Problems and Solutions

Albert Natian, Section Editor

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This section of the Journal offers readers an opportunity to exchange interesting mathematical problems and solutions. Please email them to Prof. Albert Natian at Department of Mathematics, Los Angeles Valley College. Please make sure every proposed problem or proposed solution is provided in both ***LaTeX*** and pdf documents. Please make sure your proposals adhere to **Formats, Styles and Requirements** noted below. Thank you!

To propose problems, email them to: problems4ssma@gmail.com

To propose solutions, email them to: solutions4ssma@gmail.com

Solutions to the problems published in this issue should be submitted *before* August 1, 2026.

• **5835** *Proposed by D.M. Bătineţu-Giurgiu, Bucharest, Romania and Neculai Stanciu, “George Emil Palade” School, Buzău, Romania.*

Find the limit $L = \lim\limits_{n\to\infty} \left(\sqrt[n]{a}-1\right) \sqrt[n]{b_nF_n}$ where $a > 0$, F_n is the n-th Fibonacci number and where $(b_n)_{n\geqslant 1}$ is a positive real sequence with $\lim\limits_{n\to\infty} \dfrac{b_{n+1}}{nb_n} = \pi$.

• **5836** *Proposed by Daniel Sitaru, National Economic College “Theodor Costescu" Drobeta Turnu - Severin, Romania.*

Show that in any $\triangle ABC$ the following inequality holds:

$$\sin^4 A + \sin^4 B + \sin^4 C + \sin^4\left(\frac{\pi}{3}+A\right) + \sin^4\left(\frac{\pi}{3}+B\right) + \sin^4\left(\frac{\pi}{3}+C\right) \leqslant \frac{27}{8}.$$

• **5837** *Proposed by Jose Luis Diaz-Barrero, Barcelona, Spain.*

Find all triples (x, y, z) of real numbers that are solutions to the equation

$$\sqrt{11^x(13^y+17^z)} + \sqrt{13^y(17^z+11^x)} + \sqrt{17^z(11^x+13^y)} = \sqrt{2}(11^x+13^y+17^z).$$

• **5838** *Proposed by Michel Bataille, Rouen, France.*

Let n be a positive integer. Prove that

$$\sum_{k=1}^{n} \frac{1}{\sin^2\left(\frac{(2k-1)\pi}{4n+2}\right)} = 2n(n+1).$$

• **5839** *Proposed by Prakash Pant, The University of Vermont, Bardiya, Nepal.*

Find all 2 × 2 matrices M with real entries that satisfy the equation:

$$M^2 - 5M = \begin{pmatrix} 0 & 9 \\ 4 & 0 \end{pmatrix}.$$

Solutions

To Formerly Published Problems

• **5815** *Proposed by Goran Conar, Varaždin, Croatia.*

Let $a, x_1, x_2, \ldots, x_n$ be positive real numbers. Prove the following inequality:

$$\left(\sum_{i=1}^{n} \frac{a^{x_i}}{x_i}\right) \geqslant \left(\sum_{i=1}^{n} \frac{1}{x_i}\right) \cdot a^{\frac{n}{\sum_{i=1}^{n} \frac{1}{x_i}}}.$$

When does equality occur?

Solution 1 by Saurab Banstola, Gandaki Boarding School, Pokhara, Nepal.

We first rewrite the inequality in a form suitable for applying Jensen's inequality:

$$\frac{\displaystyle\sum_{i=1}^{n} \frac{1}{x_i} a^{x_i}}{\displaystyle\sum_{i=1}^{n} \frac{1}{x_i}} \geqslant a^{\frac{n}{\sum_{i=1}^{n} \frac{1}{x_i}}}.$$

Let $f(t) = a^t$. For $t > 0$,

$$f''(t) = a^t (\ln a)^2 > 0,$$

so f is strictly convex. Define the weights $w(x) = \dfrac{1}{x}$, which are all positive. Jensen's inequality then gives

$$\frac{\sum_{i=1}^{n} w_i f(x_i)}{\sum_{i=1}^{n} w_i} \geqslant f\left(\frac{\sum_{i=1}^{n} w_i x_i}{\sum_{i=1}^{n} w_i}\right).$$

Substituting $f(t) = a^t$ and $w_i = \dfrac{1}{x_i}$ yields exactly the desired inequality.

