Problems and Solutions

Albert Natian, Section Editor

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This section of the Journal offers readers an opportunity to exchange interesting mathematical problems and solutions. Please email them to Prof. Albert Natian at Department of Mathematics, Los Angeles Valley College. Please make sure every proposed problem or proposed solution is provided in both ***LaTeX*** and pdf documents. Thank you!

To propose problems, email them to: problems4ssma@gmail.com

To propose solutions, email them to: solutions4ssma@gmail.com

Solutions to previously published problems can be seen at <www.ssma.org/publications>.

Solutions to the problems published in this issue should be submitted *before* August 1, 2025.

• **5805** *Proposed by Ovidiu Furdui and Alina Sîntămărian, Technical University of Cluj-Napoca, Cluj-Napoca, Romania.*

Calculate

$$\lim_{n\to\infty} \frac{1}{\sqrt{n}} \int_0^n \frac{\sin^2 x}{1 + n\cos^2(nx)} dx.$$

• **5806** *Proposed by Paolo Perfetti, dipartimento di matematica Universitą di "Tor Vergata", Rome, Italy.*

Calculate

$$\text{a)} \quad \sum_{n=1}^{\infty} \left[\frac{1}{2n-1} - \left(\frac{1}{n} - \frac{1}{n+1} + \frac{1}{n+2} - \cdots \right) \right]^2$$

and

$$\text{b)} \quad \sum_{n=1}^{\infty} (-1)^n \left[\frac{1}{2n-1} - \left(\frac{1}{n} - \frac{1}{n+1} + \frac{1}{n+2} - \cdots \right) \right]^2 .$$

• **5807** *Proposed by Michael Brozinsky, Central Islip, New York.*

Right triangle RST (labeled clockwise with right angle at S) is formed by three lines as follows: line L1 containing RT , line L2 containing ST, and line L3 containing RS. Points P1, P2 and P3 are such that the distances from each of P1, P2 and P3 to L1, L2 and L3 respectively are proportional to 1:2:3. And there are no other points with these (preceding) three properties. Show that the area of triangle △P1P2P3 is half the area of triangle RST and determine angle R.

• **5808** *Proposed by Mihaly Bencze, Braşov, Romania and Neculai Stanciu, Buzău, Romania.*

Solve the following equation for real x:

$$\left(x^2+1\right)\cdot\left[4^{x/\left(x^2+1\right)}-\log_4\left(x^4-4x+5\right)\right]=x^6+x^4-4x^3+5x^2-5x+5.$$

• **5809** *Proposed by Toyesh Prakash Sharma (Student) St. C.F. Andrews School, Agra, India.*

(a) If $\dfrac{\ln b\ln c}{\text{bc}}+\dfrac{\ln a\ln c}{\text{ac}}+\dfrac{\ln b\ln a}{\text{ba}}\geqslant 1$ for $a,b,c>0$ and $a,b,c\neq 1$, then show that

$$\frac{\ln^2 b+\ln^2 c}{bc\cdot(a-a^3)}+\frac{\ln^2 a+\ln^2 c}{ac\cdot(b-b^3)}+\frac{\ln^2 b+\ln^2 a}{ab\cdot(c-c^3)}\geqslant 3\sqrt{3}.$$

(b) If $0<b\leqslant a<\dfrac{\pi}{2}$, then show that

$$\ln\left(\sqrt{\frac{\tan a}{\tan b}}\right)\geqslant a-b.$$

Solutions

To Formerly Published Problems

• **5781** *Proposed by Daniel Sitaru, National Economic College "Theodor Costescu" Drobeta Turnu - Severin, Romania..*

Let $m,n,p,q,r,s\in\mathbb{N}\backslash\{0\}$ and define

$$H_n^{(m)}=\frac{1}{1^m}+\frac{1}{2^m}+\ldots+\frac{1}{n^m}.$$

Prove that

$$\left(H_n^{(2p)}+H_n^{(2q)}\right)\left(H_n^{(2r)}+H_n^{(2s)}\right)\geqslant\left(H_n^{(p+r)}+H_n^{(q+s)}\right)^2.$$

Solution 1 by Michel Bataille, Rouen, France.

Let $a_j=\dfrac{1}{j^p}$ for $j=1,2,\ldots,n$ and $a_j=\dfrac{1}{(j-n)^q}$ for $j=n+1,n+2,\ldots,2n$. Similarly, let $b_j=\dfrac{1}{j^r}$ for $j=1,2,\ldots,n$ and $b_j=\dfrac{1}{(j-n)^s}$ for $j=n+1,n+2,\ldots,2n$. Then the Cauchy-Schwarz inequality gives

$$\left(\sum_{j=1}^{2n}a_j^2\right)\left(\sum_{j=1}^{2n}b_j^2\right)\geqslant\left(\sum_{j=1}^{2n}a_jb_j\right)^2,$$

which is nothing else than

$$(H_n^{(2p)}+H_n^{(2q)})(H_n^{(2r)}+H_n^{(2s)}) \geqslant (H_n^{(p+r)}+H_n^{(q+s)})^2.$$

Solution 2 by Perfetti Paolo, dipartimento di matematica Universitą di "Tor Vergata", Roma, Italy.

Cauchy–Schwarz yields

$$(H_n^{(2p)}+H_n^{(2q)})(H_n^{(2r)}+H_n^{(2s)}) \geqslant \left(\sqrt{H_n^{(2p)}H_n^{(2r)}}+\sqrt{H_n^{(2q)}H_n^{(2s)}}\right)^2$$

hence we come to

$$\sqrt{H_n^{(2p)}H_n^{(2r)}}+\sqrt{H_n^{(2q)}H_n^{(2s)}} \geqslant H_n^{(p+r)}+H_n^{(q+s)} \tag{1}$$

By Cauchy–Schwarz again

$$H_n^{(2p)}H_n^{(2r)}=\sum_{k=1}^n\frac{1}{k^{2p}}\sum_{k=1}^n\frac{1}{k^{2r}} \geqslant \left(\sum_{k=1}^n\frac{1}{k^p}\frac{1}{k^r}\right)^2=\left(\sum_{k=1}^n\frac{1}{k^{r+p}}\right)^2=(H_n^{(r+p)})^2$$

$$H_n^{(2q)}H_n^{(2s)}=\sum_{k=1}^n\frac{1}{k^{2q}}\sum_{k=1}^n\frac{1}{k^{2s}} \geqslant \left(\sum_{k=1}^n\frac{1}{k^q}\frac{1}{k^s}\right)^2=\left(\sum_{k=1}^n\frac{1}{k^{q+s}}\right)^2=(H_n^{(q+s)})^2$$

and (1) clearly follows.

Also solved by Albert Stadler, Herrliberg, Switzerland and the problem proposer.

• **5782** *Proposed by Toyesh Prakash Sharma and Etisha Sharma, Agra College, Agra, India..*

If $a,b,c \geqslant 1$, then prove that

$$\frac{a^2}{b+c}+\frac{b^2}{c+a}+\frac{c^2}{a+b} \geqslant \frac{a+b+c}{2}\left(\frac{a^2+b^2+c^2}{ab+bc+ca}\right).$$

Solution 1 by Henry Ricardo, Westchester Area Math Circle, Purchase, New York.

