Problems and Solutions

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This section of the Journal offers readers an opportunity to exchange interesting mathematical problems and solutions. Please email them to Prof. Albert Natian at Department of Mathematics, Los Angeles Valley College. Please make sure every proposed problem or proposed solution is provided in both ***LaTeX*** and pdf documents. Thank you!

To propose problems, email them to: problems4ssma@gmail.com

To propose solutions, email them to: solutions4ssma@gmail.com

Solutions to previously published problems can be seen at <www.ssma.org/publications>.

Solutions to the problems published in this issue should be submitted *before* February 1, 2026.

• **5820** *Proposed by Michel Bataille, Rouen, France.*

Let r, s be integers with $r \geqslant s \geqslant 0$ and let the real number x satisfy $|x| < 1$. Show that

$$\sum_{n=0}^{\infty} \binom{r+n}{s} x^n = \sum_{j=0}^{s} \binom{r}{s-j} \frac{x^j}{(1-x)^{j+1}}.$$

• **5821** *Proposed by D.M. Bătineţu-Giurgiu, Matei Basarab National College, Bucharest, Romania and Neculai Stanciu, George Emil Palade Secondary School, Buzău, Romania.*

Where $a > 0$, $x > 0$, and $\mathcal{E}(x) := \left(1 + \dfrac{1}{x}\right)^x$, find

$$\lambda := \lim_{n\to\infty} \left[\lim_{x\to\infty} \left(\frac{x}{n}\right)^2 \left(-n\mathcal{E}(x) + \sum_{k=1}^{n} \mathcal{E}(x+ka) \right) \right].$$

• **5822** *Proposed by Paolo Perfetti, dipartimento di matematica Universitą di "Tor Vergata", Rome, Italy.*

Find the value of

$$S := \sum_{n=0}^{\infty} \left(\frac{1}{n+1} - \frac{1}{n+2} + \frac{1}{n+3} - \dots \right)^3.$$

• **5823** *Proposed by Daniel Sitaru, National Economic College "Theodor Costescu" Drobeta Turnu - Severin, Romania.*

Prove that if $0 < a \leqslant b$, then:

$$\left(\int_a^b e^{-x^2}dx\right)^2 \geqslant \left(\int_{\frac{a+b}{2}}^b e^{-x^2}dx + \int_{\sqrt{ab}}^b e^{-x^2}dx\right)\left(\int_a^{\frac{a+b}{2}} e^{-x^2}dx + \int_a^{\sqrt{ab}} e^{-x^2}dx\right).$$

• **5824** *Proposed by Toyesh Prakash Sharma, Agra College, Agra, India.*

Calculate

$$I := \int_2^3 \left(\frac{\ln(x+1)\ln x}{x-1} + \frac{\ln x \ln(x-1)}{x+1} + \frac{\ln(x+1)\ln(x-1)}{x}\right) dx.$$

Solutions

To Formerly Published Problems

• **5799** *Proposed by Syed Shahabudeen, Ernakulam, Kerala, India.*

Prove that

$$\int_0^1 \sqrt{x}\mathbf{K}(\sqrt{x})dx = G + \frac{1}{2}$$

where G is the Catalan's Constant and $\mathbf{K}$ is the complete elliptic integral of the first kind defined as

$$\mathbf{K}(t) = \int_0^{\frac{\pi}{2}} \frac{d\theta}{\sqrt{1-t^2\sin^2\theta}}.$$

Solution 1 by Brian Bradie, Department of Mathematics, Christopher Newport University, Newport News, VA.

First, note

$$\begin{aligned} G &= \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)^2} = \sum_{k=0}^{\infty} \frac{(-1)^k}{2k+1} \int_0^1 x^{2k}\,dx = \int_0^1 \sum_{k=0}^{\infty} \frac{(-1)^k}{2k+1} x^{2k}\,dx \\ &= \int_0^1 \frac{\tan^{-1} x}{x}\,dx. \end{aligned}$$

With the substitution $x = \tan(\theta/2)$, this becomes

$$G = \frac{1}{4}\int_0^{\pi/2} \frac{\theta}{\tan\frac{\theta}{2}} \sec^2\frac{\theta}{2}\,d\theta = \frac{1}{4}\int_0^{\pi/2} \frac{\theta}{\sin\frac{\theta}{2}\cos\frac{\theta}{2}}\,d\theta = \frac{1}{2}\int_0^{\pi/2} \frac{\theta}{\sin\theta}\,d\theta.$$

Returning to the desired integral,

$$\begin{aligned}\int_0^1 \sqrt{x}\mathbf{K}(\sqrt{x})\,dx &= \int_0^1 \sqrt{x}\int_0^{\pi/2} \frac{1}{\sqrt{1-x\sin^2\theta}}\,d\theta\,dx\\ &= \int_0^{\pi/2}\int_0^1 \frac{\sqrt{x}}{\sqrt{1-x\sin^2\theta}}\,dx\,d\theta.\end{aligned}$$

For the inner integral, make the substitution $x\sin^2\theta = \sin^2\varphi$. This yields

$$\begin{aligned}\int_0^1 \frac{\sqrt{x}}{\sqrt{1-x\sin^2\theta}}\,dx &= \int_0^{\theta} \frac{\sin\varphi/\sin\theta}{\cos\varphi}\cdot\frac{2\sin\varphi\cos\varphi}{\sin^2\theta}\,d\varphi\\ &= \frac{1}{\sin^3\theta}\int_0^{\theta}(1-\cos 2\varphi)\,d\varphi\\ &= \frac{\theta-\sin\theta\cos\theta}{\sin^3\theta}.\end{aligned}$$

Thus,

$$\begin{aligned}\int_0^1 \sqrt{x}\mathbf{K}(\sqrt{x})\,dx &= \int_0^{\pi/2}\frac{\theta-\sin\theta\cos\theta}{\sin^3\theta}\,d\theta\\ &= \int_0^{\pi/2}\frac{\theta(1-\cos^2\theta)+\cos\theta(\theta\cos\theta-\sin\theta)}{\sin^3\theta}\,d\theta\\ &= \int_0^{\pi/2}\frac{\theta}{\sin\theta}\,d\theta + \int_0^{\pi/2}\frac{\cos\theta}{\sin^3\theta}(\theta\cos\theta-\sin\theta)\,d\theta.\end{aligned}$$

For the remaining integral on the right side, integrate by parts to obtain

$$\begin{aligned}\int_0^1 \sqrt{x}\mathbf{K}(\sqrt{x})\,dx &= \int_0^{\pi/2}\frac{\theta}{\sin\theta}\,d\theta + \left.\frac{\sin\theta-\theta\cos\theta}{2\sin^2\theta}\right|_0^{\pi/2} - \frac{1}{2}\int_0^{\pi/2}\frac{\theta}{\sin\theta}\,d\theta\\ &= \frac{1}{2}\int_0^{\pi/2}\frac{\theta}{\sin\theta}\,d\theta + \frac{1}{2} - \lim_{\theta\to 0^+}\frac{\sin\theta-\theta\cos\theta}{2\sin^2\theta}\\ &= G+\frac{1}{2},\end{aligned}$$

where L'Hôpital's Rule was used to determine

$$\lim_{\theta\to 0^+}\frac{\sin\theta-\theta\cos\theta}{2\sin^2\theta} = \lim_{\theta\to 0^+}\frac{\theta}{4\cos\theta} = 0.$$

Solution 2 by Devis Alvarado, UNAH and UPNFM, Tegucigalpa, Hoduras.

Let I the integral

$$\begin{aligned}
I &= \int_0^1 \sqrt{x}K(\sqrt{x})\,dx = \int_0^1 \sqrt{x}\int_0^{\frac{\pi}{2}} \frac{d\theta}{\sqrt{1-x\sin^2(\theta)}}\,dx \\
&= \int_0^1 \int_0^{\frac{\pi}{2}} \sqrt{\frac{x}{1-x\sin^2(\theta)}}\,d\theta\,dx = \int_0^{\frac{\pi}{2}} \int_0^1 \sqrt{\frac{x}{1-x\sin^2(\theta)}}\,dx\,d\theta \\
&= \int_0^{\frac{\pi}{2}} \int_0^{\sec\theta} \frac{2y^2}{\left(1+y^2\sin^2(\theta)\right)^2}\,dy\,d\theta, \quad y = \sqrt{\frac{x}{1-x\sin^2(\theta)}} \\
&= \int_0^{\frac{\pi}{2}} \int_0^{\theta} \frac{2\sin^2\alpha}{\sin^3\theta}\,d\alpha\,d\theta, \quad y\sin\theta = \tan\alpha \\
&= \int_0^{\frac{\pi}{2}} \frac{\theta-\sin\theta\cos\theta}{\sin^3\theta}\,d\theta \\
&= \int_0^{\frac{\pi}{2}} \frac{\theta-\sin\theta\cos\theta}{\sin^3\theta}\,d\theta, \quad u = \frac{\theta-\sin\theta\cos\theta}{\sin\theta}, \; v = \frac{1}{\sin^2\theta}\,d\theta \\
&= -\frac{\theta-\sin\theta\cos\theta}{\sin\theta}\cot\theta\Big|_0^{\frac{\pi}{2}} + \int_0^{\frac{\pi}{2}} \frac{\sin\theta+\sin^2\theta-\theta\cos\theta}{\sin^2\theta}\cot\theta\,d\theta \\
&= \int_0^{\frac{\pi}{2}} \left[\frac{\sin\theta\cos\theta-\theta}{\sin^3\theta} + \frac{\theta}{\sin\theta} + \cos\theta\right] d\theta \\
&= -I + 2G + 1 \\
\Longrightarrow I &= G + \frac{1}{2}.
\end{aligned}$$

