Problems and Solutions

Albert Natian, Section Editor

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This section of the Journal offers readers an opportunity to exchange interesting mathematical problems and solutions. Please email them to Prof. Albert Natian at Department of Mathematics, Los Angeles Valley College. Please make sure every proposed problem or proposed solution is provided in both ***LaTeX*** and pdf documents. Please make sure your proposals adhere to **Formats, Styles and Requirements** noted below. Thank you!

To propose problems, email them to: problems4ssma@gmail.com

To propose solutions, email them to: solutions4ssma@gmail.com

Solutions to previously published problems can be seen at <www.ssma.org/publications>.

Solutions to the problems published in this issue should be submitted *before* June 1, 2026.

• **5830** *Proposed by Vasile Mircea Popa, Lucian Blaga University Sibiu, Romania.*

Prove the equality

$$\int_0^\infty \frac{x\sqrt{x}\ln(x)}{x^4+1}dx = \frac{1}{2}\pi^2\sin^3\left(\frac{\pi}{8}\right).$$

• **5831** *Proposed by Vasile Cirtoaje, Petroleum-Gas University of Ploiesti, Romania.*

Suppose $0 \leqslant a \leqslant b \leqslant c \leqslant d \leqslant e$ and $ab + bc + cd + de + ea = 5$. Prove that

$$(2a+1)^2 + (2b+1)^2 + (2c+1)^2 \leqslant 27.$$

• **5832** *Proposed by Toyesh Prakash Sharma, Agra College, Agra, India.*

If $z = z(x, y)$, then solve the differential equation

$$\left(y^2+1\right)\left(x^4+1\right)\frac{\partial z}{\partial x} = \left(x^2-1\right)\left(y^4+1\right)\frac{\partial z}{\partial y}.$$

• **5833** *Proposed by Paolo Perfetti, dipartimento di matematica Universitą di "Tor Vergata", Rome, Italy.*

Let $(a_k)_{k\geqslant 1}$ be a sequence of positive real numbers with $a_{k+1} \geqslant a_k(1 + c/k)^{-1}$ for some real $c > 0$. Let $S_n = \sum_{k=1}^{n} a_k$. Prove that the series $\sum_{k=1}^{\infty} \frac{S_k}{(S_{2k} - S_k)(k+1)^{1+c}(\ln(k+1))^2}$ converges.

• **5834** *Proposed by Daniel Sitaru, National Economic College "Theodor Costescu" Drobeta Turnu - Severin, Romania.*

Evaluate

$$\Omega = \int_0^{\pi} \frac{(x+2)\sin x}{1+3\cos^2 x}dx.$$

Solutions

To Formerly Published Problems

• **5810** *Proposed by Daniel Sitaru, National Economic College "Theodor Costescu" Drobeta Turnu - Severin, Romania.*

If $x, y, z \in (0,1)$ and $xy + yz + zx = 1$, then show that:

$$(x^4+y^4)(y^4+z^4)(z^4+x^4) \geqslant \frac{8}{729}$$

Solution 1 by Songen Tang (undergraduate student) and the Eagle Problem Solvers, Georgia Southern University, Savannah, GA and Statesboro, GA.

We begin by proving the following lemmas.

Lemma 1: If x, y, and z are real numbers, then

$$(x^4+y^4)(y^4+z^4)(z^4+x^4) \geqslant \frac{8}{9}(x^4+y^4+z^4)(x^4y^4+y^4z^4+z^4x^4).$$

Proof:

$$\begin{aligned}
&9(x^4+y^4)(y^4+z^4)(z^4+x^4) - 8(x^4+y^4+z^4)(x^4y^4+y^4z^4+z^4x^4)\\
&= 9(x^4y^4+y^4z^4+z^4x^4+y^8)(z^4+x^4) - 8(x^4+y^4+z^4)(x^4y^4+y^4z^4+z^4x^4)\\
&= 9y^8(z^4+x^4) + (x^4+y^4+z^4)(z^4+x^4) - 8y^4(x^4y^4+y^4z^4+z^4x^4)\\
&= y^8(z^4+x^4) + (x^4+y^4+z^4)(z^4+x^4) - 8x^4y^4z^4\\
&= (x^4+y^4)(y^4+z^4)(z^4+x^4) - 8x^4y^4z^4\\
&\geqslant (2x^2y^2)(2y^2z^2)(2z^2x^2) - 8x^4y^4z^4\\
&= 0.
\end{aligned}$$

Lemma 2: If a, b, and c are real numbers, then $a^2 + b^2 + c^2 \geqslant \dfrac{(a+b+c)^2}{3}$.

Proof:

$3(a^2+b^2+c^2)-(a+b+c)^2 = 2(a^2+b^2+c^2)-2(ab+bc+ca) = (a-b)^2+(b-c)^2+(c-a)^2 \geqslant 0.$

From Lemma **??**, using x^2, y^2, and z^2 for a, b, and c, we have

$$x^4 + y^4 + z^4 \geqslant \frac{\left(x^2+y^2+z^2\right)^2}{3} = \frac{1}{3}\left(\frac{x^2+y^2}{2} + \frac{y^2+z^2}{2} + \frac{z^2+x^2}{2}\right)^2 \geqslant \frac{1}{3}(xy+yz+zx)^2 = \frac{1}{3}.$$

Meanwhile, using Lemma **??** twice more, we get

$$x^4y^4 + y^4z^4 + z^4x^4 \geqslant \frac{\left(x^2y^2+y^2z^2+z^2x^2\right)^2}{3} \geqslant \frac{1}{3}\left(\frac{(xy+yz+zx)^2}{3}\right)^2 = \frac{1}{3}\cdot\frac{1}{9} = \frac{1}{27}.$$

Finally, using Lemma **??**, we have

$$(x^4+y^4)(y^4+z^4)(z^4+x^4) \geqslant \frac{8}{9}(x^4+y^4+z^4)(x^4y^4+y^4z^4+z^4x^4) \geqslant \frac{8}{9}\cdot\frac{1}{3}\cdot\frac{1}{27} = \frac{8}{729}.$$

Solution 2 by Henry Ricardo, Westchester Area Math Circle, Purchase, NY.

We apply the Power Mean inequality (P) and Maclaurin's inequality (M):

$$\left(\frac{x^4+y^4}{2}\right)^{1/4} \overset{\text{P}}{\geqslant} \frac{x+y}{2} \overset{\text{M}}{\geqslant} \sqrt{\frac{xy+yz+zx}{3}} \Rightarrow x^4+y^4 \geqslant 2\left(\frac{xy+yz+zx}{3}\right)^2 = \frac{2}{9}.$$

It follows that

$$(x^4+y^4)(y^4+z^4)(z^4+x^4) \geqslant \left(\frac{2}{9}\right)^3 = \frac{8}{729}.$$

We note that only $x, y, z > 0$ is required and that equality holds if and only if $x = y = z = 1/\sqrt{3}$.

Solution 3 by Michel Bataille, Rouen, France.

For any real numbers a, b, we have $2(a^2+b^2)-(a+b)^2 = (a-b)^2 \geqslant 0$, hence $a^4+b^4 \geqslant \dfrac{(a^2+b^2)^2}{2}$. It follows that the required inequality will certainly hold if

$$(x^2+y^2)(y^2+z^2)(z^2+x^2) \geqslant \frac{8}{27}.$$

Again we have $x^2+y^2 \geqslant \dfrac{(x+y)^2}{2}$, etc, hence the latter will hold if

$$(x+y)(y+z)(z+x) \geqslant \frac{8}{3\sqrt{3}}. \tag{1}$$

Now, from

$$(x+y+z)^2 = x^2+y^2+z^2+2(xy+yz+zx) \geqslant 3(xy+yz+zx) = 3$$

and

$$1 = xy+yz+zx \geqslant 3\sqrt[3]{xy\cdot yz\cdot zx} = 3\sqrt[3]{(xyz)^2}$$

we obtain $x+y+z \geqslant \sqrt{3}$ and $xyz \leqslant \dfrac{1}{3\sqrt{3}}$. We deduce that

$$(x+y)(y+z)(z+x) = (x+y+z)(xy+yz+zx) - xyz = x+y+z-xyz \geqslant \sqrt{3} - \frac{1}{3\sqrt{3}} = \frac{8}{3\sqrt{3}}.$$

Thus, (1) holds and the proof is complete.

Solution 4 by Moti Levy, Rehovot, Israel.