Noting that $(a+b+c)^3=(a+b+c)(a^2+b^2+c^2+2(ab+bc+ca))$, we have

$$\frac{(a+b+c)^3}{2(ab+bc+ca)}=\frac{(a+b+c)(a^2+b^2+c^2)}{2(ab+bc+ca)}+a+b+c.$$

This identity implies that

$$\sum_{cyclic} \frac{a^2}{b+c} \geqslant \frac{a+b+c}{2}\left(\frac{a^2+b^2+c^2}{ab+bc+ca}\right) \iff \sum_{cyclic}\left(\frac{a^2}{b+c}+a\right) \geqslant \frac{(a+b+c)^3}{2(ab+bc+ca)}$$
$$\iff \sum_{cyclic} \frac{a}{b+c} \geqslant \frac{(a+b+c)^2}{2(ab+bc+ca)}.$$

But the Cauchy-Schwarz inequality gives us

$$\sum_{cyclic} \frac{a}{b+c} = \sum_{cyclic} \frac{a^2}{a(b+c)} \geqslant \frac{(a+b+c)^2}{2(ab+bc+ca)}.$$

Note that this inequality is true for a, b, c positive. Equality holds if and only if $a = b = c$.

Solution 2 by Perfetti Paolo, dipartimento di matematica Università di "Tor Vergata", Roma, Italy.

The inequality is

$$\frac{1}{(a+b)(b+c)(c+a)} \sum_{\text{cyc}} a^2(a+b)(a+c) \geqslant \frac{a+b+c}{2}\frac{a^2+b^2+c^2}{ab+bc+ca}$$

or

$$\frac{1}{(a+b)(b+c)(c+a)} \sum_{\text{cyc}} (a^4 + a^2(ab+bc+ca)) \geqslant \frac{a+b+c}{2}\frac{a^2+b^2+c^2}{ab+bc+ca}$$

Let's introduce the new variables $a+b+c = 3u$, $ab+bc+ca = 3v^2$, $abc = w^3$. We have

$$ab+bc+ca = 9u^2 - 6v^2, \quad (a+b)(b+c)(c+a) = 9uv^2 - w^3,$$

$$a^4+b^4+c^4 = 81u^4 - 108u^2v^2 + 18v^4 + 12uw^3$$

In terms of the new variables the inequality is

$$\frac{81u^4 - 108u^2v^2 + 18v^4 + 12uw^3 + (9u^2-6v^2)3v^2}{9uv^2 - w^3} - \frac{3u}{2}\frac{9u^2-6v^2}{3v^3} \geqslant$$

which after trivial simplifications becomes

$$\frac{9u}{2}\frac{2v^2w^3 + u^2w^3 + 9v^2u^3 - 12v^4u}{(-w^3+9uv^2)v^2} \geqslant 0$$

Form the AGM we have $u \geqslant v \geqslant w$ thus $9uv^2 - w^3 \geqslant 0$ hence we need to prove

$$2v^2w^3 + u^2w^3 + 9v^2u^3 - 12v^4u \geqslant 0$$

This is an increasing function of w^3 hence we must check it for the minimum values of w^3 once fixed the values of u and v. We know that the minimum value occurs when $abc = 0$ or when $c = b$ (or cyclic).

First case $abc = 0$ hence $c = 0$. The inequality becomes

$$\frac{a^3 + b^3 - ab^2 - ba^2}{2ab} \geqslant 0$$

which is clearly true by the AGM $(a^3 + a^3 + b^3)/3 \geqslant a^2b$ and $(a^3 + b^3 + b^3)/3 \geqslant ab^2$.

Second case $b = c$. The inequality becomes

$$\frac{a(a^3 - 3ab^2 + 2b^3)}{2b(a + b)(2a + b)}$$

which follows by

$$b^3 + b^3 + a^3 \geqslant 3ab^2$$

and this concludes the proof making non necessary the condition $a, b.c \geqslant 1$.

Solution 3 by Michel Bataille, Rouen, France.

We prove that the inequality even holds for any $a, b, c > 0$.
Assume that $a, b, c > 0$. Then, by homogeneity, we can suppose that $a + b + c = 1$; the inequality to be proved becomes

$$\frac{a^2}{1 - a} + \frac{b^2}{1 - b} + \frac{c^2}{1 - c} \geqslant \frac{a^2 + b^2 + c^2}{2(ab + bc + ca)}.$$

Since the function $f(x) = \dfrac{1}{1 - x}$ is convex on $(0, 1)$, we have

$$\frac{a^2}{a^2 + b^2 + c^2} \cdot f(a) + \frac{b^2}{a^2 + b^2 + c^2} \cdot f(b) + \frac{c^2}{a^2 + b^2 + c^2} \cdot f(c) \geqslant f\left(\frac{a^3 + b^3 + c^3}{a^2 + b^2 + c^2}\right),$$

which writes as

$$\frac{a^2}{1 - a} + \frac{b^2}{1 - b} + \frac{c^2}{1 - c} \geqslant \frac{(a^2 + b^2 + c^2)^2}{(a^2 + b^2 + c^2) - (a^3 + b^3 + c^3)}.$$

It follows that it is sufficient to prove that

$$\frac{a^2 + b^2 + c^2}{(a^2 + b^2 + c^2) - (a^3 + b^3 + c^3)} \geqslant \frac{1}{2(ab + bc + ca)}.$$

This inequality is successively equivalent to

$$a^3 + b^3 + c^3 \geqslant (a^2 + b^2 + c^2)\left((a + b + c)^2 - 2(ab + bc + ca)\right)$$

$$(a^3+b^3+c^3)(a+b+c) \geqslant (a^2+b^2+c^2)^2$$
$$(a^3b+ab^3)+(b^3c+bc^3)+(c^3a+ca^3) \geqslant 2(a^2b^2+b^2c^2+c^2a^2).$$

We are done because the last inequality obviously holds (since $x^3y+xy^3 \geqslant 2\sqrt{x^3y\cdot xy^3} = 2x^2y^2$ for all positive x, y).

Solution 4 by Devis Alvarado, UNAH and UPNFM, Tegucigalpa, Honduras.

We observe that

$$\frac{1}{s}\cdot\frac{a^2}{b+c} = \frac{a}{s}\cdot\frac{a}{s-a} = \frac{a}{s}\cdot\frac{\frac{a}{s}}{1-\frac{a}{s}}$$

and $f(x) = \dfrac{x}{1-x}$ is convex $\forall x \in (0,1)$, given that $f'(x) = \dfrac{1}{(1-x)^2}$ and $f''(x) = \dfrac{2}{(1-x)^3} > 0$. Applying Jensen's inequality

$$
\begin{aligned}
& f\left(\frac{a}{s}\cdot\frac{a}{s}+\frac{b}{s}\cdot\frac{b}{s}+\frac{c}{s}\cdot\frac{c}{s}\right) \leqslant \frac{a}{s}f\left(\frac{a}{s}\right)+\frac{b}{s}f\left(\frac{b}{s}\right)+\frac{c}{s}f\left(\frac{c}{s}\right)\\
\Rightarrow\quad & f\left(\frac{a^2+b^2+c^2}{s^2}\right) \leqslant \frac{a}{s}f\left(\frac{a}{s}\right)+\frac{b}{s}f\left(\frac{b}{s}\right)++\frac{c}{s}f\left(\frac{c}{s}\right)\\
\Rightarrow\quad & \frac{\frac{a^2+b^2+c^2}{s^2}}{1-\frac{a^2+b^2+c^2}{s^2}} \leqslant \frac{a}{s}\cdot\frac{\frac{a}{s}}{1-\frac{a}{s}}+\frac{b}{s}\cdot\frac{\frac{a}{s}}{1-\frac{a}{s}}+\frac{c}{s}\frac{\frac{c}{s}}{1-\frac{c}{s}}\\
\Rightarrow\quad & \frac{a^2+b^2+c^2}{s^2-(a^2+b^2+c^2)} \leqslant \frac{1}{s}\left(\frac{a^2}{b+c}+\frac{b^2}{c+a}+\frac{c^2}{a+b}\right)\\
\Rightarrow\quad & (a+b+c)\left(\frac{a^2+b^2+c^2}{2(ab+bc+ca)}\right) \leqslant \frac{a^2}{b+c}+\frac{b^2}{c+a}+\frac{c^2}{a+b}\\
\Rightarrow\quad & \frac{a^2}{b+c}+\frac{b^2}{c+a}+\frac{c^2}{a+b} \geqslant \frac{a+b+c}{2}\left(\frac{a^2+b^2+c^2}{ab+bc+ca}\right).
\end{aligned}
$$

Comments: As $\dfrac{a}{a+b+c}, \dfrac{b}{a+b+c}, \dfrac{c}{a+b+c} \in (0,1)$, for all $a, b, c > 0$. The inequality is satisfied for all $a, b, c > 0$.

