Problems and Solutions

Albert Natian, Section Editor

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This section of the Journal offers readers an opportunity to exchange interesting mathematical problems and solutions. Please email them to Prof. Albert Natian at Department of Mathematics, Los Angeles Valley College. Please make sure every proposed problem or proposed solution is provided in both ***LaTeX*** and pdf documents. Please make sure your proposals adhere to **Formats, Styles and Requirements** noted below. Thank you!

To propose problems, email them to: problems4ssma@gmail.com

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Solutions to previously published problems can be seen at <www.ssma.org/publications>.

Solutions to the problems published in this issue should be submitted *before* April 1, 2026.

• **5825** *Proposed by Daniel Sitaru, National Economic College "Theodor Costescu" Drobeta Turnu-Severin, Romania.*

Let $(f_n)_{n=0}^{\infty}$ be the Fibonacci sequence; i.e., $f_0 = 0, f_1 = 1, f_{n+2} = f_{n+1} + f_n$ for $n \geqslant 0$. Prove that for all $n \geqslant 1$:

$$8\left(\sum_{k=1}^{n} f_k \sin\frac{1}{k}\right)\left(\sum_{k=1}^{n} f_k \cos\frac{1}{k}\right) < n f_{n+2}^2.$$

• **5826** *Proposed by Jose Luis Diaz-Barrero, Barcelona, Spain.*

Let $A(z) = \sum_{k=0}^{n} a_k z^k$ $(a_k \neq 0\ \forall k)$ be a non-constant polynomial with complex coefficients. Show that all its zeros lie in the annulus $C = \{z \in \mathbb{C} : r_1 \leqslant |z| \leqslant r_2\}$, where

$$r_1 = \min_{1\leqslant k\leqslant n}\left\{\frac{5^k\binom{n}{k}}{6^n-1}\left|\frac{a_0}{a_k}\right|\right\}^{1/k} \quad \text{and} \quad r_2 = \max_{1\leqslant k\leqslant n}\left\{\frac{6^n-1}{5^k\binom{n}{k}}\left|\frac{a_{n-k}}{a_n}\right|\right\}^{1/k}.$$

• **5827** *Proposed by Shivam Sharma, Delhi University, New Delhi, India.*

Find the limits L and M:

$$L = \lim_{n\to\infty}\left(\sqrt[n]{\prod_{k=2}^{n+1}\left(\frac{k-1}{k+1}\right)^{k+1}}\right),$$

$$M = \lim_{n\to\infty}\frac{n}{\ln(n)}\left(L - \sqrt[n]{\prod_{k=2}^{n+1}\left(\frac{k-1}{k+1}\right)^{k+1}}\right).$$

• **5828** *Proposed by Toyesh Prakash Sharma, Agra College, Agra, India.*

If $a, b, c \geqslant 5$ and $a + b + c = 18$ then show that

$$a^{1/a}b^{1/b} + b^{1/b}c^{1/c} + c^{1/c}a^{1/a} \geqslant 3\sqrt[3]{6}.$$

• **5829** *Proposed by Vasile Cîrtoaje, Petroleum-Gas University of Ploiesti, Romania.*

Find the largest positive value of the constant k such that the inequality

$$\frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2} \geqslant a^k + b^k + c^k$$

holds for any positive numbers a, b, c, with $ab + bc + ca = 3$, such that at most one of the numbers a, b, c is less than 1.

Solutions

To Formerly Published Problems

• **5805** *Proposed by Ovidiu Furdui and Alina Sîntămărian, Technical University of Cluj-Napoca, Cluj-Napoca, Romania.*

Calculate

$$\lim_{n\to\infty} \frac{1}{\sqrt{n}} \int_0^n \frac{\sin^2 x}{1 + n\cos^2(nx)} \mathrm{d}x.$$

Solution 1 by Ulrich Abel, Technische Hochschule Mittelhessen, Friedberg, Germany.

We show the following generalization. For a continuous function $f : \mathbb{R} \to \mathbb{R}$ with period π, we claim that

$$L := \lim_{n\to\infty} \frac{1}{\sqrt{n}} \int_0^n \frac{f(x)}{1 + n\cos^2(nx)} dx = \frac{1}{\pi} \int_0^\pi f(x)\, dx.$$

This problem is a special case of $f(x) = \sin^2 x$. Given that $\int_0^\pi (\sin x)^2\, dx = \pi/2$, then

$$\lim_{n\to\infty} \frac{1}{\sqrt{n}} \int_0^n \frac{\sin^2 x}{1 + n\cos^2(nx)} dx = \frac{1}{2}.$$

Proof of the Above Claim:

Define $g_n(x) := \dfrac{1}{1+n\cos^2(nx)}$. We have

$$\int_{-\pi/(2n)}^{\pi/(2n)} g_n(x)\,dx = \frac{1}{n\sqrt{n+1}} \lim_{\varepsilon\to 0+} \arctan\left(\frac{\tan(nx)}{\sqrt{n+1}}\right)\Bigg|_{-\pi/(2n)+\varepsilon}^{\pi/(2n)-\varepsilon} = \frac{\frac{\pi}{2}-\left(-\frac{\pi}{2}\right)}{n\sqrt{n+1}} = \frac{\pi}{n\sqrt{n+1}}.$$

Let G_n be an anti-derivative of g_n. By symmetry and periodicity of g_n, we conclude that, for all real numbers a,

$$G_n\left(a+\frac{\pi}{n}\right) - G_n\left(\frac{\pi}{n}\right) = \int_a^{a+\pi/n} g_n(x)\,dx = \frac{\pi}{n\sqrt{n+1}}.$$

The Riemannian sum of the function f on the interval $[0,\pi]$ is given by

$$\frac{\pi}{n}\sum_{k=0}^{n} f\left(\frac{k}{n}\pi\right) = \sqrt{n+1}\sum_{k=0}^{n} f\left(\frac{k}{n}\pi\right)\left[G_n\left(\frac{k+1}{n}\pi\right) - G_n\left(\frac{k}{n}\pi\right)\right].$$

Passing to the limit $n\to\infty$ we conclude that

$$\int_0^{\pi} f(x)\,dx \sim \sqrt{n+1}\int_0^{\pi} f(x)\,dG_n(x) = \sqrt{n+1}\int_0^{\pi} f(x)\,g_n(x)\,dx \qquad (n\to\infty).$$

By periodicity of f and g_n, we have

$$L = \lim_{n\to\infty}\frac{1}{\sqrt{n}}\int_0^{\lfloor n/\pi\rfloor\pi} f(x)\,g_n(x)\,dx = \lim_{n\to\infty}\frac{\lfloor n/\pi\rfloor}{\sqrt{n}}\int_0^{\pi} f(x)\,g_n(x)\,dx.$$

Comparison with the forelast relation yields

$$L = \lim_{n\to\infty}\frac{\sqrt{n}}{\pi}\int_0^{\pi} f(x)\,g_n(x)\,dx = \frac{1}{\pi}\int_0^{\pi} f(x)\,dx.$$

Solution 2 by Moti Levy, Rehovot, Israel.

