Problems and Solutions

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This section of the Journal offers readers an opportunity to exchange interesting mathematical problems and solutions. Please email them to Prof. Albert Natian at Department of Mathematics, Los Angeles Valley College. Please make sure every proposed problem or proposed solution is provided in both ***LaTeX*** and pdf documents. Thank you!

To propose problems, email them to: problems4ssma@gmail.com

To propose solutions, email them to: solutions4ssma@gmail.com

Solutions to the problems published in this issue should be submitted *before* December 1, 2025.

• **5815** *Proposed by Goran Conar, Varaždin, Croatia.*

Let $a, x_1, x_2, \ldots, x_n$ be positive real numbers. Prove the following inequality:

$$\left(\sum_{i=1}^{n} \frac{a^{x_i}}{x_i}\right) \geqslant \left(\sum_{i=1}^{n} \frac{1}{x_i}\right) \cdot a^{\frac{n}{\sum_{i=1}^{n} \frac{1}{x_i}}}.$$

When does equality occur?

• **5816** *Proposed by Daniel Sitaru, National Economic College "Theodor Costescu" Drobeta Turnu - Severin, Romania.*

For $0 < a \leqslant b$, prove:

$$\int_a^b \frac{xe^{x^2}\sqrt{e^{x^2}}}{e^{3x^2}+1}dx \leqslant \frac{1}{2}\tan^{-1}\left(\frac{e^{b^2}-e^{a^2}}{1+e^{a^2+b^2}}\right).$$

• **5817** *Proposed by Michel Bataille, Rouen, France.*

Let $(F_n)_{n\geqslant 0}$ be the Fibonacci sequence defined by $F_0 = 0, F_1 = 1$ and $F_n = F_{n-1} + F_{n-2}$ for all $n \geqslant 2$. If m, n are integers such that $m \geqslant n \geqslant 0$, prove that

$$\sum_{j=0}^{n} \binom{m+1}{j}(2^{n+1}-2^j)F_j = \sum_{j=0}^{n} \binom{m}{j}(2^{n+1}F_{j+2} - 2^j F_{j+3}).$$

• **5818** *Proposed by Paolo Perfetti, dipartimento di matematica Universitą di “Tor Vergata", Rome, Italy.*

Evaluate

$$\int_0^{\infty} \left(\arctan\frac{1}{1+z^2}\right)^2 dz.$$

• **5819** *Proposed by Toyesh Prakash Sharma, Agra College, Agra, India.*

For $a, b, c \geqslant 0$, show that

$$\frac{a+b}{a^2+b^2}\sqrt{ab}+\frac{b+c}{b^2+c^2}\sqrt{bc}+\frac{c+a}{c^2+a^2}\sqrt{ca}\leqslant 3.$$

Solutions

To Formerly Published Problems

• **5793** *Proposed by Daniel Sitaru, National Economic College "Theodor Costescu" Drobeta Turnu - Severin, Romania.*

Suppose $f:[a,b]\to[1,\infty)$ is a continuous function with $0<a\leqslant b$. Then:

$$n(b-a)^{n-1}\int_a^b f(x)dx\leqslant(n-1)(b-a)^n+\left(\int_a^b f(x)dx\right)^n.$$

Solution 1 by Ángel Plaza, Universidad de Las Palmas de Gran Canaria, Spain.

The proposed inequality may be written as

$$\frac{\int_a^b f(x)dx}{b-a}\leqslant\frac{(n-1)+\left(\frac{\int_a^b f(x)dx}{b-a}\right)^n}{n},$$

which follows by the AM-GM inequality.

Solution 2 by Brian Bradie, Department of Mathematics, Christopher Newport University, Newport News, VA.

Let x and y be non-negative real numbers. By the arithmetic mean - geometric mean inequality,

$$(n-1)x+y=\underbrace{x+x+\cdots+x}_{n-1\text{ terms}}+y\geqslant n\sqrt[n]{x^{n-1}y}. \qquad (1)$$

Because $f:[a,b]\to[1,\infty)$ is a continuous function and $a\leqslant b$,

$$b-a\geqslant 0\quad\text{and}\quad\int_a^b f(x)\,dx\geqslant 0.$$

Substituting

$$x=(b-a)^n\quad\text{and}\quad y=\left(\int_a^b f(x)\,dx\right)^n$$

into (1) then yields

$$\begin{aligned}(n-1)(b-a)^n + \left(\int_a^b f(x)\,dx\right)^n &\geqslant n\sqrt[n]{(b-a)^{n(n-1)}\left(\int_a^b f(x)\,dx\right)^n}\\ &= n(b-a)^{n-1}\int_a^b f(x)\,dx.\end{aligned}$$

Solution 3 by David A. Huckaby, Angelo State University, San Angelo, TX.

If $a = b$, both sides of the inequality are zero, so we assume that $a < b$. Dividing both sides by $(b-a)^n$, the desired inequality is

$$n \cdot \frac{1}{b-a}\int_a^b f(x)\,dx \leqslant n - 1 + \left(\frac{1}{b-a}\int_a^b f(x)\,dx\right)^n.$$

We first note that this inequality does not hold for all real n. For example, letting $a = 0$, $b = 1$, $f(x) = 4$, and $n = \frac{1}{2}$, we have $\frac{1}{2}(4) \not\leqslant \frac{1}{2} - 1 + (4)^{1/2}$. We will show that the inequality holds for $n \geqslant 1$ and for $n \leqslant 0$. That the inequality holds for $n = 0$ and for $n = 1$ can be seen by inspection, so we will consider the cases $n > 1$ and $n < 0$.

By the Mean Value Theorem for Integrals, there is a $c \in [a, b]$ such that $f(c) = \int_a^b f(x)\,dx$, so that the desired inequality is $nf(c) \leqslant n - 1 + [f(c)]^n$, that is, $n(f(c) - 1) \leqslant [f(c)]^n - 1$. If $f(c) = 1$, then both sides of this inequality are zero. Otherwise, $f(c) > 1$ by assumption, and we consider the equivalent inequality

$$n \leqslant \frac{[f(c)]^n - 1}{f(c) - 1}. \tag{2}$$

Let $g(x) = \dfrac{x^n - 1}{x - 1}$ for $x > 1$. By l'Hôpital's Rule we have

$$\lim_{x\to 1^+} g(x) = \lim_{x\to 1^+} \frac{x^n - 1}{x - 1} = \lim_{x\to 1^+} \frac{nx^{n-1}}{1} = n.$$

So if we show that $g(x)$ is an increasing function for $x > 1$, then inequality (2) follows. To this end we note that $g'(x) = \dfrac{(n-1)x^{n+1} - nx^n + x}{(x-1)^2 x}$.

For the case $n > 1$, we rewrite $g'(x) = \dfrac{(n-1)x^n - nx^{n-1} + 1}{(x-1)^2}$. The denominator is clearly positive. Denoting the numerator by $h(x) = (n-1)x^n - nx^{n-1} + 1$, we have $h'(x) = n(n-1)x^{n-1} - n(n-1)x^{n-2} = n(n-1)\left(x^{n-1} - x^{n-2}\right) > 0$. Since $h(1) = 0$, this shows that $h(x) > 0$ for $x > 1$, so that $g'(x) > 0$ for $x > 1$. Thus inequality (2) holds for $n > 1$.

(When $n > 1$ is an integer, a shorter route is to note that inequality (2) is simply $n \leqslant \frac{\left(f(c)-1\right)\left([f(c)]^{n-1}+[f(c)]\right.}{f(c)-1}$
that is, $n \leqslant [f(c)]^{n-1} + [f(c)]^{n-2} + \cdots + f(c) + 1$, which is clearly true for $f(c) > 1$.)

For the case $n < 0$, we note that the denominator of $g'(x) = \dfrac{(n-1)x^{n+1} - nx^n + x}{(x-1)^2 x}$ is positive. Denoting the numerator by $k(x) = (n-1)x^{n+1} - nx^n + x$, we have $k'(x) = (n+1)(n-1)x^n - n^2x^{n-1} + 1 = (n^2-1)x^n - n^2x^{n-1} + 1 = n^2(x^n - x^{n-1}) - x^n + 1 > 0$, since $x^n < 1$ for $n < 0$ and $x > 1$. Since $k(1) = 0$, this shows that $k(x) > 0$ for $x > 1$, so that $g'(x) > 0$ for $x > 1$. Thus inequality (2) also holds for $n < 0$.

So the original inequality holds for $n \geqslant 1$ and for $n \leqslant 0$.

Solution 4 by Michel Bataille, Rouen, France.

Let $I = \displaystyle\int_a^b f(x)\,dx$. Since $b - a \geqslant 0$ and $I \geqslant 0$, we can apply the arithmetic mean-geometric mean inequality as follows:

$$(n-1)(b-a)^n + I^n = (b-a)^n + \cdots + (b-a)^n + I^n \geqslant n\left((b-a)^n \cdots (b-a)^n \cdot I^n\right)^{1/n}$$

and deduce that

$$(n-1)(b-a)^n + I^n \geqslant n\left((b-a)^{n(n-1)} \cdot I^n\right)^{1/n} = n(b-a)^{n-1}I,$$

as desired.

Solution 5 by Perfetti Paolo, dipartimento di matematica Università di "Tor Vergata", Roma, Italy.

$$\left(\int_a^b f(x)dx\right)^n + \underbrace{(b-a)^n + \ldots (b-a)^n}_{n-1\, times} \geqslant n\left(\left(\int_a^b f(x)dx\right)^n (b-a)^{n(n-1)}\right)^{1/n} =$$

$$= n(b-a)^{n-1}\int_a^b f(x)dx$$

Solution 6 by Ulrich Abel, Technische Hochschule Mittelhessen, Friedberg, Germany.

As $f(x) \geqslant 1$, we have $\displaystyle\int_a^b f(x)\,dx = (b-a)(1+y)$ with a real number $y \geqslant 0$. We have to show that

$$n(b-a)^n(1+y) \leqslant (n-1)(b-a)^n + (b-a)^n(1+y)^n.$$

If $a = b$ the inequality is obvious. In the case $a < b$ the inequality is equivalent to

$$n(1+y) \leqslant n - 1 + (1+y)^n$$

or, after an elementary simplification, to

$$1 + ny \leqslant (1+y)^n .$$

This is just the Bernoulli inequality.

Solution 7 by Albert Stadler, Herrliberg, Switzerland.