Solution 5 by David A. Huckaby, Angelo State University, San Angelo, TX.

The Cauchy-Schwarz inequality applied to the vectors $\left\langle \sqrt{\dfrac{a^2}{ab+ac}}, \sqrt{\dfrac{b^2}{bc+ba}}, \sqrt{\dfrac{c^2}{ca+cb}} \right\rangle$ and $\langle \sqrt{ab+ac}, \sqrt{bc+ba}, \sqrt{ca+cb} \rangle$ gives

$$\left(\frac{a^2}{ab+ac}+\frac{b^2}{bc+ba}+\frac{c^2}{ca+cb}\right)(ab+ac+bc+ba+ca+cb) \geqslant (a+b+c)^2,$$

that is,

$$\frac{a}{b+c}+\frac{b}{c+a}+\frac{c}{a+b} \geqslant \frac{(a+b+c)^2}{2(ab+bc+ca)}. \tag{1}$$

Note that

$$\begin{aligned}
(a+b+c)\left(\frac{a}{b+c}+\frac{b}{c+a}+\frac{c}{a+b}\right) &= \frac{a^2}{b+c}+\frac{ab}{c+a}+\frac{ac}{a+b} \\
&\quad+\frac{ba}{b+c}+\frac{b^2}{c+a}+\frac{bc}{a+b}+\frac{ca}{b+c}+\frac{cb}{c+a}+\frac{c^2}{a+b} \\
&= \frac{a^2}{b+c}+\frac{b^2}{c+a}+\frac{c^2}{a+b}+\frac{ba+ca}{b+c}+\frac{ab+cb}{c+a}+\frac{ac+bc}{a+b} \\
&= \frac{a^2}{b+c}+\frac{b^2}{c+a}+\frac{c^2}{a+b}+a+b+c.
\end{aligned}$$

So

$$\begin{aligned}
\frac{a^2}{b+c}+\frac{b^2}{c+a}+\frac{c^2}{a+b} &= (a+b+c)\left(\frac{a}{b+c}+\frac{b}{c+a}+\frac{c}{a+b}\right)-(a+b+c) \\
&= (a+b+c)\left(\frac{a}{b+c}+\frac{b}{c+a}+\frac{c}{a+b}-1\right) \\
&\geqslant (a+b+c)\left(\frac{(a+b+c)^2}{2(ab+bc+ca)}-1\right) \\
&= (a+b+c)\left(\frac{a^2+b^2+c^2+2ab+2bc+2ca}{2(ab+bc+ca)}-1\right) \\
&= (a+b+c)\left(\frac{a^2+b^2+c^2}{2(ab+bc+ca)}+1-1\right) \\
&= \frac{a+b+c}{2}\left(\frac{a^2+b^2+c^2}{ab+bc+ca}\right),
\end{aligned}$$

where the inequality on line 3 follows from (1). (Note that Nesbitt's inequality, $\frac{a}{b+c}+\frac{b}{c+a}+\frac{c}{a+b} \geqslant \frac{3}{2}$ for $a,b,c>0$, ensures that the second factor on line 2 is positive.)

Solution 6 by Albert Stadler, Herrliberg, Switzerland.

The equation holds true even if a, b, c >0. We note that

$$\frac{a^2}{b+c}+\frac{b^2}{c+a}+\frac{c^2}{a+b}=\frac{(a+b+c)\left(a^3+b^3+c^3+abc\right)}{(a+b)(a+c)(b+c)}.$$

Hence we need to prove that

$$\frac{2\left(a^3+b^3+c^3+abc\right)}{(a+b)(a+c)(b+c)} \geqslant \frac{a^2+b^2+c^2}{ab+bc+ca}$$

which after clearing denominators is seen to be equivalent to

$$a^4b + ab^4 + b^4c + bc^4 + c^4a + ca^4 \geqslant a^3b^2 + a^2b^3 + b^3c^2 + b^2c^3 + c^3a^2 + c^2a^3.$$

Note that if x, y > 0 then by the AM-GM inequality,

$$x^4y + xy^4 = \left(\frac{2}{3}x^4y + \frac{1}{3}xy^4\right) + \left(\frac{1}{3}x^4y + \frac{2}{3}xy^4\right) \geqslant x^3y^2 + x^2y^3.$$

Above inequality then follows by replacing (x,y) by (a,b), (b,c), (c,a), respectively and then adding the resulting inequalities.

Also solved by and the problem proposer.

• **5783** *Proposed by Goran Conar, Varaždin, Croatia.*

Let $x_1, \ldots, x_n > 0$ be real numbers and set $s = \sum_{i=1}^{n} x_i$. Prove

$$\prod_{i=1}^{n} x_i^{x_i} \geqslant \left(\frac{s}{n+s}\right)^s \prod_{i=1}^{n}(1 + x_i)^{x_i}.$$

When does equality occur?

Editor's note: Problem #5783 was formerly published as problem #5753. The editor regrets this repetition. Solutions to problem #5753 from Albert Stadler, Perfetti Paolo, Prakash Pant, Toyesh Prakash Sharma, Henry Ricardo were published or acknowledged in a previous issue of the P&S (Problems and Solutions).

Also solved by Devis Alvarado, UNAH and UPNFM, Tegucigalpa, Honduras and the problem proposer.

• **5784** *Proposed by Ivan Hadinata, Senior High School 1 Jember, Jember, Indonesia.*

Find all postive integers n for which there exist n pairwise-distinct positive integers $x_1, x_2, \ldots\ldots, x_n$ satisfying the equation:

$$\ln x_1 + \ln x_2 + \cdots + \ln x_n = \ln(x_1 + x_2 + \cdots + x_n).$$

where ln denotes natural logarithm.

Solution 1 by Ulrich Abel, Technische Hochschule Mittelhessen, Friedberg, Germany.

Since $\ln : (0, \infty) \to R$ is a bijective function, the equation is equivalent to

$$x_1 x_2 \cdots x_n = x_1 + x_2 + \cdots + x_n.$$

In the case $n = 1$ this equation is obviously valid for each integer $x_1 \geqslant 1$. For fixed $n \geqslant 2$, define $x_0 = \max\{x_1, x_2, \ldots, x_n\}$. Since the numbers are pairwise-distinct positive integers it follows that

$$1 \cdot 2 \cdots (n-1)\, x_0 \leqslant x_1 x_2 \cdots x_n = x_1 + x_2 + \cdots + x_n \leqslant (n-1)\,(x_0 - 1) + x_0 < n x_0,$$

which implies that $1 \cdot 2 \cdots (n-1) < n$. This inequality is not satisfied for all $n \geqslant 4$.