We aim to evaluate the limit:

$$\lim_{n\to\infty}\frac{1}{n\sqrt{n}}\int_0^{n^2}\frac{\sin^2(x/n)}{1+n\cos^2(x)}\,dx = \frac{1}{2}.$$

We observe that the integrand contains a denominator $\dfrac{1}{1+n\cos^2(x)}$, which becomes "sharply peaked" at the zeros of $\cos(x)$. As $n\to\infty$, these peaks act like "Dirac delta functions" that "samples" the numerator $\sin^2(x/n)$ at specific points. Define:

$$K_n(x) := \frac{\sqrt{n}}{\pi}\cdot\frac{1}{1+n\cos^2(x)}.$$

Near a zero $x_k = \frac{\pi}{2} + k\pi$ of $\cos(x)$, we expand:

$$\cos(x) = \cos(x_k + \epsilon) \approx (-1)^{k+1}\epsilon, \quad \text{as } \epsilon \to 0.$$

Hence,

$$K_n(x) \approx \frac{\sqrt{n}}{\pi} \cdot \frac{1}{1 + n(x - x_k)^2}.$$

This is a "Cauchy-type kernel" centered at x_k with width $\sim 1/\sqrt{n}$. Let

$$K_n^{(k)}(x) := \frac{\sqrt{n}}{\pi} \cdot \frac{1}{1 + n(x - x_k)^2}.$$

We observe that $K_n^{(k)}(x)$ is approximate identity since it satisfies the conditions:

1. Normalization:
$$\int_{-\infty}^{\infty} K_n^{(k)}(x)\, dx = 1.$$
2. Concentration:
$$\int_{|x - x_k| > \delta} K_n^{(k)}(x)\, dx \to 0 \quad \text{as } n \to \infty.$$
3. Sampling:
$$\int f(x) K_n^{(k)}(x)\, dx \to f(x_k) \text{ for continuous } f.$$

Now we express the original integral in terms of K_n:

$$\frac{1}{n\sqrt{n}} \int_0^{n^2} \frac{\sin^2(x/n)}{1 + n\cos^2(x)}\, dx = \frac{\pi}{n^2} \int_0^{n^2} \sin^2(x/n) \cdot K_n(x)\, dx.$$

As $n \to \infty$, the function $K_n(x)$ becomes sharply peaked near the zeros of $\cos(x)$, effectively sampling $\sin^2(x/n)$ at:

$$x_k = \frac{\pi}{2} + k\pi \quad \Rightarrow \quad \frac{x_k}{n} \in [0, n].$$

Total number of such points in $[0, n^2]$ is roughly $\frac{n^2}{\pi}$. Each k-th peak contributes approximately

$$\sin^2\left(\frac{x_k}{n}\right).$$

Thus

$$\frac{\pi}{n^2} \int_0^{n^2} \sin^2(x/n) \cdot K_n(x)\, dx \approx \frac{\pi}{n^2} \sum_k \sin^2(x_k/n).$$

We now have

$$\frac{\pi}{n^2} \int_0^{n^2} \sin^2(x/n) K_n(x)\, dx \approx \frac{\pi}{n^2} \sum_{k=0}^{\lfloor n^2/\pi \rfloor} \sin^2\left(\frac{x_k}{n}\right).$$

This is a Riemann sum:

$$\frac{1}{\lfloor n^2/\pi \rfloor} \sum_{k=0}^{\lfloor n^2/\pi \rfloor} \sin^2\left(\frac{\pi}{2n} + \frac{k\pi}{n}\right) \to \frac{1}{\pi}\int_0^{\pi} \sin^2(u)\, du = \frac{1}{2}.$$

Hence

$$\lim_{n\to\infty} \frac{1}{n\sqrt{n}} \int_0^{n^2} \frac{\sin^2(x/n)}{1+n\cos^2(x)}\, dx = \frac{1}{2}.$$

Also solved Albert Stadler, Herrliberg, Switzerland and by the problem proposer.

• **5806** *Proposed by Paolo Perfetti, dipartimento di matematica Universitą di "Tor Vergata", Rome, Italy.*

Calculate

$$\text{a)} \quad \sum_{n=1}^{\infty} \left[\frac{1}{2n-1} - \left(\frac{1}{n} - \frac{1}{n+1} + \frac{1}{n+2} - \cdots\right)\right]^2$$

and

$$\text{b)} \quad \sum_{n=1}^{\infty} (-1)^n \left[\frac{1}{2n-1} - \left(\frac{1}{n} - \frac{1}{n+1} + \frac{1}{n+2} - \cdots\right)\right]^2.$$

Solution 1 by Albert Stadler, Herrliberg, Switzerland.

a) We have

$$\frac{1}{n} - \frac{1}{n+1} + \frac{1}{n+2} - \cdots = \sum_{k=n}^{\infty} (-1)^{k-n} \int_0^1 t^{k-1} dt = \int_0^1 \frac{t^{n-1}}{1+t} dt = 2\int_0^1 \frac{t^{2n-1}}{1+t^2} dt.$$

Then

$$\sum_{n=1}^{\infty} \left(\frac{1}{2n-1} - \left(\frac{1}{n} - \frac{1}{n+1} + \frac{1}{n+2} - \ldots\right)\right)^2 =$$

$$= \sum_{n=1}^{\infty} \frac{1}{(2n-1)^2} - 4\sum_{n=1}^{\infty} \frac{1}{2n-1} \int_0^1 \frac{t^{2n-1}}{1+t^2} dt + \sum_{n=1}^{\infty} \int_0^1 \frac{t^{n-1}}{1+t} dt \int_0^1 \frac{u^{n-1}}{1+u} du =$$

$$= \frac{\pi^2}{8} - 2\int_0^1 \frac{\ln\left(\frac{1+t}{1-t}\right)}{1+t^2} dt + \int_0^1 \int_0^1 \frac{1}{(1+t)(1+u)(1-ut)} du dt.$$

The change of variables $u = \dfrac{1-t}{1+t}$, $t = \dfrac{1-u}{1+u}$, $dt = -\dfrac{2}{(1+u)^2} du$ gives

$$-2\int_0^1 \frac{\ln\left(\frac{1+t}{1-t}\right)}{1+t^2} dt = -2\int_0^1 \frac{\ln(u)}{1+u^2} du = -2G,$$

where G is Catalan's constant. Furthermore

$$\int_0^1 \frac{1}{(1+u)(1-ut)}du = \int_0^1 \left(\frac{1}{(1+t)(1+u)} + \frac{t}{(1+t)(1-ut)}\right) du = \frac{\ln 2}{1+t} - \frac{\ln(1-t)}{1+t}$$

and

$$\int_0^1 \frac{1}{1+t}\left(\frac{\ln 2}{1+t} - \frac{\ln(1-t)}{1+t}\right) dt = \frac{1}{2}\ln 2 + \left(\frac{1}{1+t} - \frac{1}{2}\right)\ln(1-t)\Big|_{t=0}^{t=1} + \int_0^1 \left(\frac{1}{1+t} - \frac{1}{2}\right)\frac{1}{1-t}dt =$$

$$= \ln 2 .$$

So $\sum_{n=1}^{\infty}\left(\frac{1}{2n-1} - \left(\frac{1}{n} - \frac{1}{n+1} + \frac{1}{n+2} - \dots\right)\right)^2 = \frac{\pi^2}{8} - 2G + \ln 2$.