Equality Case: Since f is strictly convex, equality in Jensen's inequality occurs only when

$$x_1 = x_2 = \cdots = x_n.$$

Solution 2 by Prakash Pant, The University of Vermont, Bardiya, Nepal.

Rewrite the problem as :

$$\frac{\sum_{i=1}^n \frac{1}{x_i} a^{x_i}}{\sum_{i=1}^n \frac{1}{x_i}} \geqslant a^{\frac{n}{\sum_{i=1}^n \frac{1}{x_i}}}$$

Consider function $f(x) = a^x$ and weights $w(x) = \dfrac{1}{x}$. Notice that $f(x) = a^x$ is convex whenever x is positive as $f''(x) = a^x \ln^2(x)$ is positive for positive x. Thus, using Jensen inequality, we can say that

$$\frac{\sum_{i=1}^n w_i f(x_i)}{\sum_{i=1}^n w_i} \geqslant f\left(\frac{\sum_{i=1}^n w_i x_i}{\sum_{i=1}^n w_i}\right)$$

Using $w_i = \dfrac{1}{x_i}$ and $f(x) = a^x$ gives us our required result. The equality occurs when $x_1 = x_2 = ... = x_n$.

Solution 3 by Michel Bataille, Rouen, France.

Clearly equality holds for all $x_1, x_2, \dots, x_n$ if $a = 1$. In what follows we suppose that $a \neq 1$.
We consider the function f defined on $(0, \infty)$ by $f(x) = a^x$. Since its second derivative ($f''(x) = (\ln a)^2 a^x$) is positive, the function f is strictly convex on the interval $(0, \infty)$. Jensen's inequality yields

$$\alpha_1 f(x_1) + \alpha_2 f(x_2) + \cdots + \alpha_n f(x_n) \geqslant (\alpha_1 + \alpha_2 + \cdots + \alpha_n) f\left(\frac{\alpha_1 x_1 + \alpha_2 x_2 + \cdots + \alpha_n x_n}{\alpha_1 + \alpha_2 + \cdots + \alpha_n}\right)$$

whenever $\alpha_1, \alpha_2, \dots, \alpha_n$ are nonnegative real numbers with $\alpha_1 + \alpha_2 + \cdots + \alpha_n \neq 0$, with equality if and only if $x_1 = x_2 = \cdots = x_n$.
Taking $\alpha_i = \dfrac{1}{x_i}$, the latter gives

$$\sum_{i=1}^n \frac{a^{x_i}}{x_i} \geqslant \left(\sum_{i=1}^n \frac{1}{x_i}\right) \cdot a^{\frac{n}{\sum_{i=1}^n \frac{1}{x_i}}}$$

with equality if and only if $x_1 = x_2 = \cdots = x_n$.

Solution 4 by Ángel Plaza, Universidad de Las Palmas de Gran Canaria, Spain.

The inequality may be written equivalentlu as

$$\sum_{i=1}^{n} \frac{\frac{1}{x_i}}{\sum_{k=1}^{n} \frac{1}{x_k}} \cdot a^{x_i} \geqslant a^{\frac{n}{\sum_{i=1}^{n} \frac{1}{x_i}}}.$$

The inequality follows because by the weighted AM-GM inequality.

$$\sum_{i=1}^{n} \frac{\frac{1}{x_i}}{\sum_{k=1}^{n} \frac{1}{x_k}} \cdot a^{x_i} \geqslant \sqrt[\sum_{k=1}^{n} \frac{1}{x_k}]{a^{\frac{x_i}{x_i}}} = a^{\frac{n}{\sum_{i=1}^{n} \frac{1}{x_i}}}.$$

Since, in the weighted AM-GM inequality, equality holds if and only if all the terms with non-zero weights are equal, it follows that for the proposed inequality, the equality occurs for $x_1 = x_2 = \ldots = x_n = x$. In this case, the inequality reads

$$n \cdot \frac{a^x}{x} = \frac{n}{x} \cdot a^{\frac{n}{n/x}}.$$

Solution 5 by Albert Stadler, Herrliberg, Switzerland.