In the case $n = 2$ there is no solution (x_1, x_2). If $x_1 = 1$ we see that $x_2 = x_1 x_2 = x_1 + x_2 = 1 + x_2$ has no solution. If $x_1 \geqslant 2$ and $x_2 = 3 + k$ with an integer $k \geqslant 0$, we infer that $x_1\,(3+k) = x_1 x_2 = x_1 + x_2 = x_1 + 3 + k$ which implies

$$x_1 = \frac{x_1}{3+k} + 1 \leqslant \frac{x_1}{3} + 1 \implies 1 \geqslant \frac{2}{3} x_1 \geqslant \frac{4}{3},$$

a contradiction.

In the case $n = 3$ we have the admissible solution $(x_1, x_2, x_3) = (1, 2, 3)$.

Summarizing, the equation has admissible solutions if and only if $n = 1$ or $n = 3$.

Solution 2 by Paul Flesher, Fort Hays State University, Hays, KS.

We first note that this problem reduces to finding all positive integers n for which there exist n pairwise-distinct positive integers where $\prod_{i=1}^{n} x_i = \sum_{i=1}^{n} x_i$ because the sum of logarithms is the logarithm of the products and the natural logarithm is injective.

Without loss of generality, we may relabel any list of finitely many pairewise-distinct positive integers so that $x_i < x_j$ when $i < j$.

When $n = 1$, the problem is of no interest. Suppose $n \geqslant 2$ and $x_1, x_2, \ldots, x_n$ form a solution. So, we have that $\prod_{i=1}^{n} x_i = \sum_{i=1}^{n} x_i$. Given our labeling, we know that x_n is the largest of the positive integers. Replacing each x_i in our sum with x_n, we have that $\sum_{i=1}^{n} x_i < x_n + x_n + \ldots + x_n = n \cdot x_n$. This subsequently leads to the conclusion that $\prod_{i=1}^{n} x_i < n \cdot x_n$. Since x_n is a positive integer, we can cancel the x_n on each side of the inequality which produces $\prod_{i=1}^{n-1} x_i < n$

Note that this product is minimized when $x_i = i$ meaning $(n-1)! \leqslant \prod_{i=1}^{n-1} x_i$. So we deduce the necessary condition on n that $(n-1)! < n$. Such a condition is only true for $n = 2, 3$.

Suppose x_1 and x_2 form a solution, then $x_1 + x_2 = x_1 \cdot x_2$ and $x_1 < 2$. So $x_1 = 1$ as it is a positive integer. Meaning $1 + x_2 = 1 \cdot x_2$. This cannot happen. So, there is no set of positive integers that form a solution for $n = 2$. When $n = 3$, the pairwise-distinct positive integers $1, 2, 3$ form a solution. Hence, the only positive integer n that admits a meaningful solution to the proposed problem is $n = 3$.

Solution 3 by Michel Bataille, Rouen, France.

We show that the solutions for n are $n = 1$ and $n = 3$.

First, $n = 1$ is an obvious solution and $n = 3$ is a solution because $1 \cdot 2 \cdot 3 = 1 + 2 + 3$.
Conversely, suppose that $n > 1$ and that the equation $x_1 \cdot x_2 \cdots x_n = x_1 + x_2 + \cdots + x_n$ (equivalent to the given equation) holds for some pairwise-distinct positive integers $x_1, x_2, \ldots, x_n$. Without loss of generality, we suppose that $x_1 < x_2 < \cdots < x_n$. Note that in consequence $x_j \geqslant j$ $(j = 1, \ldots, n)$.
Let $p = x_1 \cdot x_2 \cdots x_n$. Then $p = x_1 + x_2 + \cdots + x_n$, $\dfrac{x_1}{p} < \dfrac{x_2}{p} < \cdots < \dfrac{x_n}{p}$ and

$$\frac{x_1}{p} + \frac{x_2}{p} + \cdots + \frac{x_n}{p} = 1. \tag{1}$$

It follows that

$$\frac{1}{x_1 \cdot x_2 \cdots x_{n-1}} = \frac{x_n}{p} \geqslant \frac{1}{n}$$

(otherwise the left side of (1) would be less than 1).
Therefore we have $n \geqslant x_1 \cdot x_2 \cdots x_{n-1} \geqslant (n-1)!$, which calls for $n < 4$; indeed, if $n \geqslant 4$, then $(n-1)! > (n-1)(n-2) > n$ (since $n^2 - 4n + 2 = n(n-4) + 2 \geqslant 2 > 0$). Thus, besides 1, the only possible solutions for n are 2 and 3. However, 2 is not a solution: $x_1x_2 = x_1 + x_2$ writes as $(x_1 - 1)(x_2 - 1) = 1$ implying $x_1 = x_2 = 2$ and contradicting $x_1 \neq x_2$.
Thus, a solution for n satisfies $n = 1$ or $n = 3$ and the proof is complete.

Solution 4 by Henry Ricardo, Westchester Area Math Circle, Purchase, NY.

We have the trivial solution $n = 1$. Now suppose that $n > 1$ and the positive integers $x_1, x_2, \ldots, x_n$ are pairwise-distinct. Then

$$\ln x_1 + \ln x_2 + \cdots + \ln x_n = \ln(x_1 + x_2 + \cdots + x_n) \iff x_1x_2 \cdots x_n = x_1 + x_2 + \cdots + x_n.$$

But it is known that the sum of $n > 1$ distinct positive integers equals their product only when the integers are $1, 2,$ and 3, so that $n = 3$ is the only nontrivial solution. (See, for example, p. 694 of Dickson's *History of the Theory of Numbers*, Volume II, where this result is attributed to Housel.)

$$\ln x_1 + \ln x_2 + \cdots + \ln x_n = \ln(x_1 + x_2 + \cdots + x_n).$$

A second solution: We have the trivial solution $n = 1$. Without loss of generality, suppose $0 < x_1 < x_2 < \cdots < x_n$ with $n > 1$ and $x_1x_2 \cdots x_n = x_1 + x_2 + \cdots + x_n$. Since the x_i are distinct integers, we have $x_n \geqslant n$. It follows that $x_1 + x_2 + \cdots + x_n < nx_n$ and $x_1x_2 \cdots x_n \geqslant (n-1)!x_n$, implying that $nx_n > (n-1)!x_n$, or $n > (n-1)!$. This, in turn, means that $n \leqslant 3$. If $n = 2$, we have $x_1 + x_2 = x_1x_2$, or $1/x_1 + 1/x_2 = 1$, which is easily seen to have only the solution $(2, 2)$, which doesn't satisfy our hypothesis that the x_i are distinct. When $n = 3$, we have the obvious solution $x_1 = 1, x_2 = 2, x_3 = 3$. Thus the only nontrivial solution is $n = 3$.

Solution 5 by the Eagle Problem Solvers, Georgia Southern University, Savannah, GA and Statesboro, GA.

The only such positive integers are $n = 1$ and $n = 3$.

If $n = 1$, then the equation is trivially satisfied for any positive integer x_1. For $n > 1$, the left side of the equation is equal to $\ln(x_1x_2\cdots x_n)$ by the product rule for logarithms. Since the natural logarithm function is one-to-one, the problem is equivalent to finding solutions of the equation

$$x_1x_2\cdots x_n = x_1 + x_2 + \cdots + x_n,$$

which we may rewrite as

$$1 = \frac{1}{\hat{x}_1} + \frac{1}{\hat{x}_2} + \cdots + \frac{1}{\hat{x}_n}, \tag{2}$$

where $\hat{x}_k = \dfrac{x_1x_2\cdots x_n}{x_k}$.