Solution 3 by Moti Levy, Rehovot, Israel.

$$I := \int_0^1 \sqrt{x}\mathbf{K}\left(\sqrt{x}\right)dx = \int_0^1 \sqrt{x}\int_0^{\frac{\pi}{2}} \frac{dt}{\sqrt{1-x\sin^2(t)}}dx$$

$$= \int_0^{\frac{\pi}{2}} \int_0^1 \frac{\sqrt{x}}{\sqrt{1-x\sin^2(t)}}dxdt = 2\int_0^{\frac{\pi}{2}} \int_0^1 \frac{u^2}{\sqrt{1-u^2\sin^2(t)}}dudt$$

$$\int_0^1 \frac{u^2}{\sqrt{1-u^2\sin^2(t)}}du = \frac{t-\sin(t)\cos(t)}{2\sin^3(t)}$$

$$I = \int_0^{\frac{\pi}{2}} \frac{t-\sin(t)\cos(t)}{\sin^3(t)}dt \tag{1}$$

We use the following identity,

$$\frac{1}{\sin^3(t)} = \frac{1}{2}\frac{d}{dt}\left(\frac{-\cos(t)}{\sin^2(t)}\right) + \frac{1}{2\sin(t)} \tag{2}$$

We substitute (2) in (1),

$$I = \int_0^{\frac{\pi}{2}} \left(\frac{1}{2} t \frac{d}{dt} \left(\frac{-\cos(t)}{\sin^2(t)} \right) - \frac{\cos(t)}{\sin^2(t)} \right) dt + \frac{1}{2} \int_0^{\frac{\pi}{2}} \frac{t}{\sin(t)} dt. \tag{3}$$

By integration by parts,

$$\int_\alpha^{\frac{\pi}{2}} \left(\frac{1}{2} t \frac{d}{dt} \left(\frac{-\cos(t)}{\sin^2(t)} \right) - \frac{\cos(t)}{\sin^2(t)} \right) dt = \frac{1}{2} + \frac{\alpha\cos(\alpha) - \sin(\alpha)}{2\sin^2(\alpha)}.$$

One can check that

$$\lim_{\alpha\to 0} \frac{\alpha\cos(\alpha) - \sin(\alpha)}{2\sin^2(\alpha)} = 0. \tag{4}$$

It follows that

$$\int_0^{\frac{\pi}{2}} \left(\frac{1}{2} t \frac{d}{dt} \left(\frac{-\cos(t)}{\sin^2(t)} \right) - \frac{\cos(t)}{\sin^2(t)} \right) dt = \frac{1}{2}. \tag{5}$$

By (3) and (1), we obtain

$$I = \frac{1}{2} + \frac{1}{2} \int_0^{\frac{\pi}{2}} \frac{t}{\sin(t)} dt.$$

To complete the solution, we recall an integral identity of the Catalan's constant

$$G = \frac{1}{2} \int_0^{\frac{\pi}{2}} \frac{t}{\sin(t)} dt.$$

Solution 4 by Perfetti Paolo, dipartimento di matematica, Universitą di "Tor Vergata", Roma, Italy.

$$\int_0^1 \sqrt{x}\mathbf{K}(\sqrt{x})dx = \int_0^1 \left(\int_0^{\frac{\pi}{2}} \frac{d\theta}{\sqrt{1 - x\sin^2\theta}} \right) \sqrt{x}dx = \int_0^{\frac{\pi}{2}} \left(\int_0^1 \frac{\sqrt{x}dx}{\sqrt{1 - x\sin^2\theta}} \right) d\theta$$

Let's call $\sin^2\theta = a$. The inner integral

$$\int_0^1 \frac{\sqrt{x}dx}{\sqrt{1-xa}} = \int_0^a \frac{1}{a\sqrt{a}} \frac{\sqrt{y}dy}{\sqrt{1-y}} \underbrace{=}_{t^2 = y/(1-y)} \int_0^{\sqrt{\frac{a}{1-a}}} \frac{1}{a\sqrt{a}} \frac{2t^2\,dt}{(1+t^2)^2} =$$

$$= \frac{1}{a\sqrt{a}} \left(\arctan t - \frac{t}{1+t^2} \right) \Big|_0^{\sqrt{\frac{a}{1-a}}} = \frac{1}{a\sqrt{a}} \left(\arctan\sqrt{\frac{a}{1-a}} - \sqrt{a(1-a)} \right) =$$

$$= \frac{\theta - \sin\theta\cos\theta}{\sin^3\theta} \to \int_0^{\frac{\pi}{2}} \frac{\theta - \sin\theta\cos\theta}{\sin^3\theta} d\theta = \lim_{\varepsilon\to 0^+} \int_\varepsilon^{\frac{\pi}{2}} \frac{\theta - \sin\theta\cos\theta}{\sin^3\theta} d\theta$$

The limit exists because the numerator goes to zero like θ^3 thus compensating the zero of order three at denominator. Upon setting $\theta = 2\arctan y$ and denoting $\arctan(\varepsilon/2) = p$ we get

$$\lim_{\varepsilon\to 0} \left[\int_p^1 \frac{4(1+y^2)^2 \arctan y\, dy}{8y^3} + \left(1 - \frac{1}{\sin\varepsilon} \right) \right] \tag{1}$$

The three integrals are

$$\frac{1}{2}\lim_{\varepsilon\to 0}\int_p^1 (y\arctan y)\,dy = \frac{1}{2}\int_0^1 y\arctan y\,dy = \frac{\pi}{8}-\frac{1}{4}$$

$$\lim_{\varepsilon\to 0}\int_p^1 \frac{\arctan y\,dy}{y} = \int_0^1 \frac{\arctan y\,dy}{y} = \sum_{k=0}^{\infty}\frac{(-1)^k}{(2k+1)^2} = G$$

$$\int_p^1 \frac{\arctan y\,dy}{2y^3} = \sum_{k=0}^{\infty}\frac{(-1)^k}{2(2k+1)(2k-1)}\left(1-p^{2k-1}\right)$$

$$\sum_{k=0}^{\infty}\frac{(-1)^k}{2(2k+1)(2k-1)} = \frac{1}{4}\sum_{k=0}^{\infty}\left(\frac{(-1)^k}{2k-1}-\frac{(-1)^k}{2k+1}\right) = \frac{-1}{4}-\frac{1}{2}\sum_{k=0}^{\infty}\frac{(-1)^k}{(2k+1)} =$$
$$= \frac{-1}{4}-\frac{\pi}{8}$$

$$-\sum_{k=0}^{\infty}\frac{(-1)^k}{2(2k+1)(2k-1)}p^{2k-1} = \frac{1}{2p}-\sum_{k=1}^{\infty}\frac{(-1)^k}{2(2k+1)(2k-1)}p^{2k-1}$$

We can bound

$$\Big|\sum_{k=1}^{\infty}\frac{(-1)^k}{2(2k+1)(2k-1)}p^{2k-1}\Big| \leqslant \sum_{k=1}^{\infty}\frac{1}{2(2k+1)(2k-1)} < +\infty$$

hence converges uniformly for $\arctan(\varepsilon/2)\leqslant 1$ and we can write

$$\lim_{\varepsilon\to 0}\sum_{k=1}^{\infty}\frac{(-1)^k}{2(2k+1)(2k-1)}p^{2k-1} = \sum_{k=1}^{\infty}\frac{(-1)^k}{2(2k+1)(2k-1)}\lim_{\varepsilon\to 0}p^{2k-1} = 0$$

The limit (1) becomes

$$\lim_{\varepsilon\to 0}\left(\frac{1}{2p}+\frac{-1}{4}-\frac{\pi}{8}+G+\frac{\pi}{8}-\frac{1}{4}+1-\frac{1}{\sin\varepsilon}\right) = G+\frac{1}{2}$$

because

$$\lim_{\varepsilon\to 0}\frac{1}{2p}-\frac{1}{\sin\varepsilon} = \lim_{\varepsilon\to 0}\frac{1}{2\arctan\frac{\varepsilon}{2}}-\frac{1}{\sin\varepsilon} = \lim_{\varepsilon\to 0}\frac{1}{\varepsilon(1+O(\varepsilon^2)}-\frac{1}{\varepsilon(1+O(\varepsilon^2))} =$$
$$= \lim_{\varepsilon\to 0}\frac{1}{\varepsilon}\left(1+O(\varepsilon^2)-1-O(\varepsilon^2)\right) = \lim_{\varepsilon\to 0}\frac{O(\varepsilon^2)}{\varepsilon} = 0$$

Solution 5 by Yunyong Zhang, Chinaunicom, Yunnan, China.

