Problems and Solutions

Albert Natian, Section Editor
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This section of the Journal offers readers an opportunity to exchange interesting mathematical problems and solutions. Please email them to Prof. Albert Natian at Department of Mathematics, Los Angeles Valley College. Please make sure every proposed problem or proposed solution is provided in both ***LaTeX*** and pdf documents. Thank you!

To propose problems, email them to: problems4ssma@gmail.com

To propose solutions, email them to: solutions4ssma@gmail.com

Solutions to the problems published in this issue should be submitted *before* October 1, 2025.

• **5810** *Proposed by Daniel Sitaru, National Economic College "Theodor Costescu" Drobeta Turnu - Severin, Romania.*

If $x, y, z \in (0,1)$ and $xy + yz + zx = 1$, then show that:

$$(x^4+y^4)(y^4+z^4)(z^4+x^4) \geqslant \frac{8}{729}$$

• **5811** *Proposed by Prakash Pant, Mathematics Initiatives in Nepal, Bardiya, Nepal.*

Let a and b be real numbers such that $0 < a \leqslant b$. Prove that:

$$\int_a^b e^{x^{2023}} dx > \left(b^{\frac{2025}{2}} a^{\frac{2023}{2}} - a^{\frac{2025}{2}} b^{\frac{2023}{2}} \right).$$

• **5812** *Proposed by D.M. Bătineţu-Giurgiu, "Matei Basarab" National College, Bucharest, Romania and Neculai Stanciu, "George Emil Palade" School, Buzău, Romania.*

The convergent sequence $(a_n)_{n\geqslant 1}$ is defined by $a_n = \sum_{k=1}^{n} \left(\frac{1}{4k-3} - \frac{1}{4k-1} \right)$, with $a = \lim_{n\to\infty} a_n$. Compute

$$\Lambda := \lim_{n\to\infty} n \left(a^{m+1} - \prod_{i=0}^{m} a_{n+i} \right).$$

• **5813** *Proposed by Ivan Hadinata, Jember, Indonesia.*

With $n = 2023$, determine the greater of A and B where

$$A = \underbrace{\int_0^1 \int_0^1 \int_0^1 \cdots \int_0^1 \int_0^1}_{n \text{ times}} \left(\prod_{j=1}^{n} \frac{1-x_j^2}{1+x_j^2} \right) \prod_{j=1}^{n} dx_j,$$

$$B = \int_0^1 \left(\frac{1-x^2}{1+x^2}\right)^n \, dx.$$

• **5814** *Proposed by Problem Section Editor, Albert Natian.*

Where $\alpha > 0$, suppose the twice-differentiable funciton $\varphi : (-\infty, \infty) \to (0, \infty)$, with $\varphi(0) = 1/2$, satisfies the differential/functional equation

$$\alpha\varphi'(x) = \left[\alpha - e^x \varphi(-x)\right] \varphi(x)$$

for all x in $(-\infty, \infty)$. Find $y = \varphi(x)$.

Solutions

To Formerly Published Problems

• **5787** *Proposed by Albert Stadler, Herrliberg, Switzerland.*

Let n be a natural number. Prove

$$\sum_{k=1}^{n} \frac{(-1)^k}{k} \begin{pmatrix} 2n \\ n+k \end{pmatrix} = \begin{pmatrix} 2n \\ n \end{pmatrix} (H_n - H_{2n}),$$

where $H_k = \sum_{j=1}^{k} \frac{1}{j}$ denotes the k^{th} harmonic number.

Solution 1 by Péter Fülöp, Gyömrő, Hungary.

Let's extend the sum in the following way:

$$LHS = \sum_{k=1}^{n} \frac{(-1)^k}{k} \frac{(2n)!}{(n+k)!(n-k)!} = \sum_{k=1}^{n} \frac{(-1)^k}{k} \underbrace{\frac{(n)!}{k!(n-k)!}}_{\binom{n}{k}} \underbrace{\frac{(2n)}{(n!)^2}}_{\binom{2n}{n}} \underbrace{\frac{n!k!}{(n+k)!}}_{k\beta(k,n+1)}$$

where

$$\frac{n!k!}{(n+k)!} = \frac{k\Gamma(k)\Gamma(n+1)}{\Gamma(n+k+1)} = k\beta(k, n+1) = k \int_0^1 t^{k-1}(1-t)^n dt$$

$$LHS = \binom{2n}{n} \sum_{k=1}^{n} (-1)^k \binom{n}{k} \int_0^1 t^{k-1}(1-t)^n dt$$

The order of integration a summation can be swapped:

$$LHS = \binom{2n}{n} \int_0^1 (1-t)^n \underbrace{\sum_{k=1}^{n} \binom{n}{k} (-1)^k t^{k-1}}_{\dfrac{(1-t)^n - 1}{t}} dt$$

$$LHS = \binom{2n}{n} \Big[\int_0^1 \frac{(1-t)^{2n}}{t} - \frac{1}{t} - \frac{(1-t)^n}{t} + \frac{1}{t} dt \Big]$$

After performing the substitution $t = 1 - x$. It can be seen that the statement is proved.

$$LHS = \binom{2n}{n} \Big[\int_0^1 \frac{1-x^n}{1-x} dx - \int_0^1 \frac{1-x^{2n}}{1-x} dx \Big] = \binom{2n}{n} \Big(H_n - H_{2n} \Big)$$

Solution 2 by Perfetti Paolo, dipartimento di matematica Università di “Tor Vergata", Roma, Italy.

Observe

$$\binom{n}{k} = \frac{1}{2\pi i} \oint_{|z|=d<1} \frac{dz}{(1-z)^{k+1} z^{n-k+1}}.$$

From now on all the complex integrals are performed over a circle surrounding the origin and with radius less than 1.

$$\sum_{k=1}^{n} \frac{(-1)^k}{k} \binom{2n}{n+k} = \sum_{k=1}^{\infty} \frac{(-1)^k}{k} \binom{2n}{n+k} = \sum_{k=1}^{\infty} \frac{(-1)^k}{k} \frac{1}{2\pi i} \oint \frac{dz}{(1-z)^{n+k+1} z^{n-k+1}} =$$

$$= \frac{1}{2\pi i} \oint \frac{-\mathrm{Ln}\left(1 + \frac{z}{1-z}\right) dz}{(1-z)^{n+1} z^{n+1}} = \frac{1}{2\pi i} \oint \frac{\mathrm{Ln}(1-z) dz}{(1-z)^{n+1} z^{n+1}} \underbrace{=}_{z - z^2 = w} \frac{1}{2\pi i} \oint \frac{\mathrm{Ln} \frac{1+\sqrt{1-4w}}{2} dw}{\sqrt{1-4w} w^{n+1}}$$

$$\left[\mathrm{Ln} \frac{1 + \sqrt{1-4w}}{2} \right]' = \frac{1}{2w} - \frac{1}{2w\sqrt{1-4w}}$$

thus

$$\frac{1}{2\pi i}\oint \frac{\mathrm{Ln}\frac{1+\sqrt{1-4w}}{2}dw}{\sqrt{1-4w}w^{n+1}} = \frac{1}{2\pi i}\oint \mathrm{Ln}\frac{1+\sqrt{1-4w}}{2}\left[1-2w\left(\mathrm{Ln}\frac{1+\sqrt{1-4w}}{2}\right)'\right]\frac{dw}{w^{n+1}} =$$

$$= \frac{1}{2\pi i}\oint\left[\mathrm{Ln}\frac{1+\sqrt{1-4w}}{2} - w\left(\left(\mathrm{Ln}\frac{1+\sqrt{1-4w}}{2}\right)^2\right)'\right]\frac{dw}{w^{n+1}} =$$

$$= [w^n]\mathrm{Ln}\frac{1+\sqrt{1-4w}}{2} - [w^{n-1}]\left(\left(\mathrm{Ln}\frac{1+\sqrt{1-4w}}{2}\right)^2\right)' =$$

$$= [w^n]\mathrm{Ln}\frac{1+\sqrt{1-4w}}{2} - n[w^n]\left(\mathrm{Ln}\frac{1+\sqrt{1-4w}}{2}\right)^2.$$

Now we make use of some results on web: https://math.stackexchange.com/questions/1148203/identity-with-harmonic-and-catalan-numbers:

$$[w^n]\mathrm{Ln}\frac{1+\sqrt{1-4w}}{2} = \frac{-1}{n}\binom{2n-1}{n}$$

$$[w^n]\left(\mathrm{Ln}\frac{1+\sqrt{1-4w}}{2}\right)^2 = \binom{2n}{n}(H_{2n-1}-H_n)\frac{1}{n}.$$

Based on this we have

$$\frac{-1}{n}\binom{2n-1}{n} + \binom{2n}{n}(H_n - H_{2n} + \frac{1}{2n}) = \binom{2n}{n}(H_n - H_{2n}).$$

Solution 3 by Michel Bataille, Rouen, France.