For $u, v \geqslant 0$,

$$\left(\frac{u^4+v^4}{2}\right)^{1/4} \geqslant \frac{u+v}{2} \quad \Longrightarrow \quad u^4+v^4 \geqslant \frac{(u+v)^4}{8}.$$

Applying this to the pairs (x,y), (y,z), (z,x) and multiplying yields

$$(x^4+y^4)(y^4+z^4)(z^4+x^4) \geqslant \frac{\big((x+y)(y+z)(z+x)\big)^4}{512}.$$

Thus it suffices to show

$$(x+y)(y+z)(z+x) \geqslant \frac{8}{3\sqrt{3}}. \tag{1}$$

$$(x+y)(y+z)(z+x) = (x+y+z)(xy+yz+zx) - xyz.$$

Write $p = x+y+z$ and $r = xyz$, and because $xy+yz+zx = 1$, then

$$(x+y)(y+z)(z+x) = p - r.$$

Let $x = \dfrac{a}{\sqrt{3}}$, $y = \dfrac{b}{\sqrt{3}}$, $z = \dfrac{c}{\sqrt{3}}$ so that $ab+bc+ca = 3$ and

$$p - r := \frac{1}{\sqrt{3}}\left(a+b+c-\frac{abc}{3}\right).$$

We claim that

$$a+b+c-\frac{abc}{3} \geqslant \frac{8}{3}. \tag{2}$$

Indeed, from $ab+bc+ca = 3$ we have

$$a^2+b^2+c^2 \geqslant ab+bc+ca = 3,$$

whence

$$(a+b+c)^2 = a^2+b^2+c^2+2(ab+bc+ca) \geqslant 3+6=9 \quad \Rightarrow$$

$$a+b+c \geqslant 3. \tag{3}$$

Also, by AM–GM applied to ab, bc, ca,

$$1 = \frac{ab+bc+ca}{3} \geqslant \sqrt[3]{(ab)(bc)(ca)} = (abc)^{2/3},$$

so

$$abc \leqslant 1. \tag{4}$$

Combining (3) and (4) gives,

$$p - r \geqslant \frac{1}{\sqrt{3}}\left(3 - \frac{1}{3}\right) = \frac{8}{3\sqrt{3}}.$$

Finally,

$$(x^4+y^4)(y^4+z^4)(z^4+x^4) \geqslant \frac{\left(\frac{8}{3\sqrt{3}}\right)^4}{512} = \frac{8}{729}.$$

Also solved by Albert Stadler, Herrliberg, Switzerland; Yunyong Zhang, Chinaunicom, Yunnan, China and the problem proposer.

• **5811** *Proposed by Prakash Pant, Mathematics Initiatives in Nepal, Bardiya, Nepal.*

Let a and b be real numbers such that $0 < a \leqslant b$. Prove that:

$$\int_a^b e^{x^{2023}} dx > \left(b^{\frac{2025}{2}} a^{\frac{2023}{2}} - a^{\frac{2025}{2}} b^{\frac{2023}{2}}\right).$$

Solution 1 by Albert Stadler, Herrliberg, Switzerland. The function $x \to e^{x^{2023}}$ is convex. So by Jensen's inequality,

$$\int_a^b e^{x^{2023}} dx = \int_a^b \frac{1}{2}\left(e^{x^{2023}} + e^{(a+b-x)^{2023}}\right) dx \geqslant e^{\left(\frac{a+b}{2}\right)^{2023}} \int_a^b dx = (b-a)\, e^{\left(\frac{a+b}{2}\right)^{2023}}$$

$$\geqslant (b-a)\, e^{\left(\sqrt{ab}\right)^{2023}} > (b-a)\left(\sqrt{ab}\right)^{2023} = b^{\frac{2025}{2}} a^{\frac{2023}{2}} - a^{\frac{2025}{2}} b^{\frac{2023}{2}}$$

since $e^x > x$ for $x > 0$.

Solution 2 by Daniel Văcaru, National Economic College „Maria Teiuleanu", Pitești, Romania.

One knows that

$$e^x \geqslant x+1, (\forall)\, x \in \mathbb{R} \tag{5}$$

It follows that

$$e^{x^{2023}} \geqslant x^{2023}+1, (\forall)\, x \in \mathbb{R} \tag{6}$$

Integrating in (2), one obtain

$$\int_a^b e^{x^{2023}} \mathrm{dx} \geqslant \int_a^b \left(x^{2023}+1\right) \mathrm{dx} = \frac{\mathrm{x}^{2024}}{2024}\Big|_{\mathrm{a}}^{\mathrm{b}} + \mathrm{x}\Big|_{\mathrm{a}}^{\mathrm{b}} \tag{7}$$

But

$$\frac{\mathrm{x}^{2024}}{2024}\Big|_{\mathrm{a}}^{\mathrm{b}} + \mathrm{x}\Big|_{\mathrm{a}}^{\mathrm{b}} = \frac{b^{2024}-a^{2024}}{2024} + (b-a) = (b-a)\cdot\left(\frac{b^{2023}+b^{2022}a+\ldots+ba^{2022}+a^{2023}}{2024}+1\right) \tag{8}$$

One must prove that

$$(b-a)\cdot\left(\frac{b^{2023}+b^{2022}a+\ldots+ba^{2022}+a^{2023}}{2024}+1\right) > a^{\frac{2023}{2}} b^{\frac{2023}{2}} (b-a) \,|:(b-a) \tag{9}$$

This relationship is equivalent to

$$\frac{b^{2023}+b^{2022}a+\ldots+ba^{2022}+a^{2023}}{2024}+1 > a^{\frac{2023}{2}} b^{\frac{2023}{2}} \tag{10}$$

By **AM-GM,** one has

$$\frac{b^{2023}+b^{2022}a+\ldots+ba^{2022}+a^{2023}}{2024} \overset{\mathbf{AM \geqslant GM}}{\geqslant} \sqrt[2024]{(ab)^{1+2+\cdots+2023}} = (ab)^{\frac{2023\cdot 2024}{2\cdot 2024}} = a^{\frac{2023}{2}} \cdot b^{\frac{2023}{2}}. \tag{11}$$

The relationship (6) follows.

Solution 3 by Songen Tang (undergraduate student) and the Eagle Problem Solvers, Georgia Southern University, Savannah, GA and Statesboro, GA.

Let $f(x) = e^{x^{2023}}$ and $t = \dfrac{a+b}{2}$. Since $f(x)$ is continuous and infinitely differentiable, then the second-order Taylor expansion of $f(x)$ on the interval $[a,b]$ is

$$f(x) = f(t) + f'(t)(x-t) + \frac{f''(\zeta)}{2}(x-t)^2,$$

with $a \leqslant \zeta \leqslant b$. Since $0 < a \leqslant \zeta \leqslant b$ and $f''(x) = 2023e^{x^{2023}}x^{2021}\left(2022+2023x^{2023}\right) > 0$ for $x > 0$, then $f(x) \geqslant f(t) + f'(t)(x-t)$, which means that

$$\int_a^b \left(f(t)+f'(t)(x-t)\right)dx = f(t)(b-a) + f'(t)\int_{\frac{a-b}{2}}^{\frac{b-a}{2}} x\,dx = f(t)(b-a) = e^{t^{2023}}(b-a).$$

Since $e^x > x$, then

$$\int_a^b e^{x^{2023}} dx > t^{2023}(b-a) = \left(\frac{a+b}{2}\right)^{2023}(b-a) \geqslant \left(\sqrt{ab}\right)^{2023}(b-a) = b^{\frac{2025}{2}}a^{\frac{2023}{2}} - a^{\frac{2025}{2}}b^{\frac{2023}{2}}.$$

Solution 4 by Michel Bataille, Rouen, France.

Let $n = 2023$ and $f(x) = e^{x^n}$. We have to prove that

$$\int_a^b f(x)\,dx > (b-a)(\sqrt{ab})^n. \tag{1}$$

From the second derivative $f''(x) = nf(x)x^{n-2}(n-1+nx^n) > 0$, we deduce that f is convex on the interval $[a,b]$. Therefore we may apply Hadamard's inequality $\int_a^b f(x)\,dx \geqslant (b-a)f\left(\frac{a+b}{2}\right)$, which here gives

$$\int_a^b f(x)\,dx \geqslant (b-a)e^{\left(\frac{a+b}{2}\right)^n}.$$

Now, we know that $e^x \geqslant x+1 > x$ for positive x and that $\frac{a+b}{2} \geqslant \sqrt{ab}$ for positive a, b. We deduce that

$$\int_a^b f(x)\,dx > (b-a)\left(\frac{a+b}{2}\right)^n \geqslant (b-a)(\sqrt{ab})^n$$

and (1) follows.

Solution 5 by Moti Levy, Rehovot, Israel.