We assume that n is a real variable with n⩾1. The inequality holds trivially true if b=a. Let b>a, and let $I := \dfrac{1}{b-a}\displaystyle\int_a^b f(x)\,dx$. The inequality then reads as

$$nI \leqslant n - 1 + I^n$$

which is exactly Bernoulli's inequality (see for instance https://en.wikipedia.org/wiki/Bernoulli%27s_inequality):

$$(1+x)^r \geqslant 1 + rx$$

for every real number r⩾1 and x⩾-1. The inequality is strict if x≠0 and r≠1.
Hence the assumption that f(x)⩾1 is not required. It suffices to assume that f is nonnegative.

Also solved by the problem proposer.

• **5794** *Proposed by Michel Bataille, Rouen, France.*

Let B_m denote the m-th Bernoulli number ($B_0 = 1$ and $(m+1)B_m + \sum_{j=0}^{m-1}\binom{m+1}{j}B_j = 0$ for $m \geqslant 1$). Evaluate

$$\lim_{n\to\infty}\sum_{k=0}^{n}\binom{2n}{2k}\frac{B_{2k}}{(2n)^{2k}}.$$

Solution 1 by Péter Fülöp, Gyömrő, Hungary.

1. Introducing the $S_n = \sum_{k=0}^{n}\binom{2n}{2k}\frac{B_{2k}}{(2n)^{2k}}$ notation.

Let's use the integral representation of the $2r$-th $(r \geqslant 1)$ Bernoulli number:

$$B_{2r} = 4r(-1)^{r+1} \int_0^\infty \frac{t^{2r-1}}{e^{2\pi t} - 1} dt$$

$$S_n = \sum_{k=0}^{n} \binom{2n}{2k} \frac{1}{(2n)^{2k}} 4k(-1)^{k+1} \int_0^\infty \frac{t^{2k-1}}{e^{2\pi t} - 1} dt$$

After swapping the order of integration and summation and we take the following facts into account:

(i) $(-1)^{k+1} = -(i)^{2k}$

(ii) $4ik\frac{(it)^{2k-1}}{(2n)^{2k}} = 2\frac{d}{dt}(\frac{it}{2n})^{2k}$

$$S_n = \int_0^\infty \frac{2}{1 - e^{2\pi t}} \sum_{k=0}^{n} \binom{2n}{2k} \frac{d}{dt}\Big\{\Big(\frac{it}{2n}\Big)^{2k}\Big\} dt$$

Swapping the order of summation and derivation.

$$S_n = \int_0^\infty \frac{2}{1 - e^{2\pi t}} \frac{d}{dt}\Big[\sum_{k=0}^{n} \binom{2n}{2k} \Big(\frac{it}{2n}\Big)^{2k}\Big] dt$$

2. Calculation of the sum inside the integral:

$$\sum_{k=0}^{n} \binom{2n}{2k} \Big(\frac{it}{2n}\Big)^{2k} = \frac{1}{2}\sum_{k=0}^{n} \binom{2n}{2k} \Big(\frac{it}{2n}\Big)^{2k} ((-1)^{2k} + 1) =$$

Let $m = 2k$

$$\frac{1}{2}\sum_{m=0}^{2n} \binom{2n}{m} \Big(\frac{it}{2n}\Big)^{m} ((-1)^{m} + 1) = \frac{1}{2}\sum_{m=0}^{2n} \binom{2n}{m} \Big(\frac{it}{2n}\Big)^{m} (-1)^{2n-m} + \frac{1}{2}\sum_{m=0}^{2n} \binom{2n}{m} \Big(\frac{it}{2n}\Big)^{m}$$

Based on the binomial formula we get:

$$\sum_{k=0}^{n} \binom{2n}{2k} \Big(\frac{it}{2n}\Big)^{2k} = \frac{1}{2}\Big\{\Big(1 - \frac{it}{2n}\Big)\Big\}^{2n} + \frac{1}{2}\Big\{1 + \Big(\frac{it}{2n}\Big)\Big\}^{2n}$$

After the derivation of the result of the sum let's turn back to the calculation of the limit:

$$\lim_{n\to\infty} i\int_0^\infty \frac{1}{1-e^{2\pi t}}\Big[\Big(1+\frac{it}{2n}\Big)^{2n-1}-\Big(1-\frac{it}{2n}\Big)^{2n-1}\Big]dt =$$

$$i\int_0^\infty \frac{1}{1-e^{2\pi t}}\underbrace{\lim_{n\to\infty}\Big[\frac{\Big(1+\frac{it}{2n}\Big)^{2n}}{\Big(1+\frac{it}{2n}\Big)}-\frac{\Big(1-\frac{it}{2n}\Big)^{2n}}{\Big(1-\frac{it}{2n}\Big)}\Big]}_{e^{it}-e^{-it}}dt =$$

$$\lim_{n\to\infty} S_n = i\int_0^\infty \frac{e^{it}-e^{-it}}{1-e^{2\pi t}}dt = I$$

3. Determining the value of the limit (by the evaluate of the integral) in four steps:

(i) Substitution $z = e^{-2\pi t}$

$$I = \frac{i}{2\pi}\int_0^1 \frac{z^{\frac{i}{2\pi}-1}}{1-z}dz - \frac{i}{2\pi}\int_0^1 \frac{z^{\frac{-i}{2\pi}-1}}{1-z}dz$$

(ii) Application of β function and the sum form of incomplete β function:

$$I = -\frac{i}{2\pi}\Big(\beta(\frac{i}{2\pi},0)-\beta(\frac{-i}{2\pi},0)\Big) = \frac{i}{2\pi}\Big(\sum_{k=0}^{\infty}\frac{1}{1-\frac{i}{2\pi}}-\sum_{k=0}^{\infty}\frac{1}{1+\frac{i}{2\pi}}\Big)$$

(iii) It can be realized that the difference of the sums equal to the digamma function at the place $\pm\frac{i}{2\pi}$.

$$I = \frac{i}{2\pi}\Big(\psi(\frac{-i}{2\pi})-\psi(\frac{i}{2\pi})\Big)$$

(iv) After the application of the recurrance relation $\psi(z+1) = \psi(z)+\frac{1}{z}$, then the reflection formula $\psi(1-z)-\psi(z) = \pi\cot(\pi z)$ we get the result:

$$I = \frac{i}{2\pi}\Big(\underbrace{\psi(1-\frac{i}{2\pi})-\psi(\frac{i}{2\pi})}_{\pi\cot(\frac{i}{2})}-\frac{2\pi}{i}\Big) = -1+\frac{i}{2}\cot(\frac{i}{2})$$

Finally

$$\lim_{n\to\infty}\sum_{k=0}^{n}\binom{2n}{2k}\frac{B_{2k}}{(2n)^{2k}} = -1+\frac{1}{2}\coth\Big(\frac{1}{2}\Big)$$

Solution 2 by Albert Stadler, Herrliberg, Switzerland.

We will prove that $\lim\limits_{n\to\infty}\sum\limits_{k=0}^{n}\binom{2n}{2k}\frac{B_{2k}}{(2n)^{2k}} = \frac{1}{2}\coth\left(\frac{1}{2}\right)$.

It is known (see for instance https://en.wikipedia.org/wiki/Bernoulli_number) that the even Bernoulli numbers can be expressed in terms of the Riemann zeta function, namely

$$B_{2n} = \frac{(-1)^{n+1}2\,(2n)!}{(2\pi)^{2n}}\,(2n)\,, \quad n \geqslant 1.$$

We use this formula to derive an integral representation of the even Bernoulli numbers, namely

$$B_{2n} = \frac{(-1)^{n+1}2\,(2n)!}{(2\pi)^{2n}}\sum_{k=1}^{\infty}\frac{1}{k^{2n}} = 4n(-1)^{n+1}\sum_{k=1}^{\infty}\int_0^{\infty} e^{-2\pi kt}t^{2n-1}dt = 4n(-1)^{n+1}\int_0^{\infty}\frac{t^{2n-1}}{e^{2\pi t}-1}dt.$$

By the Binomial theorem,

$$(1+x)^{2n} = \sum_{k=0}^{2n}\binom{2n}{k}x^k$$

and

$$2nx(1+x)^{2n-1} = x\frac{d}{dx}(1+x)^{2n} = x\frac{d}{dx}\sum_{k=0}^{2n}\binom{2n}{k}x^k = \sum_{k=1}^{2n}\binom{2n}{k}kx^k.$$

So

$$\sum_{k=0}^{n}\binom{2n}{2k}\frac{B_{2k}}{(2n)^{2k}} = 1-\sum_{k=1}^{n}\binom{2n}{2k}\frac{4k(-1)^k}{(2n)^{2k}}\int_0^{\infty}\frac{t^{2k-1}}{e^{2\pi t}-1}dt =$$

$$= 1-2\int_0^{\infty}\frac{1}{t\left(e^{2\pi t}-1\right)}\left(\sum_{k=1}^{n}\binom{2n}{2k}\frac{2k(it)^{2k}}{(2n)^{2k}}\right)dt =$$

$$= 1-2\int_0^{\infty}\frac{1}{t\left(e^{2\pi t}-1\right)}\left(\sum_{k=1}^{2n}\frac{1+(-1)^k}{2}\binom{2n}{k}\frac{k(it)^k}{(2n)^k}\right)dt =$$

$$= 1-\int_0^{\infty}\frac{i}{e^{2\pi t}-1}\left(\left(1+\frac{it}{2n}\right)^{2n-1}-\left(1-\frac{it}{2n}\right)^{2n-1}\right)dt.$$