From $\binom{2n}{n+k} = \binom{2n}{n}\cdot\frac{\binom{n}{k}}{\binom{n+k}{k}}$ (readily checked) and $H_{2n} - H_n = \sum_{j=1}^{n}\frac{1}{n+j}$, we deduce that all boils down to prove

$$\sum_{k=1}^{n}\binom{n}{k}(-1)^{k+1}\frac{1}{k\binom{n+k}{k}} = \sum_{j=1}^{n}\frac{1}{n+j}. \qquad (1)$$

Let L_n denote the sum on the left hand-side of (1). Since

$$\frac{1}{k\binom{n+k}{k}} = \frac{(k-1)!n!}{(n+k)!} = \frac{\Gamma(k)\Gamma(n+1)}{\Gamma(n+1+k)} = B(n+1,k) = \int_0^1 x^n(1-x)^{k-1}\,dx$$

we obtain that

$$L_n = \int_0^1\left(\sum_{k=1}^{n}\binom{n}{k}(-1)^{k+1}x^n(1-x)^{k-1}\right)\,dx = \int_0^1\left(x^n\sum_{k=1}^{n}\binom{n}{k}(x-1)^{k-1}\right)\,dx,$$

hence

$$L_n = \int_0^1 x^n(1 + x + \cdots + x^{n-1})\,dx$$

(because of the polynomial identity $1 + x + \cdots + x^{n-1} = \sum_{k=1}^{n} \binom{n}{k} (x-1)^{k-1}$, a consequence of

$$\begin{aligned}(x-1)(1 + x + \cdots + x^{n-1}) &= x^n - 1 = (1 + (x-1))^n - 1 = \sum_{k=1}^{n} \binom{n}{k} (x-1)^k \\ &= (x-1) \sum_{k=1}^{n} \binom{n}{k} (x-1)^{k-1}.)\end{aligned}$$

Thus

$$L_n = \int_0^1 (x^n + x^{n+1} + \cdots + x^{2n-1})\,dx = \sum_{j=1}^{n} \frac{1}{n+j}$$

and (1) follows.

Also solved by the problem proposer.

• **5788** *Proposed by Ángel Plaza, Universidad de Las Palmas de Gran Canaria, Spain.*

Evaluate $L := \lim_{n\to\infty} \left(n^2\, x_n\right)$ where the sequence $(x_n)_{n\geqslant 2}$ is defined by

$$x_n = n\,\left(\frac{\sin\frac{1}{n}}{\sin\frac{1}{n^2}} - n\right) + \frac{1}{6}.$$

Solution 1 by Ulrich Abel, Technische Hochschule Mittelhessen, Friedberg, Germany.

It is a matter of calculus to calculate that

$$x\frac{\sin x}{\sin x^2} = 1 - \frac{x^2}{6} + \frac{7}{40}x^4 + O\left(x^6\right) \qquad (x \to 0)\,.$$

It follows that

$$\frac{1}{x}\left(\frac{\sin x}{\sin x^2} - \frac{1}{x}\right) + \frac{1}{6} = \frac{7}{40}x^2 + O\left(x^4\right) \qquad (x \to 0)\,.$$

Substituting $x = 1/n$, we obtain

$$\lim_{n\to\infty} n^2 x_n = \frac{7}{40}.$$

Solution 2 by Brian D. Beasley, Simpsonville, SC.

We note the similar problem #5769 and follow the methods used in its solutions (December 2024 issue, pages 2-8). Since

$$\sin\frac{1}{n} = \frac{1}{n} - \frac{1}{6n^3} + \frac{1}{120n^5} - \frac{1}{5040n^7} + O\left(\frac{1}{n^9}\right)$$

and

$$\sin\frac{1}{n^2} = \frac{1}{n^2} - \frac{1}{6n^6} + O\left(\frac{1}{n^{10}}\right),$$

we have

$$\sin\frac{1}{n} - n\sin\frac{1}{n^2} = -\frac{1}{6n^3} + \frac{7}{40n^5} - \frac{1}{5040n^7} + O\left(\frac{1}{n^9}\right).$$

This in turn yields

$$\begin{aligned} x_n \sin\frac{1}{n^2} &= n\left[-\frac{1}{6n^3} + \frac{7}{40n^5} - \frac{1}{5040n^7} + O\left(\frac{1}{n^9}\right)\right] + \frac{1}{6}\left[\frac{1}{n^2} - \frac{1}{6n^6} + O\left(\frac{1}{n^{10}}\right)\right] \\ &= \frac{7}{40n^4} - \frac{47}{1680n^6} + O\left(\frac{1}{n^8}\right). \end{aligned}$$

Hence we obtain

$$L = \lim_{n\to\infty} n^2 x_n = \lim_{n\to\infty} \frac{\frac{7}{40n^2} - \frac{47}{1680n^4} + O\left(\frac{1}{n^6}\right)}{\frac{1}{n^2} - \frac{1}{6n^6} + O\left(\frac{1}{n^{10}}\right)} = \frac{7}{40}.$$

Solution 3 by Henry Ricardo, Westchester Area Math Circle, Purchase, NY.

We have

$$\begin{aligned} \frac{\sin\frac{1}{n}}{\sin\frac{1}{n^2}} &= \frac{\frac{1}{n} - \frac{1}{6n^3} + \frac{1}{120n^5} + O\left(\frac{1}{n^7}\right)}{\frac{1}{n^2} - \frac{1}{6n^6} + \frac{1}{120n^{10}} + O\left(\frac{1}{n^{14}}\right)} = n - \frac{1}{6n} + \frac{7}{40n^3} + O\left(\frac{1}{n^5}\right), \\ x_n &= n\left(\frac{\sin\frac{1}{n}}{\sin\frac{1}{n^2}} - n\right) + \frac{1}{6} = \frac{7}{40n^2} + O\left(\frac{1}{n^4}\right), \\ n^2 x_n &= \frac{7}{40} + O\left(\frac{1}{n^2}\right), \end{aligned}$$

which has the limit $7/40$ as $n \to \infty$.

Solution 4 by Albert Stadler, Herrliberg, Switzerland.

We have

$$n^2 x_n = n^3\left(\frac{\sin\frac{1}{n}}{\sin\frac{1}{n^2}} - n\right) + \frac{n^2}{6} = n^3\left(\frac{\frac{1}{n} - \frac{1}{6n^3} + \frac{1}{120n^5} + O\left(\frac{1}{n^7}\right)}{\frac{1}{n^2} - \frac{1}{6n^6} + O\left(\frac{1}{n^{10}}\right)} - n\right) + \frac{n^2}{6} =$$

$$= n^3\left(\frac{-\frac{1}{6n^3} + \frac{1}{120n^5} + \frac{1}{6n^5} + O\left(\frac{1}{n^7}\right)}{\frac{1}{n^2} - \frac{1}{6n^6} + O\left(\frac{1}{n^{10}}\right)}\right) + \frac{n^2}{6} = n^3\left(\frac{-\frac{1}{6n} + \frac{7}{40n^3} + O\left(\frac{1}{n^5}\right)}{1 - \frac{1}{6n^4} + O\left(\frac{1}{n^8}\right)}\right) + \frac{n^2}{6} =$$

$$= n^3\left(-\frac{1}{6n} + \frac{7}{40n^3} + O\left(\frac{1}{n^5}\right)\right) + \frac{n^2}{6} = \frac{7}{40} + O\left(\frac{1}{n^2}\right).$$

So L=7/40.

