Problems and Solutions

Albert Natian, Section Editor

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This section of the Journal offers readers an opportunity to exchange interesting mathematical problems and solutions. Please email them to Prof. Albert Natian at Department of Mathematics, Los Angeles Valley College. Please make sure every proposed problem or proposed solution is provided in both ***LaTeX*** and pdf documents. Thank you!

To propose problems, email them to: problems4ssma@gmail.com

To propose solutions, email them to: solutions4ssma@gmail.com

Solutions to previously published problems can be seen at <www.ssma.org/publications>.

Solutions to the problems published in this issue should be submitted *before* February 1, 2025.

• **5787** *Proposed by Albert Stadler, Herrliberg, Switzerland.*

Let n be a natural number. Prove

$$\sum_{k=1}^{n} \frac{(-1)^k}{k} \binom{2n}{n+k} = \binom{2n}{n} (H_n - H_{2n}),$$

where $H_k = \sum_{j=1}^{k} \frac{1}{j}$ denotes the k^{th} harmonic number.

• **5788** *Proposed by Ángel Plaza, Universidad de Las Palmas de Gran Canaria, Spain.*

Evaluate $L := \lim_{n\to\infty} \left(n^2\, x_n\right)$ where the sequence $(x_n)_{n\geqslant 2}$ is defined by

$$x_n = n\left(\frac{\sin\frac{1}{n}}{\sin\frac{1}{n^2}} - n\right) + \frac{1}{6}.$$

• **5789** *Proposed by Daniel Sitaru, National Economic College "Theodor Costescu" Drobeta Turnu - Severin, Romania.*

Let $0 < a \leqslant b$. Suppose $f : [a,b] \to (0,\infty)$ is a continuous function. Then

$$\int_a^b\int_a^b\int_a^b \left(f^2(x)+f^2(y)+f^2(z)\right)^2 dxdydz \geqslant 9(b-a)\left(\int_a^b f(x)dx\right)\left(\int_a^b f^3(x)dx\right).$$

• **5790** *Proposed by Michel Bataille, Rouen, France.*

Here $\llbracket \cdot \rrbracket$ denotes the floor (greatest integer value) function. Solve for $n \in \mathbb{N}$:

$$\sum_{k=1}^{n} \left\llbracket \sqrt{n^2 + 2k - 1} + \sqrt{n^2 + 2k} \right\rrbracket = 205.$$

• **5791** *Proposed by Vasile Cirtoaje, Petroleum-Gas University of Ploiesti, Romania.*

Given any non-negative real numbers $a_1, a_2, \ldots, a_9$ with $a_1 \geqslant a_2 \geqslant \cdots \geqslant a_9$, prove that 5 is the least positive value of k for which

$$\left(\frac{ka_1 + a_2 + \cdots + a_8}{k + 7} \right)^2 \geqslant \frac{a_1^2 + a_2^2 + \cdots + a_9^2}{9}.$$

• **5792** *Proposed by Shivam Sharma, Delhi University, New Delhi, India.*

Let $s_n = -2\sqrt{n} + \sum_{k=1}^{n} \frac{1}{\sqrt{k}}$ with $\lim_{n\to\infty} s_n = s$ (Ioachimescu Constant). For non-negative intger m, evaluate:

$$L = \lim_{n\to\infty} \left(s^{m+1} - \prod_{j=n}^{n+m} s_j \right) \sqrt{n}.$$

Solutions

To Formerly Published Problems

• **5763** *Proposed by Rafael Jakimczuk, Departamento de Ciencias Básicas, División Matemática, Universidad Nacional de Luján, Buenos Aires, Argentina..*

Consider the Diophantine equation in the unknown positive integer n:

$$b_1 (n + a_1)^{dn} + b_2 (n + a_2)^{dn} + \cdots + b_k (n + a_k)^{dn} = c$$

where k, d, c and a_i, b_i $(i = 1, 2, \ldots, k)$ are given fixed integers with $k \geqslant 2$, $d > 0$, $b_i \neq 0$ and $a_1 < a_2 < \cdots < a_k$. Prove that the solution set (possibly empty) of the above Diophantine equation is finite.

Solution 1 by the problem proposer. We will show that the Diophantine equation

$$b_1 (n + a_1)^{dn} + b_2 (n + a_2)^{dn} + \cdots + b_k (n + a_k)^{dn} = c, \tag{1}$$

either has no solution in n or has only a finite number of solutions in n.

Let us consider the equality

$$b_1 e^{da_1} + b_2 e^{da_2} + \cdots + b_k e^{da_k} = 0. \tag{2}$$

We shall prove that the above equality is impossible. First, suppose that $a_1 \geqslant 0$. Then e is a solution of the polynomial equation $b_1 x^{da_1} + b_2 x^{da_2} + \cdots + b_k x^{da_k} = 0$ with integer coefficients b_i, which is an impossibility since e is not algebraic, but transcendental (Hermite in 1873). Second, suppose that $a_1 < 0$. Upon multiplying both sides of (2) by e^{-da_1}, we see that again e (impossibly) satisfies a polynomial equation with integer coefficients. Therefore we conclude

$$b_1 e^{da_1} + b_2 e^{da_2} + \cdots + b_k e^{da_k} \neq 0. \tag{3}$$

Now, equation (1) is equivalent to the equation

$$A_n := \frac{b_1 (n + a_1)^{dn} + b_2 (n + a_2)^{dn} + \cdots + b_k (n + a_k)^{dn} - c}{n^{dn}} = 0$$

or

$$A_n = b_1 \left(\left(1 + \frac{a_1}{n} \right)^n \right)^d + b_2 \left(\left(1 + \frac{a_2}{n} \right)^n \right)^d + \cdots + b_k \left(\left(1 + \frac{a_k}{n} \right)^n \right)^d - \frac{c}{n^{dn}} = 0.$$

Since for $\alpha > 0$

$$\lim_{n\to\infty} \left(1 + \frac{\alpha}{n} \right)^n = e^{\alpha},$$

then, by (3),

$$\lim_{n\to\infty} A_n = b_1 e^{da_1} + b_2 e^{da_2} + \cdots + b_k e^{da_k} \neq 0.$$

Therefore, there exists n_0 such that for all $n \geqslant n_0$: $A_n \neq 0$ and thus (1) does not hold for $n \geqslant n_0$. Hence, (1) holds for at most a finite number of values of n.

• **5764** *Proposed by D.M. Bătineţu-Giurgiu, "Matei Basarab" National College, Bucharest, Romania and Neculai Stanciu, "George Emil Palade" School, Buzău, Romania.*

Let F_n denote the nth term of the Fibonacci sequence defined by $F_0 = 0$, $F_1 = 1$, $F_{n+2} = F_{n+1} + F_n$ for $n \geqslant 0$. Let $(a_n)_{n\geqslant 1}$ be any sequence of positive real numbers with $\lim_{n\to\infty} \frac{a_{n+1}}{n^2 a_n} = 2\pi$. Evaluate :

$$\Lambda = \lim_{n\to\infty} \left(\sqrt[n+1]{\frac{a_{n+1} F_{n+1}}{(2n+1)!!}} - \sqrt[n]{\frac{a_n F_n}{(2n-1)!!}} \right).$$

Solution 1 by Ángel Plaza, Universidad de Las Palmas de Gran Canaria, Spain.

By the Stolz-Cezaro Lemma, the Stirling approximation to $n!$ and the root-quotient criterion at calculating limits, it follows that

$$\begin{aligned}
\Lambda &= \lim_{n\to\infty} \frac{\sqrt[n+1]{\dfrac{a_{n+1}F_{n+1}}{(2n+1)!!}}}{n+1} \\
&= \lim_{n\to\infty} \sqrt[n+1]{\frac{a_{n+1}F_{n+1}2^{n+1}(n+1)!}{(2n+2)!(n+1)^{n+1}}} \\
&= \lim_{n\to\infty} \frac{a_{n+1}}{a_n}\,\frac{F_{n+1}}{F_n}\,\frac{2^{n+1}}{2^n}\,\frac{(n+1)!}{n!}\,\frac{(2n)!}{(2n+2)!}\,\frac{n^n}{(n+1)^{n+1}} \\
&= \lim_{n\to\infty} \frac{a_{n+1}}{n^2 a_n}\,\alpha\,2\,(n+1)\,\frac{n^2}{2(n+1)(2n+1)}\,\left(\frac{n}{n+1}\right)^n\,\frac{1}{n+1} \\
&= 2\pi\alpha\frac{1}{2}\frac{1}{e} \\
&= \frac{\alpha\pi}{e}
\end{aligned}$$

where $\alpha = \dfrac{1+\sqrt{5}}{2}$ is the golden ratio.

Solution 2 by G. C. Greubel, Newport News, VA.

Given the property

$$\lim_{n\to\infty} \frac{a_{n+1}}{n^2\,a_n} = 2\pi$$

then a reasonable guess to the form of a_n is $(p\,n)^{2n}$. In this view the limit takes the form

$$\lim_{n\to\infty} \frac{a_{n+1}}{n^2\,a_n} = p^2\,\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^{n+2} = (p\,e)^2 = 2\pi.$$

This yields

$$a_n \approx \left(\frac{\sqrt{2\pi}\,n}{e}\right)^{2n}.$$

Using

$$\begin{aligned}
F_n &\approx 5^{-1/2}\,\alpha^n \\
(2n+1)!! &\approx e^{-(n+1)}\,n^{n+1}\,2^{n+3/2} \\
(2n-1)!! &\approx e^{-n}\,n^n\,2^{n+1/2},
\end{aligned}$$

where *alpha* is the golden ratio, then

$$\begin{aligned}
\sqrt[n+1]{\frac{a_{n+1}\,F_{n+1}}{(2n+1)!!}} &\approx \frac{\pi\,\alpha}{e}\,\left(n+2+\frac{1}{n}\right) \\
\sqrt[n+1]{\frac{a_n\,F_n}{(2n-1)!!}} &\approx \frac{\pi\,\alpha}{e}\,n
\end{aligned}$$

and

$$\Lambda = \lim_{n\to\infty}\left(\sqrt[n+1]{\frac{a_{n+1}\,F_{n+1}}{(2n+1)!!}} - \sqrt[n]{\frac{a_n\,F_n}{(2n-1)!!}}\right) = \frac{\pi\,\alpha}{e}\lim_{n\to\infty}\left(2+\frac{1}{n}\right) = \frac{2\,\pi\,\alpha}{e}.$$

Solution 3 by Michel Bataille, Rouen, France.