Define $f(x) = e^{x^{2023}}$ on $(0,\infty)$. Since $x \mapsto x^{2023}$ is convex and e^x is convex and increasing, the composition f is strictly convex. By Jensen's inequality in integral form, for $a < b$,

$$\frac{1}{b-a}\int_a^b e^{x^{2023}}\,dx > e^{\left(\frac{a+b}{2}\right)^{2023}},$$

hence

$$\int_a^b e^{x^{2023}}\,dx > (b-a)\,e^{\left(\frac{a+b}{2}\right)^{2023}}. \tag{12}$$

It remains to show

$$e^{\left(\frac{a+b}{2}\right)^{2023}} > (ab)^{\frac{2023}{2}}. \tag{13}$$

Taking logarithms, this is equivalent to

$$\left(\frac{a+b}{2}\right)^{2023} > \frac{2023}{2}\ln(ab).$$

By AM–GM,

$$\frac{a+b}{2} \geqslant \sqrt{ab},$$

so

$$\left(\frac{a+b}{2}\right)^{2023} \geqslant (ab)^{\frac{2023}{2}}.$$

Thus it suffices to prove that

$$t^{\frac{2023}{2}} > \frac{2023}{2} \ln t \qquad (t > 0). \tag{14}$$

Define $g(t) = t^{\alpha} - \alpha \ln t$ with $\alpha = \frac{2023}{2}$. Then

$$g'(t) = \frac{\alpha}{t}\left(t^{\alpha} - 1\right), \qquad g''(t) = \alpha\left((\alpha-1)t^{\alpha-2} + \frac{1}{t^2}\right) > 0.$$

Thus g is strictly convex, with unique minimum at $t = 1$, where $g(1) = 1 > 0$. Hence $g(t) > 0$ for all $t > 0$, proving(14). This implies (13).

Substituting (13) into (12) yields

$$\int_a^b e^{x^{2023}}\, dx \; > \; (b-a)(ab)^{\frac{2023}{2}} = b^{\frac{2025}{2}} a^{\frac{2023}{2}} - a^{\frac{2025}{2}} b^{\frac{2023}{2}}.$$

Solution 6 by Saurab Banstola, Gandaki Boarding School, Pokhara, Nepal.

From the series expansion, $e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots$, we can say that for positive x, $e^x \geqslant 1 + x$ which implies $e^{x^{2023}} \geqslant 1 + x^{2023}$. Thus,

$$\int_a^b e^{x^{2023}} dx \geqslant \int_a^b 1 + x^{2023} dx = (b-a) + \frac{b^{2024} - a^{2024}}{2024}$$

$$= (b-a)\left(1 + \frac{\sum_{n=0}^{2023} b^{2023-n} a^n}{2024}\right)$$

Using AM-GM inequality, we have

$$\frac{\sum_{n=0}^{2023} b^{2023-n} a^n}{2024} \geqslant \sqrt[2024]{b^{2023+2022+\ldots+1} a^{1+2+\ldots+2023}} = \sqrt[2024]{b^{\frac{2023\times 2024}{2}} a^{\frac{2023\times 2024}{2}}} = b^{\frac{2023}{2}} a^{\frac{2023}{2}}$$

Moreover,

$$1 + \frac{\sum_{n=0}^{2023} b^{2023-n} a^n}{2024} > b^{\frac{2023}{2}} a^{\frac{2023}{2}}$$

Since $(b-a) > 0$, so,

$$\int_a^b e^{x^{2023}} dx \geqslant (b-a)\left(1 + \frac{\sum_{n=0}^{2023} b^{2023-n} a^n}{2024}\right) > (b-a) b^{\frac{2023}{2}} a^{\frac{2023}{2}} = b^{\frac{2025}{2}} a^{\frac{2023}{2}} - b^{\frac{2023}{2}} a^{\frac{2025}{2}}$$

This gives proof to the above statement.

Solution 7 by Yunyong Zhang, Chinaunicom, Yunnan, China.

Let $k = \dfrac{2023}{2}$, $LHS = a^k b^k (b-a)$,
$\because \quad e^{2023} > x^{2023}$,
$\therefore \quad LHS > \displaystyle\int_a^b x^{2023} = \frac{b^{2024} - a^{2024}}{2024}$,
that is, it suffices to prove: $b^{2024} - a^{2024} > 2024 a^{\frac{2023}{2}} b^{\frac{2023}{2}} (b-a)$,
let $t = \dfrac{b}{a} > 1$,
it remains to show: $h(t) = t^{2024} - 1 > 2024 t^{\frac{2023}{2}} (t-1)$,
$h(1) = 0$,
$h'(t) = 2024 t^{\frac{2023}{2}} - 1 \left(t^{\frac{2025}{2}} - \dfrac{2025}{2} t + \dfrac{2023}{2} \right)$.
Let $u = t^{\frac{2025}{2}} - \dfrac{2025}{2} t + \dfrac{2023}{2}$,
$u(1) = 0$,
$u'(t) = \dfrac{2025}{2} \left(t^{\frac{2023}{2}} - 1 \right)$,
when $t > 1, u'(t) > 0$,
$\therefore \quad h'(t) > 0 (t > 1)$,
$\therefore \quad$ when $t > 1 h(t)$ is strictly increasing,
$\therefore \quad t^{2024} - 1 > 2024 + \dfrac{2023}{2} (t-1)(t > 1)$.

Also solved by the problem proposer.

• **5812** *Proposed by D.M. Bătineţu-Giurgiu, "Matei Basarab" National College, Bucharest, Romania and Neculai Stanciu, "George Emil Palade" School, Buzău, Romania.*

The convergent sequence $(a_n)_{n \geqslant 1}$ is defined by $a_n = \displaystyle\sum_{k=1}^{n} \left(\frac{1}{4k-3} - \frac{1}{4k-1} \right)$, with $a = \displaystyle\lim_{n \to \infty} a_n$. Compute

$$\Lambda := \lim_{n \to \infty} n \left(a^{m+1} - \prod_{i=0}^{m} a_{n+i} \right).$$

Solution 1 by Yunyong Zhang, Chinaunicom, Yunnan, China.

$\because \quad$ when $a_n = \displaystyle\sum_{k=a}^{n} \left(\frac{1}{4k-3} - \frac{1}{4k-1} \right), \lim_{n \to \infty} a_n = a$,

$$\therefore \quad a = \frac{1}{4}\left[-\psi^{(0)}(a-\frac{3}{4}) + \psi^{(0)}(a-\frac{1}{4}) + \psi^{(0)}(n+\frac{1}{4}) - \psi^{(0)}(a+\frac{3}{4})\right],$$

when $n \to 160$,

$a = \frac{1}{4}(\psi^{(0)}(a-\frac{1}{4}) - \psi^{(0)}(a-\frac{3}{4}))$,

if $a_n = \sum_{k=1}^{n}\left(\frac{1}{4k-3} - \frac{1}{4k-1}\right)$, then

$a = \sum_{k=1}^{\infty}\left(\frac{1}{4k-3} - \frac{1}{4k-1}\right) = \frac{\pi}{4}$,

$\prod_{i=0}^{m} a_{n+i} = a_n a_{n+1} a_{n+2} \cdots a_{n+m}$,

let $a - a_n = \frac{\pi}{4} - a_n = r_n = \sum_{k=n+1}^{\infty}\left(\frac{1}{4k-3} - \frac{1}{4k-1}\right) \sim \frac{1}{8n} - \frac{1}{128n^3}$ be the remainder term,

$a - r_{n+i} = a_{n+i} = \frac{\pi}{4} - r_{n+1}$,

$$\therefore \quad \prod_{i=0}^{m} a_{n+i} = \prod_{i=0}^{m}(\frac{\pi}{4} - r_{n+i}) = a^{m+1}\prod_{i=0}^{m}(1 - \frac{r_{n+i}}{a}),$$

$$\therefore \prod_{i=0}^{m}(1-x_i) = 1 - \sum_{i=0}^{m} x_i + \sum_{0\leqslant i<j\leqslant m} x_i x_j \cdots,$$

$$\therefore \quad a^{m+1} - \prod_{i=0}^{\infty} a_{n+i} \sim a^{m+1}(\sum_{i=0}^{m}\frac{r_{n+i}}{a}) = a^m \sum_{i=0}^{m} r_{n+i} \sim a^m(\frac{m+1}{8n}),$$

$$\therefore \quad s = \lim_{n\to\infty} n\frac{a^m}{8n}(m+1) = \frac{m+1}{8}a^m = \frac{m+1}{8}(\frac{\pi}{4})^m.$$

Solution 2 by Wanlong Han, Xuchang University, Xuchang, China.

By Stolz theorem:

$$\lim_{n\to\infty}\frac{a_{n+1}-a_n}{1/n} = -\frac{1}{32} \Rightarrow \lim_{n\to\infty}\frac{a_n - a}{1/n} = -\frac{1}{32}.$$

So $a_n = a - \frac{1}{32n} + o\left(\frac{1}{n}\right)$ $(n\to\infty) \Rightarrow a_{n+k} = a - \frac{1}{32(n+k)} + o\left(\frac{1}{n+k}\right)$ $(n\to\infty)$. So $a_{n+k} = a - \frac{1}{32n} + o\left(\frac{1}{n}\right)$ $(n\to\infty)$. So

$$\prod_{i=0}^{m} a_{n+i} = \left(a - \frac{1}{32n} + o\left(\frac{1}{n}\right)\right)^{m+1} = a^{m+1}\left(1 - \frac{1}{32na} + o\left(\frac{1}{n}\right)\right)^{m+1} = a^{m+1} - \frac{(m+1)a^m}{32n} + o\left(\frac{1}{n}\right) \ (n\to\infty)$$

So $\Lambda = \lim_{n\to\infty} n\left(a^{m+1} - \prod_{i=0}^{m} a_{n+i}\right) = \frac{(m+1)a^m}{32}$.