b) We have

$$\sum_{n=1}^{\infty}(-1)^n\left(\frac{1}{2n-1} - \left(\frac{1}{n} - \frac{1}{n+1} + \frac{1}{n+2} - \dots\right)\right)^2 =$$

$$= \sum_{n=1}^{\infty}\frac{(-1)^n}{(2n-1)^2} - 4\sum_{n=1}^{\infty}\frac{(-1)^n}{2n-1}\int_0^1 \frac{t^{2n-1}}{1+t^2}dt + \sum_{n=1}^{\infty}(-1)^n\int_0^1 \frac{t^{n-1}}{1+t}dt\int_0^1 \frac{u^{n-1}}{1+u}du$$

$$= -G + 4\int_0^1 \frac{\arctan t}{1+t^2}dt - \int_0^1\int_0^1 \frac{1}{(1+t)(1+u)(1+ut)}dudt.$$

Then

$$4\int_0^1 \frac{\arctan t}{1+t^2}dt = 2\arctan^2 t\Big|_{t=0}^{t=1} = \frac{\pi^2}{8}$$

and

$$\int_0^1\int_0^1 \frac{1}{(1+t)(1+u)(1+ut)}dudt = \int_0^1 \frac{1}{1+t}\int_0^1 \left(\frac{1}{(1-t)(1+u)} - \frac{t}{(1-t)(1+tu)}\right) dudt =$$

$$= \int_0^1 \frac{1}{1+t}\left(\frac{\ln 2}{1-t} - \frac{\ln(1+t)}{1-t}\right) dt \overbrace{=}^{t=\frac{1-u}{1+u}} \int_0^1 \frac{1}{1-\left(\frac{1-u}{1+u}\right)^2}\left(\ln 2 - \ln\left(1+\frac{1-u}{1+u}\right)\right)\frac{2du}{(1+u)^2} =$$

$$= \int_0^1 \frac{\ln(1+u)}{2u}du = \frac{1}{2}\sum_{k=1}^{\infty}\frac{(-1)^{k+1}}{k}\int_0^1 u^{k-1}du = \frac{1}{2}\sum_{k=1}^{\infty}\frac{(-1)^{k+1}}{k^2} = \frac{1}{2}\sum_{k=1}^{\infty}\frac{1}{k^2}\left(1-\frac{1}{2}\right) = \frac{\pi^2}{24}.$$

So $\sum_{n=1}^{\infty}(-1)^n\left(\frac{1}{2n-1} - \left(\frac{1}{n} - \frac{1}{n+1} + \frac{1}{n+2} - \dots\right)\right)^2 = -G + \frac{\pi^2}{12}.$

Solution 2 by Michel Bataille, Rouen, France.

a) Let $R_n = \sum_{k=0}^{\infty} \frac{(-1)^k}{n+k}$. We are looking for the value of

$$S = \sum_{n=1}^{\infty} \left(\frac{1}{2n-1} - R_n \right)^2 = \sum_{n=1}^{\infty} \left(\frac{1}{(2n-1)^2} + R_n^2 - \frac{2R_n}{2n-1} \right).$$

We know that $\sum_{n=1}^{\infty} \frac{1}{(2n-1)^2} = \frac{\pi^2}{8}$ and that $\sum_{n=1}^{\infty} R_n^2 = \ln(2)$ (see *The College Mathematics Journal*, Problem 997, Vol. 45, No 2, March 2014, p. 147). Recalling that $R_n = \int_0^1 \frac{x^{n-1}}{1+x}\,dx$, we have

$$\begin{aligned}
\sum_{n=1}^{\infty} \frac{R_n}{2n-1} &= \int_0^1 \left(\sum_{n=1}^{\infty} \frac{x^{n-1}}{2n-1} \right) \frac{dx}{1+x} \\
&= \int_0^1 \frac{\ln(1+\sqrt{x}) - \ln(1-\sqrt{x})}{2\sqrt{x}} \cdot \frac{dx}{1+x} = \int_0^1 \frac{\ln(1+u) - \ln(1-u)}{1+u^2}\,du \\
&= -\int_0^{\pi/4} \ln\left(\frac{1-\tan\theta}{1+\tan\theta} \right) d\theta = -\int_0^{\pi/4} \ln\left(\frac{\pi}{4} - \theta \right) d\theta = -\int_0^{\pi/4} \ln(\tan t)\,dt.
\end{aligned}$$

Thus, $\sum_{n=1}^{\infty} \frac{R_n}{2n-1} = G$, the Catalan constant and consequently, $S = \frac{\pi^2}{8} + \ln(2) - 2G$.

b) Similarly, the desired sum is

$$T = \sum_{n=1}^{\infty} (-1)^n \left(\frac{1}{2n-1} - R_n \right)^2 = -\sum_{n=1}^{\infty} \left(\frac{(-1)^{n-1}}{(2n-1)^2} + (-1)^{n-1} R_n^2 - \frac{2(-1)^{n-1} R_n}{2n-1} \right).$$

We know that $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{(2n-1)^2} = G$. Also,

$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1} R_n}{2n-1} = \sum_{n=1}^{\infty} \int_0^1 \frac{(-1)^{n-1} x^{n-1}}{(2n-1)(1+x)}\,dx.$$

Since

$$\sum_{n=1}^{\infty} \int_0^1 \left| \frac{(-1)^{n-1} x^{n-1}}{(2n-1)(1+x)} \right| dx \leqslant \sum_{n=1}^{\infty} \int_0^1 \frac{x^{n-1}}{(2n-1)}\,dx = \sum_{n=1}^{\infty} \frac{1}{n(2n-1)} < \infty$$

we obtain

$$\begin{aligned}
\sum_{n=1}^{\infty} \frac{(-1)^{n-1} R_n}{2n-1} &= \int_0^1 \left(\sum_{n=1}^{\infty} \frac{(-1)^{n-1} x^{n-1}}{2n-1} \right) \frac{dx}{1+x} \\
&= \int_0^1 \frac{\arctan(\sqrt{x})}{\sqrt{x}(1+x)}\,dx = 2\int_0^1 \frac{\arctan(u)}{1+u^2}\,du = 2\left[\frac{1}{2}(\arctan(u))^2 \right]_0^1 = \frac{\pi^2}{16}.
\end{aligned}$$

Lastly, since $\lim_{n\to\infty} R_n = 0$, we have $\sum_{n=1}^{\infty}(-1)^{n-1}R_n^2 = \lim_{N\to\infty}\sum_{n=1}^{2N}(-1)^{n-1}R_n^2$, with

$$\begin{aligned}\sum_{n=1}^{2N}(-1)^{n-1}R_n^2 &= \sum_{n=1}^{N}(R_{2n-1}^2 - R_{2n}^2) = \sum_{n=1}^{N}(R_{2n-1}+R_{2n})(R_{2n-1}-R_{2n})\\ &= 2\sum_{n=1}^{N}\frac{R_{2n-1}}{2n-1} - \sum_{n=1}^{N}\frac{1}{(2n-1)^2}\end{aligned}$$