We have

$$\left|\int_0^{\infty}\frac{i}{e^{2\pi t}-1}\left(\left(1+\frac{it}{2n}\right)^{2n-1}-\left(1-\frac{it}{2n}\right)^{2n-1}\right)dt-\int_0^{\infty}\frac{i\left(e^{it}-e^{-it}\right)}{e^{2\pi t}-1}dt\right| \leqslant$$

$$\leqslant\left|\int_0^{\sqrt{n}}\frac{\left(1+\frac{it}{2n}\right)^{2n-1}-e^{it}}{e^{2\pi t}-1}dt\right|+\left|\int_0^{\sqrt{n}}\frac{\left(1-\frac{it}{2n}\right)^{2n-1}-e^{-it}}{e^{2\pi t}-1}dt\right|+\left|\int_{\sqrt{n}}^{\infty}\frac{\left(1+\frac{it}{2n}\right)^{2n-1}}{e^{2\pi t}-1}dt\right|+\left|\int_{\sqrt{n}}^{\infty}\frac{\left(1-\frac{it}{2n}\right)^{2n-1}}{e^{2\pi t}-1}dt\right|+\left|\int_{\sqrt{}}^{\infty}\right.$$

and

$$\left|\int_0^{\sqrt{n}} \frac{\left(1+\frac{it}{2n}\right)^{2n-1}-e^{it}}{e^{2\pi t}-1}dt\right| = \left|\int_0^{\sqrt{n}} \frac{\left(1-\frac{it}{2n}\right)^{2n-1}-e^{-it}}{e^{2\pi t}-1}dt\right| = \left|\int_0^{\sqrt{n}} \frac{e^{(2n-1)\ln\left(1+\frac{it}{2n}\right)-it}-1}{e^{2\pi t}-1}dt\right| =$$

$$= \left|\int_0^{\sqrt{n}} \frac{e^{-\frac{it}{2n}+O\left(\frac{t^2}{n}\right)}-1}{e^{2\pi t}-1}dt\right| = O\left(\frac{1}{n}\int_0^{\sqrt{n}} \frac{t+t^2}{e^{2\pi t}-1}dt\right) = O\left(\frac{1}{n}\right),$$

$$\left|\int_{\sqrt{n}}^{\infty} \frac{\left(1+\frac{it}{2n}\right)^{2n-1}}{e^{2\pi t}-1}dt\right| = \left|\int_{\sqrt{n}}^{\infty} \frac{\left(1-\frac{it}{2n}\right)^{2n-1}}{e^{2\pi t}-1}dt\right| \leqslant \int_{\sqrt{n}}^{\infty} \frac{\left(1+\frac{t}{2n}\right)^{2n}}{e^{2\pi t}-1}dt \leqslant \int_{\sqrt{n}}^{\infty} \frac{e^t}{e^{2\pi t}-1}dt = O\left(e^{-(2\pi-1)\sqrt{n}}\right),$$

$$\left|\int_{\sqrt{n}}^{\infty} \frac{e^{it}}{e^{2\pi t}-1}dt\right| = \left|\int_{\sqrt{n}}^{\infty} \frac{e^{-it}}{e^{2\pi t}-1}dt\right| = O\left(e^{-2\pi\sqrt{n}}\right).$$

In addition, if a>0,

$$-\int_0^{\infty} \frac{i}{e^{at}-1}\left(e^{it}-e^{-it}\right)dt = -i\sum_{k=1}^{\infty}\int_0^{\infty} e^{-akt}\left(e^{it}-e^{-it}\right)dt = -i\sum_{k=1}^{\infty}\left(\frac{1}{ak-i}-\frac{1}{ak+i}\right) =$$

$$= 2\sum_{k=1}^{\infty}\frac{1}{a^2k^2+1} = -1+\frac{\pi}{a}\coth\left(\frac{\pi}{a}\right),$$

by the partial fraction decomposition of the hyperbolic cotangent function, namely

$$\coth(x) = \frac{1}{x}+2x\sum_{n=1}^{\infty}\frac{1}{x^2+\pi^2n^2}.$$

Hence

$$\lim_{n\to\infty}\sum_{k=0}^{n}\binom{2n}{2k}\frac{B_{2k}}{(2n)^{2k}} = 1-\lim_{n\to\infty}\int_0^{\infty}\frac{i}{e^{2\pi t}-1}\left(\left(1+\frac{it}{2n}\right)^{2n-1}-\left(1-\frac{it}{2n}\right)^{2n-1}\right)dt =$$

$$= 1-\int_0^{\infty}\frac{i}{e^{2\pi t}-1}\left(e^{it}-e^{-it}\right)dt = \frac{1}{2}\coth\frac{1}{2}.$$

Also solved by and the problem proposer.

• **5795** *Proposed by Ovidiu Furdui and Alina Sîntămărian, Technical University of Cluj-Napoca, Cluj-Napoca, Romania.*

Calculate the integral

$$I=\int_0^{\infty}\frac{\arctan x}{(1+x)\sqrt{1+x^2}}dx.$$

Solution 1 by Akash Chaudhary, Pulchowk Campus, Kapilvastu, Nepal.

In

$$I=\int_0^{\infty}\frac{\arctan(x)}{(1+x)\sqrt{1+x^2}}\,dx$$

put

$$\arctan(x)=t \text{ so that } \quad x=\tan(t), \quad dx=\sec^2(t)\,dt$$

We have

$$(x\to 0:\ t\to 0)\,, \quad \left(x\to\infty:\ t\to\frac{\pi}{2}\right).$$

Then the integral becomes

$$I=\int_0^{\frac{\pi}{2}}\frac{t\sec^2(t)}{(1+\tan(t))\sqrt{1+\tan^2(t)}}\,dt.$$

Simplifying, we get

$$I=\int_0^{\frac{\pi}{2}}\frac{t\sec(t)}{1+\tan(t)}\,dt \tag{3}$$

By King's rule

$$\int_a^b f(x)\,dx=\int_a^b f(a+b-x)\,dx,$$

we have

$$I=\int_0^{\frac{\pi}{2}}\frac{\left(\frac{\pi}{2}-t\right)\sec\left(\frac{\pi}{2}-t\right)}{1+\tan\left(\frac{\pi}{2}-t\right)}\,dt$$

$$I=\int_0^{\frac{\pi}{2}}\frac{\left(\frac{\pi}{2}-t\right)\csc(t)}{1+\cot(t)}\,dt$$

$$I=\int_0^{\frac{\pi}{2}}\frac{\left(\frac{\pi}{2}-t\right)\sec(t)}{1+\tan(t)}\,dt. \tag{4}$$

Adding (3) and (4), we obtain

$$2I=\int_0^{\frac{\pi}{2}}\frac{\left(\frac{\pi}{2}-t+t\right)\sec(t)}{1+\tan(t)}\,dt$$

$$2I = \frac{\pi}{2}\int_0^{\frac{\pi}{2}} \frac{\sec(t)}{1+\tan(t)}\, dt.$$

Thus

$$I = \frac{\pi}{4}\int_0^{\frac{\pi}{2}} \frac{1}{\sin(t)+\cos(t)}\, dt$$

Applying the half-angle formula for sine and cosine, we simplify the expression:

$$I = \frac{\pi}{4}\int_0^{\frac{\pi}{2}} \frac{\sec^2\left(\frac{t}{2}\right)}{1+2\tan\left(\frac{t}{2}\right)-\tan^2\left(\frac{t}{2}\right)}\, dt.$$

Put

$$\tan\left(\frac{t}{2}\right) = m, \quad \text{so that} \quad \sec^2\left(\frac{t}{2}\right)\, dt = 2\, dm.$$

We have

$$(t \to 0 : m \to 0), \quad \left(t \to \frac{\pi}{2} : m \to 1\right).$$

The integral becomes

$$I = \frac{\pi}{4}\int_0^1 \frac{2}{1+2m-m^2}\, dm$$

$$I = \frac{-\pi}{2}\int_0^1 \frac{1}{(m-1)^2-(\sqrt{2})^2}\, dm$$

$$I = \frac{-\pi}{2}\left[\frac{1}{2\sqrt{2}}\ln\left|\frac{m-1-\sqrt{2}}{m-1+\sqrt{2}}\right|\right]_0^1.$$

Upon substituting the limits and simplifying, we get

$$I = \frac{\pi}{4\sqrt{2}}\ln\left|\frac{\sqrt{2}+1}{\sqrt{2}-1}\right|.$$

Solution 2 by Albert Stadler, Herrliberg, Switzerland.

We perform the change of variables x→1/x and see that

$$I = \int_0^{\infty} \frac{\arctan\left(\frac{1}{x}\right)}{(1+x)\sqrt{1+x^2}}dx = \int_0^{\infty} \frac{\frac{\pi}{2}}{(1+x)\sqrt{1+x^2}}dx - I.$$

So

$$I = \frac{\pi}{4}\int_0^{\infty} \frac{1}{(1+x)\sqrt{1+x^2}}dx.$$

We have

$$\int_0^{\infty} \frac{1}{(1+x)\sqrt{1+x^2}}dx = \frac{1}{\sqrt{2}}\ln\left(\frac{\sqrt{2}+1+x-\sqrt{1+x^2}}{\sqrt{2}-1-x+\sqrt{1+x^2}}\right) + C,$$

since

$$\frac{d}{dx}\frac{1}{\sqrt{2}}\ln\left(\frac{\sqrt{2}+1+x-\sqrt{1+x^2}}{\sqrt{2}-1-x+\sqrt{1+x^2}}\right)=\frac{1-\frac{x}{\sqrt{1+x^2}}}{\sqrt{2}\left(\sqrt{2}+1+x-\sqrt{1+x^2}\right)}-\frac{-1+\frac{x}{\sqrt{1+x^2}}}{\sqrt{2}\left(\sqrt{2}-1-x+\sqrt{1+x^2}\right)}=$$

$$=\frac{1}{(1+x)\sqrt{1+x^2}}.$$

So

$$I=\frac{\pi}{4}\cdot\frac{1}{\sqrt{2}}\ln\left(\frac{\sqrt{2}+1+x-\sqrt{1+x^2}}{\sqrt{2}-1-x+\sqrt{1+x^2}}\right)\Bigg|_{x=0}^{x=\infty}=\frac{\pi}{4}\cdot\frac{1}{\sqrt{2}}\ln\left(\frac{\sqrt{2}+1}{\sqrt{2}-1}\right)=\frac{\pi}{4}\sqrt{2}\ln\left(\sqrt{2}+1\right).$$

Solution 3 by Arslene Ghodbane, University of M'Hamed BOUGARA, Institute of Electrical and Electronic Engineering, Boumerdes, Algeria.

We have

$$I=\int_0^\infty\frac{\arctan(x)}{(1+x)\sqrt{(1+x^2)}}dx=\lim_{\epsilon\to\infty}\int_0^\epsilon\frac{\arctan(x)}{(1+x)\sqrt{(1+x^2)}}dx.$$

Let $\arctan(x)=t$ so that $dx=\sec^2(t)dt$, $x\in[0;\infty[$, $t\in\left[0;\frac{\pi}{2}\right]$.