Solution 3 by Ulrich Abel, Technische Hochschule Mittelhessen, Friedberg, Germany.

We have

$$4a_n = \sum_{k=1}^{n} \left(\frac{1}{k - 3/4} - \frac{1}{k - 1/4} \right) = \left(\Psi\left(n + \frac{1}{4}\right) - \Psi\left(\frac{1}{4}\right) \right) - \left(\Psi\left(n + \frac{3}{4}\right) - \Psi\left(\frac{3}{4}\right) \right),$$

where Ψ denotes the digamma function, defined as the derivative $\Psi(z) = d\left(\ln \Gamma(z)\right)/dz$ of the logarithm of the gamma function Γ. Using the well-known asymptotic relation $\Psi(z) = \ln z - 1/(2z) + O\left(z^{-2}\right)$ as $z \to \infty$, we infer that

$$4a_n = A + \Psi\left(n + \frac{1}{4}\right) - \Psi\left(n + \frac{3}{4}\right) = A + \ln \frac{n + \frac{1}{4}}{n + \frac{3}{4}} - \frac{1}{2} \left(\frac{1}{n + \frac{1}{4}} - \frac{1}{n + \frac{3}{4}} \right) + O\left(n^{-2}\right)$$

as $n \to \infty$, where

$$A := \Psi\left(\frac{3}{4}\right) - \Psi\left(\frac{1}{4}\right).$$

Since $\left(n + \frac{1}{4}\right)^{-1} - \left(n + \frac{3}{4}\right)^{-1} = O\left(n^{-2}\right)$ and

$$\ln\left(\left(n + \frac{1}{4}\right) / \left(n + \frac{3}{4}\right) \right) = \ln\left(1 + \frac{1}{4n}\right) - \ln\left(1 + \frac{3}{4n}\right) = \frac{1}{4n} - \frac{3}{4n} + O\left(n^{-2}\right) = -\frac{1}{2n} + O\left(n^{-2}\right)$$

as $n \to \infty$, it follows $a := \lim_{n\to\infty} a_n = A/4$ and the more precise relation

$$a_n = a - \frac{1}{8n} + O\left(n^{-2}\right) \qquad (n \to \infty).$$

Noting that, for fixed integer i, it holds $a_{n+i} - a_n = O\left(n^{-2}\right)$ as $n \to \infty$, we obtain

$$\begin{aligned} \prod_{i=0}^{m} a_{n+i} &= \prod_{i=0}^{m} \left(a - \frac{1}{8n} + O\left(n^{-2}\right) \right) = \left(a - \frac{1}{8n} \right)^{m+1} + O\left(n^{-2}\right) \\ &= a^{m+1} - \binom{m+1}{1} \frac{1}{8n} a^m + O\left(n^{-2}\right) \\ &= a^{m+1} - \frac{m+1}{8n} a^m + O\left(n^{-2}\right) \qquad (n \to \infty). \end{aligned}$$

This implies

$$\Lambda = \frac{m+1}{8} a^m.$$

Taking advantage of the reflection formula

$$\Psi(1 - z) = \Psi(z) + \pi \cot(\pi z)$$

with $z = 1/4$ we conclude that $a = (\pi/4)\cot(\pi/4) = \pi/4$. Finally, we obtain

$$\Lambda = \frac{m+1}{2^{2m+3}} \pi^m.$$

Solution 4 by Michel Bataille, Rouen, France.

We claim that $\Lambda = \dfrac{(m+1)a^m}{8}$.

Let $b_n = \sum_{k=1}^{n} \dfrac{(-1)^{k-1}}{2k-1}$ and $I_n = \int_0^{\pi/4} (\tan x)^n \, dx$. For $n \geqslant 0$, we have

$$I_n + I_{n+2} = \int_0^{\pi/4} (\tan x)^n (1 + (\tan x)^2) \, dx = \left[\frac{(\tan x)^{n+1}}{n+1} \right]_0^{\pi/4} = \frac{1}{n+1}.$$

We deduce that the decreasing, positive sequence $(I_n)_{n\geqslant 0}$ converges to 0 with $2I_{n+2} \leqslant \dfrac{1}{n+1} \leqslant 2I_n$, so that $I_n \sim \dfrac{1}{2n}$ as $n \to \infty$. In addition, an easy induction shows that

$$I_{2n} = (-1)^n \left(\frac{\pi}{4} - \sum_{k=1}^{n} \frac{(-1)^{k-1}}{2k-1} \right).$$

It follows that $\lim_{n\to\infty} b_n = \dfrac{\pi}{4}$ (a well-known result). Since $a_n = b_{2n}$ we have $a = \dfrac{\pi}{4}$ and

$$a - a_n = \frac{\pi}{4} - b_{2n} = \frac{\pi}{4} - \sum_{k=1}^{2n} \frac{(-1)^{k-1}}{2k-1} = I_{4n} \sim \frac{1}{8n}.$$

From $a_n = a - \dfrac{1}{8n} + o(1/n)$, we obtain $a_{n+i} = a - \dfrac{1}{8(n+i)} + o(1/n) = a - \dfrac{1}{8n} + o(1/n)$ if i is a nonnegative integer and therefore

$$\prod_{i=0}^{m} a_{n+i} = \prod_{i=0}^{m} (a - \frac{1}{8n} + o(1/n)) = a^{m+1} \left(1 - \frac{1}{8an} + o(1/n) \right)^{m+1} = a^{m+1} \left(1 - \frac{m+1}{8an} + o(1/n) \right)$$

so that

$$a^{m+1} - \prod_{i=0}^{m} a_{n+i} \sim \frac{a^m(m+1)}{8} \cdot \frac{1}{n}$$

as $n \to \infty$. The claim immediately follows.

Solution 5 by David Huckaby, Angelo State University, San Angelo, TX.

We first note that $a_n = \left(1 - \frac{1}{3}\right) + \left(\frac{1}{5} - \frac{1}{7}\right) + \cdots = \sum_{k=1}^{n} \dfrac{(-1)^{k+1}}{2k-1}$. So using the well-known result $\sum_{k=1}^{\infty} \dfrac{(-1)^{k+1}}{2k-1} = \dfrac{\pi}{4}$ (see, for example, `https://en.wikipedia.org/wiki/List_of_mathematical_series`), we have $a = \dfrac{\pi}{4}$.

Define $(b_n)_{n\geqslant 1}$ by $b_n = \dfrac{1}{n}$ and $(c_n)_{n\geqslant 1}$ by $c_n = a^{m+1} - \prod_{i=0}^{m} a_{n+i}$. Then

$$\begin{aligned}
\lim_{n\to\infty} \frac{c_{n+1}-c_n}{b_{n+1}-b_n} &= \lim_{n\to\infty} \frac{\left(a^{m+1} - \prod_{i=0}^{m} a_{n+1+i}\right) - \left(a^{m+1} - \prod_{i=0}^{m} a_{n+i}\right)}{\frac{1}{1+n} - \frac{1}{n}} \\
&= \lim_{n\to\infty} \frac{\prod_{i=0}^{m} a_{n+1+i} - \prod_{i=0}^{m} a_{n+i}}{\frac{1}{n} - \frac{1}{n+1}} = \lim_{n\to\infty} \frac{a_{n+1}\cdot a_{n+2}\cdots a_{n+m}(a_{n+1+m} - a_n)}{\frac{1}{n(n+1)}} \\
&= \lim_{n\to\infty} \frac{a_{n+1}\cdot a_{n+2}\cdots a_{n+m}\left(\sum_{k=n+1}^{n+1+m}\left(\frac{1}{4k-3} - \frac{1}{4k-1}\right)\right)}{\frac{1}{n(n+1)}} \\
&= \lim_{n\to\infty} \frac{a_{n+1}\cdot a_{n+2}\cdots a_{n+m}\left(\sum_{k=n+1}^{n+1+m}\frac{2}{(4k-3)(4k-1)}\right)}{\frac{1}{n(n+1)}} \\
&= \lim_{n\to\infty} a_{n+1}\cdot a_{n+2}\cdots a_{n+m}\left[n(n+1)\left(\sum_{k=n+1}^{n+1+m}\frac{2}{(4k-3)(4k-1)}\right)\right] \\
&= a^m\left[(m+1)\cdot\frac{1}{8}\right] = \left(\frac{\pi}{4}\right)^m\left(\frac{m+1}{8}\right).
\end{aligned}$$

Now $(b_n)_{n\geqslant 1}$ is strictly decreasing, and $\lim_{n\to\infty} b_n = 0$. Also, $\lim_{n\to\infty} c_n = \left(\frac{\pi}{4}\right)^{m+1} - \left(\frac{\pi}{4}\right)^{m+1} = 0$. So by the Stolz-Cesàro Theorem (see `https://en.wikipedia.org/wiki/StolzâĂŞCesaro_theorem`),

$$\lim_{n\to\infty}\frac{c_n}{b_n} = \lim_{n\to\infty}\frac{c_{n+1}-c_n}{b_{n+1}-b_n}.$$

That is,

$$\begin{aligned}
\Lambda := \lim_{n\to\infty} n\left(a^{m+1} - \prod_{i=0}^{m} a_{n+i}\right) &= \lim_{n\to\infty}\frac{\left(a^{m+1} - \prod_{i=0}^{m} a_{n+i}\right)}{\frac{1}{n}} \\
&= \lim_{n\to\infty}\frac{c_n}{b_n} = \lim_{n\to\infty}\frac{c_{n+1}-c_n}{b_{n+1}-b_n} = \left(\frac{\pi}{4}\right)^m\left(\frac{m+1}{8}\right).
\end{aligned}$$

Solution 6 by Albert Stadler, Herrliberg, Switzerland.