The function $x \to a^x$ is convex, since its second derivative is positive. Hence, by Jensen's inequality, if $r_1, r_2, \ldots, r_n$ are positive numbers with $r_1 + r_2 + \cdots + r_n = 1$ then

$$\sum_{i=1}^{n} r_i a^{x_i} \geqslant a^{\sum_{i=1}^{n} r_i x_i}.$$

The claim of the problem statement follows by choosing $r_i = \left(x_i \sum_{i=1}^{n} \frac{1}{x_i}\right)^{-1}$. Equality occurs if and only if $x_1 = x_2 = \cdots = x_n$.

Also solved by the problem proposer.

• **5816** *Proposed by Daniel Sitaru, National Economic College "Theodor Costescu" Drobeta Turnu - Severin, Romania.*

For $0 < a \leqslant b$, prove: $\displaystyle\int_a^b \frac{xe^{x^2}\sqrt{e^{x^2}}}{e^{3x^2}+1}dx \leqslant \frac{1}{2}\tan^{-1}\left(\frac{e^{b^2}-e^{a^2}}{1+e^{a^2+b^2}}\right).$

Solution 1 by Saurab Banstola, Gandaki Boarding School, Pokhara, Nepal.

We begin by rewriting the right–hand side. Observe that

$$\frac{1}{2}\tan^{-1}\left(\frac{e^{b^2}-e^{a^2}}{1+e^{a^2+b^2}}\right) = \frac{1}{2}\big(\tan^{-1}(e^{b^2}) - \tan^{-1}(e^{a^2})\big).$$

This difference can be expressed as an integral:

$$\frac{1}{2}\tan^{-1}(e^{x^2})\Big|_a^b = \frac{1}{2}\int_a^b \frac{d}{dx}\big(\tan^{-1}(e^{x^2})\big)\,dx = \int_a^b \frac{xe^{x^2}}{1+e^{2x^2}}\,dx.$$

Thus, the original inequality reduces to comparing the two integrands:

$$\frac{xe^{x^2}\sqrt{e^{x^2}}}{e^{3x^2}+1} \leqslant \frac{xe^{x^2}}{1+e^{2x^2}} \quad\iff\quad \frac{e^{x^2/2}}{e^{3x^2}+1} \leqslant \frac{1}{1+e^{2x^2}}.$$

Since all quantities involved are positive, we may cross-multiply safely, obtaining the equivalent inequality

$$e^{\frac{x^2}{2}} + e^{\frac{5x^2}{2}} \leqslant 1 + e^{3x^2}. \tag{1}$$

To justify (1), we apply Karamata's inequality. Consider the two sequences

$$(3,0) \quad\text{and}\quad \left(\tfrac{5}{2},\tfrac{1}{2}\right).$$

It is easy to check that

$$(3,0) \succ \left(\tfrac{5}{2},\tfrac{1}{2}\right),$$

meaning that the former majorizes the latter. Because the function

$$f(t) = e^{tx^2}$$

is convex for every fixed x (as an exponential function), Karamata's inequality yields

$$f(3)+f(0) \geqslant f\left(\tfrac{5}{2}\right) + f\left(\tfrac{1}{2}\right).$$

Substituting back $f(t) = e^{tx^2}$ gives exactly inequality (1). This verifies the required comparison of integrands, and therefore the given integral inequality follows.

Solution 2 by Ángel Plaza, Universidad de Las Palmas de Gran Canaria, Spain.

Let I denote the integral. Then

$$I = \frac{1}{3}\int_a^b \frac{3xe^{\frac{3}{2}x^2}}{\left(e^{\frac{3}{2}x^2}\right)^2+1}dx = \frac{1}{3}\tan^{-1} e^{\frac{3}{2}x^2}\Big]_a^b = \frac{1}{3}\tan^{-1} e^{\frac{3}{2}b^2} - \frac{1}{3}\tan^{-1} e^{\frac{3}{2}a^2}.$$

Also, $\tan^{-1}\left(\dfrac{e^{b^2}-e^{a^2}}{1+e^{a^2+b^2}}\right) = \tan^{-1} e^{b^2} - \tan^{-1} e^{a^2}$. Let us change variables by doing $x = e^{a^2}$, $y = e^{b^2}$.
So the inequality to be proved reads now as:
For $1 < x \leqslant y$, prove

$$\frac{1}{2}\tan^{-1} x - \frac{1}{3}\tan^{-1} x^{\frac{3}{2}} \leqslant \frac{1}{2}\tan^{-1} y - \frac{1}{3}\tan^{-1} y^{\frac{3}{2}}.$$

Let f be the function defined by $f(x) = \frac{1}{2}\tan^{-1} x - \frac{1}{3}\tan^{-1} x^{\frac{3}{2}}$, for $x > 1$. Then, for $x > 1$:

$$f'(x) = \frac{\left(-1+\sqrt{x}\right)^2\left(1+\sqrt{x}+x+x^{3/2}+x^2\right)}{2(1+x)(1+x^2)(1-x+x^2)} > 0,$$

so function f is increasing for $x > 1$ and the problem is done.