We have

$$I=\int_0^{\frac{\pi}{2}}\frac{t\sec^2(t)}{(1+\tan(t))\sqrt{(1+\tan^2(t))}}dt=\int_0^{\frac{\pi}{2}}\frac{t\sec(t)}{(1+\tan(t))}dt$$

$$I=\int_0^{\frac{\pi}{2}}\frac{t\sec(t)}{\sin(t)+\cos(t)}dt=\int_0^{\frac{\pi}{2}}\frac{t}{\sin(t)+\cos(t)}dt$$

$$I=\int_0^{\frac{\pi}{2}}\frac{t}{\cos(t)+\sin(t)}dt=\int_0^{\frac{\pi}{2}}\frac{t}{\sqrt{2}\cos\left(\frac{\pi}{4}\right)\left[\cos(t)+\sin(t)\right]}dt=\int_0^{\frac{\pi}{2}}\frac{t}{\sqrt{2}\cos\left(\frac{\pi}{4}-t\right)}dt.$$

Let $u=t-\frac{\pi}{4}$, $t\in\left[0;\frac{\pi}{2}\right]$, $u\in\left[-\frac{\pi}{4};\frac{\pi}{4}\right]$. Then

$$I=\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}}\frac{u+\frac{\pi}{4}}{\sqrt{2}\cos(u)}du$$

$$I=\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}}\frac{u}{\sqrt{2}\cos(u)}du+\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}}\frac{\frac{\pi}{4}}{\sqrt{2}\cos(u)}du.\quad(*)$$

The first integrand is an odd function because

$$f(-u)=\frac{-u}{\sqrt{2}\cos(-u)}=\frac{-u}{\sqrt{2}\cos(u)}=-f(u),$$

hence the first integral is zero. Regarding the second integral in (*), since

$$\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \sec(u)du = \ln|\sec(u) + \tan(u)|_{-\frac{\pi}{4}}^{\frac{\pi}{4}} = \ln\left(\frac{\sqrt{2}+1}{\sqrt{2}-1}\right)$$

then

$$I = \frac{\pi}{4\sqrt{2}} \ln\left(\frac{\sqrt{2}+1}{\sqrt{2}-1}\right).$$

Solution 4 by Brian Bradie, Department of Mathematics, Christopher Newport University, Newport News, VA.

With the change of variable $x \to \dfrac{1}{x}$ and the identity $\arctan x + \arctan \dfrac{1}{x} = \dfrac{\pi}{2}$,

$$\int_1^\infty \frac{\arctan x}{(1+x)\sqrt{1+x^2}}\,dx = \int_1^0 \frac{\arctan \frac{1}{x}}{\left(1+\frac{1}{x}\right)\sqrt{1+\frac{1}{x^2}}} \cdot -\frac{1}{x^2}\,dx = \int_0^1 \frac{\frac{\pi}{2} - \arctan x}{(1+x)\sqrt{1+x^2}}\,dx.$$

Thus,

$$\begin{aligned}
\int_0^\infty \frac{\arctan x}{(1+x)\sqrt{1+x^2}}\,dx &= \int_0^1 \frac{\arctan x}{(1+x)\sqrt{1+x^2}}\,dx + \int_1^\infty \frac{\arctan x}{(1+x)\sqrt{1+x^2}}\,dx \\
&= \frac{\pi}{2}\int_0^1 \frac{1}{(1+x)\sqrt{1+x^2}}\,dx \qquad (\text{let } x = \tan\theta) \\
&= \frac{\pi}{2}\int_0^{\pi/4} \frac{1}{\cos\theta + \sin\theta}\,d\theta \qquad (\text{let } \theta = 2\arctan t) \\
&= \pi \int_0^{\sqrt{2}-1} \frac{1}{1+2t-t^2}\,dt \\
&= \frac{\pi}{2}\int_0^{\sqrt{2}-1} \frac{1}{1-\left(\frac{t-1}{\sqrt{2}}\right)^2}\,dt \qquad \left(\text{let } w = \frac{t-1}{\sqrt{2}}\right) \\
&= \frac{\pi\sqrt{2}}{2}\int_{-1/\sqrt{2}}^{1-\sqrt{2}} \frac{1}{1-w^2}\,dw \\
&= \frac{\pi\sqrt{2}}{4} \ln\frac{1+w}{1-w}\bigg|_{-1/\sqrt{2}}^{1-\sqrt{2}} \\
&= \frac{\pi\sqrt{2}}{4}\left(\ln\frac{2-\sqrt{2}}{\sqrt{2}} - \ln\frac{1-1/\sqrt{2}}{1+1/\sqrt{2}}\right) \\
&= \frac{\pi\sqrt{2}}{4}\ln(\sqrt{2}+1).
\end{aligned}$$

Solution 5 by Devis Alvarado, UNAH y UPNFM, Tegucigalpa, Honduras.

Let I the integral

$$
\begin{aligned}
I &= \int_0^{\infty} \frac{\arctan(x)}{(1+x)\sqrt{1+x^2}}\,dx \\
&= \int_0^{\infty} \frac{\arctan\left(\frac{1}{y}\right)}{\left(1+\frac{1}{y}\right)\sqrt{1+\frac{1}{y^2}}}\,\frac{dy}{y^2},\ y=\frac{1}{x} \\
&= \int_0^{\infty} \frac{\frac{\pi}{2}-\arctan(y)}{(1+y)\sqrt{1+y^2}}\,dy, \quad \arctan(y)+\arctan\left(\frac{1}{y}\right)=\frac{\pi}{2} \\
&= \frac{\pi}{2}\int_0^{\infty} \frac{1}{(1+y)\sqrt{1+y^2}}\,dy - \int_0^{\infty} \frac{\arctan(y)}{(1+y)\sqrt{1+y^2}}\,dy \\
&= \frac{\pi}{2}\int_0^{\infty} \frac{1}{(1+y)\sqrt{1+y^2}}\,dy - I \\
\Longrightarrow I &= \frac{\pi}{4}\int_0^{\infty} \frac{1}{(1+y)\sqrt{1+y^2}}\,dy \\
&= \frac{\pi}{4}\int_0^{\frac{\pi}{4}} \frac{\sec^2(\theta)\,d\theta}{\left(1+\tan(\theta)\right)\sqrt{1+\tan^2(\theta)}},\ y=\tan(\ theta) \\
&= \frac{\pi}{4}\int_0^{\frac{\pi}{4}} \frac{1}{\cos(\theta)+\sin(\theta)}\,d\theta = \frac{\pi}{4\sqrt{2}}\int_0^{\frac{\pi}{4}} \frac{1}{\cos\left(\theta-\frac{\pi}{4}\right)}\,d\theta \\
&= \frac{\pi}{4\sqrt{2}}\ln\left(\sec\left(\theta-\frac{\pi}{4}\right)+\tan\left(\theta-\frac{\pi}{4}\right)\right)\Bigg|_0^{\frac{\pi}{4}} \\
&= \frac{\pi}{2\sqrt{2}}\ln\left(\sqrt{2}+1\right).
\end{aligned}
$$

Solution 6 by Michel Bataille, Rouen, France.

The change of variables $x=\dfrac{1}{u}$ yields

$$I=\int_0^{\infty}\frac{\arctan(1/u)}{(1+u)\sqrt{1+u^2}}\,du=\int_0^{\infty}\frac{\frac{\pi}{2}-\arctan u}{(1+u)\sqrt{1+u^2}}\,du,$$

hence $I=\dfrac{\pi}{4}J$ where

$$J=\int_0^{\infty}\frac{du}{(1+u)\sqrt{1+u^2}}.$$

The successive change of variables $u = \sinh t = \dfrac{e^t - e^{-t}}{2}$, $t = \ln v$ give

$$J = \int_0^\infty \frac{dt}{1 + \sinh t} = 2\int_1^\infty \frac{dv}{v^2 + 2v - 1}.$$

Lastly, with $w = \dfrac{v+1}{\sqrt{2}}$, that is, $v = w\sqrt{2} - 1$, we obtain

$$\int_1^\infty \frac{dv}{v^2 + 2v - 1} = \sqrt{2}\int_{\sqrt{2}}^\infty \frac{dw}{(w-1)(w+1)} = \sqrt{2}\int_{\sqrt{2}}^\infty \left(\frac{1}{w-1} - \frac{1}{w+1}\right)\frac{dw}{2}$$
$$= \frac{\sqrt{2}}{2}\left[\ln\frac{w-1}{w+1}\right]_{\sqrt{2}}^\infty = \sqrt{2}\ln(1 + \sqrt{2})$$

and consequently,

$$I = \frac{\pi\sqrt{2}}{4}\ln(1 + \sqrt{2}).$$

Solution 7 by Mingcan Fan, Huizhou University, Huizhou, China.

Let $x = 1/t$, and note that $\arctan(1/t) + \arctan t = \pi/2$. Then

$$I = \int_0^\infty \frac{\arctan(1/t)}{(1+t)\sqrt{1+t^2}}dt = \frac{\pi}{4}\int_0^\infty \frac{1}{(1+t)\sqrt{1+t^2}}dt.$$

Let $t = \tan y$, then

$$\begin{aligned}
\int_0^\infty \frac{1}{(1+t)\sqrt{1+t^2}}dt &= \int_0^{\pi/2} \frac{1}{\sin y + \cos y}dy \\
&= \frac{1}{\sqrt{2}}\int_0^{\pi/2} \csc(y + \pi/4)dy \\
&= \frac{1}{\sqrt{2}}\ln(\tan(y/2 + \pi/8))\Big|_0^{\pi/2} \\
&= \frac{1}{\sqrt{2}}[\ln(\cot(\pi/8)) - \ln(\tan(\pi/8))] \\
&= -\sqrt{2}\ln(\tan(\pi/8)).
\end{aligned}$$

Hence, $I = -\dfrac{\pi}{4}\cdot\sqrt{2}\ln(\tan(\pi/8)) = \dfrac{-\pi}{2\sqrt{2}}\ln(\tan(\pi/8))$.

Solution 8 by Perfetti Paolo, dipartimento di matematica Universitą di "Tor Vergata", Roma, Italy.

$$\int_0^\infty \frac{\arctan x}{(1+x)\sqrt{1+x^2}}dx = \int_0^\infty \frac{\frac{\pi}{2} - \arctan\frac{1}{x}}{(1+x)\sqrt{1+x^2}}dx$$

$$\int_0^\infty \frac{\arctan\frac{1}{x}}{(1+x)\sqrt{1+x^2}}dx \underbrace{=}_{x=1/t} \int_0^\infty \frac{\arctan t}{(1+t)\sqrt{1+t^2}}dt$$

hence

$$\int_0^\infty \frac{\arctan x}{(1+x)\sqrt{1+x^2}}dx = \frac{\pi}{4}\int_0^\infty \frac{dx}{(1+x)\sqrt{1+x^2}} \underbrace{=}_{x=\tan u} \frac{\pi}{4}\int_0^{\frac{\pi}{2}} \frac{du}{\cos u + \sin u} =$$

$$= \frac{\pi}{4\sqrt{2}}\int_0^{\frac{\pi}{2}} \frac{du}{\sin(u+\pi/4)} = \frac{\pi}{4\sqrt{2}} \ln\tan\left(\frac{u}{2}+\frac{\pi}{8}\right)\Big|_0^{\frac{\pi}{2}} = \frac{\pi}{4\sqrt{2}} \ln\frac{\tan\left(\frac{\pi}{4}+\frac{\pi}{8}\right)}{\tan\frac{\pi}{8}} =$$

$$= \frac{\pi}{4\sqrt{2}} \ln\frac{\tan\left(\frac{\pi}{2}-\frac{\pi}{8}\right)}{\tan\frac{\pi}{8}} = \frac{\pi}{4\sqrt{2}} \ln\frac{1}{\tan^2\frac{\pi}{8}} = \frac{\pi}{4\sqrt{2}} \ln\frac{1+\cos\frac{\pi}{4}}{1-\cos\frac{\pi}{4}}$$

$$= \frac{\pi}{4\sqrt{2}} \ln\frac{2+\sqrt{2}}{2-\sqrt{2}} = \frac{\pi\ln(1+\sqrt{2})}{2\sqrt{2}}.$$

Solution 9 by Péter Fülöp, Gyömrő, Hungary.