It is well-known that for $n \geqslant 0$, we have $F_n = \dfrac{\alpha^n - \beta^n}{\sqrt{5}}$ where $\alpha = \dfrac{1+\sqrt{5}}{2}$, $\beta = \dfrac{1-\sqrt{5}}{2}$. It follows that $F_n \sim \dfrac{\alpha^n}{\sqrt{5}}$ as $n \to \infty$ so that $\lim\limits_{n\to\infty}\dfrac{F_{n+1}}{F_n} = \alpha$. Let $u_n = \dfrac{a_nF_n}{(2n-1)!!}$ and $v_n = \sqrt[n]{u_n}$. We will show that the required limit is $\lim\limits_{n\to\infty}(v_{n+1}-v_n) = \dfrac{\pi\alpha}{e}$.

First we will prove that $\dfrac{u_{n+1}}{u_n} \sim n\pi\alpha$ and $v_n \sim \dfrac{n\pi\alpha}{e}$ as $n\to\infty$. As $n\to\infty$, we have

$$\frac{u_{n+1}}{u_n} = \frac{a_{n+1}}{a_n}\cdot\frac{F_{n+1}}{F_n}\cdot\frac{1}{2n+1} \sim (2\pi n^2)\cdot\alpha\cdot\frac{1}{2n} = n\pi\alpha$$

and then

$$\frac{u_{n+1}}{(n+1)^{n+1}}\frac{n^n}{u_n} = \frac{u_{n+1}}{u_n}\cdot\frac{1}{n+1}\cdot\frac{1}{\left(1+\frac{1}{n}\right)^n} \sim n\pi\alpha\cdot\frac{1}{n}\cdot\frac{1}{e} = \frac{\pi\alpha}{e}.$$

From the known result: $\lim\limits_{n\to\infty}\dfrac{x_{n+1}}{x_n} = \ell$ implies $\lim\limits_{n\to\infty}\sqrt[n]{x_n} = \ell$, which holds for any positive real sequence (x_n), we deduce that

$$\frac{\pi\alpha}{e} = \lim_{n\to\infty}\sqrt[n]{\frac{u_n}{n^n}} = \lim_{n\to\infty}\frac{v_n}{n}$$

and so $v_n \sim \dfrac{n\pi\alpha}{e}$ as $n\to\infty$. As a result, we first obtain

$$\left(\frac{v_{n+1}}{v_n}\right)^n = \frac{u_{n+1}^{1-\frac{1}{n+1}}}{u_n} = \frac{u_{n+1}}{u_n}\cdot\frac{1}{v_{n+1}} \sim (n\pi\alpha)\cdot\frac{1}{\frac{n\pi\alpha}{e}} = e$$

as $n\to\infty$ so that $\lim\limits_{n\to\infty} n\ln\left(\dfrac{v_{n+1}}{v_n}\right) = 1$. Now, from $\ln\left(\dfrac{v_{n+1}}{v_n}\right) \sim \dfrac{1}{n}$ and $\lim\limits_{n\to\infty}\dfrac{v_{n+1}}{v_n} = 1$ we deduce

$$v_{n+1}-v_n = v_n\left(\frac{v_{n+1}}{v_n}-1\right) = v_n\cdot\ln\left(\frac{v_{n+1}}{v_n}\right)\cdot\frac{\frac{v_{n+1}}{v_n}-1}{\ln\left(\frac{v_{n+1}}{v_n}\right)} \sim \frac{n\pi\alpha}{e}\cdot\frac{1}{n}\cdot 1 = \frac{\pi\alpha}{e}$$

as $n\to\infty$ (since $\lim\limits_{t\to 1}\dfrac{t-1}{\ln t} = 1$). Thus, $\lim\limits_{n\to\infty}(v_{n+1}-v_n) = \dfrac{\pi\alpha}{e}$ and we are done.

Solution 4 by Moti Levy, Rehovot, Israel.

We will show the following asymptotic approximation:

$$\sqrt[n]{\frac{a_nF_n}{(2n-1)!!}} = \pi\varphi\frac{n}{e}\left(1+\frac{\ln\left(2\pi^2\right)}{2n}+\frac{\ln n}{n}\right)+O\left(\frac{\ln^2(n)}{n}\right),\ \ \varphi=\frac{1+\sqrt{5}}{2}. \tag{4}$$

Suppose the sequence $(a_n)_{n\geqslant 1}$ is expressed as follows, where $(k_n)_{n\geqslant 1}$ is a sequence of positive real numbers:

$$a_n = k_n(2\pi)^n(n!)^2. \tag{5}$$

Then the condition $\lim\limits_{n\to\infty}\frac{a_{n+1}}{n^2a_n}=2\pi$ implies that

$$2\pi=\lim_{n\to\infty}\frac{k_{n+1}(2\pi)^{n+1}\left((n+1)!\right)^2}{n^2k_n^n(2\pi)(n!)^2}=\lim_{n\to\infty}\frac{k_{n+1}(n+1)^2}{k_nn^2}2\pi=2\pi\lim_{n\to\infty}\frac{k_{n+1}}{k_n}.$$

It follows that $a_n=k_n(2\pi)^n(n!)^2$ if and only if

$$\lim_{n\to\infty}\frac{k_{n+1}}{k_n}=1. \tag{6}$$

By definition,

$$(2n-1)!!=\frac{(2n)!}{2^nn!}. \tag{7}$$

$$\frac{a_nF_n}{(2n-1)!!}=\frac{k_n(2\pi)^n(n!)^2\,2^nn!}{(2n)!}=(2\pi)^nF_nk_n\frac{2^n(n!)^3}{(2n)!},$$

$$\sqrt[n]{\frac{a_nF_n}{(2n-1)!!}}=4\pi F_n^{\frac{1}{n}}k_n^{\frac{1}{n}}\frac{(n!)^{\frac{3}{n}}}{\left((2n)!\right)^{\frac{1}{n}}}. \tag{8}$$

The closed form of the Fibonacci numbers is

$$F_n=\frac{\varphi^n-(1-\varphi)^n}{\sqrt{5}},\quad \varphi=\frac{1+\sqrt{5}}{2}.$$

Since $\left|\frac{1-\varphi}{\varphi}\right|<1$ then

$$F_n^{\frac{1}{n}}=\frac{\varphi}{\left(\sqrt{5}\right)^{\frac{1}{n}}}\left(1-\left(\frac{1-\varphi}{\varphi}\right)^n\right)^{\frac{1}{n}}=\varphi\frac{1}{\left(\sqrt{5}\right)^{\frac{1}{n}}}\left(1-\left(\frac{1-\frac{1+\sqrt{5}}{2}}{\frac{1+\sqrt{5}}{2}}\right)^n\right)^{\frac{1}{n}}\implies\lim_{n\to\infty}F_n^{\frac{1}{n}}=\varphi. \tag{9}$$

Lemma: Let $(b_n)_{n\geqslant 0}$ be a sequence of positive real number such that $\lim\limits_{n\to\infty}\frac{b_{n+1}}{b_n}=1$ then $\lim\limits_{n\to\infty}\sqrt[n]{b_n}=1$.

Proof of lemma: $\lim_{n\to\infty} \frac{b_{n+1}}{b_n} = 1$ implies that for every $1 > \varepsilon > 0$, there is a positive integer N such that

$$n > N \implies 1-\varepsilon < \frac{b_{n+1}}{b_n} < 1+\varepsilon,$$

or that

$$b_n\left(1-\varepsilon\right) < b_{n+1} < b_n\left(1+\varepsilon\right). \tag{10}$$

Let $n_0 > N$. Then for a any positive integer m

$$b_{n_0}\left(1-\varepsilon\right)^m < b_{n_0+m} < b_{n_0}\left(1+\varepsilon\right)^m.$$

$$b_{n_0}^{\frac{1}{n_0+m}}\left(1-\varepsilon\right)^{\frac{m}{n_0+m}} < \sqrt[n_0+m]{b_{n_0+m}} < b_{n_0}^{\frac{1}{n_0+m}}\left(1+\varepsilon\right)^{\frac{m}{n_0+m}} \tag{11}$$

Since $(1-\varepsilon)^{\frac{m}{n_0+m}} > 1-\varepsilon$ and $(1+\varepsilon)^{\frac{m}{n_0+m}} < 1+\varepsilon$, we rewrite (11) as follows,

$$b_{n_0}^{\frac{1}{n_0+m}}\left(1-\varepsilon\right) < \sqrt[n_0+m]{b_{n_0+m}} < b_{n_0}^{\frac{1}{n_0+m}}\left(1+\varepsilon\right). \tag{12}$$

Now, we have

$$\lim_{m\to\infty} b_{n_0}^{\frac{1}{n_0+m}} = 1,$$

then taking limits of the sides of (12) we get,

$$1-\varepsilon < \lim_{m\to\infty} \sqrt[n_0+m]{b_{n_0+m}} < 1+\varepsilon$$

or

$$\left|\lim_{m\to\infty} \sqrt[n_0+m]{b_{n_0+m}} - 1\right| < \varepsilon. \tag{13}$$