Solution 3 by Péter Fülöp, Gyömrő, Hungary.

1. Regarding the left hand side:

(i) - Substitution $x^2 = t$:

(ii) - Substitution $e^t = z$:

(iii) - Substitution $z^3 = -r$:

$$\frac{1}{2}\int_{a^2}^{b^2} \frac{e^{\frac{3t}{2}}}{1+e^{3t}}dt = \frac{1}{2}\int_{e^{a^2}}^{e^{b^2}} \frac{z^{\frac{1}{2}}}{1+z^3}dz = \frac{i}{6}\int_{-e^{3b^2}}^{-e^{3a^2}} \frac{r^{-\frac{1}{2}}}{1-r}dr$$

(iv) - Introduction of the incomplete β function:

$$\frac{i}{6}\beta_{-e^{3a^2}}\left(\frac{1}{2},0\right) - \frac{i}{6}\beta_{-e^{3b^2}}\left(\frac{1}{2},0\right)$$

(v) - Applied the summation form of incomplete β function:

$$\frac{i}{6}\sum_{k=0}^{\infty}\frac{(-e^{3a^2})^{(k+\frac{1}{2})}}{k+\frac{1}{2}} - \frac{i}{6}\sum_{k=0}^{\infty}\frac{(-e^{3b^2})^{(k+\frac{1}{2})}}{k+\frac{1}{2}} =$$

$$-\frac{1}{3}\sum_{k=0}^{\infty}\frac{(-1)^k(e^{\frac{3a^2}{2}})^{(2k+1)}}{2k+1} + \frac{1}{3}\sum_{k=0}^{\infty}\frac{(-1)^k(e^{\frac{3b^2}{2}})^{(2k+1)}}{2k+1}$$

(vi) - Known that $\tan^{-1}(x) = \sum_{k=0}^{\infty}\frac{(-1)^k}{2k+1}x^{2k+1}$ we get:

$$\frac{1}{3}\tan^{-1}(e^{\frac{3b^2}{2}}) - \frac{1}{3}\tan^{-1}(e^{\frac{3a^2}{2}})$$

(vii) - Transformation back to integral form,

(viii) - Substitution $x = \frac{3t}{2}$:

$$LHS = \frac{1}{3}\int\limits_{e^{\frac{3a^2}{2}}}^{e^{\frac{3b^2}{2}}} \frac{1}{1+x^2}dx = \underline{\frac{1}{2}\int\limits_{e^{a^2}}^{e^{b^2}} \frac{1}{1+(\frac{3t}{2})^2}dt}$$

2. Regarding the left hand side:

(i) - Application of the following identity: $\tan^{-1}\left(\frac{x-y}{1+xy}\right) = \tan^{-1}(x) - \tan^{-1}(y)$

In this case

$$RHS = \frac{1}{2}\Big(\tan^{-1}(e^{b^2}) - \tan^{-1}(e^{a^2})\Big) = \underline{\frac{1}{2}\int\limits_{e^{a^2}}^{e^{b^2}} \frac{1}{1+t^2}dt}$$

3. The follownig inequality is true for the integrands:

$$0 \leqslant \frac{1}{1+(\frac{3t}{2})^2} \leqslant \frac{1}{1+t^2} \text{ for } \forall t$$

So it is also true for it's integrals in the same domain:

$$\frac{1}{2}\int\limits_{e^{a^2}}^{e^{b^2}} \frac{1}{1+(\frac{3t}{2})^2}dt \leqslant \frac{1}{2}\int\limits_{e^{a^2}}^{e^{b^2}} \frac{1}{1+t^2}dt$$

Statement is proved.

Solution 4 by Prakash Pant, The University of Vermont, Bardiya, Nepal.