Let's devide the integral into two parts based on the domain $[0,\infty)$:

$$I = \int_0^1 \frac{\arctan(x)}{(1+x)\sqrt{1+x^2)}}dx + \int_1^\infty \frac{\arctan(x)}{(1+x)\sqrt{1+x^2)}}dx$$

Performing the $t = \frac{1}{x}$ substitution in second integral and using the $\arctan(\frac{1}{x}) = \frac{\pi}{2} - \arctan(x)$ equality $(x > 0)$, we get:

$$I = \int_0^1 \frac{\arctan(x)}{(1+x)\sqrt{1+x^2)}}dx + \int_0^1 \frac{1}{t^2}\frac{\frac{\pi}{2}-\arctan(t)}{(1+\frac{1}{t})\sqrt{1+\frac{1}{t^2})}}dt = \frac{\pi}{2}\int_0^1 \frac{1}{(1+x)\sqrt{1+x^2)}}dx$$

Applying further substitutions $x = sh(z)$ and $z = ln(t)$ we get the followings:

$$I = \frac{\pi}{2}\int_0^{\sinh^{-1}(1)} \frac{1}{1+\sinh(z)}dz$$

where $\sinh^{-1}(1) = \ln(1+\sqrt{2})$

$$I = \frac{\pi}{2}\int_0^{\ln(1+\sqrt{2})} \frac{2e^z}{e^{2z}+2e^z-1}dz$$

after the second substitution:

$$I = \frac{\pi}{2} \int_{1}^{(1+\sqrt{2})} \frac{2}{(t+1)^2 - 2} dt$$

Applying the partial fraction decomposition for the integrand:

$$I = \frac{\pi}{4} \int_{1}^{(1+\sqrt{2})} \frac{1}{\frac{t+1}{\sqrt{2}} - 1} - \frac{1}{\frac{t+1}{\sqrt{2}} + 1} dt = \frac{\pi}{4} \Big| \ln\left(\frac{t+1}{\sqrt{2}} - 1\right) - \ln\left(\frac{t+1}{\sqrt{2}} + 1\right) \Big|_{1}^{(1+\sqrt{2})}$$

We get the result:

$$I = -\frac{\pi\sqrt{2}}{4} \ln(\sqrt{2} - 1) = \frac{\pi\sqrt{2}}{4} \ln(\sqrt{2} + 1)$$

Also solved by Bruno Salgueiro Fanego, Viveiro, Lugo, Spain and the problem proposer.

• **5796** *Proposed by Shivam Sharma, Delhi University, New Delhi, India, and Surjeet Singh, Indian Institute of Technology Kanpur, India.*

Here, ζ denotes the zeta function. Prove that:

$$\sum_{k=0}^{\infty} \sum_{p=k}^{\infty} \frac{1}{(p+1)(p+2)\binom{p}{k}} = \zeta(2).$$

Solution 1 by Michel Bataille, Rouen, France.

First, we consider the internal sum:

$$\begin{aligned}
\sum_{p=k}^{\infty} \frac{1}{(p+1)(p+2)\binom{p}{k}} &= \sum_{j=0}^{\infty} \frac{1}{(k+j+1)(k+j+2)\binom{k+j}{k}} \\
&= \sum_{j=0}^{\infty} \frac{1}{(k+j+1)(k+j+2)\binom{k+j}{j}} \\
&= \sum_{j=0}^{\infty} \frac{j!}{(k+j+2)(k+j+1)(k+j)\cdots(k+1)}.
\end{aligned}$$

It follows that

$$\sum_{k=0}^{\infty}\sum_{p=k}^{\infty}\frac{1}{(p+1)(p+2)\binom{p}{k}} = \sum_{k=0}^{\infty}\sum_{j=0}^{\infty}\frac{j!}{(k+j+2)(k+j+1)(k+j)\cdots(k+1)}$$
$$= \sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\frac{j!}{(k+j+2)(k+j+1)(k+j)\cdots(k+1)}$$
$$= \sum_{j=0}^{\infty} j!\sum_{k=1}^{\infty}\frac{1}{k(k+1)\cdots(k+j+1)}$$
$$= \sum_{j=0}^{\infty} j!\sum_{k=1}^{\infty} a_{k,j+2}$$

where $a_{k,n}$ denotes the ratio $\dfrac{1}{k(k+1)\cdots(k+n-1)} = \dfrac{(k-1)!}{(k+n-1)!}$ for $k,n\in\mathbb{N}$.

It is readily checked that $a_{k,n+1} = \dfrac{1}{n}\left(a_{k,n} - a_{k+1,n}\right)$ and we deduce that

$$\sum_{k=1}^{\infty} a_{k,n+1} = \frac{1}{n(n!)}.$$

Thus,

$$\sum_{k=0}^{\infty}\sum_{p=k}^{\infty}\frac{1}{(p+1)(p+2)\binom{p}{k}} = \sum_{j=0}^{\infty} j!\frac{1}{(j+1)(j+1)!} = \sum_{j=0}^{\infty}\frac{1}{(j+1)^2} = \zeta(2).$$

Solution 2 by Perfetti Paolo, dipartimento di matematica Università di “Tor Vergata", Roma, Italy.

We know that

$$\frac{1}{(p+1)(p+2)\binom{p}{k}} = \frac{1}{p+2}\int_0^1 x^k(1-x)^{p-k}dx,$$

hence

$$\sum_{k=0}^{\infty}\sum_{p=k}^{\infty}\frac{1}{(p+1)(p+2)\binom{p}{k}} = \sum_{p=0}^{\infty}\sum_{k=0}^{p}\frac{1}{(p+1)(p+2)\binom{p}{k}} =$$

$$= \sum_{p=0}^{\infty}\sum_{k=0}^{p}\frac{1}{p+2}\int_0^1 x^k(1-x)^{p-k}dx = \sum_{p=0}^{\infty}\frac{1}{p+2}\int_0^1 (1-x)^p\frac{1-\left(\frac{x}{1-x}\right)^{p+1}}{1-\frac{x}{1-x}}dx =$$

$$= \sum_{p=0}^{\infty}\frac{1}{p+2}\int_0^1\frac{(1-x)^{p+1}-x^{p+1}}{1-2x}dx = \int_0^1\left[\frac{\ln(1-x)}{x(1-2x)} - \frac{\ln x}{(1-x)(1-2x)}\right]dx =$$

$$= \int_0^1 \ln(1-x)\left(\frac{1}{x}+\frac{2}{1-2x}\right) - \ln x\left(\frac{2}{1-2x}-\frac{1}{1-x}\right)dx =$$

$$= \int_0^1\left(\frac{\ln(1-x)}{x}+\frac{\ln x}{1-x}\right)dx - 2\int_0^1\frac{\ln\frac{x}{1-x}}{1-2x}dx = \int_0^1\frac{2\ln x}{1-x}dx - 2\int_0^1\frac{\ln\frac{x}{1-x}}{1-2x}dx.$$

It is known that

$$\int_0^1 \frac{\ln x}{1-x}dx = \sum_{k=0}^{\infty}\int_0^1 x^k \ln x dx = \sum_{k=0}^{\infty}\frac{-1}{k^2} = -\zeta(2).$$

As for the second integral let's change $t = x/(1-x)$ whence

$$\int_0^1 \frac{2\ln\frac{x}{1-x}}{1-2x}dx = \int_0^{\infty}\frac{-2\ln t\,dt}{1-t^2} = \int_0^1 \frac{-2\ln t\,dt}{1-t^2} + \int_1^{\infty}\frac{-2\ln t\,dt}{1-t^2}.$$

By changing $t = 1/y$ in the second integral we get $\int_0^1 \frac{-2\ln t\,dt}{1-t^2}$, hence

$$\int_0^{\infty}\frac{-2\ln t\,dt}{1-t^2} = \int_0^1 \frac{-4\ln t\,dt}{1-t^2} = \sum_{k=0}^{\infty}\frac{4}{(2k+1)^2} = \frac{\pi^2}{2} = 3\zeta(2).$$

Finally we get

$$\sum_{k=0}^{\infty}\sum_{p=k}^{\infty}\frac{1}{(p+1)(p+2)\binom{p}{k}} = -2\zeta(2) + 3\zeta(2) = \zeta(2).$$

Solution 3 by Péter Fülöp, Gyömrő, Hungary.