We have

$$a_n = \sum_{k=1}^{n}\left(\frac{1}{4k-3} - \frac{1}{4k-1}\right) = \sum_{k=1}^{\infty}\left(\frac{1}{4k-3} - \frac{1}{4k-1}\right) - \sum_{k=n+1}^{\infty}\left(\frac{1}{4k-3} - \frac{1}{4k-1}\right)$$

$$= \frac{\pi}{4} - \sum_{k=n+1}^{\infty} \frac{2}{(4k-3)(4k-1)}$$

$$= \frac{\pi}{4} - \sum_{k=n+1}^{\infty} \left(\frac{1}{8(k-1)} - \frac{1}{8k} \right) + \sum_{k=n+1}^{\infty} \left(\frac{1}{8(k-1)} - \frac{1}{8k} - \frac{2}{(4k-3)(4k-1)} \right)$$

$$= \frac{\pi}{4} - \sum_{k=n+1}^{\infty} \left(\frac{1}{8(k-1)} - \frac{1}{8k} \right) + \frac{3}{8} \sum_{k=n+1}^{\infty} \frac{1}{(k-1)k(4k-3)(4k-1)}$$

$$= \frac{\pi}{4} - \frac{1}{8n} + O\left(\frac{1}{n^3}\right).$$

As a result

$$\prod_{i=0}^{m} a_{n+i} = \left(\frac{\pi}{4}\right)^{m+1} \prod_{i=0}^{m} \left(1 - \frac{1}{2\pi(n+i)} + O\left(\frac{1}{(n+i)^3}\right)\right)$$

$$\prod_{i=0}^{m} a_{n+i} = \left(\frac{\pi}{4}\right)^{m+1} \prod_{i=0}^{m} \left(1 - \frac{1}{2\pi n} + \frac{i}{2\pi n^2} + O\left(\frac{i^2}{n^3}\right)\right)$$

$$= \left(\frac{\pi}{4}\right)^{m+1} \exp\left(\sum_{i=0}^{m} \log\left(1 - \frac{1}{2\pi n} + \frac{i}{2\pi n^2} + O\left(\frac{i^2}{n^3}\right)\right)\right)$$

$$= \left(\frac{\pi}{4}\right)^{m+1} \exp\left(\sum_{i=0}^{m} \left(-\frac{1}{2\pi n} - \frac{1}{8\pi^2 n^2} + \frac{i}{2\pi n^2}\right) + O_m\left(\frac{1}{n^3}\right)\right)$$

$$= \left(\frac{\pi}{4}\right)^{m+1} \exp\left(-\frac{m+1}{2\pi n} - \frac{m+1}{8\pi^2 n^2} + \frac{m(m+1)}{4\pi n^2} + O_m\left(\frac{1}{n^3}\right)\right)$$

$$= \left(\frac{\pi}{4}\right)^{m+1} \left(1 - \frac{m+1}{2\pi n} - \frac{m+1}{8\pi^2 n^2} + \frac{m(m+1)}{4\pi n^2} + \frac{1}{2}\left(-\frac{m+1}{2\pi n}\right)^2 + O_m\left(\frac{1}{n^3}\right)\right)$$

$$= \left(\frac{\pi}{4}\right)^{m+1} \left(1 - \frac{m+1}{2\pi n} + \frac{m(m+1)}{4\pi n^2} + \frac{m(m+1)}{8\pi^2 n^2} + O_m\left(\frac{1}{n^3}\right)\right),$$

where $O_m(.)$ means that the implied constant depends on m. This asymptotic expansion is stronger than what is actually required. We get in particular

$$= \lim_{n\to\infty} n \left(a^{m+1} - \prod_{i=0}^{m} a_{n+i}\right) = \left(\frac{\pi}{4}\right)^{m+1} \frac{m+1}{2\pi} = \frac{(m+1)\pi^m}{2^{2m+3}}.$$

Solution 7 by Moti Levy, Rehovot, Israel.

$$a = \sum_{k=1}^{\infty} \left(\frac{1}{4k-3} - \frac{1}{4k-1} \right) = \sum_{k=1}^{\infty} (-1)^{k-1} \frac{1}{2k-1} = \frac{\pi}{4}.$$

We write the tail $a - a_n$ as

$$a - a_n = \sum_{k=n+1}^{\infty} \frac{2}{(4k-3)(4k-1)}.$$

$$\frac{1}{4k-3} - \frac{1}{4k-1} = \frac{2}{16k^2 - 16k + 3} = \frac{2}{16k^2 \left(1 - \frac{1}{k} + \frac{3}{16k^2}\right)}.$$

For large k

$$\frac{1}{1 - \frac{1}{k} + \frac{3}{16k^2}} = 1 + \frac{1}{k} + \frac{1}{k^2} + O\left(\frac{1}{k^2}\right),$$

hence,

$$\frac{2}{(4k-3)(4k-1)} = \frac{1}{8k^2} + \frac{1}{8k^3} + O\left(\frac{1}{k^4}\right).$$

Standard tails are

$$\sum_{k=n+1}^{\infty} \frac{1}{k^2} = \frac{1}{n} - \frac{1}{2n^2} + O\left(\frac{1}{n^3}\right), \quad \sum_{k=n+1}^{\infty} \frac{1}{k^3} = \frac{1}{2n^2} + O\left(\frac{1}{n^3}\right).$$

Hence

$$a - a_n = \frac{1}{8}\left(\frac{1}{n} - \frac{1}{2n^2}\right) + \frac{1}{8}\left(\frac{1}{2n^2}\right) + O(n^{-3}) = \frac{1}{8n} + O(n^{-3}).$$

For fixed m and large n,

$$a_{n+i} = a - \frac{1}{8(n+i)} + O(n^{-3}) \qquad (0 \leqslant i \leqslant m).$$

Thus

$$\begin{aligned} \prod_{i=0}^{m} a_{n+i} &= a^{m+1} \prod_{i=0}^{m} \left(1 - \frac{1}{8a(n+i)} + O(n^{-3})\right) \\ &= a^{m+1} \left(1 - \frac{1}{8a} \sum_{i=0}^{m} \frac{1}{n+i} + O(n^{-2})\right). \end{aligned}$$

Since $\sum_{i=0}^{m} \frac{1}{n+i} = \frac{m+1}{n} + O(n^{-2})$,

$$a^{m+1} - \prod_{i=0}^{m} a_{n+i} = \frac{(m+1)a^m}{8n} + O(n^{-2}).$$

Multiplying by n and taking $n \to \infty$ gives

$$\Lambda = \frac{(m+1)a^m}{8}.$$

Also solved by the problem proposer.

• **5813** *Proposed by Ivan Hadinata, Jember, Indonesia.*

With $n = 2023$, determine the greater of A and B where

$$A = \underbrace{\int_0^1 \int_0^1 \int_0^1 \cdots \int_0^1 \int_0^1}_{n \text{ times}} \left(\prod_{j=1}^{n} \frac{1 - x_j^2}{1 + x_j^2} \right) \prod_{j=1}^{n} dx_j,$$

$$B = \int_0^1 \left(\frac{1 - x^2}{1 + x^2} \right)^n dx.$$

Solution 1 by Ulrich Abel, Technische Hochschule Mittelhessen, Friedberg, Germany.

Let n be a positive integer. By Jensen's inequality, we have

$$A = \prod_{j=1}^{n} \left(\int_0^1 \frac{1 - x_j^2}{1 + x_j^2} dx_j \right) = \left(\int_0^1 \frac{1 - x^2}{1 + x^2} dx \right)^n = f\left(\int_0^1 g(x)\, dx \right) \leqslant \left(\int_0^1 f\left(g(x) \right) dx \right) = B,$$

because $f(x) := x^n$ defines a convex function on $[0,1]$ and $g(x) := \left(1 - x^2\right) / \left(1 + x^2\right) \in [0,1]$ for all $x \in [0,1]$.