(since $R_{2n-1}+R_{2n} = \frac{1}{2n-1}$). Thus, $\sum_{n=1}^{\infty}(-1)^{n-1}R_n^2 = 2\sum_{n=1}^{\infty}\frac{R_{2n-1}}{2n-1} - \frac{\pi^2}{8}$. We calculate

$$2\sum_{n=1}^{\infty}\frac{R_{2n-1}}{2n-1} = 2\int_0^1\left(\sum_{n=1}^{\infty}\frac{x^{2n-2}}{2n-1}\right)\frac{dx}{1+x} = \int_0^1\frac{\ln(1+x)-\ln(1-x)}{x(1+x)}\,dx$$

and substitute $x = \frac{1-u}{1+u}$ in the integral to obtain

$$2\sum_{n=1}^{\infty}\frac{R_{2n-1}}{2n-1} = -\int_0^1\frac{\ln u}{1-u}\,du = -\int_0^1\frac{\ln(1-v)}{v}\,dv = \frac{\pi^2}{6}.$$

Gathering the obtained results gives

$$T = -\left[G + \frac{\pi^2}{6} - \frac{\pi^2}{8} - \frac{\pi^2}{8}\right] = \frac{\pi^2}{12} - G.$$

Solution 3 by Moti Levy, Rehovot, Israel.

$$\frac{1}{n} - \frac{1}{n+1} + \frac{1}{n+2} - \cdots = \int_0^1\frac{x^{n-1}}{1+x}dx \tag{1}$$

$$\begin{aligned}&\sum_{n=1}^{\infty}\left[\frac{1}{2n-1} - \left(\frac{1}{n} - \frac{1}{n+1} + \frac{1}{n+2} - \cdots\right)\right]^2\\ &= \sum_{n=1}^{\infty}\frac{1}{(2n-1)^2} + \sum_{n=1}^{\infty}\left(\int_0^1\frac{x^{n-1}}{1+x}dx\right)^2 - \sum_{n=1}^{\infty}\frac{2}{2n-1}\int_0^1\frac{x^{n-1}}{1+x}dx.\end{aligned}$$

1. The first term
$$\sum_{n=1}^{\infty}\frac{1}{(2n-1)^2} = \frac{1}{8}\pi^2.$$

2. The second term
$$\begin{aligned}\sum_{n=1}^{\infty}\left(\int_0^1\frac{x^{n-1}}{1+x}dx\right)^2 &= \sum_{n=1}^{\infty}\int_0^1\frac{x^{n-1}}{1+x}dx\int_0^1\frac{y^{n-1}}{1+y}dy = \int_0^1\int_0^1\sum_{n=1}^{\infty}\frac{x^{n-1}y^{n-1}}{(1+x)(1+y)}dxdy\\ &= \int_0^1\int_0^1\frac{1}{(1+x)(1+y)(1-xy)}dxdy = \ln(2).\end{aligned}$$

3. The third term

$$\sum_{n=1}^{\infty}\frac{2}{2n-1}\int_0^1\frac{x^{n-1}}{1+x}dx=\int_0^1\frac{2}{1+x}\sum_{n=1}^{\infty}\frac{x^{n-1}}{2n-1}dx$$
$$=\int_0^1\frac{1}{1+x}\frac{1}{\sqrt{x}}\ln\frac{1+\sqrt{x}}{1-\sqrt{x}}dx=2\mathbf{G},$$

where **G** is Catalan constant.

$$\sum_{n=1}^{\infty}\left[\frac{1}{2n-1}-\left(\frac{1}{n}-\frac{1}{n+1}+\frac{1}{n+2}-\cdots\right)\right]^2=\frac{1}{8}\pi^2+\ln(2)-2\mathbf{G}.$$

b)

$$\sum_{n=1}^{\infty}(-1)^n\left[\frac{1}{2n-1}-\left(\frac{1}{n}-\frac{1}{n+1}+\frac{1}{n+2}-\cdots\right)\right]^2$$
$$=\sum_{n=1}^{\infty}\frac{(-1)^n}{(2n-1)^2}+(-1)^n\sum_{n=1}^{\infty}\left(\int_0^1\frac{x^{n-1}}{1+x}dx\right)^2-\sum_{n=1}^{\infty}\frac{2(-1)^n}{2n-1}\int_0^1\frac{x^{n-1}}{1+x}dx.$$

1. The first term

$$\sum_{n=1}^{\infty}\frac{(-1)^n}{(2n-1)^2}=-\mathbf{G}.$$

2. The second term

$$\sum_{n=1}^{\infty}(-1)^n\left(\int_0^1\frac{x^{n-1}}{1+x}dx\right)^2=\sum_{n=1}^{\infty}(-1)^n\int_0^1\frac{x^{n-1}}{1+x}dx\int_0^1\frac{y^{n-1}}{1+y}dy=\int_0^1\int_0^1\sum_{n=1}^{\infty}\frac{(-1)^n x^{n-1}y^{n-1}}{(1+x)(1+y)}dxdy$$
$$=-\int_0^1\int_0^1\frac{1}{(1+x)(1+y)(1+xy)}dxdy=-\frac{\pi^2}{24}.$$

3. The third term

$$\sum_{n=1}^{\infty}\frac{2(-1)^n}{2n-1}\int_0^1\frac{x^{n-1}}{1+x}dx=\int_0^1\frac{2}{1+x}\sum_{n=1}^{\infty}\frac{(-1)^n x^{n-1}}{2n-1}dx$$
$$=-\int_0^1\frac{2\arctan\left(\sqrt{x}\right)}{\sqrt{x}(1+x)}dx=-\frac{\pi^2}{8}.$$

$$\sum_{n=1}^{\infty}(-1)^n\left[\frac{1}{2n-1}-\left(\frac{1}{n}-\frac{1}{n+1}+\frac{1}{n+2}-\cdots\right)\right]^2=-\mathbf{G}+\frac{\pi^2}{12}.$$

Also solved by Devis Alvarado, UNAH and UPNFM, Tegucigalpa, Hoduras and the problem proposer.

• **5807** *Proposed by Michael Brozinsky, Central Islip, New York.*

Right triangle RST (labeled clockwise with right angle at S) is formed by three lines as follows: line L1 containing RT , line L2 containing ST, and line L3 containing RS. Points P1, P2 and P3 are such that the distances from each of P1, P2 and P3 to L1, L2 and L3 respectively are proportional to 1:2:3. And there are no other points with these (preceding) three properties. Show that the area of triangle △P1P2P3 is half the area of triangle RST and determine angle R.

Solution 1 by Albert Stadler, Herrliberg, Switzerland.