Observe that :

$$\frac{1}{2}\tan^{-1}\left(\frac{e^{b^2}-e^{a^2}}{1+e^{a^2+b^2}}\right) = \frac{1}{2}\left(\tan^{-1}(e^{b^2}) - \tan^{-1}(e^{a^2})\right) = \frac{1}{2}(\tan^{-1}(e^{x^2}))|_a^b$$

$$= \frac{1}{2}\int_a^b \frac{d}{dx}(\tan^{-1}(e^{x^2}))dx = \int_a^b \frac{xe^{x^2}}{1+e^{2x^2}}dx$$

Now, the problem reduces to proving

$$\int_a^b \frac{xe^{x^2}\sqrt{e^{x^2}}}{e^{3x^2}+1}\,dx \leqslant \int_a^b \frac{xe^{x^2}}{1+e^{2x^2}}dx$$

for which it suffices to prove

$$\frac{\sqrt{e^{x^2}}}{e^{3x^2}+1} \leqslant \frac{1}{1+e^{2x^2}}$$

Since the denominators are clearly positive, we can cross multiply

$$e^{\frac{x^2}{2}} + e^{\frac{5x^2}{2}} \leqslant e^{3x^2} + 1 \tag{1}$$

To prove this statement, we use Karamata Inequality.
Since sequence $(3,0) \succ (\frac{5}{2},\frac{1}{2})$ and $f(a) = e^{ax^2}$ is convex whenever a is positive, thus by Karamata inequality,

$$f(3) + f(0) \geqslant f(\frac{5}{2}) + f(\frac{1}{2})$$

Solution 5 by David A. Huckaby, Angelo State University, San Angelo, TX.

Both sides of the inequality are 0 when $a = b$, so we assume $a < b$. The left-hand side of the inequality is

$$\int_a^b \frac{xe^{x^2}\sqrt{e^{x^2}}}{e^{3x^2}+1}\,dx = \int_a^b \frac{x(e^{x^2})^{\frac{3}{2}}}{e^{3x^2}+1}\,dx = \int_a^b \frac{x(e^{(\sqrt{3}x)^2})^{\frac{1}{2}}}{e^{(\sqrt{3}x)^2}+1}\,dx.$$

Let $u = e^{(\sqrt{3}x)^2}$, so that $x = \sqrt{\frac{1}{3}\ln u}$. Then $du = e^{(\sqrt{3}x)^2}\cdot 2(\sqrt{3}x)\cdot\sqrt{3}\,dx = 6\sqrt{\frac{1}{3}\ln u}\cdot u\,dx$, so that $dx = \dfrac{du}{6u\sqrt{\frac{1}{3}\ln u}}$. The integral is then

$$\int_{e^{(\sqrt{3}a)^2}}^{e^{(\sqrt{3}b)^2}} \frac{\sqrt{\frac{1}{3}\ln u}\,u^{\frac{1}{2}}}{u+1}\cdot\frac{du}{6u\sqrt{\frac{1}{3}\ln u}} = \frac{1}{6}\int_{e^{(\sqrt{3}a)^2}}^{e^{(\sqrt{3}b)^2}} \frac{du}{u^{\frac{1}{2}}(u+1)} = \frac{1}{6}\int_{e^{(\sqrt{3}a)^2}}^{e^{(\sqrt{3}b)^2}} \frac{u^{-\frac{1}{2}}\,du}{(u^{\frac{1}{2}})^2+1}$$

$$= \frac{1}{6}\left[2\tan^{-1}(u^{\frac{1}{2}})\right]_{e^{(\sqrt{3}a)^2}}^{e^{(\sqrt{3}b)^2}} = \frac{1}{3}\left[\tan^{-1}(e^{(\sqrt{3}b)^2})^{\frac{1}{2}} - \tan^{-1}(e^{(\sqrt{3}a)^2})^{\frac{1}{2}}\right]$$

$$= \frac{1}{3}(\tan^{-1}e^{\frac{3}{2}b^2} - \tan^{-1}e^{\frac{3}{2}a^2}).$$

Now from the identity $\tan^{-1}x - \tan^{-1}y = \dfrac{x-y}{1+xy}$, the right-hand side of the original inequality is $\dfrac{1}{2}\tan^{-1}\left(\dfrac{e^{b^2}-e^{a^2}}{1+e^{a^2+b^2}}\right) = \dfrac{1}{2}(\tan^{-1}e^{b^2} - \tan^{-1}e^{a^2})$. So the original inequality is

$$\frac{1}{3}(\tan^{-1}e^{\frac{3}{2}b^2} - \tan^{-1}e^{\frac{3}{2}a^2}) \leqslant \frac{1}{2}(\tan^{-1}e^{b^2} - \tan^{-1}e^{a^2}),$$