Solution 2 by Albert Stadler, Herrliberg, Switzerland.

By Hölder's inequality,

$$\sqrt[n]{A} = \int_0^1 \frac{1 - x^2}{1 + x^2} dx < \left(\int_0^1 1^{\frac{n}{n-1}} dx \right)^{1 - \frac{1}{n}} \left(\int_0^1 \left(\frac{1 - x^2}{1 + x^2} \right)^n dx \right)^{\frac{1}{n}} = \sqrt[n]{B}.$$

The inequality is strict, since 1 and $\left(\frac{1 - x^2}{1 + x^2} \right)^n$ are linearly independent for n>0. So A<B.

Mathematica finds

$$A = \left(\frac{\pi}{2} - 1 \right)^{2023} \approx 2.298297 \bullet 10^{-493},$$

$$B = \frac{1}{2}\sqrt{\pi}\text{Gamma[2024]Hypergeometric2F1Regularized}[\frac{1}{2}, 2023, \frac{3}{2} + 2023, -1] \approx 0.013932594.$$

Solution 3 by Michel Bataille, Rouen, France.

Note that $A = \left(\int_0^1 \frac{1-x^2}{1+x^2}\,dx\right)^n$. Holder's inequality gives

$$\left(\int_0^1 \left(\frac{1-x^2}{1+x^2}\right)^n dx\right)^{\frac{1}{n}} \left(\int_0^1 1 \cdot dx\right)^{\frac{n-1}{n}} > \int_0^1 \left(\left(\frac{1-x^2}{1+x^2}\right)^n\right)^{\frac{1}{n}} \cdot 1^{\frac{n-1}{n}}\,dx,$$

that is,

$$\left(\int_0^1 \left(\frac{1-x^2}{1+x^2}\right)^n dx\right)^{\frac{1}{n}} > \int_0^1 \frac{1-x^2}{1+x^2}\,dx.$$

The conclusion $B > A$ immediately follows.

Solution 4 by Moti Levy, Rehovot, Israel.

Because the integrand factors over independent variables:

$$A = \prod_{j=1}^{n} \int_0^1 \frac{1-x_j^2}{1+x_j^2}\,dx_j = \left(\int_0^1 \frac{1-x^2}{1+x^2}\,dx\right)^n.$$

Let $g(x) = \dfrac{1-x^2}{1+x^2}$ and $\phi(t) = t^n$ with $n > 1$. Since ϕ is convex on $[0,1]$, Jensen's inequality for integrals gives:

$$\phi\left(\int_0^1 g(x)\,dx\right) \leqslant \int_0^1 \phi(g(x))\,dx.$$

That is:

$$\left(\int_0^1 g(x)\,dx\right)^n \leqslant \int_0^1 g(x)^n\,dx.$$

The left side is A, the right side is B. Therefore $A < B$ (strict since g is not constant).
We conclude that $A < B$.

Also solved by Yunyong Zhang, Chinaunicom, Yunnan, China and the problem proposer.

• **5814** *Proposed by Problem Section Editor, Albert Natian.*

Where $\alpha > 0$, suppose the twice-differentiable funciton $\varphi : (-\infty, \infty) \to (0, \infty)$, with $\varphi(0) = 1/2$, satisfies the differential/functional equation

$$\alpha\varphi'(x) = \left[\alpha - e^x\varphi(-x)\right]\varphi(x)$$

for all x in $(-\infty, \infty)$. Find $y = \varphi(x)$.

Solution 1 by Songen Tang (undergraduate student) and the Eagle Problem Solvers, Georgia Southern University, Savannah, GA and Statesboro, GA.

Since $\alpha\varphi'(x) = \left[\alpha - e^x\varphi(-x)\right]\varphi(x)$, then $\alpha\left(\varphi(x) - \varphi'(x)\right) = e^x\varphi(x)\varphi(-x)$ and

$$-\alpha\left[e^{-x}\varphi(x)\right]' = \varphi(x)\varphi(-x). \tag{15}$$

Then

$$e^{-x}\varphi(x) = -\frac{1}{\alpha}\int \varphi(x)\varphi(-x)dx. \tag{16}$$

Substituting $y = -x$ into Equation (16) gives $e^y\varphi(-y) = \dfrac{1}{\alpha}\displaystyle\int \varphi(y)\varphi(-y)dx$, or

$$-e^x\varphi(-x) = -\frac{1}{\alpha}\int \varphi(x)\varphi(-x)dx. \tag{17}$$

Combining Equations (16) and (17), we get $-e^x\varphi(-x) = e^{-x}\varphi(x)$, which means that

$$\varphi(-x) = -e^{-2x}\varphi(x). \tag{18}$$

Substituting Equation (18) into Equation (15) gives

$$-\alpha\left[e^{-x}\varphi(x)\right]' = -e^{-2x}\varphi^2(x) = -\left[e^{-x}\varphi(x)\right]^2.$$

Let $f(x) = e^{-x}\varphi(x)$; then $\alpha\dfrac{df}{dx} = f^2$ and $\dfrac{df}{f^2} = \dfrac{dx}{\alpha}$. Integrating both sides gives

$$-\frac{1}{f(x)} = \frac{x}{\alpha} + C.$$

Since $f(0) = e^0\varphi(0) = \dfrac{1}{2}$, then $C = -2$ and $f(x) = e^{-x}\varphi(x) = \dfrac{\alpha}{2\alpha - x}$. Therefore,

$$y = \varphi(x) = \frac{\alpha e^x}{2\alpha - x}.$$

Solution 2 by Devis Alvarado, UNAH and UPNFM, Tegucigalpa, Hoduras.

Let us make the substitution $\phi(x) = e^{-x}\varphi(x)$ and $\phi(0) = \dfrac{1}{2}$, then

$$\begin{aligned} & \alpha\left[e^x\phi(x) + e^x\phi'(x)\right] = \left[\alpha - e^x\left(e^{-x}\phi(-x)\right)\right]e^x\phi(x) \\ \Longrightarrow \quad & \alpha\left[\phi(x) + \phi'(x)\right] = \left[\alpha - \phi(-x)\right]\phi(x) \\ \Longrightarrow \quad & \alpha\phi'(x) = -\phi(x)\phi(-x) \end{aligned}$$

Replacing $-x$ by x in the last equality we obtain

$$\alpha\phi'(-x) = -\phi(-x)\phi(x) = \alpha\phi'(x),$$

hence $\phi'(-x) = \phi'(x)$, that is, $\phi'(x)$ is an even function.

From this we have $\phi(-x) = -\phi(x) + l$, with l constant, and evaluating at $x = 0$ we get $l = 1$.

Substituting this last expression we obtain

$$\begin{aligned} & \alpha\phi'(x) = -\phi(x)\left(-\phi(x) + 1\right) = \phi(x)\left(\phi(x) - 1\right) \\ \Longrightarrow\ & \alpha\ln\left|\frac{\phi(x) - 1}{\phi(x)}\right| = x + c \\ \Longrightarrow\ & \alpha\ln\left|\frac{e^{-x}\varphi(x) - 1}{e^{-x}\varphi(x)}\right| = x + c \\ \Longrightarrow\ & \alpha\ln\left|\frac{\varphi(x) - e^x}{\varphi(x)}\right| = x + c \end{aligned}$$

To find c we substitute $x = 0$ and obtain $c = 0$. Then

$$\alpha\ln\left|\frac{\varphi(x) - e^x}{\varphi(x)}\right| = x \implies \varphi(x) = \frac{e^x}{1 \mp e^{\frac{x}{\alpha}}}$$

Since $\varphi(x) > 0, \ \ \forall x \in \mathbb{R}$, the negative solution is discarded and therefore the unique solution is

$$\varphi(x) = \frac{e^x}{1 + e^{\frac{x}{\alpha}}}.$$

Solution 3 by Michel Bataille, Rouen, France.