But $\lim_{m\to\infty} \sqrt[n_0+m]{b_{n_0+m}} = \lim_{m\to\infty} \sqrt[m]{b_m}$ hence

$$\left|\lim_{m\to\infty} \sqrt[m]{b_m} - 1\right| < \varepsilon.$$

Since ε can be arbitrarily close to 0 then

$$\lim_{m\to\infty} \sqrt[m]{b_m} = 1. \quad \blacksquare$$

By the lemma and (6)

$$\lim_{n\to\infty} \sqrt[n]{k_n} = 1. \tag{14}$$

The Stirling's asymptotic formula for $\ln(n!)$ is

$$\ln(n!) = n\ln(n) - n + \frac{1}{2}\ln(2\pi n) + \frac{1}{12n} + O\left(\frac{1}{n^3}\right) \tag{15}$$

After division of both sides of (15) by n and taking the exponent, we get

$$\frac{1}{n}\ln(n!) = \ln(n) - 1 + \frac{1}{2n}\ln(2\pi n) + \frac{1}{12n^2} + O\left(\frac{1}{n^4}\right),$$

$$
\begin{aligned}
(n!)^{\frac{1}{n}} &= \frac{n}{e}\exp\left(\frac{1}{2n}\ln(2\pi n) + +\frac{1}{12n^2} + O\left(\frac{1}{n^4}\right)\right) \\
&= \frac{n}{e}\left(1 + \frac{\ln(2\pi n)}{2n} + O\left(\frac{\ln^2(n)}{n^2}\right)\right). \qquad (16)
\end{aligned}
$$

Now we use (16) to obtain asymptotic expressions for $\left((2n)!\right)^{\frac{1}{n}}$ and $(n!)^{\frac{3}{n}}$,

$$
\left((2n)!\right)^{\frac{1}{n}} = \frac{4n^2}{e^2}\left(1 + \frac{\ln(4\pi n)}{2n} + O\left(\frac{\ln^2(n)}{n^2}\right)\right), \qquad (17)
$$

$$
(n!)^{\frac{3}{n}} = \frac{n^3}{e^3}\left(1 + \frac{3\ln(2\pi n)}{2n} + O\left(\frac{\ln^2(n)}{n^2}\right)\right). \qquad (18)
$$

$$
\begin{aligned}
\sqrt[n]{\frac{a_n F_n}{(2n-1)!!}} &= 4\pi\sqrt[n]{F_n}\sqrt[n]{k_n}\frac{(n!)^{\frac{3}{n}}}{\left((2n)!\right)^{\frac{1}{n}}} \\
&= 4\pi\sqrt[n]{F_n}\sqrt[n]{k_n}\frac{\frac{n^3}{e^3}\left(1 + \frac{3\ln(2\pi n)}{2n} + O\left(\frac{\ln^2(n)}{n^2}\right)\right)}{\frac{4n^2}{e^2}\left(1 + \frac{\ln(4\pi n)}{2n} + O\left(\frac{\ln^2(n)}{n^2}\right)\right)} \\
&= \pi\sqrt[n]{F_n}\sqrt[n]{k_n}\frac{n}{e}\left(\frac{1 + \frac{3\ln(2\pi n)}{2n}}{1 + \frac{\ln(4\pi n)}{2n}} + O\left(\frac{\ln^2(n)}{n^2}\right)\right)
\end{aligned}
$$

Now, since $\lim_{n\to\infty}\sqrt[n]{k_n} = 1$ and $\lim_{n\to\infty}\sqrt[n]{F_n} = \varphi$, we have

$$
\begin{aligned}
\sqrt[n]{\frac{a_n F_n}{(2n-1)!!}} &= \pi\varphi\frac{n}{e}\left(\frac{1 + \frac{3\ln(2\pi n)}{2n}}{1 + \frac{\ln(4\pi n)}{2n}} + O\left(\frac{\ln^2(n)}{n^2}\right)\right) \qquad (19) \\
&= \pi\varphi\frac{n}{e}\left(1 + \frac{3\ln(2\pi n)}{2n} - \frac{\ln(4\pi n)}{2n} + O\left(\frac{\ln^2(n)}{n^2}\right)\right) \qquad (20) \\
&= \pi\varphi\frac{n}{e}\left(1 + \frac{\ln\left(2\pi^2\right)}{2n} + \frac{\ln n}{n}\right) + O\left(\frac{\ln^2(n)}{n}\right) \qquad (21)
\end{aligned}
$$

$$\sqrt[n+1]{\frac{a_{n+1}F_{n+1}}{(2n+1)!!}} - \sqrt[n]{\frac{a_nF_n}{(2n-1)!!}}$$
$$= \pi\varphi\frac{n+1}{e}\left(1+\frac{\ln\left(2\pi^2\right)}{2\,(n+1)}+\frac{\ln\,(n+1)}{n+1}\right) - \pi\varphi\frac{n}{e}\left(1+\frac{\ln\left(2\pi^2\right)}{2n}+\frac{\ln n}{n}\right)+O\left(\frac{\ln^2(n)}{n}\right)$$
$$= \frac{\pi\varphi}{e}\left(\ln\left(1+\frac{1}{n}\right)+1\right)+O\left(\frac{\ln^2(n)}{n}\right)$$

Since $\lim\limits_{n\to\infty}\frac{\ln^2(n)}{n} = 0$, then

$$\lim_{n\to\infty}\left(\sqrt[n+1]{\frac{a_{n+1}F_{n+1}}{(2n+1)!!}} - \sqrt[n]{\frac{a_nF_n}{(2n-1)!!}}\right) = \frac{\pi\varphi}{e}.$$

Solution 5 by Yunyong Zhang, Chinaunicom, Yunnan, China.

$$\lim_{n\to\infty}\frac{\frac{a_{n+1}}{(2n+1)!}}{\frac{a_n}{(2n-1)!}} = \lim_{n\to\infty}\frac{a_{n+1}}{a_n}\frac{(2n-1)!}{(2n+1)!} = \lim_{n\to\infty}\frac{a_{n+1}}{a_n}\frac{1}{4n^2} = \frac{\pi}{2},$$
$$\Rightarrow \quad \lim_{n\to\infty}\sqrt[n]{\frac{a_nF_n}{(2n-1)!}} = \frac{\pi}{2}.$$
$$\because \quad (2n-1)!! = \frac{(2n)!}{2^n n!},$$
$$\therefore \quad \lim_{n\to\infty}\sqrt[n]{\frac{a_n}{(2n-1)!!}} = \lim_{n\to\infty}\sqrt[n]{\frac{a_n n!2^n}{(2n)!}},$$
$$= \lim_{n\to\infty}\sqrt[n]{\frac{a_n(n-1)!2^n}{2(2n-1)!}} = \pi\sqrt[n]{(n-1)!} \sim \pi\sqrt[n]{n!}.$$

Similarly, $\lim\limits_{n\to\infty}\sqrt[n+1]{\frac{a_{n+1}}{(2n+1)!!}} = \pi\sqrt[n+1]{n!} \sim \pi\sqrt[n+1]{(n+1)!}$.

$$\therefore \quad A \sim \lim_{n\to\infty}\left[\sqrt[n+1]{F_{n+1}(n+1)!} - \sqrt[n]{F_n n!}\right].$$
$$\because \quad \sqrt[n]{n!} \sim \frac{n}{e}+\frac{1}{2e}\ln(2\pi n), \qquad \sqrt[n]{F_n} \sim \alpha(1-\frac{\ln 5}{2n}),$$
$$\therefore \quad A \sim \lim_{n\to\infty}\pi\left[\frac{\alpha}{e}+\frac{\alpha}{2e}\ln(\frac{n+1}{n})\right] = \frac{\pi\alpha}{e}.$$

Also solved by the problem proposer.

• **5765** *Proposed by Shivam Sharma, Delhi University, New Delhi, India.*

Let $s_n = -2\sqrt{n} + \sum_{k=1}^{n} \frac{1}{\sqrt{k}}$ with $\lim_{n\to\infty} s_n = s$ (Ioachimescu Constant). For non-negative intger m, evaluate:

$$L = \lim_{n\to\infty}\left(s^{m+1} - \prod_{j=n}^{n+m} s_j\right).$$

Solution 1 by Albert Stadler, Herrliberg, Switzerland.

Assuming that m⩾0 is a fixed integer, we have

$$L = s^{m+1} - \lim_{n\to\infty}\left(\prod_{j=n}^{n+m} s_j\right) = s^{m+1} - \lim_{n\to\infty}\left(\prod_{j=1}^{m+1} s_{n-1+j}\right) = s^{m+1} - s^{m+1} = 0.$$

Solution 2 by Michel Bataille, Rouen, France.

Clearly, $L = 0$ if $m = 0$. Suppose that $m \geqslant 1$. Then, for $i = 1, 2, \ldots, m$, the subsequence $(s_{n+i})_{n\geqslant 1}$ converges and $\lim_{n\to\infty} s_{n+i} = \lim_{n\to\infty} s_n = s$. It follows that

$$\lim_{n\to\infty}\prod_{j=n}^{n+m} s_j = \lim_{n\to\infty}(s_n \cdot s_{n+1} \cdots s_{n+m}) = s^{m+1}$$

and therefore $L = 0$.