(i) Starting with partial fraction decomposition and introducing β function.

$$S = \sum_{k=0}^{\infty}\sum_{p=k}^{\infty}\left[\frac{1}{p+1} - \frac{1}{p+2}\right]\frac{1}{\binom{p}{k}}$$

$$\frac{1}{\binom{p}{k}} = \frac{(p-k)!k!}{p!} = \frac{\Gamma(p-k+1)\Gamma(k+1)}{\Gamma(p+1)} = (p+1)\frac{\Gamma(p-k+1)\Gamma(k+1)}{\Gamma(p+2)}$$

$= (p+1)\beta(p-k+1, k+1)$ turn back to the sum:

$$S = \sum_{k=0}^{\infty}\sum_{p=k}^{\infty}\left[\frac{1}{p+1} - \frac{1}{p+2}\right](p+1)\beta(p-k+1, k+1)$$

$$S = \sum_{k=0}^{\infty}\sum_{p=k}^{\infty}\frac{\beta(p-k+1, k+1)}{p+2} = \sum_{k=0}^{\infty}\sum_{p=k}^{\infty}\frac{1}{p+2}\int_0^1 t^{p-k}(1-t)^k dt$$

(ii) Swapping the order of summations and integration

$$S = \sum_{k=0}^{\infty}\int_0^1 \left(\frac{1-t}{t}\right)^k \underbrace{\sum_{p=k}^{\infty}\frac{t^p}{p+2}}_{t^k\sum_{p=0}^{\infty}\frac{t^p}{p+k+2}} dt$$

(iii) Introducing the Lerch transcendent and its integral representation:

$$\Phi(z,s,\alpha) = \sum_{n=0}^{\infty} \frac{z^n}{(n+\alpha)^s} = \frac{1}{\Gamma(s)} \int_0^{\infty} \frac{x^{s-1}e^{-\alpha x}}{1-ze^{-x}} dx$$

Where, in this case $z = t, s = 1, \alpha = k+2$ we get:

$$S = \sum_{k=0}^{\infty} \int_0^1 \int_0^{\infty} (1-t)^k \frac{e^{-(k+2)x}}{1-te^{-x}} dxdt$$

(iv) Performing the $r = e^{-x}$ substitution and calculating the sum:

$$S = \sum_{k=0}^{\infty} \int_0^1 \int_0^1 (1-t)^k \frac{r^{k+1}}{1-tr} drdt = \int_0^1 \int_0^1 \frac{r}{(1-tr)(1-(1-t)r)} drdt$$

(v) Substitution $t = \frac{x}{r}$ then partial fraction decomposition.

$$S = \int_0^1 \int_0^r \frac{1}{1-x} \frac{1}{1-r+x} dxdr = \int_0^1 \frac{1}{2-r} \underbrace{\int_0^r \left(\frac{1}{1-x} + \frac{1}{1-r+x} \right) dx}_{-\ln(1-x)+-\ln(1-r+x)\Big|_0^r} dr$$

$$S = -2 \int_0^1 \frac{\ln(1-r)}{2-r} dr = -2 \int_0^1 \frac{\ln(t)}{1+t} dt$$

where $t = 1-r$ substitution.

(vi) Proving that $-2 \int_0^1 \frac{\ln(t)}{1+t} dt = \zeta(2)$

Using the facts: $\ln(t) = \frac{d(t^a)}{dt}\Big|_{a=0}$ and $\frac{1}{1+t} = \sum_{n=0}^{\infty} (-t)^n$.

$$S = -2 \frac{d}{dt} \Big[\sum_{k=0}^{\infty} (-1)^k \int_0^1 t^{k+a} dt \Big] = -2 \frac{d}{dt} \Big[\sum_{k=0}^{\infty} \frac{(-1)^k}{k+a+1} \Big]_{a=0} = 2 \sum_{k=0}^{\infty} \frac{(-1)^k}{(k+1)^2} = -2Li_2(-1)$$

where $Li_2(-1)$ is the dilogarithm function at the place -1.

(vii) Applying the $Li_2(z) + Li_2(-z) = \frac{1}{2} Li_2(z^2)$ dilogarithm identity than we get the solution:

$$S = -2(\frac{1}{2}Li_2[(-1)^2] - Li_2(1)) = Li_2(1) = \zeta(2)$$

Finally we can write:

$$S = \sum_{k=0}^{\infty}\sum_{p=k}^{\infty}\frac{1}{(p+1)(p+2)\binom{p}{k}} = -2\int_0^1 \frac{\ln(t)}{1+t}dt = \zeta(2)$$

The statement is proved.

Solution 4 by Albert Stadler, Herrliberg, Switzerland.

We interchange the order of summation which is permitted, since all terms of the double sum are positive, and obtain

$$S := \sum_{k=0}^{\infty}\sum_{p=k}^{\infty}\frac{1}{(p+1)(p+2)\binom{p}{k}} = \sum_{p=0}^{\infty}\sum_{k=0}^{p}\frac{1}{(p+1)(p+2)\binom{p}{k}}.$$

By Euler's evaluation of the beta function,

$$\frac{1}{(p+1)(p+2)\binom{p}{k}} = \frac{(p-k)!k!}{(p+2)!} = \frac{1}{p+2}\int_0^1 t^{p-k}(1-t)^k dt.$$

Hence

$$S = \sum_{p=0}^{\infty}\sum_{k=0}^{p}\frac{1}{p+2}\int_0^1 t^{p-k}(1-t)^k dt = \sum_{p=0}^{\infty}\frac{1}{p+2}\int_0^1 t^p \frac{1-\left(\frac{1}{t}-1\right)^{p+1}}{1-\left(\frac{1}{t}-1\right)}dt =$$

$$= \sum_{p=0}^{\infty}\frac{1}{p+2}\int_0^1 \frac{t^{p+1}-(1-t)^{p+1}}{2t-1}dt = \int_0^1 \frac{-\frac{\log(1-t)}{t}+\frac{\log t}{1-t}}{2t-1}dt,$$

using the fact that

$$\sum_{k=2}^{\infty}\frac{x^{k-1}}{k} = -\frac{\log(1-x)}{x} - 1, \quad |x| < 1.$$

We next exploit the fact that the integrand is invariant under the substitution t→1-t and get

$$S = \int_0^1 \frac{\frac{\log\left(\frac{1}{2}\right)}{\frac{1}{2}} - \frac{\log(1-t)}{t}}{2t-1}dt + \int_0^1 \frac{\frac{\log t}{1-t} - \frac{\log\left(\frac{1}{2}\right)}{\frac{1}{2}}}{2t-1}dt = 2\int_0^1 \frac{\frac{\log\left(\frac{1}{2}\right)}{\frac{1}{2}} - \frac{\log(1-t)}{t}}{2t-1}dt =$$

$$= -\int_0^1 \frac{(4\log 2)\,t + 2\log(1-t)}{(2t-1)\,t}dt = \int_0^1 \left((4\log 2)\,t + 2\log(1-t)\right)\left(\frac{1}{t} - \frac{2}{2t-1}\right)dt =$$

$$= 4\log 2 \; + 2\int_0^1 \frac{\log(1-t)}{t}dt - 2\int_0^1 \frac{(4\log 2\;)\,t + 2\log(1-t)}{2t-1}dt =$$

$$= 4\log 2 \; + 2\int_0^1 \frac{\log(1-t)}{t}dt - \int_{-1}^1 \frac{(4\log 2\;)\,\frac{u+1}{2} + 2\log\left(\frac{1-u}{2}\right)}{u}du =$$

$$= 2\int_0^1 \frac{\log(1-t)}{t}dt - 2\int_{-1}^1 \frac{\log(1-u)}{u}du = -2\int_{-1}^0 \frac{\log(1-u)}{u}du =$$

$$= 2\int_0^1 \frac{\log(1+u)}{u}du = 2\sum_{k=1}^{\infty}\frac{(-1)^{k-1}}{k}\int_0^1 u^{k-1}du = 2\sum_{k=1}^{\infty}\frac{(-1)^{k-1}}{k^2} =$$

$$= 2\sum_{k=1}^{\infty}\frac{1}{k^2} - 4\sum_{k=1}^{\infty}\frac{1}{(2k)^2} = \sum_{k=1}^{\infty}\frac{1}{k^2} = \zeta(2)\,.$$

Solution 5 by Ángel Plaza, Universidad de Las Palmas de Gran Canaria, Spain.

It is enough to prove that

$$\sum_{p=k}^{\infty}\frac{1}{(p+1)(p+2)\binom{p}{k}} = \frac{1}{(k+1)^2}.$$

To this end, note first that

$$\sum_{p=k}^{n}\frac{1}{(p+1)(p+2)\binom{p}{k}} = \frac{(n+2)\binom{n+1}{k} - k - 1}{(k+1)^2(n+2)\binom{n+1}{k}},$$

which may be proved by induction. Therefore,

$$\sum_{p=k}^{\infty}\frac{1}{(p+1)(p+2)\binom{p}{k}} = \lim_{n\to\infty}\frac{(n+2)\binom{n+1}{k} - k - 1}{(k+1)^2(n+2)\binom{n+1}{k}} = \frac{1}{(k+1)^2}.$$

Solution 6 by Brian Bradie, Department of Mathematics, Christopher Newport University, Newport News, VA.

Interchange the order of summation to obtain

$$\sum_{k=0}^{\infty}\sum_{p=k}^{\infty}\frac{1}{(p+1)(p+2)\binom{p}{k}} = \sum_{p=0}^{\infty}\frac{1}{(p+1)(p+2)}\sum_{k=0}^{p}\frac{1}{\binom{p}{k}}.$$

Now,

$$\frac{1}{\binom{p}{k}} = (p+1)\int_0^1 x^k(1-x)^{p-k}\,dx,$$

so

$$\sum_{k=0}^{p}\frac{1}{\binom{p}{k}} = (p+1)\int_0^1\sum_{k=0}^{p}x^k(1-x)^k\,dx = (p+1)\int_0^1\frac{x^{p+1}-(1-x)^{p+1}}{2x-1}\,dx,$$

and

$$\begin{aligned}\sum_{k=0}^{\infty}\sum_{p=k}^{\infty}\frac{1}{(p+1)(p+2)\binom{p}{k}} &= \sum_{p=0}^{\infty}\frac{1}{p+2}\int_0^1\frac{x^{p+1}-(1-x)^{p+1}}{2x-1}\,dx\\ &= \int_0^1\frac{1}{2x-1}\sum_{p=0}^{\infty}\frac{x^{p+1}-(1-x)^{p+1}}{p+2}\,dx\\ &= \int_0^1\frac{1}{2x-1}\left(\frac{\ln x}{1-x}-\frac{\ln(1-x)}{x}\right)dx.\end{aligned}$$

By partial fractions, this becomes

$$\begin{aligned}\sum_{k=0}^{\infty}\sum_{p=k}^{\infty}\frac{1}{(p+1)(p+2)\binom{p}{k}} &= \int_0^1\left(\frac{\ln(1-x)}{x}-2\frac{\ln(1-x)}{2x-1}+\frac{\ln x}{1-x}+2\frac{\ln x}{2x-1}\right)dx\\ &= -2\mathrm{Li}_2(x)\Big|_0^1+2\int_0^1\frac{\ln x-\ln(1-x)}{2x-1}\,dx\\ &= -2\zeta(2)+2\int_0^1\frac{\ln x-\ln(1-x)}{2x-1}\,dx,\end{aligned}$$

where $\mathrm{Li}_2(x)$ is the dilogarithm function with $\mathrm{Li}_2(1)=\zeta(2)$ and $\mathrm{Li}_2(0)=0$. For the remaining integral, let $u=2x-1$. Then

$$\begin{aligned}\int_0^1\frac{\ln x-\ln(1-x)}{2x-1}\,dx &= \frac{1}{2}\int_{-1}^1\frac{\ln\left(\frac{1+u}{2}\right)-\ln\left(\frac{1-u}{2}\right)}{u}\,du\\ &= \frac{1}{2}\int_{-1}^1\frac{\ln(1+u)-\ln(1-u)}{u}\,du\\ &= \frac{1}{2}\big(\mathrm{Li}_2(u)-\mathrm{Li}_2(-u)\big)\Big|_{-1}^1\\ &= \frac{1}{2}\big(\zeta(2)-\mathrm{Li}_2(-1)-\mathrm{Li}_2(-1)+\zeta(2)\big)\\ &= \zeta(2)-\mathrm{Li}_2(-1)=\frac{3}{2}\zeta(2),\end{aligned}$$

as $\mathrm{Li}_2(-1)=-\frac{1}{2}\zeta(2)$. Finally,

$$\sum_{k=0}^{\infty}\sum_{p=k}^{\infty}\frac{1}{(p+1)(p+2)\binom{p}{k}}=-2\zeta(2)+2\left(\frac{3}{2}\zeta(2)\right)=\zeta(2).$$

Also solved by the problem proposer.