Without loss of generality, we may assume that $1 \leqslant x_1 < x_2 < \cdots < x_n$. Because the x_k are distinct positive integers, then $x_k \geqslant k$ for each positive integer $k \leqslant n$.

If $n = 2$, then Equation (2) becomes

$$1 = \frac{1}{x_1} + \frac{1}{x_2}.$$

If $x_1 = 1$, then $\dfrac{1}{x_1} + \dfrac{1}{x_2} = 1 + \dfrac{1}{x_2} > 1$. Meanwhile, if $x_1 > 1$, then

$$\frac{1}{x_1} + \frac{1}{x_2} \leqslant \frac{1}{2} + \frac{1}{3} = \frac{5}{6} < 1.$$

Thus, there are no solutions if $n = 2$.

If $n = 3$, then Equation (2) is

$$1 = \frac{1}{x_1x_2} + \frac{1}{x_1x_3} + \frac{1}{x_2x_3}.$$

Since $x_k \geqslant k$ for each positive integer k, then

$$\frac{1}{x_1x_2} + \frac{1}{x_1x_3} + \frac{1}{x_2x_3} \leqslant \frac{1}{2} + \frac{1}{3} + \frac{1}{6} = 1,$$

with equality if and only if $(x_1, x_2, x_3) = (1, 2, 3)$. Thus, there is a unique solution (up to permutation) if $n = 3$.

Finally, suppose that $n \geqslant 4$. If $1 \leqslant k < n$, then either $k \neq 2$ or $k \neq 3$, so $\hat{x}_k \geqslant x_2 x_n \geqslant 2n > n$ and $\dfrac{1}{\hat{x}_k} < \dfrac{1}{n}$. In addition, for $n \geqslant 4$,

$$(n-1)! \geqslant (n-1)(n-2) = n^2 - 3n + 2 > n(n-3) \geqslant n,$$

so that $(n-1)! > n$. Thus,

$$\frac{1}{\hat{x}_n} = \frac{1}{x_1 x_2 \cdots x_{n-1}} \leqslant \frac{1}{(n-1)!} < \frac{1}{n},$$

and

$$\sum_{k=1}^{n} \frac{1}{\hat{x}_k} < \frac{n}{n} = 1,$$

so there are no solutions to the given equation when $n \geqslant 4$.

Solution 6 by Devis Alvarado, UNAH and UPNFM, Tegucigalpa, Honduras.

The equation is equivalent to

$$x_1 x_2 \cdots x_n = x_1 + x_2 + \cdots + x_n$$

Since the equation is symmetric, we can assume without loss of generality that $x_1 < x_2 < \cdots < x_n$. Like $x_1 \geqslant 1$ then $x_i \geqslant i$ for $i = 1, 2, \cdots, n$. Thus for $n = 1$ it is true for every non-negative integer x_1, now let's analyze if $n > 1$

$$\begin{aligned} & x_1 x_2 \cdots x_n = x_1 + x_2 + \cdots + x_n \\ \Rightarrow \quad & (n-1)! x_n \leqslant x_1 x_2 \cdots x_{n-1} x_n = x_1 + x_2 + \cdots + x_n < n x_n \\ \Rightarrow \quad & (n-1)! < n \end{aligned}$$

From the last inequality we have that $n = 2, 3$, which means that $(n-1)! > n$ for all $n \geqslant 4$. For the cases $n = 2$, the equation $x_1 x_2 = x_1 + x_2$ does not have different positive integer solutions, while for the case $n = 3$ the numbers $x_1 = 1, x_2 = 2, x_3 = 3$ satisfy the equation. In conclusion, the only n that satisfy the conditions are $n = 1, 3$.

Solution 7 by Daniel Văcaru, "Maria Teiuleanu" National Economic College, Pitești, Romania.

For $n = 2$,one obtain

$$\ln x_1 + \ln x_2 = \ln (x_1 + x_2) \tag{3}$$

It follows that

$$x_1 x_2 = x_1 + x_2 \Leftrightarrow (x_1 - 1)(x_2 - 1) = 0$$

It follows $x_1 = 1$ or $x_2 = 1$, and one obtain

$$\ln x = \ln (1 + x)\,, \tag{4}$$

which is a contradiction.

If $n \geqslant 3$, consider equation

$$\overset{[}{k}= 1]n - 1\sum \ln k + \ln x_n = \ln\left(\overset{[}{k}= 1]n - 1\sum k + x_n\right), \tag{5}$$

which is equivalent to

$$(n-1)! \cdot x_n = \frac{(n-1)\,n}{2} + x_n, \tag{6}$$

which is equivalent to

$$\left[(n-1)! - 1\right] \cdot x_n = \frac{n\,(n-1)}{2}, \tag{7}$$

and one obtain

$$x_n = \frac{n\,(n-1)}{2\left[(n-1)! - 1\right]}.$$

Solution 8 by Albert Stadler, Herrliberg, Switzerland.

The given equation is equivalent to

$$x_1 x_2 \cdots x_n = x_1 + x_2 + \cdots + x_n.$$

We claim that the set of feasible n is $\{1, 3\}$. We have 1=1 and 1 2 3=1+2+3. We note that $x_1 x_2 = x_1 + x_2$ is equivalent to $(x_1 - 1)(x_2 - 1) = 1$ implying $x_1 = x_2 = 2$, and x_1, x_2 are not pairwise distinct. We may therefore assume that $n \geqslant 4$.

Lemma: Let $n \geqslant 1$. Then

$$x_1 x_2 \cdots x_n \geqslant x_1 + x_2 + \cdots + x_n + n! - \frac{n\,(n+1)}{2},$$

if $x_1, x_2, \ldots, x_n$ are pairwise-distinct positive integers.

Proof of the Lemma: By symmetry we may assume that

$$1 \leqslant x_1 < x_2 < \cdots < x_n.$$

In particular $x_k \geqslant k$. The claim is true for n=1 (trivial), for n=2, noting that

$$x_1 x_2 - x_1 - x_2 + 1 = (x_1 - 1)(x_2 - 1) \geqslant 0,$$

and for n=3, noting that

$$x_1x_2x_3 - x_1 - x_2 - x_3 =$$
$$= (x_1 - 1)(x_2 - 2)(x_3 - 3) + 3(x_1 - 1)(x_2 - 2) + 2(x_1 - 1)(x_3 - 3) +$$
$$+ (x_2 - 2)(x_3 - 3) + 5(x_1 - 1) + 2(x_2 - 2) + x_3 - 3 \geqslant 0.$$

Then, if n⩾4,

$$x_1x_2\cdots x_n \geqslant \left(x_1 + x_2 + \cdots + x_{n-1} + (n-1)! - \frac{(n-1)n}{2}\right)x_n =$$
$$= \sum_{k=1}^{n-1}\big((x_k - 1)(x_n - 1) + x_k + x_n - 1\big) + \left((n-1)! - \frac{(n-1)n}{2}\right)x_n \geqslant$$
$$= \sum_{k=1}^{n-1}\big((k - 1)(n - 1) + x_k + x_n - 1\big) + \left((n-1)! - \frac{(n-1)n}{2}\right)n =$$
$$= (n-1)x_n + \sum_{k=1}^{n-1} x_k + n! - \frac{3}{2}(n-1)n \geqslant (n-2)n + \sum_{k=1}^{n} x_k + n! - \frac{3}{2}(n-1)n =$$
$$= x_1 + x_2 + \cdots + x_n + n! - \frac{n(n+1)}{2},$$

and the proof by induction is complete.