Solution 3 by Moti Levy, Rehovot, Israel.

$$\prod_{j=n}^{n+m} s_j = s_n s_{n+1} \cdots s_{n+m}$$

$$\lim_{n\to\infty} s_n s_{n+1} \cdots s_{n+m} = \left(\lim_{n\to\infty} s_n\right)\left(\lim_{n\to\infty} s_{n+1}\right)\cdots\left(\lim_{n\to\infty} s_{n+m}\right)$$

Since $\lim_{n\to\infty} s_{n+j} = \lim_{n\to\infty} s_n = s$, then

$$\lim_{n\to\infty} s_n s_{n+1} \cdots s_{n+m} = s^{m+1}.$$

It follows that

$$L = \lim_{n\to\infty}\left(s^{m+1} - \prod_{j=n}^{n+m} s_j\right) = s^{m+1} - \lim_{n\to\infty} s_n s_{n+1} \cdots s_{n+m} = s^{m+1} = 0.$$

Also solved by the problem proposer.

• **5766** *Proposed by Toyesh Prakash Sharma, Agra College, Agra, India.*

Show that

$$\left(\frac{2}{\pi}\int_0^{\pi}\frac{\sin^2 x}{x^2}\text{dx}\right)\left(\frac{2}{\pi}\int_{2\pi}^{3\pi}\frac{\sin^2 x}{x^2}\text{dx}\right)\left(\frac{2}{\pi}\int_{4\pi}^{5\pi}\frac{\sin^2 x}{x^2}\text{dx}\right) < \left(\frac{1}{3}\right)^3.$$

Solution 1 by G. C. Greubel, Newport News, VA.

Using the integral result, where $\text{Si}(x)$ is the Sine Integral,

$$\int \left(\frac{\sin x}{x}\right)^2 dx = \text{Si}(2x) - \frac{\sin^2(x)}{x}$$

then

$$\phi := \left(\frac{2}{\pi}\int_0^{\pi}\frac{\sin^2 x}{x^2}\,dx\right)\left(\frac{2}{\pi}\int_{2\pi}^{3\pi}\frac{\sin^2 x}{x^2}\,dx\right)\left(\frac{2}{\pi}\int_{4\pi}^{5\pi}\frac{\sin^2 x}{x^2}\,dx\right),$$

$$\phi = \left(\frac{2}{\pi}\right)^3 \text{Si}(2\pi)\,\left(\text{Si}(6\pi) - \text{Si}(4\pi)\right)\left(\text{Si}(10\pi) - \text{Si}(8\pi)\right).$$

Since

$$\begin{aligned}\text{Si}(2\pi) &< \frac{\pi}{2}\\ \text{Si}(6\pi) - \text{Si}(4\pi) &< \frac{\pi}{2}\cdot\frac{1}{50}\\ \text{Si}(10\pi) - \text{Si}(8\pi) &< \frac{\pi}{2}\cdot\frac{1}{150}\end{aligned}$$

then

$$\phi < \frac{1}{3\,(50)^2} < \frac{1}{3^3}.$$

Solution 2 by Brian D. Beasley, Simpsonville, SC.

For $x > 0$, let $f(x) = \sin^2 x/x^2$ and define

$$f(0) = \lim_{x\to 0}\frac{\sin^2 x}{x^2} = 1.$$

Let $I_1 = \displaystyle\int_0^{\pi} f(x)dx$, $I_2 = \displaystyle\int_{2\pi}^{3\pi} f(x)dx$, and $I_3 = \displaystyle\int_{4\pi}^{5\pi} f(x)dx$. Since $0 \leqslant f(x) \leqslant 1$ on $[0,\pi]$, we have

$$0 \leqslant I_1 \leqslant \pi(1) = \pi.$$

Since $0 \leqslant f(x) \leqslant 1/(2\pi)^2$ on $[2\pi, 3\pi]$, we have

$$0 \leqslant I_2 \leqslant \pi\left(\frac{1}{4\pi^2}\right) = \frac{1}{4\pi}.$$

Since $0 \leqslant f(x) \leqslant 1/(4\pi)^2$ on $[4\pi, 5\pi]$, we have

$$0 \leqslant I_3 \leqslant \pi\left(\frac{1}{16\pi^2}\right) = \frac{1}{16\pi}.$$

Thus we obtain

$$0 \leqslant \left(\frac{2}{\pi}\right)^3 I_1 I_2 I_3 \leqslant \left(\frac{8}{\pi^3}\right)\left(\frac{1}{64\pi}\right) = \frac{1}{8\pi^4} < \frac{1}{27}.$$

Solution 3 by Michel Bataille, Rouen, France.

Let $f(x) = \dfrac{\sin^2 x}{x^2}$ and, for short, let us write $\displaystyle\int_a^b f$ for $\displaystyle\int_a^b f(x)\,dx$. We recall that $\displaystyle\int_0^\infty f = \frac{\pi}{2}$.
We have

$$\int_0^\pi f + \int_{2\pi}^{3\pi} f + \int_{4\pi}^{5\pi} f \leqslant \int_0^{5\pi} f < \int_0^\infty f = \frac{\pi}{2}$$

(since $f(x) \geqslant 0$) and by AM-GM,

$$\left(\frac{2}{\pi}\int_0^\pi f\right)\left(\frac{2}{\pi}\int_{2\pi}^{3\pi} f\right)\left(\frac{2}{\pi}\int_{4\pi}^{5\pi} f\right) \leqslant \left(\frac{\frac{2}{\pi}\left(\int_0^\pi f + \int_{2\pi}^{3\pi} f + \int_{4\pi}^{5\pi} f\right)}{3}\right)^3.$$

Combining the two results readily leads to

$$\left(\frac{2}{\pi}\int_0^\pi f\right)\left(\frac{2}{\pi}\int_{2\pi}^{3\pi} f\right)\left(\frac{2}{\pi}\int_{4\pi}^{5\pi} f\right) < \left(\frac{1}{3}\right)^3.$$

Solution 4 by Yunyong Zhang, Chinaunicom, Yunnan, China.

Let $a = \dfrac{2}{\pi}\displaystyle\int_0^\pi \frac{\sin^2 x}{x^2}\mathrm{d}x, \quad b = \frac{2}{\pi}\int_{2\pi}^{3\pi} \frac{\sin^2 x}{x^2}\mathrm{d}x, \quad c = \frac{2}{\pi}\int_{4\pi}^{5\pi} \frac{\sin^2 x}{x^2}\mathrm{d}x.$

$$\because \quad \sqrt[3]{abc} \leqslant \frac{1}{3}(a+b+c) = \frac{1}{3} \times \frac{2}{\pi}\int_0^\infty \frac{\sin^2 x}{x^2}\mathrm{d}x = \frac{1}{3} \times \frac{2}{\pi} \times \frac{\pi}{2} = \frac{1}{3},$$

$$\therefore \quad abc < \left(\frac{1}{3}\right)^3.$$

Note:

$$\int_0^\infty \frac{\sin^n x}{x^n}\mathrm{d}x = \frac{\pi}{2^n(n-1)!}\sum_{[\frac{n-1}{2}]}^{k=0}(-1)^k\binom{n}{k}(n-2k)^{n-1}.$$

When $n = 2$,

$$\frac{2}{\pi}\int_0^\infty \frac{\sin^2 x}{x^2}\mathrm{d}x = \frac{\pi}{4}\sum_0^{k=0}(-1)^k\binom{n}{k}2^1 = \frac{\pi}{2}.$$

Solution 5 by Perfetti Paolo, Università di "Tor Vergata", Roma, Italy.

$$\begin{aligned}
&\left(\frac{2}{\pi}\int_0^\pi \frac{\sin^2 x}{x^2}dx\right)\left(\frac{2}{\pi}\int_{2\pi}^{3\pi} \frac{\sin^2 x}{x^2}dx\right)\left(\frac{2}{\pi}\int_{4\pi}^{5\pi} \frac{\sin^2 x}{x^2}dx\right) = \\
&= \frac{8}{\pi^3}\int_0^\pi \frac{\sin^2 x}{x^2}dx\cdot\int_0^\pi \frac{\sin^2(x+2\pi)}{(x+2\pi)^2}dx\cdot\int_0^\pi \frac{\sin^2(x+4\pi)}{(x+4\pi)^2}dx < \\
&= \frac{8}{\pi^3}\int_0^\pi \frac{\sin^2 x}{x^2}dx\cdot\int_0^\pi \frac{\sin^2 x}{(x+2\pi)^2}dx\cdot\int_0^\pi \frac{\sin^2 x}{(x+4\pi)^2}dx < \\
&< \frac{8}{\pi^3}\int_0^\pi \frac{\sin^2 x}{x^2}dx\cdot\int_0^\pi \frac{\sin^2 x}{4\pi^2}dx\cdot\int_0^\pi \frac{\sin^2 x}{16\pi^2}dx < \\
&< \frac{8}{\pi^3}\int_0^\infty \frac{\sin^2 x}{x^2}dx\cdot\int_0^\pi \frac{\sin^2 x}{4\pi^2}dx\cdot\int_0^\pi \frac{\sin^2 x}{16\pi^2}dx = \\
&= \frac{8}{\pi^3}\left(\frac{\pi}{2}\right)\left(\frac{\pi}{2}\frac{1}{4\pi^2}\right)\left(\frac{\pi}{2}\frac{1}{16\pi^2}\right) = \frac{1}{64\pi^4} < \left(\frac{1}{3}\right)^3.
\end{aligned}$$

Above we have used the results

$$\int_0^\infty \frac{\sin^2 x}{x^2}dx = \frac{-\sin^2 x}{x}\Big|_0^\infty + \int_0^\infty \frac{\sin(2x)}{x}dx = \frac{\pi}{2}\int_0^\infty \frac{\sin y}{y}dy = \frac{\pi}{2}$$

and $\displaystyle\int_0^\pi \sin^2 x\,dx = 1/2.$

Solution 6 by Moti Levy, Rehovot, Israel.