• **5797** *Proposed by Toyesh Prakash Sharma and Etisha Sharma, Agra College, Agra, India.*

Solve the following differential equation without the aid of computers:

$$\left(x^2 \ln^2 x\right) \frac{d^2y}{dx^2} - (2x \ln x) \frac{dy}{dx} + (2 + \ln x)\, y + \ln^3 x = 0.$$

Solution 1 by Albert Stadler, Herrliberg, Switzerland.

We look for solutions within the function set $\left\{ (ax+b)\ln x + (cx+d)\ln^2 x \,\middle|\, a,b,c,d \in \mathbb{C} \right\}$. Then

$$\begin{aligned} 0 = &\left(x^2 \ln^2 x\right) \frac{d^2}{dx^2}\left((ax+b)\ln x + (cx+d)\ln^2 x\right) \\ &- (2x\ln x)\frac{d}{dx}\left((ax+b)\ln x + (cx+d)\ln^2 x\right) \\ &+ (2+\ln x)\left((ax+b)\ln x + (cx+d)\ln^2 x\right) \\ &+ \ln^3 x \\ &= (1-d+cx)\ln^3 x\,. \end{aligned}$$

Hence c=0, d=1, and the general solution is given by

$$y = (ax+b)\ln x + \ln^2 x\,, \quad a,b \in \mathbb{C}.$$

Solution 2 by Ángel Plaza, Universidad de Las Palmas de Gran Canaria, Spain.

By the change of variables $\ln x = t$, or $x = e^t$ the equation becomes

$$t^2\frac{d^2y}{dt^2} - (t^2+2t)\frac{dy}{dt} + (2+t)y + t^3 = 0. \tag{5}$$

First, we solve the homogeneous equation

$$t^2\frac{d^2y}{dt^2} - (t^2+2t)\frac{dy}{dt} + (2+t)y = 0. \tag{6}$$

Let us assume that $\frac{d^2y}{dt^2} = 0$, then equation (6) becomes $-t\frac{dy}{dt} + y = 0$, which is a first order *ODE* of separable variables with solution $y = t + C$. Since $\frac{d}{dt}(te^t) = (t+1)e^t$, and $\frac{d^2}{dt^2}(te^t) = (t+2)e^t$, we may check that te^t is also a solution to equation (6). Therefore, the general solution of (6) is $y_h = C_1 t + C_2 te^t$.

It remains to find a particular solution of equation (5) and undo the initial change of variables.

Because of the term t^3, we may tray with $y = t^2$: $\dfrac{d}{dt}(t^2) = 2t$, and $\dfrac{d^2}{dt^2}(t^2) = 2$, so, putting $y = t^2$ in equation (5) results:

$$2t^2 - 2t(t^2 + 2t) + (2 + t)t^2 + t^3 = 2t^2 - 2t^3 - 4t^2 + 2t^2 + t^3 + t^3 = 0$$

So the general solution to equation (5) is $y = t^2 + C_1 t + C_2 t e^t$.

Finally, after undoing the initial change of variables, we obtain the solution of the proposed equation as

$$y = \ln^2 x + C_1 \ln x + C_2 x \ln x.$$

Solution 3 by Brian D. Beasley, Simpsonville, SC.

We start by solving the associated homogeneous equation

$$(x^2 \ln^2 x)\frac{d^2y}{dx^2} - (2x \ln x)\frac{dy}{dx} + (2 + \ln x)y = 0.$$

To determine the values of r for which $y = x^r \ln x$ is a solution of this equation, we substitute $y' = x^{r-1} + r x^{r-1} \ln x$ and $y'' = (2r - 1)x^{r-2} + r(r - 1)x^{r-2} \ln x$ to obtain

$$(x^2 \ln^2 x)[(2r - 1)x^{r-2} + r(r - 1)x^{r-2} \ln x] - (2x \ln x)(x^{r-1} + r x^{r-1} \ln x) + (2 + \ln x)x^r \ln x = 0.$$

This implies

$$r(r - 1)x^r \ln^3 x = 0$$

and hence $r = 0$ or $r = 1$. Since $y_1 = \ln x$ and $y_2 = x \ln x$ are linearly independent, we conclude that the complementary solution is $y_c = c_1 \ln x + c_2 x \ln x$.

Next, we find a particular solution for the original equation. Given the powers of $\ln x$ in the equation, we let $y_p = A \ln^2 x + B \ln^3 x$. Then $y_p' = 2A \ln x/x + 3B \ln^2 x/x$ and $y_p'' = (2A - 2A \ln x)/x^2 + (6B \ln x - 3B \ln^2 x)/x^2$, which yields

$$(x^2 \ln^2 x)\left(\frac{2A - 2A \ln x + 6B \ln x - 3B \ln^2 x}{x^2}\right) - (2x \ln x)\left(\frac{2A \ln x + 3B \ln^2 x}{x}\right)$$

$$+(2 + \ln x)(A \ln^2 x + B \ln^3 x) + \ln^3 x = 0.$$

This implies

$$-2B \ln^4 x + (1 - A + 2B) \ln^3 x = 0,$$

so $A = 1$ and $B = 0$. Hence $y_p = \ln^2 x$.

Thus we conclude that the solution of the given differential equation is

$$y = y_c + y_p = c_1 \ln x + c_2 x \ln x + \ln^2 x.$$

Solution 4 by Brian Bradie, Department of Mathematics, Christopher Newport University, Newport News, VA.

Rewrite the differential equation as

$$\frac{d^2y}{dx^2} - \frac{2}{x\ln x}\frac{dy}{dx} + \frac{2+\ln x}{x^2\ln^2 x}y = -\frac{\ln x}{x^2}. \tag{7}$$

We now seek an integrating factor $\mu(x)$ such that

$$\mu(x)\left(\frac{d^2y}{dx^2} - \frac{2}{x\ln x}\frac{dy}{dx} + \frac{2+\ln x}{x^2\ln^2 x}y\right) = \frac{d^2}{dx^2}(\mu(x)y).$$

This requires

$$\frac{d\mu}{dx} = -\frac{1}{x\ln x}\mu \quad \text{and} \quad \frac{d^2\mu}{dx^2} = \frac{2+\ln x}{x^2\ln^2 x}\mu.$$

The first of these conditions yields

$$\mu(x) = \frac{1}{\ln x},$$

so that

$$\frac{d\mu}{dx} = -\frac{1}{x\ln^2 x}$$

and

$$\frac{d^2\mu}{dx^2} = \frac{2\ln x + \ln^2 x}{x^2\ln^4 x} = \frac{2+\ln x}{x^2\ln^3 x} = \frac{2+\ln x}{x^2\ln^2 x}\mu,$$

thus confirming the second condition. Next, multiply (7) by $\mu(x)$ to obtain

$$\frac{d^2}{dx^2}\left(\frac{y}{\ln x}\right) = -\frac{1}{x^2}.$$

Two integrations yields

$$\frac{y}{\ln x} = \ln x + c_1x + c_2,$$

or

$$y(x) = \ln^2 x + c_1x\ln x + c_2\ln x.$$

Solution 5 by Michel Bataille, Rouen, France.

Consider the function $f_0 : x \mapsto f_0(x) = (\ln x)^2$. For $x > 0$, we have

$$f_0'(x) = \frac{2\ln x}{x}, \qquad f_0''(x) = \frac{2(1-\ln x)}{x^2}$$

from which it is easily checked that

$$\left(x^2\ln^2 x\right) f_0''(x) - (2x\ln x)f_0'(x) + (2+\ln x)f_0(x) + \ln^3 x = 0$$

for all $x > 0$. Thus, f_0 is a solution of the given linear equation (E).
Now, consider the homogenous equation (H) associated with the equation (E):

$$\left(x^2 \ln^2 x\right) \frac{d^2y}{dx^2} - (2x \ln x)\frac{dy}{dx} + (2 + \ln x)y = 0.$$

The function $f_1 : x \mapsto f_1(x) = \ln x$ and $f_2 : x \mapsto f_2(x) = x \ln x$ are solutions to (H) (readily checked) and are independent, hence the general solution to (H) is $x \mapsto Ax \ln x + B \ln x$ where A, B are real constants.
As a result, the general solution of the given equation (E) is

$$x \mapsto f_{A,B}(x) = Ax \ln x + B \ln x + (\ln x)^2$$

where A, B are real constants.

Note. A priori $x \mapsto f_{A,B}(x)$ is the form of the solutions defined on $(0, 1)$ or on $(1, \infty)$. However, it is easy to see that a solution $x \mapsto f_{A_1,B_1}(x)$ defined on $(0, 1)$ and a solution $x \mapsto f_{A_2,B_2}(x)$ defined on $(1, \infty)$ join up to a solution on $(0, \infty)$ if and only if $A_1 = A_2$ and $B_1 = B_2$. Therefore the functions $f_{A,B}$ are the maximal solutions on $(0, \infty)$.

Solution 6 by Perfetti Paolo, dipartimento di matematica Università di "Tor Vergata", Roma, Italy.

Let's define $t \doteq \ln x$. $y(x) = y(e^t) = Y(t) = Y(\ln x)$

$$\frac{dy}{dx} = \frac{dY}{dt}\frac{dt}{dx} = \frac{Y'}{x} = e^{-t}Y', \qquad \frac{d^2y}{dx^2} = \frac{d^2Y}{dt^2}\frac{1}{x^2} - \frac{dY}{dt}\frac{1}{x^2} = Y''e^{-2t} - Y'e^{-t}.$$

The equation becomes

$$e^{2t}t^2(Y'' - Y')e^{-2t} - 2e^t tY'e^{-t} + (2 + t)Y + t^3 = 0$$

$$t^2Y'' - t(2 + t)Y' + (2 + t)Y + t^3 = 0. \tag{1}$$

The homogeneous equation is

$$t^2Z'' - t(2 + t)Z' + (2 + t)Z = 0. \tag{2}$$

By defining $Z(t) \doteq tQ(t)$ the equation (2) becomes

$$t^2(2Q' + tQ'') - (2t + y^2)(Q + tQ') + (2 + t)tQ = 0 \iff Q'' = Q'.$$

hence $Z(t) = c_1te^t + c_2t$. And for the inhomogeneous equation let's try $Y(t) = at^b$ yielding

$$at^{b+1}(1 - b) + t^b(ab^2 - 3ab + 2a) + t^3 = 0 \implies a = 1, b = 2.$$

Thus, the solution of (1) is $Y(t) = c_1te^t + c_2t + t^2$, from which

$$y(x) = c_1x \ln x + c_2 \ln x + (\ln x)^2.$$

Solution 7 by Ulrich Abel, Technische Hochschule Mittelhessen, Friedberg, Germany.