The unique solution is φ_0 defined by $\varphi_0(x) = \dfrac{e^x}{1 + e^{x/\alpha}}$.
First, this function φ_0 is a solution because for any real number x, in addition to $\varphi_0(x) > 0$ and $\varphi(0) = 1/2$, we have

$$\varphi_0(x)\varphi_0(-x) = \frac{1}{(1 + e^{x/\alpha})(1 + e^{-x/\alpha})} = \frac{e^{x/\alpha}}{(1 + e^{x/\alpha})^2}$$

and

$$\varphi_0(x) - \varphi_0'(x) = \frac{e^x(1/\alpha)e^{x/\alpha}}{(1+e^{x/\alpha})^2}$$

so that $\alpha(\varphi_0(x) - \varphi_0'(x)) = e^x\varphi_0(x)\varphi_0(-x)$.
Conversely, let φ be a solution. Then, from

$$(\varphi(x)e^{-x})' = e^{-x}(\varphi'(x) - \varphi(x)) = -\frac{\varphi(x)\varphi(-x)}{\alpha}$$

and

$$(\varphi(-x)e^x)' = e^x\varphi(-x) - e^x\varphi'(-x)) = e^x(\varphi(-x) - \varphi'(-x)) = \frac{\varphi(-x)\varphi(x)}{\alpha},$$

we deduce that $(\varphi(x)e^{-x})' + (\varphi(-x)e^x)' = 0$ for all x. It follows that $\varphi(x)e^{-x} + \varphi(-x)e^x$ is a constant; since $\varphi(0) = 1/2$, we see that $\varphi(x)e^{-x} + \varphi(-x)e^x = 1$ for all x. In consequence, the function u defined by $u(x) = \varphi(x)e^{-x} - \frac{1}{2}$ is odd and such that $\varphi(x) = e^x\left(u(x) + \frac{1}{2}\right)$ for all x. Note that we must have $u(x) + \frac{1}{2} > 0$.
Expressing the equality $\alpha(\varphi(x) - \varphi'(x)) = e^x\varphi(x)\varphi(-x)$ readily leads to $\alpha u'(x) = (u(x))^2 - \frac{1}{4}$. Setting $v(x) = u(x) + \frac{1}{2}$, we see that $v(x) > 0$ and $\alpha v'(x) + v(x) = (v(x))^2$. Therefore z defined by $z(x) = \frac{1}{v(x)}$ satisfies $\alpha z'(x) - z(x) = -1$. It follows that $z(x) = 1 + Ae^{x/\alpha}$ for some constant A. Note that $A \geqslant 0$ [$A < 0$ and $x \to \infty$ contradicts $z(x) > 0$ for all x]. Thus, $u(x) = \frac{1}{1+Ae^{x/\alpha}} - \frac{1}{2} = \frac{1-Ae^{x/\alpha}}{2(1+Ae^{x/\alpha})}$. The relation $u(x) + u(-x) = 0$ easily gives $A^2 = 1$, hence $A = 1$. Finally, we obtain $\varphi(x) = e^x\left(\frac{1-e^{x/\alpha}}{2(1+e^{x/\alpha})} + \frac{1}{2}\right) = \frac{e^x}{1+e^{x/\alpha}} = \varphi_0(x)$ for all x and the proof is complete.

Solution 4 by Moti Levy, Rehovot, Israel.

Let $u(x) = \varphi(x)$ and $v(x) = \varphi(-x)$. Then

$$\alpha u'(x) = \left(\alpha - e^x v(x)\right) u(x). \tag{19}$$

Replacing x by $-x$ yields

$$\alpha v'(x) = -\left(\alpha - e^{-x}u(x)\right) v(x).$$

Now compute:

$$\frac{d}{dx}\left(e^x v(x)\right) = e^x v(x) + e^x v'(x) = \frac{1}{\alpha}u(x)v(x),$$
$$\frac{d}{dx}\left(e^{-x}u(x)\right) = -e^{-x}u(x) + e^{-x}u'(x) = -\frac{1}{\alpha}u(x)v(x).$$

Thus,
$$\frac{d}{dx}\left(e^{x}v(x)+e^{-x}u(x)\right)=0,$$
It follows that
$$e^{x}\varphi(-x)+e^{-x}\varphi(x)=C.$$
At $x=0$, $\varphi(0)=\frac{1}{2}$, so $C=1$. Therefore,
$$e^{x}\varphi(-x)=1-e^{-x}\varphi(x), \tag{20}$$
Subsisting in (19) we get
$$\alpha\varphi^{'}(x)=\Big(\alpha-\big(1-e^{-x}\varphi(x)\big)\Big)\,\varphi(x)$$
or
$$\varphi^{'}(x)=\frac{\alpha-1}{\alpha}\varphi(x)+\frac{1}{\alpha}e^{-x}\varphi^{2}(x).$$
Let $z(x)=\dfrac{1}{\varphi(x)}$. Then $\varphi^{'}(x)=-\dfrac{z'}{z^{2}}$, and the equation becomes
$$-\frac{z'}{z^{2}}=\frac{\alpha-1}{\alpha}\frac{1}{z}+\frac{1}{\alpha}e^{-x}\frac{1}{z^{2}},$$
or
$$z'=-\frac{\alpha-1}{\alpha}z-\frac{1}{\alpha}e^{-x}.$$
The integrating factor to solve the linear differential equation is $e^{\frac{\alpha-1}{\alpha}x}$. Hence,
$$\left(z(x)e^{\frac{\alpha-1}{\alpha}x}\right)'=-\tfrac{1}{\alpha}e^{-x/\alpha}.$$
Integrating:
$$z(x)e^{\frac{\alpha-1}{\alpha}x}=e^{-x/\alpha}+C.$$
So
$$z(x)=e^{-x}+Ce^{-\frac{\alpha-1}{\alpha}x}.$$
At $x=0$, $\varphi(0)=\frac{1}{2}$, hence $z(0)=2$. Thus $2=1+C$, so $C=1$. Therefore
$$z(x)=e^{-x}+e^{-\frac{\alpha-1}{\alpha}x},\qquad \varphi(x)=\frac{1}{z(x)}.$$
We conclude that
$$\varphi(x)=\frac{e^{x}}{1+e^{\frac{x}{\alpha}}}.$$

Solution 5 by Yunyong Zhang, Chinaunicom, Yunnan, China.

Let $g(x)=\ln\varphi(x)$, $g'(x)=\dfrac{\varphi'(x)}{\varphi(x)}$, then

$ag'(x) = a - e^x\varphi(-x) \Rightarrow e^x\varphi(-x) = a\left(1 - g'(x)\right),$

similarly, $e^{-x}\varphi(x) = a\left(1 - g'(-x)\right),$

$\varphi(-x) = e^{g(-x)}, \varphi(x) = e^{g(x)},$

$e^{x+g(-x)} = a\left(1 - g'(x)\right)$, noted as $p(x)$,

$e^{-x+g(x)} = a\left(1 - g'(-x)\right)$, noted as $q(x)$,

then, $p'(x) = p(x)\left(1 - g'(-x)\right) = p(x)\left(\dfrac{q(x)}{a}\right),$

$q'(x) = q(x)\left(-1 + g'(x)\right) = q(x)\dfrac{-p(x)}{a},$

$\therefore \quad p'(x) + q'(x) = 0 \Rightarrow p(x) + q(x) = C(\text{constant}),$

$\because \quad p'(x) = \dfrac{1}{a}p(x)(C - p(x)) \Rightarrow \dfrac{\mathrm{d}p}{p(C-p)} = \dfrac{1}{a}\mathrm{d}x,$

$\displaystyle\int \frac{\mathrm{d}p}{p(C-p)} = \frac{1}{C}\ln\left|\frac{p}{C-p}\right| = \frac{x}{a} + D,$

$\dfrac{p}{C-p} = ke^{\frac{Cx}{a}}, \ (k = e^{CD}),$

$\therefore \quad p(x) = \dfrac{Cke^{\frac{Cx}{a}}}{1 + ke^{\frac{Cx}{a}}}$, then $q = \dfrac{C}{1 + ke^{\frac{Cx}{a}}},$

$\because \quad x > 0, \varphi(0) = \dfrac{1}{2},$

$\therefore \quad g(0) = \ln 2,$

$p(0) = e^{0+g(0)} = e^{-\ln 2} = \dfrac{1}{2},$

$q(0) = e^{0+g(0)} = \dfrac{1}{2}, C = 1,$

therefore,

$\dfrac{k}{1+k} = \dfrac{1}{2} \quad \Rightarrow \quad k = 1,$

$\therefore \quad p(x) = \dfrac{e^{\frac{x}{a}}}{1 + e^{\frac{x}{a}}},$

$q(x) = \dfrac{1}{1 + e^{\frac{x}{a}}},$

$\because \quad q(x) = e^{-x+g(x)},$

$\therefore \quad e^{g(x)} = q(x)e^x,$

$\varphi(x) = e^{g(x)} = q(x)e^x = \dfrac{e^x}{1 + e^{\frac{x}{a}}}$,

i.e. $y = \dfrac{e^x}{1 + e^{\frac{x}{a}}}$.

Solution 6 by Problem Section Editor, Albert Natian.