$$\because \quad \mathbf{K}(\sqrt{x}) = \int_0^{\frac{\pi}{2}}\frac{\mathrm{d}\theta}{\sqrt{1-t^2\sin^2\theta}},$$

$$\therefore \quad I = \int_0^1 \sqrt{x}\mathbf{K}(\sqrt{x})\mathrm{d}x = \int_0^1 \int_0^{\frac{\pi}{2}} \frac{\sqrt{x}}{\sqrt{1 - x\sin^2\theta}} \mathrm{d}\theta\mathrm{d}x,$$

$$= \int_0^{\frac{\pi}{2}} \int_0^1 \frac{\sqrt{x}}{\sqrt{1 - x\sin^2\theta}} \mathrm{d}x\mathrm{d}\theta,$$

$$\text{where } \int_0^1 \frac{\sqrt{x}}{\sqrt{1 - x\sin^2\theta}} \mathrm{d}x = \frac{1}{\sin^2\theta}\left[\frac{\arcsin(\sqrt{x}\sin\theta)}{\sin\theta} - \sqrt{x}\sqrt{1 - x\sin^2\theta}\right]\Bigg|_0^1,$$

$$= \frac{1}{\sin^2\theta}\left(\frac{\theta}{\sin\theta} - \cos\theta\right),$$

$$= \frac{\theta}{\sin^r\theta} - \frac{\cos\theta}{\sin^2\theta},$$

$$\therefore \quad I = \int_0^{\frac{\pi}{2}} \left(\frac{\theta}{\sin^3 theta} - \frac{\cos\theta}{\sin^2\theta}\right) \mathrm{d}\theta,$$

$$\because \quad G = \frac{1}{2}\int_0^{\frac{\pi}{2}} \frac{\theta}{\sin\theta} \mathrm{d}\theta,$$

$$\text{now it's equivalent to proving } I = \int_0^{\frac{\pi}{2}} \left(\frac{\theta}{\sin^3\theta} - \frac{\theta}{\sin^2\theta} - \frac{\cos\theta}{\sin^2\theta}\right) \mathrm{d}\theta = \frac{1}{2},$$

$$\therefore \quad \int_0^{\frac{\pi}{2}} \left(\frac{\theta}{\sin^3\theta} - \frac{\theta}{\sin^2\theta} - \frac{\cos\theta}{\sin^2\theta}\right) \mathrm{d}\theta = -\frac{1}{2}\frac{(\theta\cot\theta - 1)}{\sin\theta}\Bigg|_0^{\frac{\pi}{2}} = \frac{1}{2}.$$

So $\int_0^1 \sqrt{x}\mathbf{K}(\sqrt{x})\mathrm{d}x = G + \frac{1}{2}$.

Also solved by the problem propposer.

• **5800** *Proposed by Michael Brozinsky, Central Islip, New York.*

Show that in any group of 8 continuous functions, each having the set of real numbers as its domain, there exist (at least) 4 whose graphs mutually intersect or (at least) 3 whose graphs mutually do not intersect.

Solution 1 by Moti Levy, Rehovot, Israel.

We rephrase the problem as follows:

Let $f_1, \ldots, f_8 : \mathbb{R} \to \mathbb{R}$ be continuous. Among these eight functions there exist either four whose graphs intersect pair–wise, or three whose graphs are pair–wise disjoint (for every two of them one graph lies strictly above the other on all of $\mathbb{R}$).

Proof: For any two continuous functions f, g either *they cross:* some x satisfies $f(x) = g(x)$; or *one dominates:* either $f(x) > g(x)$ for all x or $g(x) > f(x)$ for all x.

A *tower of height k* is a chain $f_1(x) > f_2(x) > \cdots > f_k(x)$ for every $x \in \mathbb{R}$.

For a function f, let $h(f)$ be the largest k such that f can stand at the bottom of a height–k tower.

$$\text{If some } f \text{ has } h(f)=3\text{, then there exist } g, h \text{ with } g(x) > h(x) > f(x) \text{ for all } x \in \mathbb{R}, \text{ giving a triple of disjoint graphs.} \tag{6}$$

Hence,

$$\textit{if no such triple exists}, \text{ all heights satisfy } h(f) \leqslant 2. \tag{7}$$

With only heights 1 or 2, by the *pigeonhole principle* there is a subset S of at least four functions sharing the same height.
Now we show that two equal–height functions must cross. Take distinct $f, g \in S$ with $h(f) = h(g)$ and assume, for contradiction, that $f(x) > g(x)$ for every x. Choose a maximal tower ending in f:

$$t_1(x) > t_2(x) > \cdots > t_{h(f)}(x) = f(x) \qquad (\forall x).$$

Appending g at the bottom yields

$$t_1(x) > t_2(x) > \cdots > t_{h(f)}(x) = f(x) > g(x) \qquad (\forall x),$$

a tower of length $h(f)+1$ terminating at g, contradicting the maximality of $h(g)$. Hence the assumption is impossible, and f and g must intersect. Thus any two distinct functions of the same height must cross.
Therefore S provides four graphs that intersect pair–wise, while (6) states the alternative of three disjoint graphs.

Also solved by and the problem proposer.

• **5801** *Proposed by Daniel Sitaru, National Economic College "Theodor Costescu" Drobeta Turnu - Severin, Romania.*

Solve for real x:

$$\frac{1}{1+x^4} + \frac{1}{2+x^6} + \frac{1}{3+x^8} + \frac{1}{4+x^{10}} = \frac{77}{60x^2}.$$

Solution 1 by Ángel Plaza, Universidad de Las Palmas de Gran Canaria, Spain.

It is enough to solve the equation for $x > 0$. By changing variable by doing $\frac{1}{x^2} = t$, the equation changes to

$$\frac{t^2}{t^2+1} + \frac{t^3}{2t^3+1} + \frac{t^4}{3t^4+1} + \frac{t^5}{4t^5+1} = \frac{77t}{60}.$$

and, since $t \neq 0$, it is equivalent to

$$\frac{t}{t^2+1} + \frac{t^2}{2t^3+1} + \frac{t^3}{3t^4+1} + \frac{t^4}{4t^5+1} = \frac{77}{60}. \tag{8}$$

Let $f_p(t) = \dfrac{t^p}{pt^{p+1}+1}$, for $p = 1.2, 3, 4$. $f_p'(t) = \dfrac{-pt^{p-1}(t^{p+1}-1)}{(pt^{p+1}+1)^2}$ whose only real root in the domain $t > 0$ is $t = 1$.

$$f_p''(t) = \frac{pt^{p-2}(-p(p+1)t^{p+1} + 2pt^{2p+2} + p - 1}{(pt^{p+1}+1)^3}$$

and

$$f_p''(1) = \frac{p(-p(p+5) + 2p + p - 1)}{(p+1)^3} = \frac{-p(p+1)^2}{(p+1)^3} < 0.$$

Therefore, functions $f_p(t)$ have respective maximum at $t = 1$, with value $f_p(1) = 1/(p+1)$ and $\sum_{p=1}^{4} 1/(p+1) = 77/60$, which implies that the only solution to equation (8) is $t = 1$, so the given equation has $x = 1$ and $x = -1$ as the only real roots.

Solution 2 by Brian D. Beasley, Simpsonville, SC.

Let $u = x^2$. Then we seek all non-negative real numbers u with

$$\frac{1}{1+u^2} + \frac{1}{2+u^3} + \frac{1}{3+u^4} + \frac{1}{4+u^5} = \frac{77}{60u}.$$

Noting that $u \neq 0$, we multiply by $f(u) = 60u(1+u^2)(2+u^3)(3+u^4)(4+u^5)$ to obtain

$$\frac{f(u)}{1+u^2} + \frac{f(u)}{2+u^3} + \frac{f(u)}{3+u^4} + \frac{f(u)}{4+u^5} = 77(1+u^2)(2+u^3)(3+u^4)(4+u^5).$$

Then tedious but straightforward algebra yields the equivalent equation $(u-1)^2 g(u) = 0$, where

$$g(u) = 77u^{12} + 94u^{11} + 128u^{10} + 256u^9 + 375u^8 + 716u^7 + 688u^6 + \\ 1070u^5 + 988u^4 + 1452u^3 + 1392u^2 + 696u + 1848.$$

Since $g(u) > 0$ for all $u \geqslant 0$, we conclude that the unique non-negative real solution is $u = 1$. Hence the original equation has the real solutions $x = \pm 1$.

Solution 3 by Daniel Văcaru, ”Maria Teiuleanu” National Economic College, Pitești, Romania.

By **AM-GM**, one has

$$x^4 + 1 \geqslant 2x^2 \Leftrightarrow \frac{1}{1+x^4} \leqslant \frac{1}{2x^2}$$

$$2 + x^6 = 1 + 1 + x^6 \geqslant 3\sqrt[3]{x^6} = 3x^2 \Leftrightarrow \frac{1}{2+x^6} \leqslant \frac{1}{3x^2}$$

$$3 + x^8 = 1 + 1 + 1 + x^8 \geqslant 4\sqrt[4]{x^8} = 4x^2 \Leftrightarrow \frac{1}{3+x^8} \leqslant \frac{1}{4x^2}$$

$$4 + x^{10} = 1 + 1 + 1 + 1 + x^{10} \geqslant 5\sqrt[5]{x^{10}} = 5x^2 \Leftrightarrow \frac{1}{4+x^{10}} \leqslant \frac{1}{5x^2}.$$

It follows that

$$\frac{77}{60x^2} = \frac{1}{1+x^4} + \frac{1}{2+x^6} + \frac{1}{3+x^8} + \frac{1}{4+x^{10}} \leqslant \frac{1}{2x^2} + \frac{1}{3x^2} + \frac{1}{4x^2} + \frac{1}{5x^2} = \frac{77}{60x^2}.$$

One obtain

$$x^2 = 1$$

and one find

$$\mathcal{S} = \{-1, 1\}$$

where $\mathcal{S}$ denote the set of solutions of equation $\frac{1}{1+x^4} + \frac{1}{2+x^6} + \frac{1}{3+x^8} + \frac{1}{4+x^{10}} = \frac{77}{60x^2}$.