The substitution $y(x) = z(x)\ln x$ leads to the equivalent differential equation

$$\left(x^2\ln^2 x\right)\left(\frac{d^2z}{dx^2}\ln x + 2\frac{dz}{dx}\frac{1}{x} - \frac{z}{x^2}\right) - (2x\ln x)\left(\frac{dz}{dx}\ln x + \frac{z}{x}\right) + (2+\ln x)\,z\ln x + \ln^3 x = 0,$$

which simplifies to

$$x^2\frac{d^2z}{dx^2} + 1 = 0,$$

with the general solution $z(x) = a + bx + \ln x$. Hence, the general solution is given by

$$y(x) = a\ln x + bx\ln x + \ln^2 x$$

with $a, b \in \mathbb{R}$.

Also solved by the problem proposer.

• **5798** *Proposed by Vasile Cirtoaje, Petroleum-Gas University of Ploiesti, Romania.*

Let a, b, c, d be positive real numbers such that $ab + bc + cd + da = 4$. Prove that if (i) $a \geqslant b \geqslant 1 \geqslant c \geqslant d$ or (ii) $a \geqslant b \geqslant c \geqslant 1 \geqslant d$, then

$$\frac{1}{a} + \frac{1}{b} + \frac{1}{c} + \frac{1}{d} + 8 \geqslant 3(a+b+c+d).$$

Solution 1 by Perfetti Paolo, dipartimento di matematica Universitą di "Tor Vergata", Roma, Italy.

Let's assume (i) true. Certainly $a \leqslant 4$ and $b \leqslant 4$ because otherwise $ab + bc + cd + da = 4$ would be violated. This means

$$\frac{1}{a} + \frac{1}{b} + \frac{1}{c} + \frac{1}{d} + 8 \geqslant \frac{1}{c} + \frac{1}{d} + 8 \geqslant 3(8+2) \geqslant 3(a+b+c+d).$$

This implies then the inequality is true as soon as c or d is small enough. Thus we assume $a, b \leqslant 4$ and $1 \geqslant c, d \geqslant \delta > 0$. If follows that the continuous function $F(a,b,c,d) = \frac{1}{a} + \frac{1}{b} + \frac{1}{c} + \frac{1}{d} + 8 - 3(a+b+c+d)$ admits the minimum in the set $V = \{(a,b,c,d) \colon 1 \leqslant a, b \leqslant 4,\ \delta \leqslant c, d \leqslant 1.\}$. Let's rewrite the inequality as

$$F(a,b,c,d) = \sum_{\text{cyc}}\frac{1}{a} + 2\sum_{\text{cyc}} ab \geqslant 3\sum_{\text{cyc}} a.$$

We will prove that $f(a,b,c,d) \geqslant f(a,x,x,d)$. Let $b' = b - x$, $c' = c + x$, $x \leqslant (b-c)/2$, $x \leqslant b - 1$, $x \leqslant 1 - c$. We have

$$\frac{1}{b} + \frac{1}{c} \geqslant \frac{1}{b'} + \frac{1}{c'} \iff (b-x)(c+x)(b+c) \geqslant bc(b+c) \iff b - c \geqslant x$$

clearly holds true. Moreover

$$ab + bc + cd \geqslant ab' + b'c' + c'd \geqslant \iff -ax + x(b-c) - x^2 + xd \leqslant 0.$$

That is, $a + x \geqslant b - c + d$ but $a + x \geqslant a \geqslant b \geqslant b - c + d$, hence it is true. With respect to the last point, observe that $b' + c' = b + c$. Thus, we come to $f(a,b,c,d) \geqslant f(a,x,x,d)$. But $b \geqslant 1 \geqslant c$ implies that $x = 1$. Then we come to

$$f(a,b,c,d) \geqslant f(a,1,1,d) = \frac{(ad-1)(2da-a-d)}{ad} \geqslant 0. \tag{1}$$

We have two possibilities. The first is $da \leqslant 1$. Moreover $2da \leqslant a + d$ if and only if $d(2a-1) \leqslant a$ and $d(2a-1) \leqslant \frac{1}{a}(2a-1) \leqslant a$ if and only if $a^2 - 2a + 1 = (a-1)^2 \geqslant 0$, which clearly holds true. The equality occurs when $a = 1$.

The second possibility is $ad \geqslant 1$. $ab + bc + cd + da = 4$ is $a + 1 + d + da = 4$ and then $a + d = 3 - da \leqslant 3 - 1 = 2$. Moreover $2da - a - d = 2(3 - d - a) - a - d = 6 - 3a - 3d \geqslant 0$ if and only if $a + d \leqslant 2$ hence (1) holds true.

It follows that the minimum of $F(a,b,c,d)$ is $F(1,1,1,1) = 0$

(ii) Let's define $a+c = 2x, ac = y^2, b+d = 2s, bd = t^2$. $ab+bc+cd+da = 4$ is $xs = 1$. Moreover 1) $x \geqslant y$ and $s \geqslant t$ by the AGM, 2) $y \geqslant 1$ and $x \leqslant 2$ from $a, b \geqslant 1$ and $ab + bc + cd + da = 4$. The inequality reads as

$$\frac{2x}{y} + \frac{2s}{t} + 8 - 6(x+s) \geqslant 0$$

which upon setting $s = 1/x$ becomes

$$\frac{2}{xy^2t^2}(x^2t^2 + y^2 + 4y^2xt^2 - 3y^2x^2t^2 - 3y^2t^2) \geqslant 0.$$

That is,

$$x^2t^2 + y^2 + 4y^2xt^2 - 3y^2x^2t^2 - 3y^2t^2 \geqslant 0. \tag{2}$$

The derivative respect to x is $2x + 4y^2 - 6xy^2 \leqslant 2x + 4y^2 - 2x - 4y^2 = 0$ (we have used (1) and (2)) above. This means that we must check (1) for the maximum of $x = 2$ which means $b + d = 1$ and then $d = 0$ because $b \geqslant 1$. It follows $t = 0$ and (1) becomes $y^2 \geqslant 0$ which clearly holds true.

Also solved by Albert Stadler, Herrliberg, Switzerland and the problem proposer.

Editor's Statement: It goes without saying that the problem proposers, as well as the solution proposers, are the *élan vital* of the Problems/Solutions Section of SSMJ. As the editor of this Section of the Journal, I consider myself fortunate to be in a position to receive, compile and organize a wealth of proposed ingenious problems and solutions intended for online publication. My unwavering gratitude goes to all the amazingly creative contributors. We come together from across continents because we find intellectual value, joy and satisfaction in mathematical problems, both in their creation as well as their solution. So that our collective efforts serve us well, I kindly ask all contributors to adhere to the following guidelines. As you peruse below, you may construe that the guidelines amount to a lot of work. But, as the samples show, there's not much to do. Your cooperation is much appreciated!

Keep in mind that the examples given below are your best guide!

Formats, Styles and Requirements

When submitting proposed problem(s) or solution(s), please send both **LaTeX** document and **pdf** document of your proposed problem(s) or solution(s). There are ways (discoverable from the internet) to convert from Word to proper LaTeX code. Porposals without a *proper* **LaTeX** document will not be published regrettably.

Regarding Proposed Solutions:

Below is the FILENAME format for all the documents of your proposed solution(s).

#ProblemNumber_FirstName_LastName_Solution_SSMJ

- FirstName stands for YOUR first name.
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Examples:

#1234_Max_Planck_Solution_SSMJ

#9876_Charles_Darwin_Solution_SSMJ

Please note that every problem number is *preceded* by the sign # .

All you have to do is copy the FILENAME format (or an example below it), paste it and then modify portions of it to your specs.

Please adopt the following structure, in the order shown, for the presentation of your solution:

1. On top of the first page of your solution, begin with the phrase:

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where the string of four astrisks represents the problem number.

2. On the second line, write

"Solution proposed by [your First Name, your Last Name]",

followed by your affiliation, city, country, all on the same linear string of words. Please see the example below. Make sure you do the same for your collaborator(s).

3. On a new line, state the problem proposer's name, affiliation, city and country, just as it appears published in the Problems/Solutions section.

4. On a new line below the above, write in bold type: "**Statement of the Problem**".

5. Below the latter, state the problem. Please make sure the statement of your problem (unlike the preceding item) is not in bold type.

6. Below the statement of the problem, write in bold type: "**Solution of the Problem**".

7. Below the latter, show the entire solution of the problem.

Here is a sample for the above-stated format for proposed solutions:

Proposed solution to #1234 SSMJ

Solution proposed by Emmy Noether, University of Göttingen, Lower Saxony, Germany.

Problem proposed by Isaac Newton, Trinity College, Cambridge, England.

Statement of the problem:

Compute $\sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}$.

Solution of the problem: . . . . . .

Regarding Proposed Problems:

For all your proposed problems, please adopt for all documents the following FILENAME format:

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If you do not have a ProblemTitle, then leave that component as it already is (i.e., ProblemTitle).

The component YourGivenNumber is any UNIQUE 3-digit (or longer) number you like to give to your problem.

Examples:

Max_Planck_ProposedProblem_SSMJ_314_HarmonicPatterns

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4. On a new line below the above, write in bold type: "**Statement of the Problem**".

5. Below the latter, state the problem. Please make sure the statement of your problem (unlike the preceding item) is not in bold type.

6. Below the statement of the problem, write in bold type: "**Solution of the Problem**".

7. Below the latter, show the entire solution of your problem.

Here is a sample for the above-stated format for proposed problems:

Problem proposed to SSMJ

Problem proposed by Isaac Newton, Trinity College, Cambridge, England.

Principia Mathematica (⟵ You may choose to not include a title.)

Statement of the problem:

Compute $\sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}$.

Solution of the problem: . . . . . .

♣ ♣ ♣ Thank You! ♣ ♣ ♣