Solution 5 by David A. Huckaby, Angelo State University, San Angelo, TX.

Letting $m = \dfrac{1}{n}$, we have

$$\begin{aligned}
\frac{\sin\frac{1}{n}}{\sin\frac{1}{n^2}} = \frac{\sin m}{\sin m^2} &= \frac{m - \frac{m^3}{6} + \frac{m^5}{120} + O(m^7)}{m^2 - \frac{m^6}{6} + O(m^{10})} \\
&= \frac{1}{m^2}\left(m - \frac{m^3}{6} + \frac{m^5}{120} + O(m^7)\right)\left(1 - \frac{m^4}{6} + O(m^8)\right)^{-1} \\
&= \frac{1}{m^2}\left(m - \frac{m^3}{6} + \frac{m^5}{120} + O(m^7)\right)\left(1 + \frac{m^4}{6} + O(m^8)\right) \\
&= \frac{1}{m^2}\left(m - \frac{m^3}{6} + \frac{m^5}{6} + \frac{m^5}{120} + O(m^7)\right) \\
&= \frac{1}{m^2}\left(m - \frac{m^3}{6} + \frac{7m^5}{40} + O(m^7)\right) \\
&= \frac{1}{m} - \frac{m}{6} + \frac{7m^3}{40} + O(m^5) \\
&= n - \frac{1}{6n} + \frac{7}{40n^3} + O\left(\frac{1}{n^5}\right).
\end{aligned}$$

So

$$
\begin{aligned}
L := \lim_{n\to\infty}\left(n^2 x_n\right) &= \lim_{n\to\infty}\left(n^2\left[n\left(\frac{\sin\frac{1}{n}}{\sin\frac{1}{n^2}} - n\right) + \frac{1}{6}\right]\right)\\
&= \lim_{n\to\infty}\left(n^2\left[n\left(n - \frac{1}{6n} + \frac{7}{40n^3} + O\left(\frac{1}{n^5}\right) - n\right) + \frac{1}{6}\right]\right)\\
&= \lim_{n\to\infty}\left(n^2\left[n\left(-\frac{1}{6n} + \frac{7}{40n^3} + O\left(\frac{1}{n^5}\right)\right) + \frac{1}{6}\right]\right)\\
&= \lim_{n\to\infty}\left(n^2\left[-\frac{1}{6} + \frac{7}{40n^2} + O\left(\frac{1}{n^4}\right) + \frac{1}{6}\right]\right)\\
&= \lim_{n\to\infty}\left(n^2\left[\frac{7}{40n^2} + O\left(\frac{1}{n^4}\right)\right]\right)\\
&= \lim_{n\to\infty}\left(\frac{7}{40} + O\left(\frac{1}{n^2}\right)\right)\\
&= \frac{7}{40}.
\end{aligned}
$$

Solution 6 by Michel Bataille, Rouen, France.

We have $L = \lim_{n\to\infty} X_n$ where

$$X_n = n^3\left(\frac{\sin\frac{1}{n}}{\sin\frac{1}{n^2}} - n\right) + \frac{n^2}{6} = \frac{n^2}{\sin\frac{1}{n^2}}\left(n\sin\frac{1}{n} - n^2\sin\frac{1}{n^2} + \frac{1}{6}\sin\frac{1}{n^2}\right).$$

From $\sin\frac{1}{n} = \frac{1}{n} - \frac{1}{6n^3} + \frac{1}{120n^5} + o(1/n^6)$, we obtain $\sin\frac{1}{n^2} = \frac{1}{n^2} - \frac{1}{6n^6} + o(1/n^6)$ and

$$n\sin\frac{1}{n} - n^2\sin\frac{1}{n^2} + \frac{1}{6}\sin\frac{1}{n^2} = \frac{1}{120n^4} + \frac{1}{6n^4} + o(1/n^4) = \frac{7}{40n^4} + o(1/n^4)$$

Since

$$\frac{n^2}{\sin\frac{1}{n^2}} \sim \frac{n^2}{\frac{1}{n^2}} = n^4$$

as $n \to \infty$ we finally obtain $L = \frac{7}{40}$.

Solution 7 by Perfetti Paolo, dipartimento di matematica Università di "Tor Vergata", Roma, Italy.

$$L = \lim_{n\to\infty} n^3 \left(\frac{n^2 \sin\frac{1}{n}}{n^2 \sin\frac{1}{n^2}} - n \right) + \frac{n^2}{6} =$$
$$= \lim_{n\to\infty} \frac{6n^5 \sin\frac{1}{n} - 6n^6 \sin\frac{1}{n^2} + n^4 \sin\frac{1}{n^2}}{6n^2 \sin\frac{1}{n^2}} =$$
$$= \frac{1}{6}\lim_{n\to\infty} \left(6n^4 - n^2 + \frac{6}{120} + O(\frac{1}{n^2}) - 6n^4 + 1 + O(\frac{1}{n^4}) + n^2 + O(\frac{1}{n^2}) \right) = \frac{7}{40}$$

Also solved by Bruno Salgueiro Fanego, Viveiro, Lugo; Akash Chaudhary, Pulchowk Campus, Kapilvastu, Nepal; Devis Alvarado, UNAH and UPNFM, Tegucigalpa, Honduras and the problem proposer.

• **5789** *Proposed by Daniel Sitaru, National Economic College "Theodor Costescu" Drobeta Turnu - Severin, Romania.*

Let $0 < a \leqslant b$. Suppose $f : [a, b] \to (0, \infty)$ is a continuous function. Then

$$\int_a^b \int_a^b \int_a^b \left(f^2(x) + f^2(y) + f^2(z) \right)^2 dxdydz \geqslant 9(b-a) \left(\int_a^b f(x)dx \right) \left(\int_a^b f^3(x)dx \right).$$

Solution 1 by Michel Bataille, Rouen, France.

We use the following lemma: if a, b, c are real numbers, then $(a^2 + b^2 + c^2)^2 \geqslant 3(a^3b + b^3c + c^3a)$. *Proof* (from Z. Cvetkovski, *Inequalities*, Springer, 2012, p. 227) Let $x = a^2 - ab + bc$, $y = b^2 - bc + ca$, $z = c^2 - ca + ab$. The inequality directly follows from the well-known $(x + y + z)^2 \geqslant 3(xy + yz + zx)$.
From this lemma, we have

$$\left(f^2(x) + f^2(y) + f^2(z) \right)^2 \geqslant 3 \left(f^3(x)f(y) + f^3(y)f(z) + f^3(z)f(x) \right)$$

for all $(x, y, z) \in [a, b]^3$. Integrating, we obtain

$$\int_a^b \int_a^b \int_a^b \left(f^2(x) + f^2(y) + f^2(z) \right)^2 dxdydz \geqslant$$
$$3 \left((b-a) \int_a^b \int_a^b f^3(x)f(y)\, dxdy + (b-a) \int_a^b \int_a^b f^3(y)f(z)\, dydz + (b-a) \int_a^b \int_a^b f^3(z)f(x)\, dxdz \right)$$

Now, if $I = \displaystyle\int_a^b f^3(x)$ and $J = \displaystyle\int_a^b f(x)\, dx$, we have

$$\int_a^b \int_a^b f^3(x)f(y)\, dxdy = \left(\int_a^b f^3(x)\, dx \right) \left(\int_a^b f(y)\, dy \right) = I \cdot J$$

and similarly,

$$\int_a^b \int_a^b f^3(y)f(z)\,dydz = I \cdot J = \int_a^b \int_a^b f^3(z)f(x)\,dxdz$$

Thus, we have

$$\int_a^b \int_a^b \int_a^b \left(f^2(x) + f^2(y) + f^2(z)\right)^2\, dxdydz \geqslant 3(3(b-a)I \cdot J) = 9(b-a)IJ,$$

as desired.

Solution 2 by Perfetti Paolo, dipartimento di matematica Università di “Tor Vergata", Roma, Italy.