We may assume that the triangle RST is given by the coordinates: S(0,0), T(b,0), R(0,a) with a, b>0. Let P be the point with coordinates: P(u,v). Then the squared distance from P to the line $\left\{\begin{matrix} L_1 \\ L_2 \\ L_3 \end{matrix}\right\}$ equals $\left\{\begin{matrix} d^2 \\ v^2 \\ u^2 \end{matrix}\right\}$ where d is the distance from (u,v) to the line $y = -\frac{a}{b}x + a$ and that distance equals $\frac{|au + bv - ab|}{\sqrt{a^2 + b^2}}$ as is easily verified. So we must determine (u, v) in such a way that

$$\frac{(au + bv - ab)^2}{a^2 + b^2} : v^2 : \ u^2 = 1 : 4 : 9.$$

This system of equations has the four solutions

$$(u, v) \in \left\{ \begin{matrix} \left(\frac{3ab}{3a - 2b - \sqrt{(a^2 + b^2)}}, -\frac{2ab}{3a - 2b - \sqrt{(a^2 + b^2)}}\right), \\ \left(\frac{3ab}{3a + 2b + \sqrt{(a^2 + b^2)}}, \frac{2ab}{3a + 2b + \sqrt{(a^2 + b^2)}}\right), \\ \left(\frac{3ab}{3a - 2b + \sqrt{(a^2 + b^2)}}, -\frac{2ab}{3a - 2b + \sqrt{(a^2 + b^2)}}\right), \\ \left(\frac{3ab}{3a + 2b - \sqrt{(a^2 + b^2)}}, \frac{2ab}{3a + 2b - \sqrt{(a^2 + b^2)}}\right) \end{matrix} \right\}$$

These four solutions collapse to three solutions exactly if either $3a - 2b - \sqrt{(a^2 + b^2)} = 0$ (case 1) or $3a - 2b + \sqrt{(a^2 + b^2)}$ (case 2), i.e. if and only if either $b = 2\left(1 - \frac{1}{\sqrt{3}}\right)a$ (case 1) or

$b = 2\left(1+\frac{1}{\sqrt{3}}\right) a$ (case 2).

Case 1: $b = 2\left(1-\frac{1}{\sqrt{3}}\right) a$

The angle at R equals $\arctan\left(\frac{b}{a}\right) = \arctan 2\left(1-\frac{1}{\sqrt{3}}\right) \approx 40.2078^0$, and the three points P_1, P_2, P_3 are:

$$P_1 : \left(\left(1-\frac{1}{\sqrt{3}}\right) a, \frac{2}{3}\left(1-\frac{1}{\sqrt{3}}\right) a\right),$$

$$P_2 : \left(\frac{3}{13}\left(-1+3\sqrt{3}\right) a, \frac{2}{13}\left(1-3\sqrt{3}\right) a\right),$$

$$P_3 : \left(\frac{3a}{4}, \frac{a}{2}\right).$$

Therefore the area of triangle $P_1P_2P_3$ equals

$$\frac{1}{2}\det\begin{pmatrix} \left(1-\frac{1}{\sqrt{3}}\right) a & \frac{2}{3}\left(1-\frac{1}{\sqrt{3}}\right) a & 1 \\ \frac{3}{13}\left(-1+3\sqrt{3}\right) a & \frac{2}{13}\left(1-3\sqrt{3}\right) a & 1 \\ \frac{3a}{4} & \frac{a}{2} & 1 \end{pmatrix} = \frac{1}{2}\left(1-\frac{1}{\sqrt{3}}\right) a^2 = \frac{1}{4}ab$$

which is half of the area of triangle RST.

Case 2: $b = 2\left(1+\frac{1}{\sqrt{3}}\right) a$

The angle at R equals $\arctan\left(\frac{b}{a}\right) = \arctan 2\left(1+\frac{1}{\sqrt{3}}\right) \approx 72.412^0$, and the three points P_1, P_2, P_3 are:

$$P_1 : \left(\left(1+\frac{1}{\sqrt{3}}\right) a, \frac{2}{3}\left(1+\frac{1}{\sqrt{3}}\right) a\right),$$

$$P_2 : \left(-\frac{3}{13}\left(1+3\sqrt{3}\right) a, \frac{2}{13}\left(1+3\sqrt{3}\right) a\right),$$

$$P_3 : \left(\frac{3a}{4}, \frac{a}{2}\right).$$

Therefore the area of triangle $P_1P_2P_3$ equals

$$\frac{1}{2}\det\begin{pmatrix} \left(1+\frac{1}{\sqrt{3}}\right)a & \frac{2}{3}\left(1+\frac{1}{\sqrt{3}}\right)a & 1 \\ -\frac{3}{13}\left(1+3\sqrt{3}\right)a & \frac{2}{13}\left(1+3\sqrt{3}\right)a & 1 \\ \frac{3a}{4} & \frac{a}{2} & 1 \end{pmatrix} = \frac{1}{2}\left(1+\frac{1}{\sqrt{3}}\right)a^2 = \frac{1}{4}ab$$

which is again half of the area of triangle RST.

Also solved by the problem proposer.

• **5808** *Proposed by Mihaly Bencze, Braşov, Romania and Neculai Stanciu, Buzău, Romania.*

Solve the following equation for real x:

$$\left(x^2+1\right)\cdot\left[4^{x/\left(x^2+1\right)} - \log_4\left(x^4-4x+5\right)\right] = x^6+x^4-4x^3+5x^2-5x+5.$$

Solution 1 by Devis Alvarado, UNAH and UPNFM, Tegucigalpa, Hoduras.

Looking at the argument of the "log" function in the given equation, we must have $x^4-4x+5>0$. We do the following substitutions: Let $a=4^{\frac{x}{x^2+1}}$ and $b=x^4-4x+5$. If follows that $a>0$, $b>0$, and $x/\left(x^2+1\right)=\log_4(a)$.

Divide both sides of the given equation by (x^2+1) (safe to do so since $x^2+1>0$ for all $x\in\mathbb{R}$) to get

$$4^{\frac{x}{x^2+1}} - \log_4\left(x^4-4x+5\right) = \frac{x^6+x^4-4x^3+5x^2-5x+5}{x^2+1}.$$

Perform the polynomial division on the right-hand side and write

$$\frac{x^6+x^4-4x^3+5x^2-5x+5}{x^2+1} = x^4-4x+5-\frac{x}{x^2+1}.$$

Substituting the preceding back into the given equation, we have

$$4^{\frac{x}{x^2+1}} - \log_4\left(x^4-4x+5\right) = x^4-4x+5-\frac{x}{x^2+1}.$$

Now, apply the above substitutions involving a and b to get

$$a-\log_4(b) = b-\log_4(a),$$
$$a+\log_4(a) = b+\log_4(b).$$

Consider the function $f(t) = t + \log_4(t)$ for $t > 0$. Since $f'(t) = 1 + \dfrac{1}{t\ln(4)} > 0$ for $t > 0$, then f is strictly increasing over its domain. That is, f is a one-to-one function.