Solution 4 by David A. Huckaby, Angelo State University, San Angelo, TX.

For $x = 1$ or $x = -1$, the left side of the equation is $\frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} = \frac{77}{60}$. So $x = 1$ and $x = -1$ are solutions of the equation. We will show that these are the only real solutions.Combining the fractions on the left side of the equation gives

$$\frac{x^{24} + x^{22} + x^{20} + 4x^{18} + 4x^{16} + 10x^{14} + 6x^{12} + 18x^{10} + 14x^8 + 19x^6 + 26x^4 + 50}{(x^4+1)(x^6+2)(x^8+3)(x^{10}+4)} = \frac{77}{60x^2},$$

so that

$$\begin{aligned} 60x^2(x^{24} + x^{22} + x^{20} + 4x^{18} + 4x^{16} + 10x^{14} + 6x^{12} + 18x^{10} + 14x^8 + 19x^6 + 26x^4 + 50) \\ = 77(x^4+1)(x^6+2)(x^8+3)(x^{10}+4). \end{aligned}$$

So

$$\begin{aligned} 60x^2(x^{24} + x^{22} + x^{20} + 4x^{18} + 4x^{16} + 10x^{14} + 6x^{12} + 18x^{10} + 14x^8 + 19x^6 + 26x^4 + 50) \\ -77(x^4+1)(x^6+2)(x^8+3)(x^{10}+4) = 0, \end{aligned}$$

that is,

$$\begin{aligned} 77x^{28} - 60x^{26} + 17x^{24} + 94x^{22} - 9x^{20} + 222x^{18} - 369x^{16} + 410x^{14} - 464x^{12} + 546x^{10} \\ -524x^8 - 636x^6 + 1848x^4 - 3000x^2 + 1848 = 0. \end{aligned}$$

Synthetic division reveals that $x = 1$ and $x = -1$ are actually double roots of this degree 28 polynomial, so we have

$$\begin{aligned} (x-1)^2(x+1)^2(77x^{24} + 94x^{22} + 128x^{20} + 256x^{18} + 375x^{16} + 716x^{14} + 688x^{12} + 1070x^{10} \\ +988x^8 + 1452x^6 + 1392x^4 + 696x^2 + 1848) = 0. \end{aligned}$$

By Descartes' rule of signs, the degree 24 polynomial has no real positive or negative real roots. So $x = 1$ and $x = -1$ are indeed the only real roots of the original equation.

Solution 5 by Devis Alvarado, UNAH and UPNFM, Tegucigalpa, Hoduras.

Note that for every natural number $n \geqslant 2$, we have

$$n - 1 + x^{2n} = \underbrace{1 + 1 + \cdots + 1}_{n-1 \ \ veces} + x^{2n} \geqslant n\sqrt[n]{x^{2n}} = nx^2,$$

by the AM-GM inequality, which is equivalent to

$$\frac{1}{n - 1 + x^{2n}} \leqslant \frac{1}{nx^2}.$$

Thus, we have

$$\sum_{k=2}^{n} \frac{1}{k - 1 + x^{2k}} \leqslant \sum_{k=2}^{n} \frac{1}{kx^2} = \left(\sum_{k=2}^{n} \frac{1}{k}\right) \frac{1}{x^2}.$$

This inequality becomes an equality when $x = \pm 1$. Thus, for the given problem, we use the previous inequality with $n = 5$, since $\frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} = \frac{77}{60}$.

$$\frac{1}{1 + x^4} + \frac{1}{2 + x^6} + \frac{1}{3 + x^8} + \frac{1}{4 + x^{10}} \leqslant \frac{77}{60x^2}.$$

Therefore, the solutions are $x = \pm 1$.

Solution 6 by Michel Bataille, Rouen, France.

We remark that $\frac{77}{60} = \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5}$, from which we deduce that 1 and -1 are solutions for x. We show that there are no other solutions.
Let x be a solution. Then the remark above yields

$$u_1(x) + u_2(x) + u_3(x) + u_4(x) = 0 \qquad (1)$$

where $u_n(x) = \frac{1}{n + x^{2n+2}} - \frac{1}{(n+1)x^2} = \frac{(n+1)x^2 - n - x^{2n+2}}{(n+1)x^2(n + x^{2n+2})}$.
Now, we have

$$\begin{aligned}(n+1)x^2 - n - x^{2n+2} &= n(x^2 - 1) + x^2(1 - (x^2)^n) \\ &= (1 - x^2)(x^2(1 + x^2 + \cdots + x^{2n-2}) - n) \\ &= (1 - x^2)((x^2 - 1) + (x^4 - 1) + \cdots + (x^{2n} - 1)) \\ &= (1 - x^2)(x^2 - 1)(1 + (x^2 + 1) + \cdots + (x^{2n-2} + \cdots + x^2 + 1)) \\ &= -(1 - x^2)^2 \cdot k(x)\end{aligned}$$

where $k(x) > 0$ for all real x. It follows that $u_n(x) \leqslant 0$ with equality if and only if $x = 1$ or $x = -1$. We see that the four terms in the left-hand side of (1) are less than or equal 0 so that (1) implies that each of them is 0, hence that $x = 1$ or $x = -1$. Thus, (1) implies that $x = 1$ or $x = -1$ and we are done.

Solution 7 by Moti Levy, Rehovot, Israel.

Set $t = x^2$, then the equation is equivalent to,

$$\frac{t}{1+t^2} + \frac{t}{2+t^3} + \frac{t}{3+t^4} + \frac{t}{4+t^5} = \frac{77}{60}.$$

Now we prove that

$$\frac{t}{1+t^2} + \frac{t}{2+t^3} + \frac{t}{3+t^4} + \frac{t}{4+t^5} \leqslant \frac{77}{60},$$

where equality is attained only or $t = 1$. Define

$$F(t) = \frac{t}{1+t^2} + \frac{t}{2+t^3} + \frac{t}{3+t^4} + \frac{t}{4+t^5}, \qquad t > 0.$$

For $k = 1, 2, 3, 4$ let

$$f_k(t) = \frac{t}{k+t^{k+1}}.$$

$$f_k'(t) = \frac{k(1-t^{k+1})}{(k+t^{k+1})^2}.$$

For any fixed $k > 0$, we have

$$\operatorname{sgn}\left\{f_k'(t)\right\} = \begin{cases} + & (0 < t < 1), \\ 0 & (t = 1), \\ - & (t > 1). \end{cases}$$

Consequently,

$$F'(t) = \sum_{k=1}^{4} f_k'(t) \begin{cases} > 0 & (0 < t < 1), \\ = 0 & (t = 1), \\ < 0 & (t > 1). \end{cases}$$

Thus F is strictly increasing on $(0, 1)$ and strictly decreasing on $(1, \infty)$. Because of that monotonicity, the *unique* global maximum of F on $(0, \infty)$ occurs at $t = 1$.

$$F(1) = \frac{1}{1+1} + \frac{1}{2+1} + \frac{1}{3+1} + \frac{1}{4+1} = \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} = \frac{30+20+15+12}{60} = \frac{77}{60}.$$

We conclude that for every $t > 0$

$$F(t) \leqslant F(1) = \frac{77}{60},$$

and the equality case of all four derivatives forces $t = 1$, so equality is attained solely at that point. Solution for x is $x = \pm 1$.

Solution 8 by Ulrich Abel, Technische Hochschule Mittelhessen, Friedberg, Germany.

We show the following generalization: The equation

$$\sum_{k=1}^{n} \frac{1}{k + x^{2k+2}} = \frac{H_{n+1} - 1}{x^2},$$

where $H_n := 1 + 1/2 + \cdots + 1/n$ denotes the harmonic number, has exactly two real solutions, namely, $x = -1$ and $x = 1$.

Problem #5801 SSMJ is the special case $n = 4$ with $H_5 - 1 = 77/60$.

Proof: The substitution $t = x^2$ leads to the equation

$$f_n(t) := \sum_{k=1}^{n} \frac{t}{k + t^{k+1}} = H_{n+1} - 1.$$

Since

$$f_n'(t) = \sum_{k=1}^{n} \frac{k\left(1 - t^{k+1}\right)}{\left(k + t^{k+1}\right)^2}$$

we conclude that f_n is strictly increasing on $[0, 1]$ and strictly decreasing on $[1, \infty)$. Hence, f unimodal function on $[0, \infty)$ with a unique maximum at $t = 1$. Observing that $f_n(1) = H_{n+1} - 1$ we infer that $t = 1$ is the unique solution of the equation $f_n(t) = H_{n+1} - 1$. Therefore,

$$\frac{f_n\left(x^2\right)}{x^2} = \sum_{k=1}^{n} \frac{1}{k + x^{2k+2}} = \frac{H_{n+1} - 1}{x^2}$$

has exactly two solutions $x = -1$ and $x = 1$ on $\mathbb{R} \setminus \{0\}$.