By the AM-GM inequality,

$$
\begin{aligned}
\phi &:= \left(\frac{2}{\pi}\int_0^{\pi}\frac{\sin^2(x)}{x^2}dx\right)\left(\frac{2}{\pi}\int_{2\pi}^{3\pi}\frac{\sin^2(x)}{x^2}dx\right)\left(\frac{2}{\pi}\int_{4\pi}^{5\pi}\frac{\sin^2(x)}{x^2}dx\right)\\
&\leqslant \left(\frac{\frac{2}{\pi}\int_0^{\pi}\frac{\sin^2(x)}{x^2}dx+\frac{2}{\pi}\int_{2\pi}^{3\pi}\frac{\sin^2(x)}{x^2}dx+\frac{2}{\pi}\int_{4\pi}^{5\pi}\frac{\sin^2(x)}{x^2}dx}{3}\right)^3\\
&\leqslant \left(\frac{\frac{2}{\pi}\int_0^{5\pi}\frac{\sin^2(x)}{x^2}dx}{3}\right)^3
\end{aligned}
$$

It is known that $\displaystyle\int_0^{\infty}\frac{\sin^2(x)}{x^2}dx=\frac{\pi}{2}$, hence $\displaystyle\int_0^{5\pi}\frac{\sin^2(x)}{x^2}dx\leqslant\frac{\pi}{2}$, which implies

$$
\phi \leqslant \left(\frac{\frac{2}{\pi}\int_0^{5\pi}\frac{\sin^2(x)}{x^2}dx}{3}\right)^3 \leqslant \left(\frac{1}{3}\right)^3.
$$

Also solved by Hossaena Tedla, ADA University, Baku, Azerbaijan; Albert Stadler, Herrliberg, Switzerland and the problem proposer.

• **5767** *Proposed by Vasile Cirtoaje, Petroleum-Gas University of Ploiesti, Romania.*

Let a,b,c,d be nonnegative real numbers such that

$$
ab+bc+cd+da=4, \qquad a\geqslant b\geqslant c\geqslant d.
$$

Prove that

$$
\frac{1}{ab+7}+\frac{1}{ac+7}+\frac{1}{ad+7}+\frac{1}{bc+7}+\frac{1}{bd+7}+\frac{1}{cd+7}\geqslant\frac{3}{4}.
$$

Solution 1 by Hong Biao Zeng, Fort Hays State University, Hays, KS.

Since Harmonic mean is less than or equal to arithmetic mean, we have

$$
\begin{aligned}
&\frac{6}{\frac{1}{ab+7}+\frac{1}{ac+7}+\frac{1}{ad+7}+\frac{1}{bc+7}+\frac{1}{bd+7}+\frac{1}{cd+7}}\\
&\leqslant \frac{(ab+7)+(ac+7)+(ad+7)+(bc+7)+(bd+7)+(cd+7)}{6}\\
&= \frac{46+ac+bd}{6}
\end{aligned}
$$

which implies that

$$\frac{1}{ab+7}+\frac{1}{ac+7}+\frac{1}{ad+7}+\frac{1}{bc+7}+\frac{1}{bd+7}+\frac{1}{cd+7}\geqslant\frac{36}{46+ac+bd}$$

If we can prove that $ac+bd\leqslant 2$, then

$$\frac{1}{ab+7}+\frac{1}{ac+7}+\frac{1}{ad+7}+\frac{1}{bc+7}+\frac{1}{bd+7}+\frac{1}{cd+7}\geqslant\frac{36}{46+ac+bd}$$
$$\geqslant\frac{36}{46+2}=\frac{3}{4}$$

To find maximum of $ac+bd$ given $ab+bc+cd+da=4$, and $a\geqslant b\geqslant c\geqslant d\geqslant 0$, we can use Method of Lagrange Multipliers. We want to find maximum $F(a,b,c,d)=ac+bd$ under the constraint $G(a,b,c,d)=ab+bc+cd+da-4=0$. We solve system

$$\left\{\begin{array}{r}\dfrac{\partial F}{\partial a}+\lambda\dfrac{\partial G}{\partial a}=0\\ \dfrac{\partial F}{\partial b}+\lambda\dfrac{\partial G}{\partial b}=0\\ \dfrac{\partial F}{\partial c}+\lambda\dfrac{\partial G}{\partial c}=0\\ \dfrac{\partial F}{\partial d}+\lambda\dfrac{\partial G}{\partial d}=0\\ G=0\end{array}\right\}$$

which is the same as system

$$\left\{\begin{array}{r}c+\lambda(b+d)=0\\ d+\lambda(a+c)=0\\ a+\lambda(b+d)=0\\ b+\lambda(c+a)=0\\ ab+bc+cd+da-4=0\end{array}\right\}.$$

Solving the latter system, we have $a=b=c=d=1,\lambda=\dfrac{1}{2}$. So the maximum of $F(a,b,c,d)=ac+bd=2$ when $a=b=c=d=1$

Solution 2 by Péter Fülöp, Gyömrő, Hungary.

Start with $ab+bc+cd+da=(a+c)(b+d)=4$

Using the AM-GM inequality:

$$\sqrt{abcd}\leqslant\frac{(a+c)}{2}\frac{(b+d)}{2}=1$$

On the other hand we can use the following fact:

$$LHS = \int_0^1 t^{ab+6} + t^{ac+6} + t^{ad+6} + t^{bc+6} + t^{bd+6} + t^{cd+6} dt$$

Let's groupping the terms:

$$LHS = \int_0^1 t^6(t^{ab} + t^{cd}) + t^6(t^{ad} + t^{bc}) + t^6(t^{ac} + t^{bd}) dt$$

After applying the AM-GM inequality twice, first we get:

$$LHS \geqslant 2\int_0^1 t^6 t^{\frac{ab+cd}{2}} + t^6 t^{\frac{ad+bc}{2}} + t^6 t^{\frac{ac+bd}{2}} dt$$

and second for the exponents:

$$LHS \geqslant 6\int_0^1 t^6 t^{\sqrt{abcd}} dt$$

After the integration and using that $\sqrt{abcd} \leqslant 1$:

$$LHS \geqslant 6\left[\frac{t^{\sqrt{abcd}+7}}{\sqrt{abcd}+7}\right]_{t=0}^{t=1} \geqslant \frac{6}{8}$$

Statement is proved!

Solution 3 by Perfetti Paolo, Università di "Tor Vergata", Roma, Italy.

$$\frac{1}{ab+7} + \frac{1}{ac+7} + \frac{1}{ad+7} + \frac{1}{bc+7} + \frac{1}{bd+7} + \frac{1}{cd+7} \doteq \sum_{\text{sym}} \frac{1}{ab+7}$$

$$(ab+7) + (ac+7) + (ad+7) + (bc+7) + (bd+7) + (cd+7) \doteq \sum_{\text{sym}} (ab+7).$$

Cauchy–Schwarz yileds

$$\sum_{\text{cyc}} \frac{1}{ab+7} \cdot \sum_{\text{sym}} (ab+7) \geqslant (1+1+1+1+1+1)^2 = 36$$

hence it suffices to show

$$\frac{36}{\sum_{\text{sym}}(ab+7)} \geqslant \frac{3}{4} \iff \sum_{\text{sym}} ab \leqslant 6 \iff ac + bd \leqslant 2$$

hence it suffices

$$ab + bc + cd + da = 4, \qquad a \geqslant b \geqslant c \geqslant d \implies ac + bd \leqslant 2$$

or

$$2(ac + bd) \leqslant 4 = ab + bc + cd + da.$$

We write the latter as

$$\begin{aligned} & ab - ac + ad - ac + bc - bd + cd - bd \geqslant 0 \iff \\ & \iff a(b - c) + a(d - c) + b(c - d) + d(c - b) = \\ & = (a - d)(b - c) + (c - d)(b - a) \geqslant 0. \end{aligned}$$

If $b - c \geqslant a - b$ the inequality is true because $a - d \geqslant c - d$.

Now let's assume $b - c \leqslant a - b$ that is $a + c \geqslant b + d$. Cauchy–Schwarz yields

$$\frac{1}{ab + 7} + \frac{1}{bc + 7} + \frac{1}{cd + 7} + \frac{1}{ad + 7} \geqslant \frac{4^2}{28 + ab + bc + cd + ca} = \frac{1}{2}$$

and the inequality reduces to show

$$\frac{1}{7 + ac} + \frac{1}{7 + bd} \geqslant \frac{3}{4} - \frac{1}{2} = \frac{1}{4} \tag{1}$$

Let's define the new variables $a + c = 2x$, $ac = y^2$, $b + d = 2s$, $bd = t^2$. $x \geqslant y$ and $s \geqslant t$ by the AGM. $a + c \geqslant b + d$ means $x \geqslant s$ while $(a + c)(b + d) = 4$ becomes $xs = 1$. The inequality (1) becomes

$$56 + 4y^2 + 4t^2 \geqslant (7 + y^2)(7 + t^2) \iff y^2t^2 + 3y^2 + 3t^2 - 7 \leqslant 0.$$

This is increasing in t thus it must holds for $t = s$

$$x^2s^2 + 3x^2 + 3s^2 - 7 \leqslant 0$$

which increases in s^2. It suffices to check it for the sup of s which is x hence

$$x^4 + 6x^2 - 7 \leqslant 0 \text{ and } x^2s^2 = x^4 = 1$$

which evidently holds true, hence completing the proof.

Solution 4 by Moti Levy, Rehovot, Israel.

Arithmetic Mean Theorem: Let $F(a_1, a_2, \ldots, a_n) \; : \; A \to R$ where $A \subseteq R^n$, be a symmetric continuous function satisfying

$$F(a_1, a_2, \ldots, a_n) \geqslant F(\frac{a_1 + a_n}{2}, a_2, \ldots, \frac{a_1 + a_n}{2})$$

for all $a_1, a_2, \ldots, a_n \in A$ such that $a_1 \geqslant a_2 \geqslant \ldots \geqslant a_n$ or $a_1 \leqslant a_2 \leqslant \ldots \leqslant a_n$. Then, for $a_1, a_2, \ldots, a_n \in A$, the following inequality holds:

$$F(a_1, a_2, \ldots, a_n) \geqslant F(A, A, \ldots, A)$$

where

$$A = \frac{a_1 + a_2 + \ldots + a_n}{n}.$$

For proof, see the book: Vasile Cirtoaje, "*Mathematical Inequalities, Volume 5*", Lambert Academic Publishing, 2018.