We solve the equation $x_1x_2\cdots x_n = x_1 + x_2 + \cdots + x_n$ for x_n and get

$$x_n = \frac{x_1 + \cdots + x_{n-1}}{x_1\cdots x_{n-1} - 1} \leqslant \frac{x_1 + \cdots + x_{n-1}}{x_1 + x_2 + \cdots + x_{n-1} + (n-1)! - \frac{(n-1)n}{2} - 1} < 2$$

for n⩾4, which contradicts $x_n \geqslant n$. So the set of feasible n is $\{1, 3\}$.

Also solved by the problem proposer.

• **5785** *Proposed by Vasile Cirtoaje, Petroleum-Gas University of Ploiesti, Romania.*

Prove that 3 is the largest positive value of the constant k such that

$$\frac{1}{a} + \frac{1}{b} + \frac{1}{c} + \frac{1}{d} - 4 \geqslant k(a + b + c + d - 4)$$

for any positive real numbers a, b, c, d with $a \geqslant b \geqslant c \geqslant 1 \geqslant d$ and $ab + bc + cd + da = 4$.

Solution 1 by Albert Stadler, Herrliberg, Switzerland.

We note that ab+bc+cd+da=(a+c)(b+d). Let k>3, b=c=1. Then a+1+d+da=(1+a)(1+d)=4 and

$$\frac{1}{a} + \frac{1}{b} + \frac{1}{c} + \frac{1}{d} - 4 - k(a + b + c + d - 4) = \frac{1}{a} + \frac{1}{d} - 2 - k(a + d - 2) =$$

$$= \frac{1}{a} + \frac{1}{\frac{4}{1+a} - 1} - 2 - k\left(a + \frac{4}{1+a} - 3\right) = \frac{(a-1)^2\left(3 + 3a - (3-a)\,ak\right)}{(3-a)\,a\,(1+a)}.$$

Note that a<3, since d>0 and (1+a)(1+d)=4. Furthermore

$$\lim_{a\to 1}\left(3 + 3a - (3-a)\,ak\right) = 6 - 2k < 0.$$

So if a is sufficiently close to 1 and b=c=1 then

$$\frac{1}{a} + \frac{1}{b} + \frac{1}{c} + \frac{1}{d} - 4 - k\,(a+b+c+d-4) < 0.$$

Let 0<k⩽3. (a+c)(b+d)=4 implies $d = \dfrac{4}{a+c} - b$, and d>0 implies b(a+c)<4. We need to prove that

$$\frac{1}{a} + \frac{1}{b} + \frac{1}{c} + \frac{1}{d} - 4 - k\,(a+b+c+d-4) =$$

$$= \frac{1}{a} + \frac{1}{b} + \frac{1}{c} + \frac{a+c}{4 - b\,(a+c)} - 4 - k\left(a + c + \frac{4}{a+c} - 4\right) \geqslant 0$$

for a⩾b⩾c⩾1, b(a+c)<4. We note that $a + c + \dfrac{4}{a+c} - 4 \geqslant 2\sqrt{(a+c)\,\dfrac{4}{a+c}} - 4 \geqslant 0.$
So we need to prove that for a⩾b⩾c⩾1, b(a+c)<4

$$\frac{1}{a} + \frac{1}{b} + \frac{1}{c} + \frac{a+c}{4 - b\,(a+c)} - 4 - 3\left(a + c + \frac{4}{a+c} - 4\right) \geqslant 0. \qquad (*)$$

Let 0⩽ϵ⩽a-c. Then

$$\frac{1}{a} + \frac{1}{c} - \frac{1}{a-\varepsilon} - \frac{1}{c+\varepsilon} = \frac{(a+c)\,(a-c-\varepsilon)\,\varepsilon}{ac\,(a-\varepsilon)\,(c+\epsilon)} \geqslant 0.$$

So if we replace (a,c) by (a-ϵ,c+ϵ) in (*) then a+c stays constant, while $\dfrac{1}{a} + \dfrac{1}{c} \geqslant \dfrac{1}{a-\varepsilon} + \dfrac{1}{c+\varepsilon}$.
As a⩾b⩾c we may assume that either a=b (case I) or b=c (case II).

Case I: a=b.

Then a=b⩾c⩾1, a(a+c)<4, and

$$\frac{2}{a} + \frac{1}{c} + \frac{a+c}{4 - a\,(a+c)} - 4 - 3\left(a + c + \frac{4}{a+c} - 4\right) = \frac{f\,(a,c)}{ac\,(a+c)\,(4 - a^2 - ac)},$$

where

$$f\,(a,c) = 4a^2 - a^4 - 36ac + 32a^2c - 3a^3c - 8a^4c + 3a^5c + 8c^2 +$$
$$+32ac^2 - 15a^2c^2 - 16a^3c^2 + 9a^4c^2 - 13ac^3 - 8a^2c^3 + 9a^3c^3 + 3a^2c^4 =$$
$$= \left(16(a-1)^2 + 20(a-1)^3 + 15(a-1)^4 + 3(a-1)^5\right) +$$
$$+ \left(56(a-1)^2 + 62(a-1)^3 + 25(a-1)^4 + 3(a-1)^5\right)(c-1) +$$

$$+\left(20\,(a-1)+66(a-1)^2+47(a-1)^3+9(a-1)^4\right)(c-1)^2+$$
$$+\left(22\,(a-1)+31(a-1)^2+9(a-1)^3\right)(c-1)^3+$$
$$+\left(3+6\,(a-1)+3(a-1)^2\right)(c-1)^4\geqslant 0.$$

Case II: b=c.

Then a⩾b=c⩾1, c(a+c)<4, and

$$\frac{1}{a}+\frac{2}{c}+\frac{a+c}{4-c\,(a+c)}-4-3\left(a+c+\frac{4}{a+c}-4\right)=\frac{g\,(a,c)}{ac\,(a+c)\,(4-c^2-ac)},$$

where

$$g\,(a,c)=8a^2-36ac+32a^2c-13a^3c+$$
$$+4c^2+32ac^2-15a^2c^2-8a^3c^2+3a^4c^2-3ac^3-16a^2c^3+$$
$$+9a^3c^3-c^4-8ac^4+9a^2c^4+3ac^5=$$
$$=\left(3(a-1)^4\right)+\left(20(a-1)^2+22(a-1)^3+6(a-1)^4\right)(c-1)+$$
$$+\left(16+56\,(a-1)+66(a-1)^2+31(a-1)^3+3(a-1)^4\right)(c-1)^2+$$
$$+\left(20+62\,(a-1)+47(a-1)^2+9(a-1)^3\right))(c-1)^3+$$
$$+\left(15+25\,(a-1)+9(a-1)^2\right)(c-1)^4+\left(3+3\,(a-1)\right)(c-1)^5\geqslant 0.$$

Hence 3 is the largest positive value of the constant k such that

$$\frac{1}{a}+\frac{1}{b}+\frac{1}{c}+\frac{1}{d}-4\geqslant k\,(a+b+c+d-4)$$

for any positive real numbers a,b,c,d with a⩾b⩾c⩾1⩾d and ab+bc+cd+da=4.

Also solved by the problem proposer.

