = 2x . This implies $\frac{du}{2} = dx$. The integral now goes from 0 to infinity.

$$= \int_0^\infty \frac{\sin^2(u)}{u^2} . \frac{du}{\left(2 - \sin^2(u)\right)}$$

Using $\sin^2(x) + \cos^2(x) = 1$, we get

$$= \int_0^\infty \frac{\sin^2(u)}{u^2} . \frac{du}{\left(1 + \cos^2(u)\right)}$$

Lobachevsky's Formula states that if $0 \leqslant u < \infty$, f(u)=f(-u) and f(u+π k) = f(u), then

$$\int_0^\infty \frac{\sin^2(u)}{u^2} f(u) du = \int_0^{\frac{\pi}{2}} f(u) du$$

Here, $f(u) = \frac{1}{1 + \cos^2(u)}$ satisfies the conditions of Lobachevsky's Formula. Therefore,

$$\int_0^\infty \frac{\sin^2(u)}{u^2} . \frac{du}{\left(1 + \cos^2(u)\right)} = \int_0^{\frac{\pi}{2}} \frac{du}{1 + \cos^2(u)}$$

Multiplying numerator and denominator by $\sec^2(u)$, we get

$$= \int_0^{\frac{\pi}{2}} \frac{\sec^2(u) du}{\sec^2(u) + 1}$$

Using $\sec^2(u) - \tan^2(u) = 1$, we get

$$= \int_0^{\frac{\pi}{2}} \frac{\sec^2(u) du}{2 + \tan^2(u)}$$

Now, we make a y-substitution such that y = tan(u). This implies dy= $\sec^2(u)$ du . The integral now goes from 0 to infinity.

$$= \int_0^\infty \frac{dy}{2 + y^2}$$

We know, $\int \frac{1}{a^2 + x^2} dx = \frac{1}{a} arctan\left(\frac{x}{a}\right) + c$

Thus,

$$\int_0^\infty \frac{dy}{2 + y^2} = \frac{1}{\sqrt{2}} arctan\left(\frac{x}{\sqrt{2}}\right) \Big|_0^\infty$$

$$= \frac{1}{\sqrt{2}}\left(\frac{\pi}{2} - 0\right) = \frac{\pi}{2\sqrt{2}}$$

Hence, $T = \int_0^\infty \frac{dx}{x^2(\tan^2(x) + \cot^2(x))} = \frac{\pi}{2\sqrt{2}}$.

Solution 4 proposed by Moti Levy, Rehovot, Israel.

Using the following trigonometric identity,

$$\frac{1}{\left(\tan^2(x) + \cot^2(x)\right)} = \frac{1 - \cos(4x)}{2\left(\cos(4x) + 3\right)},$$

$$T = \frac{2}{3} \int_0^\infty \frac{1}{x^2} \frac{1 - \cos(x)}{1 + \frac{1}{3}\cos(x)} dx.$$

Let us define

$$\Upsilon(p) := \int_0^\infty \frac{1}{x^2} \frac{1 - \cos(x)}{1 + p\cos(x)} dx, \quad |p| < 1 \tag{25}$$

and

$$I_k := \int_0^\infty \frac{1 - \cos(x)}{x^2} \cos^k(x)\, dx \tag{26}$$

$$J_k := \int_0^\infty \frac{\sin(x)\cos^k(x)}{x} dx \tag{27}$$

1) Evaluation of J_k :

The following trigonometric identity is well known:

$$\cos^k(x) = \frac{1}{2^k} \sum_{m=0}^{k} \binom{k}{m} \cos\left((k - 2m)x\right) \tag{28}$$

$$J_k := \int_0^\infty \frac{\sin(x)\cos^k(x)}{x} dx = \int_0^\infty \frac{\sin(x)}{x} \frac{1}{2^k} \sum_{m=0}^{k} \binom{k}{m} \cos\left((k - 2m)x\right) dx \tag{29}$$

$$= \frac{1}{2^k} \sum_{m=0}^{k} \binom{k}{m} \int_0^\infty \cos\left((k - 2m)x\right) \frac{\sin(x)}{x} dx. \tag{30}$$

Now we evaluate the definite integral $\int_0^\infty \cos(\alpha x) \frac{\sin(x)}{x} dx$:

Let $f(s)$ be defined as follows, α is real,

$$f(s) := \int_0^\infty \cos(\alpha x) \frac{\sin(x)}{x} e^{-sx} dx.$$

$$\begin{aligned}
\frac{df}{ds} &= \int_0^\infty \cos(\alpha x) \sin(x)\, e^{-sx} dx \\
&= -\frac{1}{2} \int_0^\infty \sin\left((\alpha + 1)x\right) e^{-sx} dx + \frac{1}{2} \int_0^\infty \sin\left((\alpha - 1)x\right) e^{-sx} dx \\
&= -\frac{1}{2} \frac{\alpha + 1}{(\alpha + 1)^2 + s^2} + \frac{1}{2} \frac{\alpha - 1}{(\alpha - 1)^2 + s^2}.
\end{aligned}$$

$$\begin{aligned} f(s) &= \int_0^s \left(-\frac{1}{2}\frac{\alpha+1}{(\alpha+1)^2+t^2} + \frac{1}{2}\frac{\alpha-1}{(\alpha-1)^2+t^2} \right) dt + A \\ &= \begin{cases} -\frac{1}{2}\arctan\left(\frac{s}{\alpha+1}\right) + \frac{1}{2}\arctan\left(\frac{s}{\alpha-1}\right) + A & \alpha \neq 1 \\ -\frac{1}{2}\arctan\left(\frac{s}{\alpha+1}\right) + A & \alpha = 1 \end{cases} \end{aligned}$$

Since $\lim f(s)_{s\to\infty} = 0$ then

$$A = \begin{cases} \lim\limits_{s\to\infty}\left(\frac{1}{2}\arctan\left(\frac{s}{\alpha+1}\right) - \frac{1}{2}\arctan\left(\frac{s}{\alpha-1}\right)\right) & \alpha \neq 1 \\ \lim\limits_{s\to\infty}\frac{1}{2}\arctan\left(\frac{s}{\alpha+1}\right) & \alpha = 1 \end{cases} = \begin{cases} \frac{\pi}{4}\left(\frac{\alpha+1}{\sqrt{(1+a)^2}} - \frac{\alpha-1}{\sqrt{(\alpha-1)^2}}\right) & \alpha \neq \\ \frac{\pi}{4} \quad \alpha = 1 \end{cases}$$

Since $\lim f(s)_{s\to 0} = \int_0^\infty \cos(\alpha x)\frac{\sin(x)}{x}dx = A$, hence

$$\begin{aligned} \int_0^\infty \cos(\alpha x)\frac{\sin(x)}{x}dx &= \begin{cases} \frac{\pi}{4}\left(\frac{\alpha+1}{\sqrt{(1+a)^2}} - \frac{\alpha-1}{\sqrt{(\alpha-1)^2}}\right) & \alpha \neq 1 \\ \frac{\pi}{4} \quad \alpha = 1 \end{cases} \\ &= \begin{cases} \frac{\pi}{2} & if \quad 0 \leqslant \alpha < 1 \\ \frac{\pi}{4} & if \quad \alpha = 1 \\ 0 & if \quad \alpha > 1 \end{cases} \end{aligned} \tag{31}$$

Plugging (31) into (30) gives

$$J_k = \frac{\pi}{2^{k+1}}\binom{k}{\left\lfloor \frac{k}{2} \right\rfloor}. \qquad \blacksquare \tag{32}$$

2) Evaluation of I_k :

$$\begin{aligned} I_k &:= \int_0^\infty \frac{1-\cos(x)}{x^2}\cos^k(x)\,dx = \int_0^\infty \frac{1}{x}\left(-k\cos^{k-1}(x) + (k+1)\cos^k(x)\right)\sin(x)\,dx \\ &= (k+1)J_k - kJ_{k-1} \end{aligned} \tag{33}$$

Applying (32) to (33) gives

$$\begin{aligned} I_{2m} &= \frac{\pi}{2^{2m+1}}\binom{2m}{m}, \\ I_{2m+1} &= 0. \qquad \blacksquare \end{aligned} \tag{34}$$

3) Evaluation of $\Upsilon(p)$:

We use (34) and $\sum_{m=0}^{\infty}\binom{2m}{m}z^m = \frac{1}{\sqrt{1-4z}}$ to obtain

$$\Upsilon(p) := \int_0^\infty \frac{1}{x^2}\frac{1-\cos(x)}{1+p\cos(x)}dx = \int_0^\infty \frac{1-\cos(x)}{x^2}\sum_{k=0}^{\infty}(-1)^k p^k \cos^k(x)\,dx$$
$$= \sum_{k=0}^{\infty}(-1)^k p^k I_k = \sum_{m=0}^{\infty} p^{2m} I_{2m} = \frac{\pi}{2}\sum_{m=0}^{\infty}\binom{2m}{m}\left(\frac{p^2}{4}\right)^m = \frac{1}{\sqrt{1-p^2}}\frac{\pi}{2}$$

We conclude that

$$T = \frac{2}{3}\Upsilon\left(\frac{1}{3}\right) = \frac{\sqrt{2}\,\pi}{4} \cong 1.1107.$$

Solution 5 proposed by Michel Bataille, Rouen, France.

Let $I_k = \int_{k\pi/2}^{(k+1)\pi/2} \frac{dx}{x^2(\tan^2 x + \cot^2 x)}$ where k is a non-negative integer. Since the change of variables $x = u + \frac{k\pi}{2}$ gives $I_k = \int_0^{\pi/2} \frac{du}{(u+(k\pi/2))^2(\cot^2 u + \tan^2 u)}$, we obtain

$$T = \sum_{k=0}^{\infty} I_k = \sum_{k=0}^{\infty}\int_0^{\pi/2} \frac{du}{(u+(k\pi/2))^2(\cot^2 u + \tan^2 u)} = 4\int_0^{\pi/2} f(u)\cdot\frac{du}{\tan^2 u + \cot^2 u} \qquad (1)$$

where $f(u) = \sum_{k=0}^{\infty}\frac{1}{(2u+k\pi)^2}$.

From

$$\int_0^{\pi/2} f(u)\cdot\frac{du}{\tan^2 u + \cot^2 u} = \int_0^{\pi/2} f((\pi/2)-v)\cdot\frac{dv}{\tan^2 v + \cot^2 v}$$

we deduce that

$$T = 2\int_0^{\pi/2}(f(u)+f((\pi/2)-u))\frac{du}{\tan^2 u + \cot^2 u}$$
$$= 2\int_0^{\pi/2}\left(\sum_{k=0}^{\infty}\frac{1}{(2u+k\pi)^2} + \sum_{k=1}^{\infty}\frac{1}{(2u-k\pi)^2}\right)\frac{du}{\tan^2 u + \cot^2 u}$$
$$= 2\int_0^{\pi/2}\frac{1}{\sin^2 2u}\cdot\frac{du}{\tan^2 u + \cot^2 u} = \frac{1}{2}\int_0^{\pi/2}\frac{du}{\sin^4 u + \cos^4 u}$$
$$= \frac{1}{2}\int_0^{\infty}\frac{1+t^2}{1+t^4}dt = \frac{1}{4}\int_{-\infty}^{\infty}\frac{1+t^2}{1+t^4}dt$$

where we have used the well-known equalities $\frac{1}{\sin^2 z} = \sum_{k=-\infty}^{\infty}\frac{1}{(z-k\pi)^2} = \sum_{k=1}^{\infty}\frac{1}{(z-k\pi)^2} + \sum_{k=0}^{\infty}\frac{1}{(z+k\pi)^2}$ for $z \in \mathbb{C} - \pi\mathbb{Z}$ and the change of variables $u = \arctan t$.

The calculation of the last integral is classical:

$$\int_{-\infty}^{\infty} \frac{1+t^2}{1+t^4}\,dt = 2\pi i(r_1 + r_2)$$

where r_1 and r_2 are the residues of $\dfrac{1+z^2}{1+z^4}$ at $e^{i\pi/4}$ and $e^{3\pi i/4}$, respectively. We easily find

$$r_1 = \left[\frac{1+z^2}{4z^3}\right]_{z=e^{i\pi/4}} = \frac{-i}{2\sqrt{2}}, \qquad r_2 = \frac{-i}{2\sqrt{2}},$$

hence

$$\int_{-\infty}^{\infty} \frac{1+t^2}{1+t^4}\,dt = \pi\sqrt{2}.$$

It follows that $T = \dfrac{\pi\sqrt{2}}{4}$.