Also solved by Bruno Salgueiro Fanego, Viveiro, Lugo, Spain and the problem proposer.

• **5802** *Proposed by Toyesh Prakash Sharma, Agra College, Agra, India.*

Let $a, b, c > 0$. Show that

$$\frac{1}{b^2}\sqrt{\frac{a^5 + b^5}{a + b}} + \frac{1}{c^2}\sqrt{\frac{b^5 + c^5}{b + c}} + \frac{1}{a^2}\sqrt{\frac{c^5 + a^5}{c + a}} \geqslant 3.$$

Solution 1 by Ángel Plaza, Universidad de Las Palmas de Gran Canaria, Spain.

By the AM-GM inequality it is enough to prove that

$$\frac{a^5 + b^5}{ab^4 + b^5} \cdot \frac{b^5 + c^5}{bc^4 + c^5} \cdot \frac{c^5 + a^5}{ca^4 + a^5} \geqslant 1$$

$$a^{10}(b^5+c^5)+b^{10}(c^5+a^5)+c^{10}(a^5+b^5)\geqslant a^5(b^5c^4+b^4c^5)+b^6(c^5a^4+a^5c^4)+c^6(a^5b^4+b^5a^4)$$

which follows by Muirhead's inequality since $[10,5,0]\succ[6,5,4]$.

Solution 2 by Daniel Văcaru, "Maria Teiuleanu" National Economic College, Pitești, Romania.

We have

$$a^5+b^5=(a+b)\left(a^4-a^3b+a^2b^2-ab^3+b^4\right)\Leftrightarrow\frac{a^5+b^5}{a+b}=a^4+a^2b^2+b^4-ab\left(a^2+b^2\right).$$

By **AM-GM,** we have

$$ab\leqslant\frac{a^2+b^2}{2}.$$

It follows that

$$\frac{a^5+b^5}{a+b}=a^4+a^2b^2+b^4-ab\left(a^2+b^2\right)\geqslant\frac{2a^4+2a^2b^2+2b^4-\left(a^2+b^2\right)^2}{2}=\frac{a^4+b^4}{2}\geqslant a^2b^2.$$

We obtain

$$\sqrt{\frac{a^5+b^5}{a+b}}\geqslant ab.$$

It follows that

$$\begin{aligned}\frac{1}{b^2}\sqrt{\frac{a^5+b^5}{a+b}}+\frac{1}{c^2}\sqrt{\frac{b^5+c^5}{b+c}}+\frac{1}{a^2}\sqrt{\frac{c^5+a^5}{c+a}}&\geqslant\frac{ab}{b^2}+\frac{bc}{c^2}+\frac{ca}{a^2}\\&=\frac{a}{b}+\frac{b}{c}+\frac{c}{a}\\&\overset{AM\geqslant GM}{\geqslant}3\sqrt[3]{\frac{a}{b}\cdot\frac{b}{c}\cdot\frac{c}{a}}=3.\end{aligned}$$

Solution 3 by Henry Ricardo, Westchester Area Math Circle, Purchase, NY.

First we see that $(x^5+y^5)/(x+y)\geqslant(x^4+y^4)/2$:

$$\begin{aligned}\frac{x^5+y^5}{x+y}-\frac{x^4+y^4}{2}&=\frac{2(x^5+y^5)-(x+y)(x^4+y^4)}{2(x+y)}\\&=\frac{x^5-xy^4+y^5-x^4y}{2(x+y)}\\&=\frac{(x-y)(x^4-y^4)}{2(x+y)}\geqslant0.\end{aligned}$$

Consequently, applying the Quadratic Mean and AGM inequalities, we have

$$\begin{aligned}\sum_{cyclic} \frac{1}{b^2}\sqrt{\frac{a^5+b^5}{a+b}} &\geqslant \sum_{cyclic} \frac{1}{b^2}\sqrt{\frac{(a^2)^2+(b^2)^2}{2}} \\ &\geqslant \sum_{cyc;ic} \frac{1}{b^3}\cdot\frac{a^2+b^2}{2} \\ &= \frac{3}{2} + \frac{1}{2}\left(\frac{a^2}{b^2}+\frac{b^2}{c^2}+\frac{c^2}{a^2}\right) \\ &\geqslant \frac{3}{2} + \frac{1}{2}\cdot 3 = 3.\end{aligned}$$

Solution 4 by Michel Bataille, Rouen, France.

Let $r(x) = \dfrac{x^5+1}{x+1}$ and $f(x) = (r(x))^{1/2}$. We are required to prove that

$$f\left(\frac{a}{b}\right) + f\left(\frac{b}{c}\right) + f\left(\frac{c}{a}\right) \geqslant 3. \tag{1}$$

We calculate $r'(x) = (x+1)^{-2}(4x^5+5x^4-1)$, $\quad r''(x) = 2(x+1)^{-3}(6x^5+15x^4+10x^3+1)$

$$f'(x) = \frac{r'(x)(r(x))^{-1/2}}{2},\ f''(x) = \frac{(2r''(x)r(x)-(r'(x))^2)(r(x))^{-3/2}}{4}.$$

An easy if long calculation gives

$$2r''(x)r(x) - (r'(x))^2 = (x+1)^{-4}(8x^{10}+20x^9+15x^8+36x^5+70x^4+40x^3+3).$$

We deduce that $f''(x) > 0$ for $x > 0$ so that f is convex on $(0,\infty)$. It follows that

$$f\left(\frac{a}{b}\right) + f\left(\frac{b}{c}\right) + f\left(\frac{c}{a}\right) \geqslant 3\cdot f\left(\frac{\frac{a}{b}+\frac{b}{c}+\frac{c}{a}}{3}\right).$$

Now, by AM-GM, $\dfrac{a}{b}+\dfrac{b}{c}+\dfrac{c}{a} \geqslant 3\sqrt[3]{\dfrac{a}{b}\cdot\dfrac{b}{c}\cdot\dfrac{c}{a}} = 3$, hence $\dfrac{\frac{a}{b}+\frac{b}{c}+\frac{c}{a}}{3} \geqslant 1$. Since $f'(x) > 0$ for $x \geqslant 1$, we obtain that $f\left(\dfrac{\frac{a}{b}+\frac{b}{c}+\frac{c}{a}}{3}\right) \geqslant f(1) = 1$ and therefore (1) holds.

Solution 5 by Moti Levy, Rehovot, Israel.

Lemma: For all $x, y > 0$:

$$\sqrt{\frac{x^5+y^5}{x+y}} \geqslant \frac{x^2+y^2}{2}. \tag{9}$$

Proof: Square (9) and clear denominators:

$$4\left(x^5+y^5\right) \geqslant (x+y)(x^2+y^2)^2.$$

The inequality is homogenous, so set $\dfrac{x}{y}=t>0$ to obtain

$$P(t)=3t^5-t^4-2t^3-2t^2-t+3.$$

Factoring gives $P(t)=(t-1)^2(3t^3+2t^2+t+3)\geqslant 0$ with equality only at $x=y$. Using Lemma (9) on $(a,b),(b,c),(c,a)$:

$$\frac{1}{b^2}\sqrt{\frac{a^5+b^5}{a+b}} \geqslant \frac{1}{b^2}\frac{a^2+b^2}{2} \tag{10}$$

$$\frac{1}{c^2}\sqrt{\frac{b^5+c^5}{b+c}} \geqslant \frac{1}{c^2}\frac{b^2+c^2}{2} \tag{11}$$

$$\frac{1}{a^2}\sqrt{\frac{c^5+a^5}{c+a}} \geqslant \frac{1}{a^2}\frac{c^2+a^2}{2} \tag{12}$$

Summing both sides of (10), (11), and (12) gives,

$$\frac{1}{b^2}\sqrt{\frac{a^5+b^5}{a+b}}+\frac{1}{c^2}\sqrt{\frac{b^5+c^5}{b+c}}+\frac{1}{a^2}\sqrt{\frac{c^5+a^5}{c+a}} \geqslant \tfrac{1}{2}\left(3+\frac{a^2}{b^2}+\frac{b^2}{c^2}+\frac{c^2}{a^2}\right). \tag{13}$$

The AM/GM inequality gives

$$\frac{1}{3}\left(\frac{a^2}{b^2}+\frac{b^2}{c^2}+\frac{c^2}{a^2}\right) \geqslant \sqrt[3]{\frac{a^2}{b^2}\frac{b^2}{c^2}\frac{c^2}{a^2}}=1,$$

hence

$$\frac{a^2}{b^2}+\frac{b^2}{c^2}+\frac{c^2}{a^2} \geqslant 3. \tag{14}$$

Inserting (14) into (13) completes the proof.

Solution 6 by Perfetti Paolo, dipartimento di matematica Università di "Tor Vergata", Roma, Italy.