• **5786** *Proposed by Ovidiu Furdui and Alina Sîntămărian, Technical University of Cluj-Napoca, Cluj-Napoca, Romania.*

Find all continuous functions $f:\mathbb{R}\to\mathbb{R}$ such that for all $x\in\mathbb{R}$:

$$f(-x)=1-2\int_0^x \mathrm{e}^{-t}f(x-t)\mathrm{d}t.$$

Solution 1 by Paul Flesher and Jonathan Rehmert, Fort Hays State University, Hays, KS.

Note that $f(0) = 1$. Using the u-substitution $u = x - t$, we have an alternative formulation of the problem.

$$f(-x) = 1 - 2\int_0^x e^{-t} f(x-t)\,dt = 1 - 2e^{-x}\int_0^x e^t f(t)\,dt$$

Note that $e^t f(t)$ is continuous on $\mathbb{R}$. As such, $\int_0^x e^t f(t)\,dt$ is differentiable on $\mathbb{R}$ by the Fundamental Theorem of Calculus. And hence, the right hand side is a differentiable function on $\mathbb{R}$ indicating that f is differentiable. We, therefore, differentiate both sides and substitute to arrive at the following.

$$-f'(-x) = 2e^{-x}\int_0^x e^t f(t)\,dt - 2e^{-x}e^x f(x) = 2e^{-x}\int_0^x e^t f(t)\,dt - 2f(x)$$

$$f'(-x) = -1 + f(-x) + 2f(x)$$
$$f'(x) = -1 + f(x) + 2f(-x)$$

Note that $f'(0) = 2$. Given such a relation, we have that f' is also differentiable.

$$f''(x) = f'(x) - 2f'(-x) = -1 + f(x) + 2f(-x) - 2(-1 + f(-x) + 2f(x)) = 1 - 3f(x)$$

The problem thus reduces to the second order autonomous differential equation with initial conditions.

$$f'' = 1 - 3f \qquad f(0) = 1 \qquad f'(0) = 2$$

As a second order, constant coefficient, linear initial value problem, a solution exists and is unique. It can be easily verified that

$$f(x) = \frac{2}{3}\cos(\sqrt{3}x) + \frac{2}{\sqrt{3}}\sin(\sqrt{3}x) + \frac{1}{3}$$

is a solution, and is hence the only solution to the proposed problem.

Solution 2 by Brian Bradie, Department of Mathematics, Christopher Newport University, Newport News, VA.

Within the integral, make the change of variables $t \to x - t$. This yields

$$f(-x) = 1 - 2\int_0^x e^{-(x-t)} f(t)\,dt. \tag{8}$$

Because f is continuous, the function $e^{-(x-t)}f(t)$ is integrable. This implies, by equation (8) and the Fundamental Theorem of Calculus, that f is differentiable. Differentiating (8), we obtain

$$\begin{aligned} -f'(-x) &= -2f(x) + 2\int_0^x e^{-(x-t)} f(t)\,dt \\ &= -2f(x) + 1 - f(-x). \end{aligned} \tag{9}$$

Next, replace x by $-x$ and multiply through by -1 to obtain

$$f'(x) = 2f(-x) - 1 + f(x). \tag{10}$$

Differentiating (10) then yields

$$f''(x) = -2f'(-x) + f'(x);$$

but twice equation (9) added to equation (10) yields

$$-2f'(-x) + f'(x) = -3f(x) + 1,$$

so

$$f''(x) + 3f(x) = 1.$$

From equation (8), we find $f(0) = 1$, while (9) gives $f'(0) = 2$. The solution of the initial value problem

$$f''(x) + 3f(x) = 1, \quad f(0) = 1, \quad f'(0) = 2$$

is

$$f(x) = \frac{1}{3} + \frac{2}{3}\cos\sqrt{3}x + \frac{2}{\sqrt{3}}\sin\sqrt{3}x.$$

Solution 3 by Michel Bataille, Rouen, France.

We show that the function f_0 defined by $f_0(x) = \frac{1}{3}\left(1 + 2\cos(x\sqrt{3}) + 2\sqrt{3}\sin(x\sqrt{3})\right)$ is the unique solution.

First, note that for any continuous function f, we have

$$\int_0^x e^{-t} f(x-t)\,dt = \int_0^x e^{u-x} f(u)\,du = e^{-x}F(x)$$

where F is the differentiable function defined by $F(x) = \int_0^x e^u f(u)\,du$ (so that $F'(x) = e^x f(x)$). In particular, we obtain

$$F_0(x) := \int_0^x e^u f_0(u)\,du = \frac{e^x - 1}{3} + \frac{2}{3}\int_0^x Re\left(e^{u(1+i\sqrt{3})}\right)\,du + \frac{2\sqrt{3}}{3}\int_0^x Im\left(e^{u(1+i\sqrt{3})}\right)\,du$$

The calculation of $\int_0^x e^{u(1+i\sqrt{3})}\,du$ is straightforward and leads to

$$\begin{aligned} F_0(x) &= \frac{e^x - 1}{3} + \frac{e^x}{6}\cos(x\sqrt{3}) - \frac{1}{6} + \frac{\sqrt{3}e^x}{6}\sin(x\sqrt{3}) + \frac{\sqrt{3}e^x}{6}\sin(x\sqrt{3}) - \frac{e^x}{2}\cos(x\sqrt{3}) + \frac{1}{2} \\ &= \frac{e^x}{3} - \frac{e^x}{3}\cos(x\sqrt{3}) + \frac{\sqrt{3}e^x}{3}\sin(x\sqrt{3}) \end{aligned}$$

so that

$$1 - 2e^{-x}F_0(x) = \frac{1}{3} + \frac{2}{3}\cos(x\sqrt{3}) - \frac{2\sqrt{3}}{3}\sin(x\sqrt{3}) = f_0(-x),$$

showing that f_0 is a solution.
Conversely, let f be any solution. Then, for all x, we have $f(x) = 1 - 2e^xF(-x)$, hence f is differentiable on $\mathbb{R}$ (as F is) and

$$f'(x) = -2e^xF(-x) + 2e^xF'(-x) = f(x) - 1 + 2f(-x). \tag{1}$$

Therefore f' is differentiable and for all x,

$$f''(x) = f'(x) - 2f'(-x) = f'(x) - 2\left(f(-x) - 1 + 2f(x)\right) = f'(x) + 2 - 4f(x) - \left(f'(x) - f(x) + 1\right),$$

that is, $f''(x) = 1 - 3f(x)$. Thus, f is a solution to the differentiable equation $y'' + 3y = 1$ and

$$f(x) = \frac{1}{3} + A\cos(x\sqrt{3}) + B\sin(x\sqrt{3})$$

for some constant A, B. Since $F(0) = 0$, we have $f(0) = 1$, hence $A = \frac{2}{3}$. From (1) we have $f'(0) = 2$, which leads to $B = \frac{2\sqrt{3}}{3}$. We see that we must have $f = f_0$, and the proof is complete.

Solution 4 by Perfetti Paolo, dipartimento di matematica Università di "Tor Vergata", Roma, Italy.