The inequality is

$$\int_a^b\int_a^b\int_a^b\Big(f^4(x) + f^4(y) + f^4(z) + 2(f(x)f(y))^2 + \\ +2(f(y)f(z))^2 + 2(f(x)f(z))^2\Big)dx\,dy\,dz \geqslant \\ \geqslant 9(b-a)\left(\int_a^b f(x)dx\right)\left(\int_a^b f^3(x)dx\right)$$

or

$$3(b-a)^2\int_a^b f^4(x)dx + 6(b-a)\left(\int_a^b f^2(x)dx\right)\left(\int_a^b f^2(y)dy\right) \geqslant \\ \geqslant 9(b-a)\left(\int_a^b f(x)dx\right)\left(\int_a^b f^3(x)dx\right)$$

$$(b-a)\int_a^b f^4(x)dx + 2\left(\int_a^b f^2(x)dx\right)\left(\int_a^b f^2(y)dy\right) \geqslant \\ \geqslant 3\left(\int_a^b f(x)dx\right)\left(\int_a^b f^3(x)dx\right)$$

This may be rewritten as

$$\int_a^b\int_a^b f^4(x)dxdy + 2\int_a^b\int_a^b f^2(x)f^2(y)dx\,dy \geqslant 3\int_a^b\int_a^b f(x)f^3(y)dxdy$$

that is

$$\int_a^b\int_a^b \left(f^4(x) + 2f^2(x)f^2(y) - 3f(x)f^3(y)\right)dxdy$$

or

$$\int_a^b \int_a^b f(x)(f(x)-f(y))(f^2(x)+f(x)f(y)+3f^2(y))dxdy$$

Now let's consider the two points (x, y) and (y, x). The sum of the two terms are

$$\begin{aligned}&f(x)(f(x)-f(y))(f^2(x)+f(x)f(y)+3f^2(y))+\\&\qquad\qquad+f(y)(f(y)-f(x))(f^2(y)+f(y)f(x)+3f^2(x))=\\&=(f(x)-f(y))^2(f^2(x)+f^2(y)+3f(x)f(y))\geqslant 0\end{aligned}$$

thus concluding the proof.

Also solved by Albert Stadler, Herrliberg, Switzerland and the problem proposer.

• **5790** *Proposed by Michel Bataille, Rouen, France.*

Here $\llbracket \cdot \rrbracket$ denotes the floor (greatest integer value) function. Solve for $n \in \mathbb{N}$:

$$\sum_{k=1}^{n} \left\llbracket \sqrt{n^2+2k-1}+\sqrt{n^2+2k} \right\rrbracket = 205.$$

Solution 1 by Albert Stadler, Herrliberg, Switzerland.

We note that for n⩾1

$$2n < \sqrt{n^2+n-1}+\sqrt{n^2+n} < 2n+1$$

and

$$2n+1 < \sqrt{n^2+n}+\sqrt{n^2+n+1} < 2n+2,$$

for

$$\left(n+\frac{1}{2}-\sqrt{n^2+n-1}\right)+\left(n+\frac{1}{2}-\sqrt{n^2+n}\right)=$$

$$=\frac{\frac{5}{4}}{n+\frac{1}{2}+\sqrt{n^2+n-1}}+\frac{\frac{1}{4}}{n+\frac{1}{2}+\sqrt{n^2+n}}>0$$

and

$$\sqrt{n^2+n}+\sqrt{n^2+n+1}-2n-1=\frac{\left(\sqrt{n^2+n}+\sqrt{n^2+n+1}\right)^2-(2n+1)^2}{\sqrt{n^2+n}+\sqrt{n^2+n+1}+2n+1}=$$

$$=\frac{2\sqrt{n^2+n}\sqrt{n^2+n+1}-2n^2-2n}{\sqrt{n^2+n}+\sqrt{n^2+n+1}+2n+1}=\frac{2\sqrt{n^2+n}\left(\sqrt{n^2+n+1}-\sqrt{n^2+n}\right)}{\sqrt{n^2+n}+\sqrt{n^2+n+1}+2n+1}>0.$$

The inequalities $2n < \sqrt{n^2+n-1}+\sqrt{n^2+n}$ and $\sqrt{n^2+n}+\sqrt{n^2+n+1} < 2n+2$ are true, since $n \leqslant \sqrt{n^2+n-1}, n < \sqrt{n^2+n}, \sqrt{n^2+n} < n+1, \sqrt{n^2+n+1} < n+1$.

In addition

$$\sqrt{n^2+2n-1}+\sqrt{n^2+2n} < n+1+n+1 = 2n+2.$$

So

$$\sum_{k=1}^{n}\left[\left[\sqrt{n^2+2k-1}+\sqrt{n^2+2k}\right]\right] = \sum_{1\leqslant k\leqslant \frac{n}{2}} 2n + \sum_{\frac{n+1}{2}\leqslant k\leqslant n} (2n+1).$$

If n=2m then $\sum_{1\leqslant k\leqslant m} 2n + \sum_{m+1\leqslant k\leqslant n} (2n+1) = n-m+\sum_{1\leqslant k\leqslant n} 2n = 2n^2+n-\frac{n}{2}$

If n=2m+1 then $\sum_{1\leqslant k\leqslant m} 2n + \sum_{m+1\leqslant k\leqslant n} (2n+1) = n-m+\sum_{1\leqslant k\leqslant n} 2n = 2n^2+n-\frac{n-1}{2}$.

In total,

$$\sum_{k=1}^{n}\left[\left[\sqrt{n^2+2k-1}+\sqrt{n^2+2k}\right]\right] = 2n^2+\left[\left[\frac{n+1}{2}\right]\right].$$

The function $n\to 2n^2+\left[\left[\frac{n+1}{2}\right]\right]$ is increasing. Therefore the only value of n for which $2n^2+\left[\left[\frac{n+1}{2}\right]\right] = 205$ is n=10.

Solution 2 by Ángel Plaza, Universidad de Las Palmas de Gran Canaria, Spain.

The solution is $n=10$. We will show that $\left\lfloor\sqrt{10^2+2k-1}+\sqrt{10^2+2k}\right\rfloor = 20$ if $1\leqslant k\leqslant 5$, while $\left\lfloor\sqrt{10^2+2k-1}+\sqrt{10^2+2k}\right\rfloor = 21$ for $6\leqslant k\leqslant 10$.

For $1\leqslant k\leqslant 5$,
$10^2 < 10^2+2k-1 < 10^2+2k < 121$, so $10 < \sqrt{n^2+2k-1} < \sqrt{n^2+2k} < 11$ and then $20\leqslant\left\lfloor\sqrt{n^2+2k-1}+\sqrt{n^2+2k}\right\rfloor$. Also, for $1\leqslant k\leqslant 5$, $\left\lfloor\sqrt{10^2+2k-1}+\sqrt{10^2+2k}\right\rfloor\leqslant\left\lfloor\sqrt{119}+\sqrt{120}\right\rfloor < 21 \Leftrightarrow 119+120+2\sqrt{119\cdot 120} < 441$, that is $\sqrt{119\cdot120} < 220$ which it is true. Therefore $\left\lfloor\sqrt{10^2+2k-1}+\sqrt{10^2+2k}\right\rfloor = 20$.

Similarly, for $6\leqslant k\leqslant 10$: $\left\lfloor\sqrt{10^2+2k-1}+\sqrt{10^2+2k}\right\rfloor = 21$.

Solution 3 by Brian D. Beasley, Simpsonville, SC.

Let $n\in\mathbb{N}$. For $1\leqslant k\leqslant n$, let $a_k = \llbracket\sqrt{n^2+2k-1}+\sqrt{n^2+2k}\rrbracket$, and let $S_n = \sum_{k=1}^{n} a_k$. We show that

$$a_k = \begin{cases} 2n & \text{if } 1\leqslant k\leqslant n/2 \\ 2n+1 & \text{if } (n+1)/2\leqslant k\leqslant n \end{cases}$$

and hence for even n,

$$S_n = \frac{n}{2}(2n) + \frac{n}{2}(2n+1) = \frac{4n^2+n}{2},$$

while for odd n,

$$S_n = \frac{n-1}{2}(2n) + \frac{n+1}{2}(2n+1) = \frac{4n^2+n+1}{2}.$$

In particular, since $\{S_n\}$ is strictly increasing, we have $S_n = 205$ if and only if $n = 10$.