By Power–Means-inequality we have $a^5+b^5 \geqslant 2^{-4}(a+b)^5$ hence we prove

$$\sum_{\text{cyc}}\frac{1}{b^2}\sqrt{\frac{(a+b)^5}{2^4(a+b)}}=\sum_{\text{cyc}}\frac{(a+b)^2}{b^2} \geqslant 12$$

hence

$$\sum_{\text{cyc}}\left(\frac{a^2}{b^2}+\frac{2a}{b}\right) \geqslant 9$$

The following AGM's conclude the proof

$$\sum_{\text{cyc}} \frac{a^2}{b^2} \geqslant 3\left(\frac{a^2}{b^2}\frac{b^2}{c^2}\frac{c^2}{a^2}\right)^{\frac{1}{3}}, \qquad \sum_{\text{cyc}} \frac{a}{b} \geqslant 3\left(\frac{a}{b}\frac{b}{c}\frac{c}{a}\right)^{\frac{1}{3}} = 3$$

Also solved by the problem proposer.

• **5803** *Proposed by Michel Bataille, Rouen, France.*

Let n be a positive integer. Prove that

$$\sum_{1\leqslant i\leqslant j\leqslant n} \frac{1}{ij} = \sum_{k=1}^{n} \binom{n}{k} \frac{(-1)^{k-1}}{k^2}.$$

Solution 1 by Henry Ricardo, Westchester Area Math Circle, Purchase, NY.

This problem is the case $m = 2$ of Corollary 3 in Karl Dilcher's paper *Some q-series identities related to divisor functions* (*Discrete Math.* **145** (1995), 83-93):

$$\sum_{k=1}^{n} \binom{n}{k} \frac{(-1)^{k-1}}{k^m} = \sum_{1\leqslant j_1\leqslant j_2\leqslant\cdots\leqslant j_m\leqslant n} \frac{1}{j_1 j_2 \cdots j_m}.$$

Furthermore, it is easily proved that

$$\sum_{1\leqslant i\leqslant j\leqslant n} \frac{1}{i\,j} = \frac{1}{2}(H_n^2 + H_n^{(2)}),$$

where $H_n = \sum_{k=1}^{n} 1/k$ and $H_n^{(2)} = \sum_{k=1}^{n} 1/k^2$.

Solution 2 by Ángel Plaza, Universidad de Las Palmas de Gran Canaria, Spain.

Solution is by induction on n. The identity holds trivially for $n = 1$. Let $A(n)$ and $B(n)$ respectively be the left-hand side and the right-hand side of the given identity. Let us assume that also holds for n, Then for $n + 1$:

$$\begin{aligned} A(n+1) &= \sum_{1\leqslant i\leqslant j\leqslant n} \frac{1}{ij} + \sum_{1\leqslant i\leqslant n+1} \frac{1}{i(n+1)} = A(n) + \frac{H_{n+1}}{n+1} \\ B(n+1) &= \sum_{k=1}^{n+1} \left(\binom{n}{k-1} + \binom{n}{k} \right) \frac{(-1)^{k-1}}{k^2} = B(n) + \sum_{k=1}^{n+1} \binom{n}{k-1} \frac{(-1)^{k-1}}{k^2}. \end{aligned}$$

So, it is enough to prove that

$$\sum_{k=1}^{n+1}\binom{n}{k-1}\frac{(-1)^{k-1}}{k^2}=\frac{H_{n+1}}{n+1},\ \text{or, equivalently,}\ \sum_{k=1}^{n+1}\binom{n+1}{k}\frac{(-1)^{k-1}}{k}=H_{n+1}.$$

$$\begin{aligned}\sum_{k=1}^{n+1}\binom{n+1}{k}\frac{(-1)^{k-1}}{k}&=\sum_{k=1}^{n+1}\binom{n}{k}\frac{(-1)^{k-1}}{k}+\sum_{k=1}^{n+1}\binom{n}{k-1}\frac{(-1)^{k-1}}{k}\\&=\sum_{k=1}^{n}\binom{n}{k}\frac{(-1)^{k-1}}{k}+\sum_{k=1}^{n+1}\binom{n}{k-1}\frac{(-1)^{k-1}}{k}\end{aligned}$$

and, again by induction, it is enough to show that

$$\sum_{k=1}^{n+1}\binom{n}{k-1}\frac{(-1)^{k-1}}{k}=\frac{1}{n+1}$$

or equivalently that

$$\sum_{k=1}^{n+1}\binom{n+1}{k}(-1)^{k-1}=1.$$

$$\sum_{k=1}^{n+1}\binom{n+1}{k}(-1)^{k-1}=-\sum_{k=1}^{n+1}\binom{n+1}{k}(-1)^{k}=-\Big((1-1)^{n+1}-1\Big)=1$$

and thus the equation (or identity) is proven.

Solution 3 by Brian Bradie, Department of Mathematics, Christopher Newport University, Newport News, VA.

By the binomial theorem,

$$\sum_{k=0}^{n}\binom{n}{k}x^k=(1+x)^n.$$

With $x=-1$, this becomes

$$\sum_{k=0}^{n}\binom{n}{k}(-1)^k=0,\quad\text{or}\quad\sum_{k=1}^{n}\binom{n}{k}(-1)^{k-1}=1.$$

Now, suppose

$$s_n=\sum_{k=1}^{n}\binom{n}{k}\frac{(-1)^{k-1}}{k}.$$

Then,

$$\begin{aligned}s_n&=\sum_{k=1}^{n-1}\left[\binom{n-1}{k-1}+\binom{n-1}{k}\right]\frac{(-1)^{k-1}}{k}+\frac{(-1)^{n-1}}{n}\\&=\sum_{k=1}^{n-1}\binom{n-1}{k-1}\frac{(-1)^{k-1}}{k}+s_{n-1}+\frac{(-1)^{n-1}}{n}.\end{aligned}$$

But
$$\frac{1}{k}\binom{n-1}{k-1} = \frac{1}{k}\cdot\frac{(n-1)!}{(k-1)!(n-k)!} = \frac{1}{n}\frac{n!}{k!(n-k)!} = \frac{1}{n}\binom{n}{k},$$
so
$$\begin{aligned} s_n &= \frac{1}{n}\sum_{k=1}^{n-1}\binom{n}{k}(-1)^{k-1} + s_{n-1} + \frac{(-1)^{n-1}}{n} \\ &= \frac{1}{n}\sum_{k=1}^{n}\binom{n}{k}(-1)^{k-1} + s_{n-1} = \frac{1}{n} + s_{n-1}. \end{aligned}$$
With $s_1 = 1$, it follows that
$$s_n = \sum_{j=1}^{n}\frac{1}{j} = H_n,$$
where H_n is the nth harmonic number. Next, suppose
$$t_n = \sum_{k=1}^{n}\binom{n}{k}\frac{(-1)^{k-1}}{k^2}.$$
Then,
$$\begin{aligned} t_n &= \sum_{k=1}^{n-1}\left[\binom{n-1}{k-1} + \binom{n-1}{k}\right]\frac{(-1)^{k-1}}{k^2} + \frac{(-1)^{n-1}}{n^2} \\ &= \sum_{k=1}^{n-1}\binom{n-1}{k-1}\frac{(-1)^{k-1}}{k^2} + t_{n-1} + \frac{(-1)^{n-1}}{n^2} \\ &= \frac{1}{n}\sum_{k=1}^{n-1}\binom{n}{k}\frac{(-1)^{k-1}}{k} + t_{n-1} + \frac{(-1)^{n-1}}{n^2} \\ &= \frac{1}{n}\sum_{k=1}^{n}\binom{n}{k}\frac{(-1)^{k-1}}{k} + t_{n-1} = \frac{s_n}{n} + t_{n-1} = \frac{H_n}{n} + t_{n-1}. \end{aligned}$$
With $t_1 = 1 = \frac{H_1}{1}$, it follows that
$$t_n = \sum_{j=1}^{n}\frac{H_j}{j} = \sum_{1\leqslant i\leqslant j\leqslant n}\frac{1}{ij}.$$

Solution 4 by Moti Levy, Rehovot, Israel.

$$\sum_{1\leqslant i\leqslant j\leqslant n}\frac{1}{ij} = \sum_{k=1}^{n}\frac{H_k}{k}. \tag{15}$$

Let

$$s_n := \sum_{k=1}^{n} \binom{n}{k} \frac{(-1)^{k-1}}{k^2}. \tag{16}$$

By (15) and (16), the original problem can be rephrased as:

Prove that

$$s_n = \sum_{k=1}^{n} \frac{H_k}{k}. \tag{17}$$

An integral representation of $\frac{1}{k^2}$ is:

$$\frac{1}{k^2} = \int_0^1 x^{k-1} \left(-\ln(x)\right) dx. \tag{18}$$

Inserting (18) into the RHS of (16) gives,

$$s_n = \sum_{k=1}^{n} (-1)^{k-1} \binom{n}{k} \int_0^1 x^{k-1} \left(-\ln(x)\right) dx.$$

Interchanging the order of summation and integration,

$$s_n = \int_0^1 \left(-\ln(x)\right) \sum_{k=1}^{n} (-1)^{k-1} \binom{n}{k} x^{k-1} dx. \tag{19}$$