Clearly $f(0) = 1$. $x - t = u$ yields

$$f(-x) = 1 - 2\int_0^x e^{u-x}f(u)du.$$

The r.h.s. is differentiable via the continuity of the integrand and the Torricelli–Barrow theorem then

$$f'(-x)(-1) = -2f(x) + 2\int_0^x e^{u-x}f(u)du = -2f(x) + 1 - f(-x) \tag{1}$$

hence

$$f'(x) = 2f(-x) - 1 + f(x) \implies f''(x) = -2f'(-x) + f'(x) \tag{2}$$

$$f''(x) - f'(x) = -2f'(-x).$$

From (1) we get

$$f''(x) - f'(x) = -2(2f(x) - 1 + f(-x)) = -4f(x) + 2 - 2f(-x)$$

and from the l.h.s. of (2) we get

$$f''(x) - f'(x) = -4f(x) + 2 - f'(x) - 1 + f(x) \iff f''(x) + 3f(x) = 1.$$

From the l.h.s. of (2) we get $f'(0) = 2$ hence we need to solve

$$f''(x) + 3f(x) = 1, \quad f(0) = 1, f'(0) = 2$$

whose solution is

$$f(x) = \frac{2}{\sqrt{3}} \sin(\sqrt{3}x) + \frac{2}{3} \cos(\sqrt{3}x) + \frac{1}{3}.$$

Solution 5 by Ulrich Abel, Technische Hochschule Mittelhessen, Friedberg, Germany.

The equation can be rewritten in the form

$$e^x - e^x f(-x) = 2 \int_0^x e^t f(t)\, dt.$$

Since the right-hand side is differentiable, we can differentiate both sides:

$$e^x + e^x f'(-x) - e^x f(-x) = 2e^x f(x).$$

Replacing x with $-x$ and substituting $f(-x)$ we infer that

$$1 + f'(x) - f(x) = 2f(-x) = 2 - 4e^{-x} \int_0^x e^t f(t)\, dt.$$

Differentiating the equivalent formula

$$e^x f'(x) - e^x f(x) - e^x = -4 \int_0^x e^t f(t)\, dt$$

leads to

$$e^x f''(x) - e^x f(x) - e^x = -4e^x f(x),$$

which is equivalent to

$$f''(x) + 3f(x) = 1.$$

This differential equation has the general solution

$$f(x) = \frac{1}{3} + A \cos\left(\sqrt{3}x\right) + B \sin\left(\sqrt{3}x\right).$$

Obviously, the above equation reveals that $f(0) = 1$, which implies that $A = 2/3$. Inserting into the above equation shows that the equation is satisfied if and only if $B = 2/\sqrt{3}$. Therefore, it has the unique continuous solution

$$f(x) = \frac{1}{3} + \frac{2}{3} \cos\left(\sqrt{3}x\right) + \frac{2}{\sqrt{3}} \sin\left(\sqrt{3}x\right).$$

Remark: The slightly more general equation

$$f(-x) = 1 - c \int_0^x e^{-t} f(x-t)\, dt,$$

for $c^2 \neq 1$, has the unique continuous solution

$$f(x) = \frac{c}{c+1}\cos\left(\sqrt{c^2-1}x\right) + \frac{c}{\sqrt{c^2-1}}\sin\left(\sqrt{c^2-1}x\right) + \frac{1}{c+1}.$$

Passing to the limit $c \to 1$ shows that, for $c = 1$, the unique continuous solution is given by $f(x) = x + 1$. Passing to the limit $c \to -1$ shows that, for $c = -1$, the unique continuous solution is given by $f(x) = 1 - x - x^2$.

Also solved by Bruno Salgueiro Fanego, Viveiro, Spain; Akash Chaudhary, Pulchowk Campus, Kapilvastu, Nepal; Albert Stadler, Herrliberg, Switzerland and the problem proposer.

Editor's Statement: It goes without saying that the problem proposers, as well as the solution proposers, are the *élan vital* of the Problems/Solutions Section of SSMJ. As the editor of this Section of the Journal, I consider myself fortunate to be in a position to receive, compile and organize a wealth of proposed ingenious problems and solutions intended for online publication. My unwavering gratitude goes to all the amazingly creative contributors. We come together from across continents because we find intellectual value, joy and satisfaction in mathematical problems, both in their creation as well as their solution. So that our collective efforts serve us well, I kindly ask all contributors to adhere to the following guidelines. As you peruse below, you may construe that the guidelines amount to a lot of work. But, as the samples show, there's not much to do. Your cooperation is much appreciated!

Keep in mind that the examples given below are your best guide!

Formats, Styles and Requirements

When submitting proposed problem(s) or solution(s), please send both **LaTeX** document and **pdf** document of your proposed problem(s) or solution(s). There are ways (discoverable from the internet) to convert from Word to proper LaTeX code. Porposals without a *proper* **LaTeX** document will not be published regrettably.

Regarding Proposed Solutions:

Below is the FILENAME format for all the documents of your proposed solution(s).

#ProblemNumber_FirstName_LastName_Solution_SSMJ

- FirstName stands for YOUR first name.

- LastName stands for YOUR last name.

Examples:

#1234_Max_Planck_Solution_SSMJ

#9876_Charles_Darwin_Solution_SSMJ

Please note that every problem number is *preceded* by the sign # .

All you have to do is copy the FILENAME format (or an example below it), paste it and then modify portions of it to your specs.

Please adopt the following structure, in the order shown, for the presentation of your solution:

1. On top of the first page of your solution, begin with the phrase:

"Proposed Solution to #**** SSMJ"

where the string of four astrisks represents the problem number.

2. On the second line, write

"Solution proposed by [your First Name, your Last Name]",

followed by your affiliation, city, country, all on the same linear string of words. Please see the example below. Make sure you do the same for your collaborator(s).

3. On a new line, state the problem proposer's name, affiliation, city and country, just as it appears published in the Problems/Solutions section.

4. On a new line below the above, write in bold type: "**Statement of the Problem**".

5. Below the latter, state the problem. Please make sure the statement of your problem (unlike the preceding item) is not in bold type.

6. Below the statement of the problem, write in bold type: "**Solution of the Problem**".

7. Below the latter, show the entire solution of the problem.

Here is a sample for the above-stated format for proposed solutions:

> *Proposed solution to #1234 SSMJ*
>
> *Solution proposed by Emmy Noether, University of Göttingen, Lower Saxony, Germany.*

Problem proposed by Isaac Newton, Trinity College, Cambridge, England.

Statement of the problem:

Compute $\sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}$.

Solution of the problem: . . . . . .

Regarding Proposed Problems:

For all your proposed problems, please adopt for all documents the following FILENAME format:

FirstName_LastName_ProposedProblem_SSMJ_YourGivenNumber_ProblemTitle

If you do not have a ProblemTitle, then leave that component as it already is (i.e., ProblemTitle).

The component YourGivenNumber is any UNIQUE 3-digit (or longer) number you like to give to your problem.

Examples:

Max_Planck_ProposedProblem_SSMJ_314_HarmonicPatterns

Charles_Darwin_ProposedProblem_SSMJ_358_ProblemTitle

Please adopt the following structure, in the order shown, for the presentation of your proposal:

1. On the top of first page of your proposal, begin with the phrase:

"Problem proposed to SSMJ"

2. On the second line, write

"Problem proposed by [your First Name, your Last Name]",

followed by your affiliation, city, country all on the same linear string of words. Please see the example below. Make sure you do the same for your collaborator(s) if any.

3. On a new line state the title of the problem, if any.

4. On a new line below the above, write in bold type: "**Statement of the Problem**".

5. Below the latter, state the problem. Please make sure the statement of your problem (unlike the preceding item) is not in bold type.

6. Below the statement of the problem, write in bold type: "**Solution of the Problem**".

7. Below the latter, show the entire solution of your problem.

Here is a sample for the above-stated format for proposed problems:

Problem proposed to SSMJ

Problem proposed by Isaac Newton, Trinity College, Cambridge, England.

Principia Mathematica (⟵ You may choose to not include a title.)

Statement of the problem:

Compute $\sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}$.

Solution of the problem: . . . . . .

♣ ♣ ♣ Thank You! ♣ ♣ ♣