To establish the claim about a_k, we first note that for $1 \leqslant k \leqslant n$,

$$n^2+1 \leqslant n^2+2k-1 \leqslant n^2+2n-1 \quad \Rightarrow \quad n < \sqrt{n^2+2k-1} < n+1$$

and

$$n^2+2 \leqslant n^2+2k \leqslant n^2+2n \quad \Rightarrow \quad n < \sqrt{n^2+2k} < n+1.$$

This yields $2n < \sqrt{n^2+2k-1} + \sqrt{n^2+2k} < 2n+2$, so $a_k = 2n$ or $a_k = 2n+1$.

Next, we must show that for $1 \leqslant k \leqslant n/2$, $\sqrt{n^2+2k-1} + \sqrt{n^2+2k} < 2n+1$. Since $k \leqslant n/2$, we have

$$\sqrt{n^2+2k-1} + \sqrt{n^2+2k} \leqslant \sqrt{n^2+n-1} + \sqrt{n^2+n},$$

and this upper bound in turn is less than $2n+1$ if and only if

$$2\sqrt{(n^2+n-1)(n^2+n)} < 2n^2+2n+2.$$

This inequality holds, as $(n^2+n-1)(n^2+n) < (n^2+n+1)^2$. Hence $a_k = 2n$.

Finally, we must show that for $(n+1)/2 \leqslant k \leqslant n$, $\sqrt{n^2+2k-1} + \sqrt{n^2+2k} \geqslant 2n+1$. Since $k \geqslant (n+1)/2$, we have

$$\sqrt{n^2+2k-1} + \sqrt{n^2+2k} \geqslant \sqrt{n^2+n} + \sqrt{n^2+n+1},$$

and this upper bound in turn is greater than or equal to $2n+1$ if and only if

$$2\sqrt{(n^2+n)(n^2+n+1)} \geqslant 2n^2+2n.$$

This inequality holds, as $(n^2+n)(n^2+n+1) \geqslant (n^2+n)^2$. Hence $a_k = 2n+1$.

Solution 4 by the Eagle Problem Solvers, Georgia Southern University, Savannah, GA and Statesboro, GA.

The solution is $n = 10$.

For each $n \in \mathbb{N}$, let

$$S_n = \sum_{k=1}^{n} \lfloor \sqrt{n^2+2k-1} + \sqrt{n^2+2k} \rfloor.$$

Notice that for each integer k with $1 \leqslant k \leqslant n$, we have

$$2n = 2\sqrt{n^2} < \sqrt{n^2+2k-1} + \sqrt{n^2+2k} \leqslant \sqrt{n^2+2n-1} + \sqrt{n^2+2n} < 2\sqrt{n^2+2n+1} = 2(n+1),$$

so that

$$2n^2 < S_n < 2n(n+1).$$

If $n \leqslant 9$, then $S_n < 2n(n+1) \leqslant 180$, while if $n \geqslant 11$, then $S_n > 2n^2 \geqslant 242$. Direct computation shows that $S_{10} = 5 \cdot 20 + 5 \cdot 21 = 205$, so that the only solution is $n = 10$.

We remark that it is not difficult to prove an explicit formula for the left side of the given equation, namely that

$$S_n = 2n^2 + \left\lceil \frac{n}{2} \right\rceil.$$

Solution 5 by Henry Ricardo, Westchester Area Math Circle, Purchase, NY.

A known property of the floor function is that $\lfloor \sqrt{m} + \sqrt{m+1} \rfloor = \lfloor \sqrt{4m+3} \rfloor$ (see property P26 in [**?**].) Applying this to each summand yields

$$\left\lfloor \sqrt{n^2+2k-1} + \sqrt{n^2+2k} \right\rfloor = \left\lfloor \sqrt{4n^2+8k-1} \right\rfloor = \left\lfloor 2n\sqrt{1+\frac{8k-1}{4n^2}} \right\rfloor.$$

It is easily proved that

$$\left\lfloor 2n\sqrt{1+\frac{8k-1}{4n^2}} \right\rfloor = \begin{cases} 2n & \text{for } 1 \leqslant k \leqslant \lfloor \frac{n}{2} \rfloor \\ 2n+1 & \text{for } \lfloor \frac{n}{2} \rfloor + 1 \leqslant k \leqslant n. \end{cases}$$

Thus we have

$$\begin{aligned} \sum_{k=1}^{n} \left\lfloor \sqrt{n^2+2k-1} + \sqrt{n^2+2k} \right\rfloor &= \sum_{k=1}^{\lfloor n/2 \rfloor} 2n + \sum_{k=\lfloor n/2 \rfloor+1}^{n} (2n+1) \\ &= (2n)\left\lfloor \frac{n}{2} \right\rfloor + \left(n - \left\lfloor \frac{n}{2} \right\rfloor\right)(2n+1) \\ &= 2n^2 + n - \left\lfloor \frac{n}{2} \right\rfloor. \end{aligned}$$

If n is even, the equation $2n^2 + \dfrac{n}{2} = 205$, or $4n^2 + n = 410$, has the positive integer solution $n = 10$. If n is odd, we have $2n^2 + n - \dfrac{n-1}{2} = 205$, or $4n^2 + n - 409$, which has no positive integer solutions. Thus $n = 10$ is the only solution satisfying the conditions of our problem.

[1] Xingbo Wang, "Frequently-Used Properties of the Floor Function," *Int. J. of Applied Physics and Math.*, **10** (2020), 135-142.

Solution 6 by Ulrich Abel, Technische Hochschule Mittelhessen, Friedberg, Germany.

For $1 \leqslant k \leqslant n/2$, we have

$$n^2 < n^2 + 2k - 1 < n^2 + 2k \leqslant n^2 + n < \left(n + 1/2\right)^2$$

and, consequently, $2n < \sqrt{n^2 + 2k - 1} + \sqrt{n^2 + 2k} < 2n + 1$. For $n/2 + 1 \leqslant k \leqslant n$, we have

$$\left(n + 1/2\right)^2 < n^2 + n + 1 = n^2 + 2\left(n + 2\right)/2 - 1 \leqslant n^2 + 2k - 1 < n^2 + 2k \leqslant n^2 + 2n < \left(n + 1\right)^2$$

and, consequently, $2n + 1 < \sqrt{n^2 + 2k - 1} + \sqrt{n^2 + 2k} < 2\left(n + 1\right)$.

For even n, we infer that

$$\left\lfloor \sqrt{n^2 + 2k - 1} + \sqrt{n^2 + 2k} \right\rfloor = \begin{cases} 2n & \left(1 \leqslant k \leqslant n/2\right), \\ 2n + 1 & \left(n/2 + 1 \leqslant k \leqslant n\right). \end{cases}$$

It follows that

$$s_n := \sum_{k=1}^{n} \left\lfloor \sqrt{n^2 + 2k - 1} + \sqrt{n^2 + 2k} \right\rfloor = \frac{n}{2} \cdot 2n + \frac{n}{2} \cdot (2n + 1) = 2n^2 + \frac{n}{2}.$$

In particular, we have $s_{10} = 205$. Observing that (s_n) is strictly increasing, the only solution is $n = 10$.

Also solved by the problem proposer.

• **5791** *Proposed by Vasile Cirtoaje, Petroleum-Gas University of Ploiesti, Romania.*

Given any non-negative real numbers $a_1, a_2, \ldots, a_9$ with $a_1 \geqslant a_2 \geqslant \cdots \geqslant a_9$, prove that 5 is the least positive value of k for which

$$\left(\frac{ka_1 + a_2 + \cdots + a_8}{k + 7}\right)^2 \geqslant \frac{a_1^2 + a_2^2 + \cdots + a_9^2}{9}.$$

Solution 1 by Perfetti Paolo, dipartimento di matematica Università di "Tor Vergata", Roma, Italy.