Let

$$x = ab,\ y = bc,\ w = da,\ z = cd$$

Now we will express

$$\frac{1}{bd+7} + \frac{1}{ac+7}$$

in terms of the variables x, y, z, w, as follows:

By Titu's lemma

$$\frac{1}{bd+7} + \frac{1}{ac+7} \geqslant \frac{4}{bd+ac+14}.$$

By the rearrangement inequality, $bd + ac \leqslant cd + ab$. Hence

$$\frac{4}{bd+ac+14} \geqslant \frac{4}{cd+ab+14}.$$

Therefore

$$\frac{1}{bd+7} + \frac{1}{ac+7} \geqslant \frac{4}{x+z+14},$$

and

$$\frac{1}{ab+7}+\frac{1}{bc+7}+\frac{1}{cd+7}+\frac{1}{da+7}+\frac{1}{bd+7}+\frac{1}{ac+7} \geqslant \frac{1}{x+7}+\frac{1}{y+7}+\frac{1}{w+7}+\frac{1}{z+7}+\frac{4}{x+z+14}$$

Let

$$F(x,y,z,w) := \frac{1}{x+7} + \frac{1}{y+7} + \frac{1}{w+7} + \frac{1}{z+7} + \frac{4}{z+x+14}, \quad x+y+w+z=4, \quad x \geqslant z$$

$$F(\frac{x+z}{2}, y, w, \frac{x+z}{2}) := \frac{1}{\frac{x+z}{2}+7} + \frac{1}{y+7} + \frac{1}{w+7} + \frac{1}{\frac{x+z}{2}+7} + \frac{4}{\frac{x+z}{2}+\frac{x+z}{2}+14}$$

$$\begin{aligned}
&F(x,y,z,w) - F(\frac{x+z}{2}, y, w, \frac{x+z}{2}) \\
&= \frac{1}{x+7} + \frac{1}{z+7} - \frac{1}{\frac{x+z}{2}+7} - \frac{1}{\frac{x+z}{2}+7} + \frac{4}{z+x+14} - \frac{4}{\frac{x+z}{2}+\frac{x+z}{2}+14} \\
&= \frac{(x-z)^2}{(x+7)(z+7)(x+z+14)} \geqslant 0.
\end{aligned}$$

By the Arithmetic Mean Theorem,

$$F(x,y,z,w) \geqslant F(1,1,1,1) = \frac{1}{1+7} + \frac{1}{1+7} + \frac{1}{1+7} + \frac{1}{1+7} + \frac{4}{1+1+14} = \frac{3}{4},$$

and so the inequality is proved.

Also solved by the problem proposer.

• **5768** *Proposed by Vasile Mircea Popa, Lucian Blaga University, Sibiu, Romania.*

Calculate the integral:

$$I = \int_0^\infty \frac{\arctan(x)\ln^2(x)}{x^2+x+1}dx.$$

Solution 1 by Albert Stadler, Herrliberg, Switzerland.

We will prove that

$$I = \frac{4\pi^4}{81\sqrt{3}}.$$

The change of variables x→1/x gives

$$I = \int_0^\infty \frac{\arctan\left(\frac{1}{x}\right)\ln^2(x)}{x^2+x+1}dx.$$

Hence

$$I = \frac{1}{2}\int_0^\infty \frac{\left(\arctan(x) + \arctan\left(\frac{1}{x}\right)\right)\ln^2(x)}{x^2+x+1}dx = \frac{\pi}{4}\int_0^\infty \frac{\ln^2(x)}{x^2+x+1}dx.$$

Let 0<r<1<R, -1<s<1. We consider the complex integral

$$f(s) := \int_{C_{r,R}} \frac{z^s}{z^2-z+1}dz,$$

where $C_{r,R}$ is a path in the slit complex plane $\mathbb{C}\_ := \mathbb{C} \setminus \{x\in\mathbb{R} / x\leqslant 0\}$ that consists of the following pieces:

1. segment [-R,-r], run through in the direction of increasing values,
2. full circle |z|=r, run through in the negative direction,
3. segment [-R,-r], run through in the direction of decreasing values,
4. full circle |z|=R, run through in the positive direction.

The main branch of z→z^s defined by

$$z^s := |z|^s e^{iArg(z)s} \ , \ -\pi < Arg\,(z) < \pi$$

is analytic in the slit plane $\mathbb{C}\_$. Therefore, by the residue theorem,

$$f\,(s) = 2\pi i \sum_{z\in\left\{e^{\frac{\pi i}{3}},\ \ e^{-\frac{\pi i}{3}}\right\}} res\left(\frac{z^s}{z^2 - z + 1}\right) = \frac{2\pi i}{\left(2e^{\frac{\pi i}{3}} - 1\right)} e^{\frac{\pi i s}{3}} + \frac{2\pi i}{\left(2e^{-\frac{\pi i}{3}} - 1\right)} e^{-\frac{\pi i s}{3}} = \frac{4\pi i}{\sqrt{3}} \sin\left(\frac{\pi s}{3}\right).$$

We let R tend to infinity and r tend to 0. The integrals over the full circles |z|=r and |z|=R tend to 0. Hence

$$f\,(s) = \int_{C_{r,R}} \frac{z^s}{z^2 - z + 1} dz = \left(e^{i\pi s} - e^{-i\pi s}\right) \int_0^\infty \frac{x^s}{x^2 + x + 1} dx.$$

We conclude that

$$\int_0^\infty \frac{x^s}{x^2 + x + 1} dx = \frac{2\pi}{\sqrt{3}} \frac{\sin\left(\frac{\pi s}{3}\right)}{\sin\,(\pi s)} = \frac{2\pi}{\sqrt{3}\left(1 + 2\cos\left(\frac{2\pi s}{3}\right)\right)}.$$

Finally,

$$\int_0^\infty \frac{\ln^2(x)}{x^2 + x + 1} dx = \frac{d^2}{ds^2} \int_0^\infty \frac{x^s}{x^2 + x + 1} dx\Bigg|_{s=0} = \frac{2\pi}{\sqrt{3}} \frac{d^2}{ds^2} \frac{1}{1 + 2\cos\left(\frac{2\pi s}{3}\right)}\Bigg|_{s=0} =$$

$$= \frac{2\pi}{\sqrt{3}} \cdot \frac{8\pi^2\left(\cos\left(\frac{2\pi s}{3}\right) + 2\cos^2\left(\frac{2\pi s}{3}\right) + 4\sin^2\left(\frac{2\pi s}{3}\right)\right)}{9\left(1 + 2\cos\left(\frac{2\pi s}{3}\right)\right)^3}\Bigg|_{s=0} = \frac{16\pi^3}{81\sqrt{3}}$$

and

$$I = \frac{\pi}{4} \int_0^\infty \frac{\ln^2(x)}{x^2 + x + 1} dx = \frac{\pi}{4} \cdot \frac{16\pi^3}{81\sqrt{3}} = \frac{4\pi^4}{81\sqrt{3}}.$$

Solution 2 by G. C. Greubel, Newport News, VA.

Consider a general form of the integral of the problem. Given the integral

$$I = \int_0^\infty \frac{\tan^{-1}(x)\ \ln^{2m}(x)}{x^2 + x + 1}\, dx$$

make the change of variable $x \to 1/t$ to obtain

$$I = \int_0^\infty \frac{\tan^{-1}\left(\frac{1}{t}\right)\ \ln^{2m}(x)}{x^2 + x + 1}\, dt.$$

Since

$$\tan^{-1}(x) + \tan^{-1}\left(\frac{1}{x}\right) = \frac{\pi}{2}$$

then adding the two integrals gives

$$I = \frac{\pi}{4} \int_0^\infty \frac{\ln^{2m}(x)\,dx}{x^2 + x + 1} = \frac{\pi}{4} \int_0^\infty f(x)\,dx.$$

The integral can be seen in the form

$$\frac{4}{\pi} I = \int_0^1 f(x)\,dx + \int_1^\infty f(x)\,dx.$$

In the second in integral make the change of variable $x \to 1/t$, and notice that $f(1/x) = f(x)$, to obtain

$$I = \frac{\pi}{2} \int_0^1 f(x)\,dx = \frac{\pi}{2} \int_0^1 \frac{\ln^{2m}(x)\,dx}{x^2 + x + 1}.$$

Although there are several ways to evaluate the last integral this solution will make use of the generating function for the Chebyshev polynomials of the second kind. Since

$$\frac{1}{1 + x + x^2} = \sum_{n=0}^\infty U_n\left(-\frac{1}{2}\right) x^n.$$

and since

$$\int_0^1 x^n \ln^{2m}(x)\,dx = \partial_n^{2m} \int_0^1 x^n\,dx = \partial_n^{2m} \left(\frac{1}{n+1}\right) = \frac{(2m)!}{(n+1)^{2m+1}},$$

then

$$\begin{aligned} I &= \frac{\pi}{2} \int_0^1 \frac{\ln^{2m}(x)\,dx}{x^2 + x + 1} \\ &= \frac{\pi}{2}\,(2m)! \sum_{n=0}^\infty U_n\left(-\frac{1}{2}\right) \frac{1}{(n+1)^{2m+1}} \\ &= \frac{\pi}{2}\,\frac{1}{3^{2m+1}} \left(\psi_{2m}\left(\frac{2}{3}\right) - \psi_{2m}\left(\frac{1}{3}\right)\right), \end{aligned}$$

where $\psi_n(x)$ is the polygamma function. The general result is

$$\int_0^\infty \frac{\tan^{-1}(x)\,\ln^{2m}(x)}{x^2 + x + 1}\,dx = \frac{\pi}{2}\,\frac{1}{3^{2m+1}} \left(\psi_{2m}\left(\frac{2}{3}\right) - \psi_{2m}\left(\frac{1}{3}\right)\right).$$

The integral of the proposed problem is the case $m = 1$ for which

$$\int_0^\infty \frac{\tan^{-1}(x)\,\ln^2(x)}{x^2 + x + 1}\,dx = \frac{4\,\pi^4}{81\,\sqrt{3}}.$$

Solution 3 by Michel Bataille, Rouen, France.