From the binomial expansion of $(1-x)^n = \sum_{k=0}^{n} (-1)^k \binom{n}{k} x^k$,

$$\sum_{k=1}^{n} (-1)^{k-1} \binom{n}{k} x^{k-1} = \frac{1-(1-x)^n}{x}. \tag{20}$$

$$\begin{aligned} s_n &= \int_0^1 \ln(x) \frac{(1-x)^n - 1}{x} dx = -\int_0^1 \ln(1-t) \frac{1-t^n}{1-t} dt = \\ &= -\int_0^1 \ln(1-t) \sum_{k=0}^{n-1} t^k dt = -\sum_{k=0}^{n-1} \int_0^1 \ln(1-t)\, t^k dt. \end{aligned} \tag{21}$$

$$\int_0^1 \ln(1-t)\, t^k dt = \int_0^1 \left(\sum_{j=1}^{\infty} \frac{t^j}{j}\right) t^k dt = -\sum_{j=1}^{\infty} \int_0^1 \frac{t^{j+k}}{j} dt = -\sum_{j=1}^{\infty} \frac{1}{(j+k+1)\, j}.$$

By telescoping the partial fraction of $\frac{1}{(j+k+1)\, j}$

$$\frac{1}{(j+k+1)\, j} = \frac{1}{k+1} \left(\frac{1}{j} - \frac{1}{j+k+1}\right),$$

we get

$$-\sum_{j=0}^{\infty}\frac{1}{(j+k+1)\,j}=-\frac{1}{k+1}H_{k+1},$$

hence

$$\int_0^1 \ln(1-t)\,t^k dt=-\frac{H_{k+1}}{k+1}. \tag{22}$$

It follows from (21) and (22) that

$$s_n=\sum_{k=0}^{n-1}\frac{H_{k+1}}{k+1}=\sum_{k=1}^{n}\frac{H_k}{k}.$$

Solution 5 by Perfetti Paolo, dipartimento di matematica, Università di "Tor Vergata", Roma, Italy.

We know that

$$\int_0^1 x^n\ln(1-x)=\frac{(x^{n+1}-1)\ln(1-x)}{n+1}\Big|_0^1+\int_0^1\frac{(x^{n+1}-1)dx}{(n+1)(1-x)}=\int_0^1\frac{(x^{n+1}-1)dx}{(n+1)(1-x)}=$$

$$=-\sum_{k=1}^{n}\int_0^1\frac{x^k}{n+1}dx=\frac{-H_{n+1}}{n+1}$$

$$\sum_{k=1}^{n}\binom{n}{k}\frac{(-1)^{k-1}}{k^2}=\sum_{k=1}^{n}\binom{n}{k}(-1)^{k-1}\int_0^\infty e^{-kt}tdt=-\int_0^\infty t\left((1-e^{-t})^n-1\right)dt$$

$1-e^{-t}=x$

$$\int_0^1\frac{-(x^n-1)\ln(1-x)dx}{1-x}=-\sum_{k=0}^{n-1}\int_0^1 x^k\ln(1-x)dx=\sum_{k=0}^{n-1}\frac{H_{k+1}}{k+1}=$$

$$=\sum_{k=1}^{n}\frac{1}{k}\sum_{j=1}^{k}\frac{1}{j}=\sum_{j=1}^{n}\sum_{k=j}^{n}\frac{1}{jk}=\sum_{1\leqslant i\leqslant j\leqslant n}\frac{1}{ij}$$

The proof is complete.

Also solved by Yunyong Zhang, Chinaunicom, Yunnan, China and the problem proposer.

• **5804** *Proposed by Vasile Mircea Popa, Lucian Blaga University, Sibiu, Romania.*

Calculate the integral:

$$J:=\int_{-\frac{1}{2}}^{\frac{1}{2}}\frac{\arccos x}{\sqrt{4x^4-5x^2+1}}dx.$$

Solution 1 by Yunyong Zhang, Chinaunicom, Yunnan, China.

$$
\begin{aligned}
J &= \int_{-\frac{1}{2}}^{0} \frac{\arccos x}{\sqrt{4x^4-5x^2+1}}\mathrm{d}x + \int_{0}^{\frac{1}{2}} \frac{\arccos x}{\sqrt{4x^4-5x^2+1}}\mathrm{d}x \\
&= \int_{0}^{\frac{1}{2}} \frac{\arccos(-x)}{\sqrt{4x^4-5x^2+1}}\mathrm{d}x + \int_{0}^{\frac{1}{2}} \frac{\arccos x}{\sqrt{4x^4-5x^2+1}}\mathrm{d}x \\
&= \int_{0}^{\frac{1}{2}} \frac{\frac{\pi}{2}+\arcsin x+\arccos x}{\sqrt{4x^4-5x^2+1}}\mathrm{d}x = \pi\int_{0}^{\frac{1}{2}} \frac{1}{\sqrt{4x^4-5x^2+1}}\mathrm{d}x \\
&= \pi\int_{0}^{\frac{1}{2}} \frac{1}{\sqrt{1-x^2}\sqrt{1-4x^2}}\mathrm{d}x = \pi\int_{0}^{\frac{\pi}{6}} \frac{1}{\sqrt{1-4\sin^2 x}}\mathrm{d}x \\
&= \pi F\left(\frac{\pi}{6},2\right) = \frac{\pi}{2}K\left(\frac{1}{2}\right) = \frac{\pi}{2}F\left(\frac{\pi}{2},\frac{1}{2}\right)
\end{aligned}
$$

where K is the elliptic integral of the first kind.

Solution 2 by Ángel Plaza, Universidad de Las Palmas de Gran Canaria, Spain.

We also consider the integral $I = \displaystyle\int_{-1/2}^{1/2} \frac{\arcsin x}{\sqrt{4x^4-5x^2+1}}\,dx$. We have

$$
I + J = \int_{-1/2}^{1/2} \frac{\arcsin x + \arccos x}{\sqrt{4x^4-5x^2+1}}\,dx = \frac{\pi}{2}\int_{-1/2}^{1/2} \frac{1}{\sqrt{4x^4-5x^2+1}}\,dx.
$$

But, we have $\displaystyle\int_{-1/2}^{1/2} \frac{1}{\sqrt{4x^4-5x^2+1}}dx = 2\underbrace{\int_{0}^{1/2} \frac{1}{\sqrt{4x^4-5x^2+1}}\,dx}_{C}$, since the function under the integral sign is even.

We will show that C can be expressed using the complete elliptic integral of the first kind. By doing $x = \dfrac{t}{2}$, we have

$$
C = \int_0^1 \frac{dt}{\sqrt{t^4-5t^2+4}} = \frac{1}{2}\int_0^1 \frac{dt}{\sqrt{(1-t^2)(1-\frac{1}{4}t^2)}} = \frac{1}{2}\int_0^{\pi/2} \frac{dx}{\sqrt{1-\frac{1}{4}\sin^2(x)}} = \frac{1}{2}K\left(\frac{1}{4}\right).
$$

Note that integral I is equal to zero because the function under the integral sign is odd. So, we have

$$J = \frac{\pi}{2} K\left(\frac{1}{4}\right).$$

Solution 3 by Brian Bradie, Department of Mathematics, Christopher Newport University, Newport News, VA.

Write

$$\begin{aligned}\int_{-\frac{1}{2}}^{\frac{1}{2}} \frac{\arccos x}{\sqrt{4x^4 - 5x^2 + 1}}\, dx &= \int_{-\frac{1}{2}}^{0} \frac{\arccos x}{\sqrt{4x^4 - 5x^2 + 1}}\, dx + \int_{0}^{\frac{1}{2}} \frac{\arccos x}{\sqrt{4x^4 - 5x^2 + 1}}\, dx \\ &= \int_{0}^{\frac{1}{2}} \frac{\arccos(-x)}{\sqrt{4x^4 - 5x^2 + 1}}\, dx + \int_{0}^{\frac{1}{2}} \frac{\arccos x}{\sqrt{4x^4 - 5x^2 + 1}}\, dx.\end{aligned}$$

But $\arccos(-x) = \pi - \arccos x$, so

$$\begin{aligned}\int_{-\frac{1}{2}}^{\frac{1}{2}} \frac{\arccos x}{\sqrt{4x^4 - 5x^2 + 1}}\, dx &= \pi \int_{0}^{\frac{1}{2}} \frac{1}{\sqrt{4x^4 - 5x^2 + 1}}\, dx \\ &= \pi \int_{0}^{\frac{1}{2}} \frac{1}{\sqrt{1 - 4x^2}\sqrt{1 - x^2}}\, dx.\end{aligned}$$

Now, let $2x = \sin\theta$. Then

$$\int_{-\frac{1}{2}}^{\frac{1}{2}} \frac{\arccos x}{\sqrt{4x^4 - 5x^2 + 1}}\, dx = \frac{\pi}{2} \int_{0}^{\pi/2} \frac{1}{\sqrt{1 - \frac{1}{4}\sin^2\theta}}\, d\theta = \frac{\pi}{2}\mathbf{K}\left(\frac{1}{2}\right),$$

where

$$\mathbf{K}(t) = \int_{0}^{\pi/2} \frac{1}{\sqrt{1 - t^2 \sin^2\theta}}\, d\theta$$

is the complete elliptic integral of the first kind.

Solution 4 by Michel Bataille, Rouen, France.