Let $a_3 = a_2 = a_1 = a$, and $a_k = 0$ for any $k \geqslant 4$. The inequality becomes

$$\frac{a^2(5k + 23)(k - 5)}{9(k + 7)^2} \geqslant 0,$$

hence $k \geqslant 5$. Let's prove the inequality for $k = 5$. Let's set

$$a_9 = a, \quad a_8 = a_9 + x, \quad a_7 = a_8 + y = a_9 + x + y, \quad a_6 = a_7 + z = a_9 + x + y + z$$

$$a_5 = a_6 + t = a_9 + x + y + z + t, \quad a_4 = a_5 + s = a_9 + x + y + z + t + s$$

$$a_3 = a_4 + r = a_9 + x + y + z + t + s + r, \quad a_2 = a_3 + u = a_9 + x + y + z + t + s + r + u$$

$$a_1 = a_2 + v = a_9 + x + y + z + t + s + r + u + v.$$

The inequality multiplied by 144 becomes

$$\begin{aligned}
&32ax + 40ay + 48az + 56at + 64as + 72ar + 80au + 88av + 16x^2 + 48xz + \\
&+56xt + 64xs + 72xr + 80xu + 9y^2 + 28yz + 38yt + 48ys + 58yr + \\
&+68yu + 4z^2 + 40xy + 32zs + 44zr + 56zu + t^2 + 16ts + 30tr + \\
&+44tu + 16sr + 32su + r^2 + 20ru + 4u^2 + 20zt + 88vx + 78vy \\
&+68vz + 58vt + 48vs + 38vr + 28vu + 9v^2 \geqslant 0
\end{aligned}$$

which is clearly true.

Solution 2 by Albert Stadler, Herrliberg, Switzerland.

Put $a_1 = a_2 = a_3 = a_4 = 1$, $a_5 = a_6 = a_7 = a_8 = a_9 = 0$. The inequality then implies

$$\left(\frac{k+3}{k+7}\right)^2 \geqslant \frac{4}{9}$$

and $k \geqslant 5$. If the inequality holds true for some k it holds true as well for $k' > k$, since inequality

$$\frac{k'a_1 + a_2 + \cdots + a_8}{k' + 7} \geqslant \frac{ka_1 + a_2 + \cdots + a_8}{k + 7}$$

is equivalent to the inequality $\left(k' - k\right)\left(7a_1 - (a_2 + \cdots + a_8)\right) \geqslant 0$. Hence it remains to prove that

$$f\left(a_1, a_2, a_3, a_4, a_5, a_6, a_7, a_8, a_9\right) := \left(\frac{5a_1 + a_2 + \cdots + a_8}{12}\right)^2 - \frac{a_1^2 + a_2^2 + \cdots + a_9^2}{9} \geqslant 0. \qquad (*)$$

By assumption there are nonnegative numbers $b_1, b_2, b_3, b_4, b_5, b_6, b_7, b_8, b_9$ such that

$$a_9 = b_9, a_8 = b_8 + b_9, a_7 = b_7 + b_8 + b_9, a_6 = b_6 + b_7 + b_8 + b_9, a_5 = b_5 + b_6 + b_7 + b_8 + b_9,$$

$$a_4 = b_4 + b_5 + b_6 + b_7 + b_8 + b_9, a_3 = b_3 + b_4 + b_5 + b_6 + b_7 + b_8 + b_9,$$

$$a_2 = b_2 + b_3 + b_4 + b_5 + b_6 + b_7 + b_8 + b_9, a_1 = b_1 + b_2 + b_3 + b_4 + b_5 + b_6 + b_7 + b_8 + b_9.$$

We expand

$$f(b_1 + b_2 + b_3 + b_4 + b_5 + b_6 + b_7 + b_8 + b_9, b_2 + b_3 + b_4 + b_5 + b_6 + b_7 + b_8 + b_9,$$

$b_3+b_4+b_5+b_6+b_7+b_8+b_9, b_4+b_5+b_6+b_7+b_8+b_9, b_5+b_6+b_7+b_8+b_9,$

$b_6+b_7+b_8+b_9, b_7+b_8+b_9, b_8+b_9, b_9)$

and note that this is a polynomial in the variables $b_j, 1 \leqslant j \leqslant 1$, all of whose coefficients are nonnegative. Hence (*) holds true.

Also solved by the problem proposer.

• **5792** *Proposed by Shivam Sharma, Delhi University, New Delhi, India.*

Let $s_n = -2\sqrt{n} + \sum_{k=1}^{n} \frac{1}{\sqrt{k}}$ with $\lim_{n\to\infty} s_n = s$ (Ioachimescu Constant). For non-negative intger m, evaluate:

$$L = \lim_{n\to\infty}\left(s^{m+1} - \prod_{j=n}^{n+m} s_j\right)\sqrt{n}.$$

Solution 1 by Perfetti Paolo, dipartimento di matematica Università di "Tor Vergata", Roma, Italy.

We employ Theorem 1 of Chao–Ping Chen, "Ioachimescu's constant"

$$\sum_{k=1}^{n} \frac{1}{\sqrt{k}} - 2\sqrt{n} = s + \frac{1}{2\sqrt{n}} - \sum_{k=1}^{p} \frac{B_{2k}}{(2k)!}\frac{(4k-3)!!}{2^{2k-1}}\frac{1}{n^{2k-\frac{1}{2}}} + O(n^{-2p-3/2})$$

B_{2k} is $2k$–th Bernoulli's number.

$$\prod_{j=n}^{n+m} s_j = \prod_{j=n}^{n+m}\left(s + \frac{1}{2\sqrt{j}} + O(\frac{1}{j^{\frac{3}{2}}})\right) =$$

$$= s^{m+1} + s^m \sum_{j=n}^{n+m} \frac{1}{2\sqrt{j}} + s^m \sum_{j=n}^{n+m} O(\frac{1}{j^{3/2}}) + s^{m-1} \sum_{j_0=n}^{n+m} \sum_{j=n, j\neq j_0}^{n+m} \frac{O(j_0^{-3/2})}{2\sqrt{j}} +$$

$$+ s^{m-1} \sum_{j_0=n}^{n+m} \sum_{j=n, j\neq j_0}^{n+m} \frac{O(j^{-3/2})}{2\sqrt{j_0}} +$$

$$+ \sum_{r=0}^{m-2} s^r \sum_{\substack{k,p=0\\ k+p=m+1-r}}^{m+1-r} \sum_{\substack{r_1,\dots,r_k=0\\ r_1<r_2<\dots<r_k}}^{m} \prod_{l_1=1}^{k} \frac{1}{2\sqrt{n+r_{l_i}}} \sum_{\substack{r_1,\dots,r_p=0\\ r_1<r_2<\dots<r_p}}^{m} \prod_{l_2=1}^{p} O((n+r_{l_2})^{-3/2})$$

hence $\left(s^{m+1} - \prod_{j=n}^{n+m} s_j\right)\sqrt{n}$ is equal to

$$-s^m\sqrt{n}\sum_{j=n}^{n+m}\frac{1}{2\sqrt{j}} - s^m\sqrt{n}\sum_{j=n}^{n+m}O(\frac{1}{j^{3/2}}) - s^{m-1}\sqrt{n}\sum_{j_0=n}^{n+m}\sum_{j=n,j\neq j_0}^{n+m}\frac{O(j_0^{-3/2})}{2\sqrt{j}} -$$
$$-s^{m-1}\sqrt{n}\sum_{j_0=n}^{n+m}\sum_{j=n,j\neq j_0}^{n+m}\frac{O(j^{-3/2})}{2\sqrt{j_0}} +$$
$$-\sqrt{n}\sum_{r=0}^{m-2}s^r\sum_{\substack{k,p=0\\k+p=m+1-r}}^{m+1-r}\sum_{\substack{r_1,\dots,r_k=0\\r_1<r_2<\cdots<r_k}}^{m}\prod_{l_1=1}^{k}\frac{1}{2\sqrt{n+r_{l_i}}}\sum_{\substack{r_1,\dots,r_p=0\\r_1<r_2<\cdots<r_p}}^{m}\prod_{l_2=1}^{p}O((n+r_{l_2})^{-3/2})$$

and we prove that the limit of all the above expressions are zero except for the first one.

$$\frac{m+1}{\sqrt{n+m}+\sqrt{n-1}} = \frac{1}{2}\int_{n-1}^{n+m}\frac{dx}{\sqrt{x}} \leqslant \sum_{j=n}^{n+m}\frac{1}{2\sqrt{j}} \leqslant \frac{1}{2}\int_{n}^{n+m+1}\frac{dx}{\sqrt{x}} = \frac{m+1}{\sqrt{n+m+1}+\sqrt{n}}$$

hence by the squeezing theorem

$$\lim_{n\to\infty} s^m\sqrt{n}\sum_{j=n}^{n+m}\frac{1}{2\sqrt{j}} = \frac{s^m(m+1)}{2}$$