The change of variables $x = \dfrac{1}{u}$ and the relation $\arctan(1/u) + \arctan(u) = \dfrac{\pi}{2}$ for positive u readily shows that $I = \dfrac{\pi}{2} \cdot J - I$, that is, $I = \dfrac{\pi}{4} \cdot J$ where

$$J = \int_0^\infty \frac{\ln^2(u)}{u^2+u+1}\, du.$$

To calculate J, we set $f(z) = \dfrac{(\log z)^3}{z^2+z+1}$ where $\log z = \ln(|z|) + i\theta$ with $0 \leqslant \theta < 2\pi$. The method detailed in **[1]** leads to

$$\int_0^\infty \frac{(\ln u)^3}{u^2+u+1}\, du + \int_\infty^0 \frac{(\ln u + 2\pi i)^3}{u^2+u+1}\, du = 2\pi i \left(\text{Res}(f, w) + \text{Res}(f, w^2)\right). \tag{1}$$

Here $\text{Res}(f, w)$ (resp. $\text{Res}(f, w^2)$) is the residue of f at $w = \exp(2\pi i/3)$ (resp. $w^2 = \exp(4\pi i/3)$). Since $\text{Res}(f, w) = \dfrac{(\log w)^3}{2w+1} = \dfrac{(2\pi i/3)^3}{i\sqrt{3}} = -\dfrac{8\pi^3}{27\sqrt{3}}$ and $\text{Res}(f, w^2) = \dfrac{(4\pi i/3)^3}{-i\sqrt{3}} = \dfrac{64\pi^3}{27\sqrt{3}}$, (1) yields

$$(-6\pi i)J + (12\pi^2)K + (8\pi^3 i)L = 2\pi i \cdot \frac{56\pi^3}{27\sqrt{3}}$$

where $K = \displaystyle\int_0^\infty \frac{\ln u}{u^2+u+1}\, du$ and $L = \displaystyle\int_0^\infty \frac{1}{u^2+u+1}\, du$. It follows that $K = 0$ and because

$$L = \frac{4}{3}\int_0^\infty \frac{du}{\left(\frac{2u+1}{\sqrt{3}}\right)^2 + 1} = \frac{2}{\sqrt{3}}\left(\frac{\pi}{2} - \frac{\pi}{6}\right) = \frac{2\pi}{3\sqrt{3}}$$

we finally obtain $J = \dfrac{16\pi^3}{81\sqrt{3}}$ and

$$I = \frac{4\pi^4}{81\sqrt{3}}.$$

[1] H. Cartan, *Elementary Theory of Analytic Functions of One or Several Complex Variables*, Dover, 1995, ch. III, p. 109.

Solution 4 by Moti Levy, Rehovot, Israel.

By change of integration variable $x = \dfrac{1}{u}$, we get

$$I = \int_0^\infty \frac{\arctan(x)\ln^2(x)}{x^2+x+1}dx = \int_0^\infty \frac{\arctan\left(\frac{1}{u}\right)\ln^2(u)}{u^2+u+1}\, du.$$

Using the trigonometric identity

$$\arctan(x) + \arctan\left(\frac{1}{x}\right) = \frac{\pi}{2},$$

we get rid of the arctan factor,

$$I = \frac{\pi}{4}\int_0^\infty \frac{\ln^2(x)}{x^2+x+1}dx$$

$$\frac{1}{x^2+x+1} = \frac{x-1}{x^3-1}$$

$$I = \frac{\pi}{4}\int_0^\infty \frac{x\ln^2(x)}{x^3-1}dx - \frac{\pi}{4}\int_0^\infty \frac{\ln^2(x)}{x^3-1}dx$$

$$\int_0^\infty \frac{x\ln^2(x)}{x^3-1}dx = \int_1^\infty \frac{x\ln^2(x)}{x^3-1}dx + \int_0^1 \frac{x\ln^2(x)}{x^3-1}dx = \int_0^1 \frac{\ln^2(x)}{1-x^3}\,dx - \int_0^1 \frac{x\ln^2(x)}{1-x^3}dx$$

$$\int_0^\infty \frac{\ln^2(x)}{x^3-1}dx = \int_1^\infty \frac{\ln^2(x)}{x^3-1}dx - \int_0^1 \frac{\ln^2(x)}{1-x^3}dx = \int_0^1 \frac{x\ln^2(x)}{1-x^3}dx - \int_0^1 \frac{\ln^2(x)}{1-x^3}dx$$

$$I = \frac{\pi}{2}\int_0^1 \frac{\ln^2(x)}{1-x^3}\,dx - \frac{\pi}{2}\int_0^1 \frac{x\ln^2(x)}{1-x^3}dx$$

$$\frac{1}{1-x^3} = \sum_{k=0}^\infty x^{3k}, \quad \text{for} \quad 0 \leqslant x < 1$$

$$I = \frac{\pi}{2}\sum_{k=0}^\infty \int_0^1 x^{3k}\ln^2(x)\;dx - \frac{\pi}{2}\sum_{k=0}^\infty \int_0^1 x^{3k+1}\ln^2(x)\,dx$$

$$\int_0^1 x^{3k}\ln^2(x)\;dx = \frac{2}{(3k+1)^3}, \quad \int_0^1 x^{3k+1}\ln^2(x)\,dx = \frac{2}{(3k+2)^3}$$

$$\begin{aligned} I &= \frac{\pi}{27}\sum_{k=0}^\infty \frac{1}{\left(k+\frac{1}{3}\right)^3} - \frac{\pi}{27}\sum_{k=0}^\infty \frac{1}{\left(k+\frac{2}{3}\right)^3} \\ &= -\frac{\pi}{54}\left(\psi^{(2)}\left(\frac{1}{3}\right) - \psi^{(2)}\left(\frac{2}{3}\right)\right). \end{aligned}$$

The Polygamma function reflection relation for $\psi^{(2)}(z)$ is

$$\psi^{(2)}(1-z) - \psi^{(2)}(z) = \pi\frac{d^2}{dz^2}\cot(\pi z) = 2\pi^3(\cot\pi z)\left(\cot^2\pi z + 1\right)$$

$$\begin{aligned} \psi^{(2)}\left(\frac{1}{3}\right) - \psi^{(2)}\left(\frac{2}{3}\right) &= 2\pi^3\left(\cot\left(\frac{2\pi}{3}\right)\right)\left(\cot^2\left(\frac{2\pi}{3}\right)+1\right) \\ &= -\frac{8}{9}\sqrt{3}\pi^3, \end{aligned}$$

$$I=\left(-\frac{\pi}{54}\right)\left(-\frac{8}{9}\sqrt{3}\pi^3\right)=\frac{4}{243}\sqrt{3}\pi^4\cong 2.7772.$$

Solution 5 by Perfetti Paolo, Universitą di "Tor Vergata", Roma, Italy.

$$\int_0^\infty \frac{\arctan(x)\ln^2(x)}{x^2+x+1}dx \underbrace{=}_{t=1/x} \int_0^\infty \frac{\left(\frac{\pi}{2}-\arctan(t)\right)\ln^2(t)}{t^2+t+1}dx$$

hence

$$\int_0^\infty \frac{\arctan(x)\ln^2(x)}{x^2+x+1}dx=\frac{\pi}{4}\int_0^\infty \frac{\ln^2(x)}{x^2+x+1}dx=\frac{\pi}{4}\int_0^\infty \frac{d^2}{da^2}\frac{x^a}{x^2+x+1}dx\Big|_{a=0}=$$
$$=\frac{\pi}{4}\frac{d^2}{da^2}\int_0^\infty \frac{x^a}{x^2+x+1}dx\Big|_{a=0}.$$

By assuming $0\leqslant a<1$, the integral $\displaystyle\int_0^\infty \frac{x^a\log^2 xdx}{x^2+x+1}$ is uniformly convergent respect to a and we can exchange the derivative and the integral by using a standard theorem. To show the uniform convergence it suffices to find an integrable a functions $M(x)$ independent of a and integrable on $[0,\infty)$ such that $\dfrac{x^a\log^2 x}{x^2+x+1}\leqslant M(x)$. To this end let's write

$$\int_0^1 \frac{x^a\log^2 xdx}{x^2+x+1}+\int_1^\infty \frac{x^a\log^2 xdx}{x^2+x+1}$$

and $\dfrac{x^a\log^2 x}{x^2+x+1}\leqslant \log^2 x$ if $x\leqslant 1$ while $\dfrac{x^a\log^2 x}{x^2+x+1}\leqslant\dfrac{x^a\log^2 x}{x^2}$ if $x>1$. hence the function $M(x)$ is equal to $\log^2 x$ if $x\leqslant 1$ and $M(x)=\dfrac{\log^2 x}{x^{2-a}}$ if $x>1$.