The change of variables $x = -u$ and the relation $\arccos(-u) = \pi - \arccos u$ give

$$J = \int_{-\frac{1}{2}}^{\frac{1}{2}} \frac{\pi - \arccos u}{\sqrt{4u^4 - 5u^2 + 1}}\, du = 2\pi \cdot I - J \tag{1}$$

where

$$I = \int_{0}^{\frac{1}{2}} \frac{du}{\sqrt{4u^4 - 5u^2 + 1}}.$$

The substitution $u = \dfrac{\sin t}{2}$ now yields

$$I = \int_0^{\frac{\pi}{2}} \frac{(1/2)\cos t}{\sqrt{(\sin^4 t)/4 - (5\sin^2 t)/4 + 1}}\,dt = \int_0^{\frac{\pi}{2}} \frac{\cos t}{\sqrt{(1-\sin^2 t)^2 + 3(1-\sin^2 t)}}\,dt$$
$$= \int_0^{\frac{\pi}{2}} \frac{dt}{\sqrt{4-\sin^2 t}} = \frac{1}{2}\int_0^{\frac{\pi}{2}} \frac{dt}{\sqrt{1-\frac{1}{4}\sin^2 t}} = \frac{1}{2}K(1/2)$$

where for $0 \leqslant k < 1$, $K(k)$ denotes the elliptic integral of the first kind $\displaystyle\int_0^{\pi/2} \frac{dt}{\sqrt{1-k^2\sin^2 t}}$.

Returning to (1), we conclude that

$$J = \pi I = \frac{\pi}{2}K(1/2).$$

Solution 5 by Moti Levy, Rehovot, Israel.

$$J = \int_0^{\frac{1}{2}} \frac{\arccos(x)}{\sqrt{4x^4-5x^2+1}}dx + \int_{-\frac{1}{2}}^0 \frac{\arccos(x)}{\sqrt{4x^4-5x^2+1}}dx$$
$$= \int_0^{\frac{1}{2}} \frac{\arccos(x)+\arccos(-x)}{\sqrt{4x^4-5x^2+1}}dx$$

Because

$$\arccos(x) + \arccos(-x) = \pi,$$

$$J = \pi \int_0^{\frac{1}{2}} \frac{1}{\sqrt{4x^4-5x^2+1}}\,dx.$$

$$4x^4 - 5x^2 + 1 = (1-x^2)(1-4x^2),$$

so

$$J = \pi \int_0^{0.5} \frac{dx}{\sqrt{(1-x^2)(1-4x^2)}}$$

Set $x = \frac{1}{2}\sin\theta$ $(0 \leqslant \theta \leqslant \pi/2)$. Then

$$dx = \tfrac{1}{2}\cos\theta\,d\theta, \qquad \sqrt{(1-x^2)(1-4x^2)} = \frac{\cos\theta}{2}\sqrt{1-\tfrac{1}{4}\sin^2\theta}.$$

$$J = \frac{\pi}{2}\int_0^{\pi/2} \frac{d\theta}{\sqrt{1-\frac{1}{4}\sin^2\theta}} = \frac{\pi}{2}K\left(\frac{1}{4}\right),$$

where

$$K(m) = \int_0^{\pi/2} \frac{d\phi}{\sqrt{1-m\sin^2\phi}},$$

is the complete elliptic integral of the first kind.

$$K\left(\tfrac{1}{4}\right) = 1.685\,750\,355\,184\ldots \Longrightarrow J = \frac{\pi}{2}K\left(\tfrac{1}{4}\right) = 2.647\,970\,465\ldots$$

Solution 6 by Perfetti Paolo, dipartimento di matematica, Universitą di "Tor Vergata", Roma, Italy.

$$\int_{-\frac{1}{2}}^{\frac{1}{2}} \frac{\arccos(x) - \pi/2}{\sqrt{4x^4 - 5x^2 + 1}}dx + \frac{\pi}{2}\int_{-\frac{1}{2}}^{\frac{1}{2}} \frac{1}{\sqrt{4x^4 - 5x^2 + 1}}dx = \frac{\pi}{2}\int_{-\frac{1}{2}}^{\frac{1}{2}} \frac{1}{\sqrt{4x^4 - 5x^2 + 1}}dx$$

thanks to the oddness of the function $\arccos(x) - \pi/2$.

$$\frac{\pi}{4}\int_{-\frac{1}{2}}^{\frac{1}{2}} \frac{dx}{\sqrt{(1-x^2)(\frac{1}{4}-x^2)}} \underbrace{=}_{x=\frac{\sin y}{2}} \frac{\pi}{4}\int_{\frac{-\pi}{2}}^{\frac{\pi}{2}} \frac{dy}{\sqrt{1-\frac{\sin^2 y}{4}}} = \frac{\pi}{2}\int_{0}^{\frac{\pi}{2}} \frac{dy}{\sqrt{1-\frac{\sin^2 y}{4}}} = \frac{\pi}{2}K(\frac{1}{2})$$

and $K(\dfrac{1}{2})$ is the complete elliptic integral of the first kind

Also solved by the problem proposer.

Editor's Statement: It goes without saying that the problem proposers, as well as the solution proposers, are the *élan vital* of the Problems/Solutions Section of SSMJ. As the editor of this Section of the Journal, I consider myself fortunate to be in a position to receive, compile and organize a wealth of proposed ingenious problems and solutions intended for online publication. My unwavering gratitude goes to all the amazingly creative contributors. We come together from across continents because we find intellectual value, joy and satisfaction in mathematical problems, both in their creation as well as their solution. So that our collective efforts serve us well, I kindly ask all contributors to adhere to the following guidelines. As you peruse below, you may construe that the guidelines amount to a lot of work. But, as the samples show, there's not much to do. Your cooperation is much appreciated!

Keep in mind that the examples given below are your best guide!

Formats, Styles and Requirements

When submitting proposed problem(s) or solution(s), please send both **LaTeX** document and **pdf** document of your proposed problem(s) or solution(s). There are ways (discoverable from the internet) to convert from Word to proper LaTeX code. Porposals without a *proper* **LaTeX** document will not be published regrettably.

Regarding Proposed Solutions:

Below is the FILENAME format for all the documents of your proposed solution(s).

#ProblemNumber_FirstName_LastName_Solution_SSMJ

- FirstName stands for YOUR first name.
- LastName stands for YOUR last name.

Examples:

#1234_Max_Planck_Solution_SSMJ

#9876_Charles_Darwin_Solution_SSMJ

Please note that every problem number is *preceded* by the sign # .

All you have to do is copy the FILENAME format (or an example below it), paste it and then modify portions of it to your specs.

Please adopt the following structure, in the order shown, for the presentation of your solution:

1. On top of the first page of your solution, begin with the phrase:

"Proposed Solution to #**** SSMJ"

where the string of four astrisks represents the problem number.

2. On the second line, write

"Solution proposed by [your First Name, your Last Name]",

followed by your affiliation, city, country, all on the same linear string of words. Please see the example below. Make sure you do the same for your collaborator(s).

3. On a new line, state the problem proposer's name, affiliation, city and country, just as it appears published in the Problems/Solutions section.

4. On a new line below the above, write in bold type: "**Statement of the Problem**".

5. Below the latter, state the problem. Please make sure the statement of your problem (unlike the preceding item) is not in bold type.

6. Below the statement of the problem, write in bold type: "**Solution of the Problem**".

7. Below the latter, show the entire solution of the problem.

Here is a sample for the above-stated format for proposed solutions:

Proposed solution to #1234 SSMJ

Solution proposed by Emmy Noether, University of Göttingen, Lower Saxony, Germany.

Problem proposed by Isaac Newton, Trinity College, Cambridge, England.

Statement of the problem:

Compute $\displaystyle\sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}$.

Solution of the problem: . . . . . .

Regarding Proposed Problems:

For all your proposed problems, please adopt for all documents the following FILENAME format:

FirstName_LastName_ProposedProblem_SSMJ_YourGivenNumber_ProblemTitle

If you do not have a ProblemTitle, then leave that component as it already is (i.e., ProblemTitle).

The component YourGivenNumber is any UNIQUE 3-digit (or longer) number you like to give to your problem.

Examples:

Max_Planck_ProposedProblem_SSMJ_314_HarmonicPatterns

Charles_Darwin_ProposedProblem_SSMJ_358_ProblemTitle

Please adopt the following structure, in the order shown, for the presentation of your proposal:

1. On the top of first page of your proposal, begin with the phrase:

"Problem proposed to SSMJ"

2. On the second line, write

"Problem proposed by [your First Name, your Last Name]",

followed by your affiliation, city, country all on the same linear string of words. Please see the example below. Make sure you do the same for your collaborator(s) if any.

3. On a new line state the title of the problem, if any.

4. On a new line below the above, write in bold type: "**Statement of the Problem**".

5. Below the latter, state the problem. Please make sure the statement of your problem (unlike the preceding item) is not in bold type.

6. Below the statement of the problem, write in bold type: "**Solution of the Problem**".

7. Below the latter, show the entire solution of your problem.

Here is a sample for the above-stated format for proposed problems:

Problem proposed to SSMJ

Problem proposed by Isaac Newton, Trinity College, Cambridge, England.

Principia Mathematica (⟵ You may choose to not include a title.)

Statement of the problem:

Compute $\sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}$.

Solution of the problem: . . . . . .

♣ ♣ ♣ Thank You! ♣ ♣ ♣