By $|O(j^{-3/2})| \leqslant Cj^{-3/2}$

$$\frac{1}{(n-1)^{3/2}} - \frac{1}{(m+n)^{3/2}} = \frac{1}{2}\int_{n-1}^{n+m}\frac{dx}{x^{3/2}} \leqslant$$
$$\leqslant \sum_{j=n}^{n+m}\frac{1}{j^{3/2}} \leqslant \int_{n}^{n+m+1}\frac{dx}{x^{3/2}} = \frac{1}{n^{3/2}} - \frac{1}{(m+n+1)^{3/2}}$$

hence $\lim_{n\to\infty} s^m\sqrt{n}\sum_{j=n}^{n+m}O(\frac{1}{j^{3/2}}) = 0$. A fortiori

$$\lim_{n\to\infty} s^{m-1}\sqrt{n}\sum_{j_0=n}^{n+m}\sum_{j=n,j\neq j_0}^{n+m}\frac{O(j_0^{-3/2})}{2\sqrt{j}} = 0$$

and

$$\lim_{n\to\infty} s^{m-1}\sqrt{n}\sum_{j_0=n}^{n+m}\sum_{j=n,j\neq j_0}^{n+m}\frac{O(j^{-3/2})}{2\sqrt{j_0}} = 0$$

Moreover it is straightforward to estimate

$$\sum_{r=0}^{m-2}s^r\sum_{\substack{k,p=0\\k+p=m+1-r}}^{m+1-r}\sum_{\substack{r_1,\dots,r_k=0\\r_1<r_2<\cdots<r_k}}^{m}\prod_{l_1=1}^{k}\frac{1}{2\sqrt{n+r_{l_i}}}\sum_{\substack{r_1,\dots,r_p=0\\r_1<r_2<\cdots<r_p}}^{m}\prod_{l_2=1}^{p}O((n+r_{l_2})^{-3/2}) \leqslant C(m)/n^{3/2}$$

hence

$$\lim_{n\to\infty} \sqrt{n} \sum_{r=0}^{m-2} s^r \sum_{\substack{k,p=0 \\ k+p=m+1-r}}^{m+1-r} \sum_{\substack{r_1,\dots,r_k=0 \\ r_1<r_2<\dots<r_k}}^{m} \prod_{l_1=1}^{k} \frac{1}{2\sqrt{n+r_{l_i}}} \sum_{\substack{r_1,\dots,r_p=0 \\ r_1<r_2<\dots<r_p}}^{m} \prod_{l_2=1}^{p} O((n+r_{l_2})^{-3/2}) = 0$$

The limit equals $-s^m(m+1)/2$.

Solution 2 by Michel Bataille, Rouen, France.

We first show that $s - s_n \sim \dfrac{-1}{2\sqrt{n}}$ as $n \to \infty$. Let $t_1 = s_1$ and for $n \geqslant 2$, $t_n = s_n - s_{n-1}$. Then, we have

$$t_n = \frac{1}{\sqrt{n}} - 2(\sqrt{n} - \sqrt{n-1}) = \frac{1}{\sqrt{n}} - \frac{2}{\sqrt{n} + \sqrt{n-1}} = \frac{1}{\sqrt{n}}\left(1 - 2\left(1 + \left(1 - \frac{1}{n}\right)^{1/2}\right)^{-1}\right)$$

and, as $n \to \infty$,

$$1 - 2\left(1 + \left(1 - \frac{1}{n}\right)^{1/2}\right)^{-1} = 1 - 2\left(1 + 1 - \frac{1}{2n} + o(1/n)\right)^{-1} = 1 - \left(1 - \frac{1}{4n} + o(1/n)\right)^{-1}$$
$$= 1 - \left(1 + \frac{1}{4n} + o(1/n)\right) = -\frac{1}{4n} + o(1/n).$$

so that $t_n \sim -\dfrac{1}{4n\sqrt{n}}$ as $n \to \infty$.

This result, besides confirming the convergence of the series $\sum_{k=1}^{\infty} t_k$ (clearly $\sum_{k=1}^{\infty} t_k = s_1 + \sum_{k=2}^{\infty}(s_k - s_{k-1}) = \lim_{n\to\infty} s_n = s$), gives

$$\sum_{k=n+1}^{\infty} t_k \sim -\frac{1}{4}\sum_{k=n+1}^{\infty} \frac{1}{n^{3/2}} \sim -\frac{1}{4} \cdot \frac{2}{\sqrt{n}} \qquad \text{as } n \to \infty$$

(since for $\alpha > 1$, $\sum_{k=n+1}^{\infty} \frac{1}{n^{\alpha}} \sim \frac{1}{(\alpha-1)n^{\alpha-1}}$). It follows that

$$s - \sum_{k=1}^{n} t_k = s - s_n \sim \frac{-1}{2\sqrt{n}} \qquad \text{as } n \to \infty.$$

Now, if r is a non-negative integer, we see that, as $n \to \infty$, $s_{n+r} = s + \dfrac{1}{2\sqrt{n+r}} + o(1/\sqrt{n+r}) =$

$s+\dfrac{1}{2\sqrt{n}}+o(1/\sqrt{n})$ so that

$$\prod_{j=n}^{n+m} s_j = \left(s+\frac{1}{2\sqrt{n}}+o(1/\sqrt{n})\right)^{m+1}$$
$$= s^{m+1}\left(1+\frac{1}{2s\sqrt{n}}+o(1/\sqrt{n})\right)^{m+1} = s^{m+1}+\frac{(m+1)s^m}{2}\frac{1}{\sqrt{n}}+o(1/\sqrt{n}).$$

It readily follows that $L=-\dfrac{(m+1)s^m}{2}$.

Solution 3 by Albert Stadler, Herrliberg, Switzerland.

It appears that this problem is a follow-up on problem 5765.
By Abel's summation formula, we have

$$\sum_{k\leqslant n} k^{-s} = n^{1-s}+s\int_1^n \frac{[u]}{u^{s+1}}du,\ \ n\geqslant 1,\ \ \mathrm{Re}(s)>0,$$

where, as usual, [x] denotes the integral part of x. This implies that

$$(s)=\sum_{k=1}^{\infty} k^{-s} = \frac{s}{s-1}+s\int_1^{\infty}\frac{[u]-u}{u^{s+1}}du,\ \ s>1.$$

Since $\left|\dfrac{[u]-u}{u^{s+1}}\right| \leqslant \dfrac{1}{u^{Re(s)+1}}$ the last integral converges uniformly for Re(s)>δ, where δ is any fixed positive number, and therefore represents a regular function of s for Re(s)>0. Above relation gives the analytic continuation of the Riemann zeta function ζ for Re(s)>0. In particular,

$$\left(\frac{1}{2}\right) = -1+\frac{1}{2}\int_1^{\infty}\frac{[u]-u}{u^{\frac{3}{2}}}du.$$

Thus we have

$$s_n = -2\sqrt{n}+\sum_{k=1}^{n}\frac{1}{\sqrt{k}} = -\sqrt{n}+\frac{1}{2}\int_1^n\frac{[u]}{u^{\frac{3}{2}}}du = -1+\frac{1}{2}\int_1^n\frac{[u]-u}{u^{\frac{3}{2}}}du =$$
$$= -1+\frac{1}{2}\int_1^{\infty}\frac{[u]-u}{u^{\frac{3}{2}}}du-\frac{1}{2}\int_n^{\infty}\frac{[u]-u}{u^{\frac{3}{2}}}du = \left(\frac{1}{2}\right)+\frac{1}{2}\int_n^{\infty}\frac{u-[u]}{u^{\frac{3}{2}}}du.$$