Solution 6 by Michel Bataille, Rouen, France. Let I be the integral on the left. The change of variables $x = \sqrt{\ln u}$ (so that $e^{x^2} = u$ and $dx = \dfrac{du}{2u\sqrt{\ln u}}$) leads to

$$I = \frac{1}{2}\int_{e^{a^2}}^{e^{b^2}} \frac{\sqrt{u}}{u^3+1}\,du = \frac{1}{2}\int_{e^{a^2}}^{e^{b^2}} \left(\frac{1}{1+u^2} - \frac{((\sqrt{u})^5-1)(\sqrt{u}-1)}{(1+u^2)(1+u^3)}\right)du.$$

Since $\sqrt{u} \geqslant 1$ when $u > 1$, hence when $e^{a^2} \leqslant u \leqslant e^{b^2}$, we see that

$$\frac{1}{1+u^2} - \frac{((\sqrt{u})^5-1)(\sqrt{u}-1)}{(1+u^2)(1+u^3)} \leqslant \frac{1}{1+u^2}$$

and it follows that

$$I \leqslant \frac{1}{2}\int_{e^{a^2}}^{e^{b^2}} \frac{du}{1+u^2} = \frac{1}{2}(\tan^{-1}(e^{b^2}) - \tan^{-1}(e^{a^2})) = \frac{1}{2}\tan^{-1}\left(\frac{e^{b^2}-e^{a^2}}{1+e^{a^2}e^{b^2}}\right) = \frac{1}{2}\tan^{-1}\left(\frac{e^{b^2}-e^{a^2}}{1+e^{a^2+b^2}}\right).$$

Also solved by Albert Stadler, Herrliberg, Switzerland: Daniel Văcaru, National Economic College „Maria Teiuleanu”, Pitești, Romania and the problem proposer.

• **5817** *Proposed by Michel Bataille, Rouen, France.*

Let $(F_n)_{n\geqslant 0}$ be the Fibonacci sequence defined by $F_0 = 0, F_1 = 1$ and $F_n = F_{n-1} + F_{n-2}$ for all $n \geqslant 2$. If m, n are integers such that $m \geqslant n \geqslant 0$, prove that

$$\sum_{j=0}^{n} \binom{m+1}{j}(2^{n+1} - 2^j)F_j = \sum_{j=0}^{n} \binom{m}{j}(2^{n+1}F_{j+2} - 2^jF_{j+3}).$$

Solution 1 by Ángel Plaza, Universidad de Las Palmas de Gran Canaria, Spain.

We will use the property $\binom{r}{k} = \binom{r-1}{k} + \binom{r-1}{k-1}$, for integer k. Applying this property to the binomial coefficient at the left hand side of the identity, it may be written as

$$\sum_{j=0}^{n} \binom{m}{j-1}(2^{n+1} - 2^j)F_j = \sum_{j=0}^{n} \binom{m}{j}(2^{n+1}(F_{j+2} - F_j) - 2^j(F_{j+3} - F_j)).$$

Now, since $F_{j+2} - F_j = F_{j+1}$ and $F_{j+3} - F_j = 2F_{j+1}$, the identity to prove reduces to

$$\sum_{j=0}^{n} \binom{m}{j-1}(2^{n+1} - 2^j)F_j = \sum_{j=0}^{n} \binom{m}{j}(2^{n+1} - 2^{j+1})F_{j+1}.$$

which it is true since for $j = 0$, $\binom{m}{-1} = 0$ at the left-hand side, and for $j = n$, $2^{n+1} - 2^{j+1} = 2^{n+1} - 2^{n+1} = 0$ at the right-hand side.

Solution 2 by Albert Stadler, Herrliberg, Switzerland.