Since $a + \log_4(a) = b + \log_4(b)$, then $f(a) = f(b)$. And since f is one-to-one, then $a = b$. We write the latter as

$$4^{\frac{x}{x^2+1}} = x^4 - 4x + 5.$$

Observe

$$x^4 - 4x + 5 = (x^2 - 1)^2 + 2(x - 1)^2 + 2.$$

Since $(x^2 - 1)^2 \geqslant 0$ and $2(x - 1)^2 \geqslant 0$ for all $x \in \mathbb{R}$, we have:

$$x^4 - 4x + 5 = \left(x^2 - 1\right)^2 + 2(x - 1)^2 + 2 \geqslant 2, \quad \forall x \in \mathbb{R}$$

Therefore, the equation $4^{\frac{x}{x^2+1}} = x^4 - 4x + 5$ implies that:

$$4^{\frac{x}{x^2+1}} \geqslant 2$$

For $4^Y \geqslant 2$, where $Y = \dfrac{x}{x^2 + 1}$, we must have $Y \geqslant \log_4(2)$. Since $\log_4(2) = \dfrac{\log_2(2)}{\log_2(4)} = \dfrac{1}{2}$, we get:

$$\frac{x}{x^2 + 1} \geqslant \frac{1}{2}$$

$$2x \geqslant x^2 + 1$$

$$0 \geqslant x^2 - 2x + 1$$

$$0 \geqslant (x - 1)^2.$$

Since the square of a real number cannot be negative, the only way for $(x - 1)^2 \leqslant 0$ to hold is that $(x - 1)^2 = 0$.

$$(x - 1)^2 = 0 \implies x - 1 = 0 \implies x = 1$$

Therefore, the only real solution for the equation is $x = 1$.

Solution 2 by David A. Huckaby, Angelo State University, San Angelo, TX.

The right-hand side of the equation is

$$\begin{aligned}
x^6 + x^4 - 4x^3 + 5x^2 - 5x + 5 &= x^4(x^2 + 1) - 4x^3 + 4x^2 + x^2 - 5x + 4 + 1 \\
&= x^4(x^2 + 1) - 4x^3 - 5x + 4(x^2 + 1) + x^2 + 1 \\
&= (x^2 + 1)(x^4 + 4 + 1) - 4x^3 - 5x \\
&= (x^2 + 1)(x^4 + 5) - 4x^3 - 5x \\
&= (x^2 + 1)(x^4 - 4x + 5) + 4x(x^2 + 1) - 4x^3 - 5x \\
&= (x^2 + 1)(x^4 - 4x + 5) + 4x^3 - 4x^3 + 4x - 5x \\
&= (x^2 + 1)(x^4 - 4x + 5) - x,
\end{aligned}$$

so that the original equation can be written as

$$\left(x^2+1\right)\left[4^{x/(x^2+1)}-\log_4(x^4-4x+5)\right]=(x^2+1)(x^4-4x+5)-x.$$

Dividing both sides of this equation by x^2+1 gives

$$4^{x/(x^2+1)}-\log_4(x^4-4x+5)=x^4-4x+5-\frac{x}{x^2+1},$$

that is,

$$4^{x/(x^2+1)}+\frac{x}{x^2+1}=x^4-4x+5+\log_4(x^4-4x+5). \tag{2}$$

Denote the left-hand and right-hand sides of equation (2) by $f(x)$ and $g(x)$, respectively. By inspection, $f(1)=\dfrac{5}{2}=g(1)$, so that $x=1$ is a solution. We proceed to show that this is the only real solution.

As a preliminary step we note that the derivative of x^4-4x+5 is $4x^3-4=4(x^3-1)$, which is negative for $x<1$ and positive for $x>1$. So x^4-4x+5 has a global minimum of 2 at $x=1$ and hence is everywhere positive.

Therefore $g'(x)=4(x^3-1)\left(1+\dfrac{1}{(x^4-4x+5)\ln 4}\right)$ is negative for $x<1$ and positive for $x>1$.

Now $f'(x)=-\dfrac{(x^2-1)4^{x/(x^2+1)}\ln 4+1}{(x^2+1)^2}$ is negative for $x>1$. Since $g'(x)$ is positive for $x>1$, equation (2) has no solutions greater than $x=1$. Also, $f'(x)$ is positive for $0<x<1$, while $g'(x)$ is negative on this interval. So equation (2) has no solutions for $0<x<1$.

Now $f(0)=1$, and for $x<0$, $f(x)=4^{x/(x^2+1)}+\dfrac{x}{x^2+1}<4^{x/(x^2+1)}<4^0=1$. So since $g(0)=5+\log_4 5>1=f(0)$ and $g'(x)<0$ for $x<1$, equation (2) has no solutions for $x\leqslant 0$.

So the only real solution of equation (2), and hence of the original equation, is $x=1$.

Solution 3 by Albert Stadler, Herrliberg, Switzerland.

We note that $\left(x^2+1\right)\left(x^4-4x+5\right)=x^6+x^4-4x^3+5x^2-4x+5$. So dividing both sides of the given equation by x^2+1 we get

$$4^{\frac{x}{x^2+1}}+\frac{x}{x^2+1}=\log_4\left(x^4-4x+5\right)+x^4-4x+5.$$

The function $u\to 4^u+u$ is monotonically increasing in $(-\infty,\infty)$ and the function $u\to\log_4 u+u$ is monotonically increasing in $(0,\infty)$.
The minimum of $x\to x^4-4x+5$ equals 2 and is assumed at x=1, since $x^4-4x+5=\left(x^2-1\right)^2+2(x-1)^2+2$.

The maximum of $x \to \dfrac{x}{x^2+1}$ equals ½ and is assumed at x=1, since $\dfrac{1}{2} - \dfrac{x}{x^2+1} \geqslant \dfrac{(x-1)^2}{2\,(1+x^2)} \geqslant 0.$
So

$$\log_4 \left(x^4 - 4x + 5\right) \ + x^4 - 4x + 5 \geqslant \log_4 (2) \ + 2 = \frac{5}{2}$$

with equality if and only if x=1, and

$$4^{\frac{x}{x^2+1}} + \frac{x}{x^2+1} \leqslant \frac{5}{2}$$

with equality if and only if x=1. So the only real value of x that satisfies the given equation is x=1.

Solution 4 by Brian D. Beasley, Simpsonville, SC.

For all real x, we have $x^2 + 1 \neq 0$ and

$$\frac{x^6 + x^4 - 4x^3 + 5x^2 - 5x + 5}{x^2 + 1} = x^4 - 4x + 5 - \frac{x}{x^2 + 1}.$$

Then the original equation is equivalent to $f(x) = g(x)$, where

$$f(x) = 4^{x/(x^2+1)} + \frac{x}{x^2+1} \quad \text{and} \quad g(x) = x^4 - 4x + 5 + \log_4(x^4 - 4x + 5).$$

Since

$$f'(x) = \left[4^{x/(x^2+1)}(\ln 4) + 1\right] \frac{1 - x^2}{(x^2+1)^2},$$

f is decreasing on $(-\infty, -1) \cup (1, \infty)$ and increasing on $(-1, 1)$ with $\lim\limits_{x \to -\infty} f(x) = 1$. Hence f has an absolute maximum point at $(1, 5/2)$. Similarly, since

$$g'(x) = \left[1 + \frac{1}{(x^4 - 4x + 5)(\ln 4)}\right](4x^3 - 4)$$

and $x^4 - 4x + 5 > 0$ for all real x, g is decreasing on $(-\infty, 1)$ and increasing on $(1, \infty)$. Hence g has an absolute minimum point at $(1, 5/2)$. Thus the unique real solution of the original equation is $x = 1$.

Solution 5 by Michel Bataille, Rouen, France.