Define the function $f : (-\infty, \infty) \to (0, \infty)$ by setting

$$f(x) := \frac{\alpha e^x}{\varphi(x)} \tag{21}$$

for all x in $(-\infty, \infty)$. It is immediate that $f(0) = 2\alpha > 0$. From (21), we have

$$\varphi(x) = \frac{\alpha e^x}{f(x)} \tag{22}$$

and

$$\varphi'(x) = \alpha \cdot \frac{f(x) - f'(x)}{\left[f(x)\right]^2} \cdot e^x.$$

Using the latter two results, we re-write the given equation $\alpha\varphi'(x) = \left[\alpha - e^x\varphi(-x)\right]\varphi(x)$ as

$$\alpha \cdot \alpha \cdot \frac{f(x) - f'(x)}{\left[f(x)\right]^2} \cdot e^x = \left[\alpha - e^x \cdot \frac{\alpha e^{-x}}{f(-x)}\right] \cdot \frac{\alpha e^x}{f(x)}$$

which simplifies to

$$f'(x) f(-x) = f(x). \tag{23}$$

We see that

$$f'(-x) \cdot f(x) = f(-x)$$

$$f'(-x) = \frac{f(-x)}{f(x)}$$

$$f'(x) = \frac{f(x)}{f(-x)}.$$

Multiplying the latter two results, we see that

$$f'(x) \cdot f'(-x) = 1. \tag{24}$$

Now

$$\frac{d}{dx}\left(f'(x) \cdot f(-x)\right) = \frac{d}{dx}\left(f(x)\right)$$

$$f''(x) \cdot f(-x) - f'(x) \cdot f'(-x) = f'(x).$$

By virtue of the result (24), the latter can be simplified to

$$f''(x) \cdot f(-x) - 1 = f'(x).$$

Multiplying both sides of the latter by $f'(x)$, we have

$$f''(x) \cdot f'(x) \cdot f(-x) - f'(x) = \left[f'(x)\right]^2.$$

By equation (23), the latter implies

$$f''(x) \cdot f(x) - f'(x) = \left[f'(x)\right]^2$$
$$y''y - y' = \left(y'\right)^2$$
$$y''y - \left(y'\right)^2 = y'$$
$$\frac{y''y - (y')^2}{y^2} = \frac{y'}{y^2}$$
$$\frac{d}{dx}\left(\frac{y'}{y}\right) = \frac{d}{dx}\left(-\frac{1}{y} + m\right)$$
$$\frac{y'}{y} = -\frac{1}{y} + m$$
$$y' = my - 1$$
$$y' = m\left(y - \frac{1}{m}\right)$$
$$\frac{y'}{y - 1/m} = m$$
$$\frac{d}{dx}\ln\left(y - 1/m\right) = \frac{d}{dx}(mx + n)$$
$$\ln\left(y - \frac{1}{m}\right) = mx + n$$
$$y - \frac{1}{m} = e^{mx+n}$$
$$y = \frac{1}{m} + e^{mx+n}$$
$$f(x) = \frac{1}{m} + e^{mx+n}$$
$$f'(x) = me^{mx+n}$$
$$f(-x) = \frac{1}{m} + e^{-mx+n}.$$

Inserting the latter three results into equation (23); namely, $f'(x) \cdot f(-x) = f(x)$, we get

$$me^{mx+n} \cdot \left(\frac{1}{m} + e^{-mx+n}\right) = \frac{1}{m} + e^{mx+n}$$

$$
\begin{aligned}
me^{mx+n} \cdot \left(1 + me^{-mx+n}\right) &= 1 + me^{mx+n} \\
me^{mx+n} + m^2 e^{2n} &= 1 + me^{mx+n} \\
me^{mx+n} + m^2 e^{2n} &= 1 + me^{mx+n} \\
(me^n)^2 &= 1 \\
me^n &= \pm 1.
\end{aligned}
$$

If $me^n = -1$, then $m < 0$, in which case $f'(x) < 0$, which would contradict equation (23); namely, $f'(x) \cdot f(-x) = f(x)$. Thus $me^n = 1$, and so $e^n = 1/m$, which allows us to write

$$
f(x) = \frac{1}{m} + e^{mx+n} = \frac{1}{m} + e^n \cdot e^{mx} = \frac{1}{m} + \frac{1}{m} \cdot e^{mx}.
$$

Since $f(0) = 2\alpha$, then

$$
2\alpha = f(0) = \frac{1}{m} + \frac{1}{m} \cdot e^{m(0)} = \frac{1}{m} + \frac{1}{m} = \frac{2}{m}
$$

$$
\frac{1}{m} = \alpha \quad \text{and} \quad m = \alpha.
$$

Thus we conclude

$$
\forall x \in (-\infty, \infty): \quad f(x) = \alpha + \alpha e^{x/\alpha}.
$$

By equation (22); namely, $\varphi(x) = f(x) - \left(\alpha + \alpha e^{x/\alpha}\right)$, we see that

$$
\varphi(x) = \frac{\alpha e^x}{f(x)} \varphi(x) = \frac{\alpha e^x}{\alpha + \alpha e^{x/\alpha}} = \frac{e^x}{1 + e^{x/\alpha}}.
$$

Also solved by Bruno Salgueiro Fanego, Viveiro, Lugo, Spain; Albert Stadler, Herrliberg, Switzerland.

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Editor's Statement: It goes without saying that the problem proposers, as well as the solution proposers, are the *élan vital* of the Problems/Solutions Section of SSMJ. As the editor of this Section of the Journal, I consider myself fortunate to be in a position to receive, compile and organize a wealth of proposed ingenious problems and solutions intended for online publication. My unwavering gratitude goes to all the amazingly creative contributors. We come together from across continents because we find intellectual value, joy and satisfaction in mathematical problems, both in their creation as well as their solution. So that our collective efforts serve us well, I kindly ask all contributors to adhere to the following guidelines. As you peruse below, you may construe that the guidelines amount to a lot of work. But, as the samples show, there's not much to do. Your cooperation is much appreciated!

Keep in mind that the examples given below are your best guide!

Formats, Styles and Requirements

When submitting proposed problem(s) or solution(s), please send both **LaTeX** document and **pdf** document of your proposed problem(s) or solution(s). There are ways (discoverable from the internet) to convert from Word to proper LaTeX code. Porposals without a *proper* **LaTeX** document will not be published regrettably.

Regarding Proposed Solutions:

Below is the FILENAME format for all the documents of your proposed solution(s).

#ProblemNumber_FirstName_LastName_Solution_SSMJ

- FirstName stands for YOUR first name.
- LastName stands for YOUR last name.

Examples:

#1234_Max_Planck_Solution_SSMJ

#9876_Charles_Darwin_Solution_SSMJ

Please note that every problem number is *preceded* by the sign # .

All you have to do is copy the FILENAME format (or an example below it), paste it and then modify portions of it to your specs.

Please adopt the following structure, in the order shown, for the presentation of your solution:

1. On top of the first page of your solution, begin with the phrase:

"Proposed Solution to #**** SSMJ"

where the string of four astrisks represents the problem number.

2. On the second line, write

"Solution proposed by [your First Name, your Last Name]",

followed by your affiliation, city, country, all on the same linear string of words. Please see the example below. Make sure you do the same for your collaborator(s).

3. On a new line, state the problem proposer's name, affiliation, city and country, just as it appears published in the Problems/Solutions section.

4. On a new line below the above, write in bold type: "**Statement of the Problem**".

5. Below the latter, state the problem. Please make sure the statement of your problem (unlike the preceding item) is not in bold type.

6. Below the statement of the problem, write in bold type: "**Solution of the Problem**".

7. Below the latter, show the entire solution of the problem.

Here is a sample for the above-stated format for proposed solutions:

Proposed solution to #1234 SSMJ

Solution proposed by Emmy Noether, University of Göttingen, Lower Saxony, Germany.

Problem proposed by Isaac Newton, Trinity College, Cambridge, England.

Statement of the problem:

Compute $\sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}.$

Solution of the problem: . . . . . .

Regarding Proposed Problems:

For all your proposed problems, please adopt for all documents the following FILENAME format:

FirstName_LastName_ProposedProblem_SSMJ_YourGivenNumber_ProblemTitle

If you do not have a ProblemTitle, then leave that component as it already is (i.e., ProblemTitle).

The component YourGivenNumber is any UNIQUE 3-digit (or longer) number you like to give to your problem.

Examples:

Max_Planck_ProposedProblem_SSMJ_314_HarmonicPatterns

Charles_Darwin_ProposedProblem_SSMJ_358_ProblemTitle

Please adopt the following structure, in the order shown, for the presentation of your proposal:

1. On the top of first page of your proposal, begin with the phrase:

"Problem proposed to SSMJ"

2. On the second line, write

"Problem proposed by [your First Name, your Last Name]",

followed by your affiliation, city, country all on the same linear string of words. Please see the example below. Make sure you do the same for your collaborator(s) if any.

3. On a new line state the title of the problem, if any.

4. On a new line below the above, write in bold type: "**Statement of the Problem**".

5. Below the latter, state the problem. Please make sure the statement of your problem (unlike the preceding item) is not in bold type.

6. Below the statement of the problem, write in bold type: "**Solution of the Problem**".

7. Below the latter, show the entire solution of your problem.

Here is a sample for the above-stated format for proposed problems:

Problem proposed to SSMJ

Problem proposed by Isaac Newton, Trinity College, Cambridge, England.

Principia Mathematica (⟵ You may choose to not include a title.)

Statement of the problem:

Compute $\displaystyle\sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}$.

Solution of the problem: . . . . . .

♣ ♣ ♣ Thank You! ♣ ♣ ♣