Using the binomial identity

$$\binom{m+1}{j} = \binom{m}{j} + \binom{m}{j-1}$$

write the left hand side as

$$L := \sum_{j=0}^{n} \binom{m+1}{j} \left(2^{n+1} - 2^{j}\right) F_j = \underbrace{\sum_{j=0}^{n} \binom{m}{j} \left(2^{n+1} - 2^{j}\right) F_j}_{S_1} + \underbrace{\sum_{j=0}^{n} \binom{m}{j-1} \left(2^{n+1} - 2^{j}\right) F_j}_{S_2},$$

where $\binom{m}{-1} = 0$.

The right hand side is

$$R := \sum_{j=0}^{n} \binom{m}{j} \left(2^{n+1} F_{j+2} - 2^{j} F_{j+3}\right).$$

Consider $R - S_1$. Using the Fibonacci recurrence and the identity

$$F_{j+3} - F_j = \left(F_{j+2} + F_{j+1}\right) - F_j = 2F_{j+1},$$

we get

$$\begin{aligned} R - S_1 &= \sum_{j=0}^{n} \binom{m}{j} \left(2^{n+1} F_{j+2} - 2^{j} F_{j+3} - \left(2^{n+1} - 2^{j}\right) F_j\right) \\ &= \sum_{j=0}^{n} \binom{m}{j} \left(2^{n+1}\left(F_{j+2} - F_j\right) - 2^{j}\left(F_{j+3} - F_j\right)\right) \\ &= \sum_{j=0}^{n} \binom{m}{j} \left(2^{n+1} F_{j+1} - 2^{j}\left(2F_{j+1}\right)\right) = \sum_{j=0}^{n} \binom{m}{j} \left(2^{n+1} - 2^{j+1}\right) F_{j+1}. \end{aligned}$$

So

$$\begin{aligned} L - R &= S_2 - (R - S_1) \\ &= \sum_{j=0}^{n} \binom{m}{j-1} \left(2^{n+1} - 2^{j}\right) F_j - \sum_{j=0}^{n} \binom{m}{j} \left(2^{n+1} - 2^{j+1}\right) F_{j+1} \\ &= \sum_{j=1}^{n+1} \binom{m}{j-1} \left(2^{n+1} - 2^{j}\right) F_j - \sum_{j=0}^{n} \binom{m}{j} \left(2^{n+1} - 2^{j+1}\right) F_{j+1} \\ &= \sum_{j=0}^{n} \binom{m}{j} \left(2^{n+1} - 2^{j+1}\right) F_{j+1} - \sum_{j=0}^{n} \binom{m}{j} \left(2^{n+1} - 2^{j+1}\right) F_{j+1} = 0. \end{aligned}$$

Solution 3 by Péter Fülöp, Gyömrő, Hungary.

- Right hand side:

$$\sum_{j=0}^{n} \binom{m}{j} (2^{n+1} F_{j+2} - 2^{j} F_{j+3})$$

Applying the definition of the Fibonacci sequence: $F_{j+2} = F_{j+1} + F_j$ and $F_{j+3} = 2F_{j+1} + F_j$

$$\sum_{j=0}^{n} \binom{m}{j} \left[F_j(2^{n+1} - 2^j) + F_{j+1}(2^{n+1} - 2^{j+1}) \right]$$

$$RHS = \sum_{j=0}^{n} \binom{m}{j} F_j(2^{n+1} - 2^j) + \sum_{j=0}^{n} \binom{m}{j} F_{j+1}(2^{n+1} - 2^{j+1})$$

- Left hand side:

Applying the following identity of the binomial coefficients:

$$\binom{m+1}{j} = \binom{m}{j-1} + \binom{m}{j}$$

$$\sum_{j=0}^{n} \binom{m}{j-1} (2^{n+1} - 2^j) F_j + \sum_{j=0}^{n} \binom{m}{j} (2^{n+1} - 2^j) F_j$$

It can be realized that in case of the first sum at $j = 0$ the sum equals to zero. So the sum can be started from 1 and then reindex it from zero.

$$\sum_{j=1}^{n} \binom{m}{j-1} (2^{n+1} - 2^j) F_j = \sum_{j=0}^{n-1} \binom{m}{j} (2^{n+1} - 2^{j+1}) F_{j+1} = \underbrace{\sum_{j=0}^{n} \binom{m}{j} (2^{n+1} - 2^{j+1}) F_{j+1}}_{\text{In case of j=n the sum equals to zero}}$$

$$LHS = \sum_{j=0}^{n} \binom{m}{j} (2^{n+1} - 2^{j+1}) F_{j+1} + \sum_{j=0}^{n} \binom{m}{j} (2^{n+1} - 2^j) F_j$$

The statement is proved.