By long division we obtain $x^6 + x^4 - 4x^3 + 5x^2 - 5x + 5 = (x^2+1)(x^4 - 4x + 5) - x$ and therefore we may arrange the given equation into $f(x) = g(x)$ where the functions f and g are defined by

$$f(x) = \frac{x}{x^2+1} + 4^{x/(x^2+1)} = \frac{x}{x^2+1} + e^{(2x\ln(2))/(x^2+1)}, \qquad g(x) = \frac{\ln(x^4 - 4x + 5)}{2\ln(2)} + (x^4 - 4x + 5).$$

A quick study of the variations of $x \mapsto x^4 - 4x + 5$ shows that $x^4 - 4x + 5 \geqslant 2$ for all real x with equality if and only if $x = 1$. We readily deduce that $g(x) \geqslant \frac{5}{2}$ with equality if and only if $x = 1$.
Studying $x \mapsto \frac{x}{x^2+1}$, we similarly obtain that $-\frac{1}{2} \leqslant \frac{x}{x^2+1} \leqslant \frac{1}{2}$. Since $t \mapsto \phi(t) = t + e^{2t\ln(2)}$ is strictly increasing on $\mathbb{R}$, we deduce that $\phi(-1/2) \leqslant f(x) \leqslant \phi(1/2)$, that is, $0 \leqslant f(x) \leqslant \frac{5}{2}$, with $f(x) = \frac{5}{2}$ if and only if $x = 1$.
As a result, the equality $f(x) = g(x)$ occurs if and only if $f(x) = g(x) = \frac{5}{2}$ if and only if $x = 1$ and we conclude that $x = 1$ is the unique solution to the given equation.

Also solved by Bruno Salgueiro Fanego, Viveiro, Lugo, Spain, and the problem proposer.

• **5809** *Proposed by Toyesh Prakash Sharma (Student) St. C.F. Andrews School, Agra, India.*

(a) If $\frac{\ln b \ln c}{\text{bc}} + \frac{\ln a \ln c}{\text{ac}} + \frac{\ln b \ln a}{\text{ba}} \geqslant 1$ for $a, b, c > 0$ and $a, b, c \neq 1$, then show that

$$\frac{\ln^2 b + \ln^2 c}{bc \cdot (a - a^3)} + \frac{\ln^2 a + \ln^2 c}{ac \cdot (b - b^3)} + \frac{\ln^2 b + \ln^2 a}{ab \cdot (c - c^3)} \geqslant 3\sqrt{3}.$$

(b) If $0 < b \leqslant a < \frac{\pi}{2}$, then show that

$$\ln\left(\sqrt{\frac{\tan a}{\tan b}}\right) \geqslant a - b.$$

Solution 1 by Michel Bataille, Rouen, France.

(a) Let $X = \frac{\ln b \ln c}{bc} + \frac{\ln a \ln c}{ac} + \frac{\ln b \ln a}{ba}$ and $Y = \frac{\ln^2 b + \ln^2 c}{bc \cdot (a - a^3)} + \frac{\ln^2 a + \ln^2 c}{ac \cdot (b - b^3)} + \frac{\ln^2 b + \ln^2 a}{ab \cdot (c - c^3)}$.
The statement of the problem has to be modified: take $a = 2.05$, $b = c = \frac{1}{a}$; then, $X = (\ln a)^2(a^2 - 2) > 1$, but $Y = \frac{2a(\ln a)^2(2a^2 - 1)}{a^2 - 1} < 3\sqrt{3}$, as it is readily checked.
We prove that the inequality $Y \geqslant 3\sqrt{3}$ holds if $X \geqslant 1$ and $0 < a, b, c < 1$.
A quick study of the function $x \mapsto \frac{1}{x - x^3}$ shows that $\frac{1}{x - x^3} \geqslant \frac{3\sqrt{3}}{2}$ for $0 < x < 1$. We deduce that

$$Y \geqslant \frac{3\sqrt{3}}{2}\left(\frac{\ln^2 b + \ln^2 c}{bc} + \frac{\ln^2 a + \ln^2 c}{ac} + \frac{\ln^2 b + \ln^2 a}{ab}\right) \geqslant \frac{3\sqrt{3}}{2}\left(\frac{2\ln b \ln c}{bc} + \frac{2\ln a \ln c}{ac} + \frac{2\ln b \ln a}{ab}\right)$$

(since $x^2 + y^2 \geqslant 2xy$). Thus, $Y \geqslant 3\sqrt{3}X$ and $Y \geqslant 3\sqrt{3}$ follows (since $X \geqslant 1$).
Note that the inequality $X \geqslant 1$ only implies that at most one of the numbers a, b, c is greater than 1.

Indeed, if $a, b, c > 1$ then $X \leqslant \left(\frac{\ln a}{a}\right)^2 + \left(\frac{\ln b}{b}\right)^2 + \left(\frac{\ln c}{c}\right)^2 \leqslant \frac{3}{e^2} < 1$ and if $a, b > 1, c < 1$, say, then $X \leqslant \frac{\ln a \ln b}{ab} \leqslant \frac{1}{e^2} < 1$ (since $0 < \frac{\ln x}{x} \leqslant \frac{1}{e}$ when $x > 1$).

(b) Let $f(x) = \ln(\tan x)$ for $0 < x < \frac{\pi}{2}$. Then, we have $f'(x) = \frac{1 + \tan^2 x}{\tan x} = \frac{1}{\tan x} + \tan x \geqslant 2$ and by the Mean Value Theorem, $f(a) - f(b) = (a - b)f'(c)$ for some $c \in (b, a)$.
This yields $\ln(\tan a) - \ln(\tan b) \geqslant 2(a - b)$, which is clearly equivalent to the desired inequality.

Solution 2 by Moti Levy, Rehovot, Israel.

(a) One can check numerically that the statement of the problem is not true if a or b or c is greater than 1. So we will proceed assuming that $a, b, c \in (0, 1)$. Using this inequality

$$\frac{x^2 + y^2}{2} \geqslant xy,$$

we have

$$\frac{\ln^2 b + \ln^2 c}{2bc} + \frac{\ln^2 a + \ln^2 c}{2ac} + \frac{\ln^2 b + \ln^2 a}{2ba} \geqslant \frac{\ln b \ln c}{bc} + \frac{\ln a \ln c}{ac} + \frac{\ln b \ln a}{ba} \geqslant 1. \qquad (3)$$

Let us define for brevity,

$$\alpha := \frac{\ln^2 b + \ln^2 c}{2bc}, \quad \beta := \frac{\ln^2 a + \ln^2 c}{2ac}, \quad \gamma := \frac{\ln^2 b + \ln^2 a}{2ba}.$$

where

$$\alpha + \beta + \gamma = k \geqslant 1.$$

Define

$$f(x) := \frac{1}{x - x^3}.$$

Then the original inequality can be rephrased as follows

$$\alpha f(a) + \beta f(b) + \gamma f(c) \geqslant \frac{3\sqrt{3}}{2},$$

or

$$\frac{\alpha}{k} f(a) + \frac{\beta}{k} f(b) + \frac{\gamma}{k} f(c) \geqslant \frac{3\sqrt{3}}{2k}. \qquad (4)$$

Since

$$\frac{d^2 f(x)}{dx^2} = \frac{2\left(6x^4 - 3x^2 + 1\right)}{x^3\left(1 - x^2\right)^3} > 0, \quad \text{for } x \in (0, 1),$$

then $f(x)$ is convex in the interval $(0, 1)$. By Jensen's inequality,

$$\frac{\alpha}{k} f(a) + \frac{\beta}{k} f(b) + \frac{\gamma}{k} f(c) \geqslant f\left(\frac{\alpha}{k} a + \frac{\beta}{k} b + \frac{\gamma}{k} c\right) \qquad (5)$$

$$\frac{df(x)}{dx} = \frac{3x^2 - 1}{(x - x^3)^2},$$

hence $f(x)$ attains a minimum at $x = \dfrac{1}{\sqrt{3}}$ and the minimum is $\dfrac{3\sqrt{3}}{2}$.