So Ioachimescu's constant s is just the value $\left(\dfrac{1}{2}\right)$. We next derive an asymptotic formula for $\dfrac{1}{2}\displaystyle\int_n^{\infty}\frac{u-[u]}{u^{\frac{3}{2}}}du$, namely

$$\frac{1}{2}\int_n^{\infty}\frac{u-[u]}{u^{\frac{3}{2}}}du = \frac{1}{2\sqrt{n}}+\frac{1}{2}\int_n^{\infty}\frac{u-[u]-\frac{1}{2}}{u^{\frac{3}{2}}}du = \frac{1}{2\sqrt{n}}+\frac{1}{2}\sum_{k=n}^{\infty}\int_0^1\frac{u-\frac{1}{2}}{(k+u)^{\frac{3}{2}}}du =$$

$$= \frac{1}{2\sqrt{n}} + \frac{1}{2}\sum_{k=n}^{\infty}\left(\left.\frac{u^2-u}{2(k+u)^{\frac{3}{2}}}\right|_0^1 + \frac{3}{4}\int_0^1 \frac{u^2-u}{(k+u)^{\frac{5}{2}}}du \right) =$$

$$= \frac{1}{2\sqrt{n}} + O\left(\sum_{k=n}^{\infty}\left(\int_0^1 \frac{1}{(k+u)^{\frac{5}{2}}}du \right)\right) = \frac{1}{2\sqrt{n}} + O\left(\frac{1}{n^{\frac{3}{2}}}\right).$$

To sum up we have

$$s_n = \left(\frac{1}{2}\right) + \frac{1}{2n^{\frac{1}{2}}} + O\left(\frac{1}{n^{\frac{3}{2}}}\right).$$

For the current purpose it is sufficient to use the weaker form $s_n = \left(\frac{1}{2}\right) + \frac{1}{2n^{\frac{1}{2}}} + O\left(\frac{1}{n}\right)$. So

$$\prod_{j=n}^{n+m} s_j =^{m+1} \left(\frac{1}{2}\right)\prod_{j=n}^{n+m}\left(1 + \frac{1}{2\left(\frac{1}{2}\right)j^{\frac{1}{2}}} + O\left(\frac{1}{j}\right)\right) =^{m+1} \left(\frac{1}{2}\right)\left(1 + \sum_{j=n}^{n+m}\frac{1}{2\left(\frac{1}{2}\right)j^{\frac{1}{2}}} + O\left(\frac{1}{n}\right)\right),$$

where the implied constant depends on m. We have

$$\sum_{j=n}^{n+m}\frac{1}{j^{\frac{1}{2}}} = \sum_{j=0}^{m}\frac{1}{(j+n)^{\frac{1}{2}}} = \frac{1}{n^{\frac{1}{2}}}\sum_{j=0}^{m}\frac{1}{\left(1+\frac{j}{n}\right)^{\frac{1}{2}}} = \frac{1}{n^{\frac{1}{2}}}\sum_{j=0}^{m}\left(1 + O\left(\frac{1}{n}\right)\right) = \frac{m+1}{n^{\frac{1}{2}}} + O\left(\frac{1}{n^{\frac{3}{2}}}\right),$$

where again the implied constant depends on m. Hence

$$s^{m+1} - \prod_{j=n}^{n+m} s_j =^{m+1} \left(\frac{1}{2}\right) -^{m+1} \left(\frac{1}{2}\right)\left(1 + \frac{m+1}{2\left(\frac{1}{2}\right)n^{\frac{1}{2}}} + O\left(\frac{1}{n}\right)\right) = -\frac{(m+1)^m\left(\frac{1}{2}\right)}{2n^{\frac{1}{2}}} + O\left(\frac{1}{n}\right)$$

and finally,

$$L = \lim_{n\to\infty}\left(s^{m+1} - \prod_{j=n}^{n+m} s_j\right)\sqrt{n} = -\frac{1}{2}(m+1)^m\left(\frac{1}{2}\right).$$

Also solved by the problem proposer.

Editor's Statement: It goes without saying that the problem proposers, as well as the solution proposers, are the *élan vital* of the Problems/Solutions Section of SSMJ. As the editor of this Section of the Journal, I consider myself fortunate to be in a position to receive, compile and organize a wealth of proposed ingenious problems and solutions intended for online publication. My unwavering gratitude goes to all the amazingly creative contributors. We come together from across continents because we find intellectual value, joy and satisfaction in mathematical problems, both in their creation as well as their solution. So that our collective efforts serve us well, I kindly ask all contributors to adhere to the following guidelines. As you peruse below, you may construe that the guidelines amount to a lot of work. But, as the samples show, there's not much to do. Your cooperation is much appreciated!

Keep in mind that the examples given below are your best guide!

Formats, Styles and Requirements

When submitting proposed problem(s) or solution(s), please send both **LaTeX** document and **pdf** document of your proposed problem(s) or solution(s). There are ways (discoverable from the internet) to convert from Word to proper LaTeX code. Porposals without a *proper* **LaTeX** document will not be published regrettably.

Regarding Proposed Solutions:

Below is the FILENAME format for all the documents of your proposed solution(s).

#ProblemNumber_FirstName_LastName_Solution_SSMJ

- FirstName stands for YOUR first name.
- LastName stands for YOUR last name.

Examples:

#1234_Max_Planck_Solution_SSMJ

#9876_Charles_Darwin_Solution_SSMJ

Please note that every problem number is *preceded* by the sign # .

All you have to do is copy the FILENAME format (or an example below it), paste it and then modify portions of it to your specs.

Please adopt the following structure, in the order shown, for the presentation of your solution:

1. On top of the first page of your solution, begin with the phrase:

"Proposed Solution to #**** SSMJ"

where the string of four astrisks represents the problem number.

2. On the second line, write

"Solution proposed by [your First Name, your Last Name]", followed by your affiliation, city, country, all on the same linear string of words. Please see the example below. Make sure you do the same for your collaborator(s).

3. On a new line, state the problem proposer's name, affiliation, city and country, just as it appears published in the Problems/Solutions section.

4. On a new line below the above, write in bold type: "**Statement of the Problem**".

5. Below the latter, state the problem. Please make sure the statement of your problem (unlike the preceding item) is not in bold type.

6. Below the statement of the problem, write in bold type: "**Solution of the Problem**".

7. Below the latter, show the entire solution of the problem.

Here is a sample for the above-stated format for proposed solutions:

> *Proposed solution to #1234 SSMJ*
>
> *Solution proposed by Emmy Noether, University of Göttingen, Lower Saxony, Germany.*
>
> *Problem proposed by Isaac Newton, Trinity College, Cambridge, England.*
>
> **Statement of the problem:**
>
> Compute $\sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}$.
>
> **Solution of the problem:** . . . . . .

Regarding Proposed Problems:

For all your proposed problems, please adopt for all documents the following FILENAME format:

FirstName_LastName_ProposedProblem_SSMJ_YourGivenNumber_ProblemTitle

If you do not have a ProblemTitle, then leave that component as it already is (i.e., ProblemTitle).

The component YourGivenNumber is any UNIQUE 3-digit (or longer) number you like to give to your problem.

Examples:

Max_Planck_ProposedProblem_SSMJ_314_HarmonicPatterns

Charles_Darwin_ProposedProblem_SSMJ_358_ProblemTitle

Please adopt the following structure, in the order shown, for the presentation of your proposal:

1. On the top of first page of your proposal, begin with the phrase:

"Problem proposed to SSMJ"

2. On the second line, write

"Problem proposed by [your First Name, your Last Name]",

followed by your affiliation, city, country all on the same linear string of words. Please see the example below. Make sure you do the same for your collaborator(s) if any.

3. On a new line state the title of the problem, if any.

4. On a new line below the above, write in bold type: "**Statement of the Problem**".

5. Below the latter, state the problem. Please make sure the statement of your problem (unlike the preceding item) is not in bold type.

6. Below the statement of the problem, write in bold type: "**Solution of the Problem**".

7. Below the latter, show the entire solution of your problem.

Here is a sample for the above-stated format for proposed problems:

Problem proposed to SSMJ

Problem proposed by Isaac Newton, Trinity College, Cambridge, England.

Principia Mathematica ($\longleftarrow$ You may choose to not include a title.)

Statement of the problem:

Compute $\displaystyle\sum_{k=0}^{n}\binom{n}{k}x^k y^{n-k}$.

Solution of the problem: . . . . . .

♣ ♣ ♣ Thank You! ♣ ♣ ♣