Also solved by Bruno Salgueiro Fanego, Viveiro, Lugo; Perfetti Paolo, dipartimento di matematica Universitą di "Tor Vergata," Roma, Italy; and the problem proposer.

Editor's Statement: It goes without saying that the problem proposers, as well as the solution proposers, are the *élan vital* of the Problems/Solutions Section of SSMJ. As the editor of this Section of the Journal, I consider myself fortunate to be in a position to receive, compile and organize a wealth of proposed ingenious problems and solutions intended for online publication. My unwavering gratitude goes to all the amazingly creative contributors. We come together from across continents because we find intellectual value, joy and satisfaction in mathematical problems, both in their creation as well as their solution. So that our collective efforts serve us well, I kindly ask all contributors to adhere to the following guidelines. As you peruse below, you may construe that the guidelines amount to a lot of work. But, as the samples show, there's not much to do. Your cooperation is much appreciated!

Keep in mind that the examples given below are your best guide!

Formats, Styles and Recommendations

When submitting proposed problem(s) or solution(s), please send both **LaTeX** document and **pdf** document of your proposed problem(s) or solution(s). There are ways (discoverable from the internet) to convert from Word to proper LaTeX code. Porposals without a *proper* **LaTeX** document will not be published regrettably.

Regarding Proposed Solutions:

Below is the FILENAME format for all the documents of your proposed solution(s).

#ProblemNumber_FirstName_LastName_Solution_SSMJ

- FirstName stands for YOUR first name.
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Examples:

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Please note that every problem number is *preceded* by the sign # .

All you have to do is copy the FILENAME format (or an example below it), paste it and then modify portions of it to your specs.

Please adopt the following structure, in the order shown, for the presentation of your solution:

1. On top of the first page of your solution, begin with the phrase:

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2. On the second line, write

"Solution proposed by [your First Name, your Last Name]",

followed by your affiliation, city, country, all on the same linear string of words. Please see the example below. Make sure you do the same for your collaborator(s).

3. On a new line, state the problem proposer's name, affiliation, city and country, just as it appears published in the Problems/Solutions section.

4. On a new line below the above, write in bold type: "**Statement of the Problem**".

5. Below the latter, state the problem. Please make sure the statement of your problem (unlike the preceding item) is not in bold type.

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7. Below the latter, show the entire solution of the problem.

Here is a sample for the above-stated format for proposed solutions:

Proposed solution to #1234 SSMJ

Solution proposed by Emmy Noether, University of Göttingen, Lower Saxony, Germany.

Problem proposed by Isaac Newton, Trinity College, Cambridge, England.

Statement of the problem:

Compute $\displaystyle\sum_{k=0}^{n}\binom{n}{k}x^k y^{n-k}$.

Solution of the problem: . . . . . .

Regarding Proposed Problems:

For all your proposed problems, please adopt for all documents the following FILENAME format:

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If you do not have a ProblemTitle, then leave that component as it already is (i.e., ProblemTitle).

The component YourGivenNumber is any UNIQUE 3-digit (or longer) number you like to give to your problem.

Examples:

Max_Planck_ProposedProblem_SSMJ_314_HarmonicPatterns

Charles_Darwin_ProposedProblem_SSMJ_358_ProblemTitle

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"Problem proposed to SSMJ"

2. On the second line, write

"Problem proposed by [your First Name, your Last Name]", followed by your affiliation, city, country all on the same linear string of words. Please see the example below. Make sure you do the same for your collaborator(s) if any.

3. On a new line state the title of the problem, if any.

4. On a new line below the above, write in bold type: "**Statement of the Problem**".

5. Below the latter, state the problem. Please make sure the statement of your problem (unlike the preceding item) is not in bold type.

6. Below the statement of the problem, write in bold type: "**Solution of the Problem**".

7. Below the latter, show the entire solution of your problem.

Here is a sample for the above-stated format for proposed problems:

Problem proposed to SSMJ

Problem proposed by Isaac Newton, Trinity College, Cambridge, England.

Principia Mathematica ($\longleftarrow$ You may choose to not include a title.)

Statement of the problem:

Compute $\displaystyle\sum_{k=0}^{n}\binom{n}{k}x^k y^{n-k}$.

Solution of the problem: . . . . . .

♣ ♣ ♣ **Thank You!** ♣ ♣ ♣