Form $0 < a, b, c < 1$, it follows that

$$0 < \frac{\alpha}{k}a + \frac{\beta}{k}b + \frac{\gamma}{k}c < 1,$$

and

$$f\left(\frac{\alpha}{k}a + \frac{\beta}{k}b + \frac{\gamma}{k}c\right) \geqslant \frac{3\sqrt{3}}{2}. \tag{6}$$

It follows from (5) and (6) that

$$\frac{\alpha}{k}f(a) + \frac{\beta}{k}f(b) + \frac{\gamma}{k}f(c) \geqslant f\left(\frac{\alpha}{k}a + \frac{\beta}{k}b + \frac{\gamma}{k}c\right) \geqslant \frac{3\sqrt{3}}{2}. \tag{7}$$

(**b**) Let us rewrite the inequality as follows

$$\frac{1}{2}\ln\big(\tan(a)\big) - \frac{1}{2}\ln\big(\tan(b)\big) \geqslant a - b,$$

or

$$\left(\frac{1}{2}\ln\big(\tan(a)\big) - a\right) - \left(\frac{1}{2}\ln\big(\tan(b)\big) - b\right) \geqslant 0.$$

Now define the function

$$f(x) := \frac{1}{2}\ln\big(\tan(x)\big) - x.$$

The original inequality is equivalent to

$$f(a) \geqslant f(b).$$

$$\frac{df}{dx} = \frac{1}{\sin(2x)} - 1 \geqslant 0 \quad for \quad 0 < x < \frac{\pi}{2}.$$

It follows the $f(x)$ is monotonically increasing for $x \in \left(0, \dfrac{\pi}{2}\right)$, which implies $f(a) \geqslant f(b)$.

Also solved by Albert Stadler, Herrliberg, Switzerland and the problem proposer.

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Editor's Statement: It goes without saying that the problem proposers, as well as the solution proposers, are the *élan vital* of the Problems/Solutions Section of SSMJ. As the editor of this Section of the Journal, I consider myself fortunate to be in a position to receive, compile and organize a wealth of proposed ingenious problems and solutions intended for online publication. My unwavering gratitude goes to all the amazingly creative contributors. We come together from across continents because we find intellectual value, joy and satisfaction in mathematical problems, both in their creation as well as their solution. So that our collective efforts serve us well, I kindly ask all contributors to adhere to the following guidelines. As you peruse below, you may construe that the guidelines amount to a lot of work. But, as the samples show, there's not much to do. Your cooperation is much appreciated!

Keep in mind that the examples given below are your best guide!

Formats, Styles and Requirements

When submitting proposed problem(s) or solution(s), please send both **LaTeX** document and **pdf** document of your proposed problem(s) or solution(s). There are ways (discoverable from the internet) to convert from Word to proper LaTeX code. Porposals without a *proper* **LaTeX** document will not be published regrettably.

Regarding Proposed Solutions:

Below is the FILENAME format for all the documents of your proposed solution(s).

#ProblemNumber_FirstName_LastName_Solution_SSMJ

- FirstName stands for YOUR first name.

- LastName stands for YOUR last name.

Examples:

#1234_Max_Planck_Solution_SSMJ

#9876_Charles_Darwin_Solution_SSMJ

Please note that every problem number is *preceded* by the sign # .

All you have to do is copy the FILENAME format (or an example below it), paste it and then modify portions of it to your specs.

Please adopt the following structure, in the order shown, for the presentation of your solution:

1. On top of the first page of your solution, begin with the phrase:

"Proposed Solution to #**** SSMJ"

where the string of four astrisks represents the problem number.

2. On the second line, write

"Solution proposed by [your First Name, your Last Name]",

followed by your affiliation, city, country, all on the same linear string of words. Please see the example below. Make sure you do the same for your collaborator(s).

3. On a new line, state the problem proposer's name, affiliation, city and country, just as it appears published in the Problems/Solutions section.

4. On a new line below the above, write in bold type: "**Statement of the Problem**".

5. Below the latter, state the problem. Please make sure the statement of your problem (unlike the preceding item) is not in bold type.

6. Below the statement of the problem, write in bold type: "**Solution of the Problem**".

7. Below the latter, show the entire solution of the problem.

Here is a sample for the above-stated format for proposed solutions:

Proposed solution to #1234 SSMJ

Solution proposed by Emmy Noether, University of Göttingen, Lower Saxony, Germany.

Problem proposed by Isaac Newton, Trinity College, Cambridge, England.

Statement of the problem:

Compute $\sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}$.

Solution of the problem: . . . . . .

Regarding Proposed Problems:

For all your proposed problems, please adopt for all documents the following FILENAME format:

FirstName_LastName_ProposedProblem_SSMJ_YourGivenNumber_ProblemTitle

If you do not have a ProblemTitle, then leave that component as it already is (i.e., ProblemTitle).

The component YourGivenNumber is any UNIQUE 3-digit (or longer) number you like to give to your problem.

Examples:

Max_Planck_ProposedProblem_SSMJ_314_HarmonicPatterns

Charles_Darwin_ProposedProblem_SSMJ_358_ProblemTitle

Please adopt the following structure, in the order shown, for the presentation of your proposal:

1. On the top of first page of your proposal, begin with the phrase:

"Problem proposed to SSMJ"

2. On the second line, write

"Problem proposed by [your First Name, your Last Name]",

followed by your affiliation, city, country all on the same linear string of words. Please see the example below. Make sure you do the same for your collaborator(s) if any.

3. On a new line state the title of the problem, if any.

4. On a new line below the above, write in bold type: "**Statement of the Problem**".

5. Below the latter, state the problem. Please make sure the statement of your problem (unlike the preceding item) is not in bold type.

6. Below the statement of the problem, write in bold type: "**Solution of the Problem**".

7. Below the latter, show the entire solution of your problem.

Here is a sample for the above-stated format for proposed problems:

Problem proposed to SSMJ

Problem proposed by Isaac Newton, Trinity College, Cambridge, England.

Principia Mathematica (⟵ You may choose to not include a title.)

Statement of the problem:

Compute $\sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}$.

Solution of the problem: . . . . . .

♣ ♣ ♣ Thank You! ♣ ♣ ♣