To evaluate $\displaystyle\int_0^\infty \frac{x^a}{x^2+x+1}dx$ I adopt complex analysis. Let's define $f(z)=\dfrac{z^a}{z^2+z+1}$ where $z=|z|e^{i\vartheta}$, $0\leqslant\vartheta\leqslant 2\pi$ and let's integrate over the curves $\gamma_1\cup\gamma_2\cup\gamma_3$

$$\gamma_1(t)=t+i\varepsilon\quad 0\leqslant t\leqslant R,\qquad \gamma_2(t)=Re^{it}\quad 0\leqslant t\leqslant 2\pi,$$
$$\gamma_3(t)=-te^{2i\pi}-i\varepsilon\ \ -R\leqslant t\leqslant 0,\qquad \gamma_4(t)=\varepsilon e^{-it},\quad -\pi\leqslant t\leqslant 0.$$

By assuming $a<1/2$ (possible because we have to perform the limit $a\to 0$ eventually) and from $\lim\limits_{x\to\infty} x^{3/2+a}f(x)=0$ it follows $\lim\limits_{R\to\infty}\displaystyle\int_{\gamma_2} f(z)dz=0$.

Moreover

$$\Big|\int_{\gamma_4}\frac{z^a}{z^2+z+1}dz\Big|\leqslant\int_{-\pi}^0\Big|\frac{\varepsilon^a e^{-ita}\,\varepsilon e^{-it}(-1)}{\varepsilon^4e^{-4it}+\varepsilon^2e^{-2it}+1}\Big|dt\underbrace{\leqslant}_{\varepsilon<1/2}\int_{-\pi}^0 2\varepsilon^3dt\underbrace{\to 0}_{\varepsilon\to 0}$$

$$\int_{\gamma_3} f(z)dz = \int_{-R}^{0} \frac{(-t)^a e^{2ia\pi} dt}{t^2+t+1} = -e^{2ia\pi} \int_0^R \frac{t^a dt}{t^2+t+1}$$

thus

$$\lim_{R\to\infty} \int_{\gamma_1\cup\gamma_2\cup\gamma_3} f(z)dz = (1-e^{2i\pi a}) \int_0^\infty \frac{x^a dx}{x^2+x+1} = 2\pi i \sum \mathrm{Res} f(z)$$

where the residues only of the poles lie in the upper semi–plane. From $x^2+x+1=0$, we have $z_1 = e^{2i\pi/3}$ and $z_2 = e^{4i\pi/3}$. We thus have

$$\int_0^\infty \frac{x^a dx}{x^2+x+1} = \frac{2\pi i}{1-e^{2i\pi a}} \left(\frac{e^{2i\pi a/3}}{2e^{i2\pi/3}+1} + \frac{e^{i4\pi a/3}}{2e^{i4\pi/3}+1} \right) =$$

$$= \frac{2\pi i}{-2i\sin(\pi a)} \frac{-1}{i\sqrt{3}} 2i \sin\frac{a\pi}{3} = \frac{2\pi}{\sqrt{3}\sin(\pi a)} \sin\frac{a\pi}{3}$$

and finally

$$\lim_{a\to 0} \frac{d^2}{da^2} \frac{2\pi}{\sqrt{3}\sin(\pi a)} \sin\frac{a\pi}{3} = \frac{2\pi}{\sqrt{3}} \frac{8\pi^2}{81} = \frac{16\pi^3\sqrt{3}}{243}.$$

Solution 6 by Yunyong Zhang, Chinaunicom, Yunnan, China.

$$\begin{aligned}
I &= \int_0^1 \frac{\arctan(x)\ln^2(x)}{x^2+x+1} \mathrm{d}x + \int_1^\infty \frac{\arctan(x)\ln^2(x)}{x^2+x+1} \mathrm{d}x, \\
&= \int_0^1 \frac{\arctan(x)\ln^2(x)}{x^2+x+1} \mathrm{d}x + \int_0^1 \frac{\arctan(\frac{1}{y})\ln^2(y)}{\frac{1}{y^2}+\frac{1}{y}+1} (-\frac{1}{y^2}) \mathrm{d}y, \\
&= \frac{\pi}{2} \int_0^1 \frac{\ln^2(x)}{x^2+x+1} \mathrm{d}x, \\
&= \frac{\pi}{2} \int_0^1 \ln^2(x) \sum_{k=0}^\infty (1-x) x^{3k} \mathrm{d}x, \\
&= \frac{\pi}{2} \sum_{k=0}^\infty \int_0^1 \left[\ln^2(x) x^{3k} - \ln^2(x) x^{3k+1} \right] \mathrm{d}x. \\
\because \quad & \int_0^1 \ln^2(x) x^{3k} \mathrm{d}x = \frac{2}{(3k+1)^3} \text{ and } \int_0^1 \ln^2(x) x^{3k+1} \mathrm{d}x = \frac{2}{(3k+2)^3}, \\
\therefore \quad & I = \pi \sum_{k=0}^\infty \frac{1}{(3k+1)^3} - \frac{1}{(3k+2)^3}, \\
&= \pi \left[\frac{117\zeta(3) + 2\sqrt{3}\pi^3}{243} - \frac{117\zeta(3) - 2\sqrt{3}\pi^3}{243} \right], \\
&= \pi \times \frac{4\sqrt{3}\pi^3}{243} = \frac{4\sqrt{3}\pi^4}{243}.
\end{aligned}$$

Also solved by Bruno Salgueiro Fanego, Viveiro, Lugo; Péter Fülöp, Gyömrő, Hungary; Prakash Pant, Mathematics Initiatives in Nepal, Bardiya, Nepal and the problem proposer.

Editor's Statement: It goes without saying that the problem proposers, as well as the solution proposers, are the *élan vital* of the Problems/Solutions Section of SSMJ. As the editor of this Section of the Journal, I consider myself fortunate to be in a position to receive, compile and organize a wealth of proposed ingenious problems and solutions intended for online publication. My unwavering gratitude goes to all the amazingly creative contributors. We come together from across continents because we find intellectual value, joy and satisfaction in mathematical problems, both in their creation as well as their solution. So that our collective efforts serve us well, I kindly ask all contributors to adhere to the following guidelines. As you peruse below, you may construe that the guidelines amount to a lot of work. But, as the samples show, there's not much to do. Your cooperation is much appreciated!

Keep in mind that the examples given below are your best guide!

Formats, Styles and Requirements

When submitting proposed problem(s) or solution(s), please send both **LaTeX** document and **pdf** document of your proposed problem(s) or solution(s). There are ways (discoverable from the internet) to convert from Word to proper LaTeX code. Porposals without a *proper* **LaTeX** document will not be published regrettably.

Regarding Proposed Solutions:

Below is the FILENAME format for all the documents of your proposed solution(s).

#ProblemNumber_FirstName_LastName_Solution_SSMJ

- FirstName stands for YOUR first name.
- LastName stands for YOUR last name.

Examples:

#1234_Max_Planck_Solution_SSMJ

#9876_Charles_Darwin_Solution_SSMJ

Please note that every problem number is *preceded* by the sign # .

All you have to do is copy the FILENAME format (or an example below it), paste it and then modify portions of it to your specs.

Please adopt the following structure, in the order shown, for the presentation of your solution:

1. On top of the first page of your solution, begin with the phrase:

"Proposed Solution to #**** SSMJ"

where the string of four astrisks represents the problem number.

2. On the second line, write

"Solution proposed by [your First Name, your Last Name]",

followed by your affiliation, city, country, all on the same linear string of words. Please see the example below. Make sure you do the same for your collaborator(s).

3. On a new line, state the problem proposer's name, affiliation, city and country, just as it appears published in the Problems/Solutions section.

4. On a new line below the above, write in bold type: "**Statement of the Problem**".

5. Below the latter, state the problem. Please make sure the statement of your problem (unlike the preceding item) is not in bold type.

6. Below the statement of the problem, write in bold type: "**Solution of the Problem**".

7. Below the latter, show the entire solution of the problem.

Here is a sample for the above-stated format for proposed solutions:

Proposed solution to #1234 SSMJ

Solution proposed by Emmy Noether, University of Göttingen, Lower Saxony, Germany.

Problem proposed by Isaac Newton, Trinity College, Cambridge, England.

Statement of the problem:

Compute $\sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}$.

Solution of the problem: . . . . . .

Regarding Proposed Problems:

For all your proposed problems, please adopt for all documents the following FILENAME format:

FirstName_LastName_ProposedProblem_SSMJ_YourGivenNumber_ProblemTitle

If you do not have a ProblemTitle, then leave that component as it already is (i.e., ProblemTitle).

The component YourGivenNumber is any UNIQUE 3-digit (or longer) number you like to give to your problem.

Examples:

Max_Planck_ProposedProblem_SSMJ_314_HarmonicPatterns

Charles_Darwin_ProposedProblem_SSMJ_358_ProblemTitle

Please adopt the following structure, in the order shown, for the presentation of your proposal:

1. On the top of first page of your proposal, begin with the phrase:

"Problem proposed to SSMJ"

2. On the second line, write

"Problem proposed by [your First Name, your Last Name]",

followed by your affiliation, city, country all on the same linear string of words. Please see the example below. Make sure you do the same for your collaborator(s) if any.

3. On a new line state the title of the problem, if any.

4. On a new line below the above, write in bold type: "**Statement of the Problem**".

5. Below the latter, state the problem. Please make sure the statement of your problem (unlike the preceding item) is not in bold type.

6. Below the statement of the problem, write in bold type: "**Solution of the Problem**".

7. Below the latter, show the entire solution of your problem.

Here is a sample for the above-stated format for proposed problems:

Problem proposed to SSMJ

Problem proposed by Isaac Newton, Trinity College, Cambridge, England.

Principia Mathematica (⟵ You may choose to not include a title.)

Statement of the problem:

Compute $\sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}$.

Solution of the problem: . . . . . .

♣ ♣ ♣ Thank You! ♣ ♣ ♣