Solution 4 by Prakash Pant, The University of Vermont, Bardiya, Nepal.

$$= \sum_{j=0}^{n} \binom{m+1}{j} (2^{n+1} - 2^j) F_j$$

$$= \sum_{j=0}^{n} \left(\binom{m}{j} + \binom{m}{j-1} \right) (2^{n+1} - 2^j) F_j$$

$$= \sum_{j=0}^{n} \binom{m}{j} 2^{n+1} F_j + \sum_{j=0}^{n} \binom{m}{j-1} 2^{n+1} F_j - \sum_{j=0}^{n} \binom{m}{j} 2^j F_j - \sum_{j=0}^{n} \binom{m}{j-1} 2^j F_j$$

$$= \sum_{j=0}^{n} \binom{m}{j} 2^{n+1} F_j + \sum_{j=0}^{n} \binom{m}{j} 2^{n+1} F_{j+1} - \sum_{j=0}^{n} \binom{m}{j} 2^j F_j - \sum_{j=0}^{n} \binom{m}{j} 2^{j+1} F_{j+1}$$

Combining first two terms and last two terms

$$= \sum_{j=0}^{n} \binom{m}{j} 2^{n+1}(F_j + F_{j+1}) - \sum_{j=0}^{n} \binom{m}{j} 2^j (F_j + 2F_{j+1})$$

Since $F_j + F_{j+1} = F_{j+2}$ and $F_j + 2F_{j+1} = F_{j+2} + F_{j+1} = F_{j+3}$, we have

$$= \sum_{j=0}^{n} \binom{m}{j} 2^{n+1} F_{j+2} - \sum_{j=0}^{n} \binom{m}{j} 2^j F_{j+3}$$

proving the initial claim.

Solution 5 by Saurab Banstola, Gandaki Boarding School, Pokhara, Nepal.

Starting from the left-hand side,

$$\sum_{j=0}^{n} \binom{m+1}{j} (2^{n+1} - 2^j) F_j,$$

we first expand the binomial coefficient using

$$\binom{m+1}{j} = \binom{m}{j} + \binom{m}{j-1}.$$

Thus,

$$\sum_{j=0}^{n} \left(\binom{m}{j} + \binom{m}{j-1} \right) (2^{n+1} - 2^j) F_j.$$

Distributing the factors gives four sums:

$$\sum_{j=0}^{n} \binom{m}{j} 2^{n+1} F_j + \sum_{j=0}^{n} \binom{m}{j-1} 2^{n+1} F_j - \sum_{j=0}^{n} \binom{m}{j} 2^j F_j - \sum_{j=0}^{n} \binom{m}{j-1} 2^j F_j.$$

Shifting indices in the sums involving $\binom{m}{j-1}$ (i.e., letting $j \mapsto j+1$) transforms them into:

$$\sum_{j=0}^{n} \binom{m}{j} 2^{n+1} F_j + \sum_{j=0}^{n} \binom{m}{j} 2^{n+1} F_{j+1} - \sum_{j=0}^{n} \binom{m}{j} 2^j F_j - \sum_{j=0}^{n} \binom{m}{j} 2^{j+1} F_{j+1}.$$

Now we group the first pair of sums and the second pair:

$$\sum_{j=0}^{n} \binom{m}{j} 2^{n+1}(F_j + F_{j+1}) - \sum_{j=0}^{n} \binom{m}{j} 2^j (F_j + 2F_{j+1}).$$

Using Fibonacci identities,

$$F_{j+2} = F_j + F_{j+1}, \qquad F_{j+3} = F_{j+2} + F_{j+1} = F_j + 2F_{j+1},$$

each expression simplifies nicely:

$$\sum_{j=0}^{n}\binom{m}{j}2^{n+1}F_{j+2}-\sum_{j=0}^{n}\binom{m}{j}2^{j}F_{j+3}.$$

This matches exactly the right-hand side of the identity to be proved, completing the argument.

Also solved by the problem proposer.

• **5818** *Proposed by Paolo Perfetti, dipartimento di matematica Universitą di "Tor Vergata", Rome, Italy.*

Evaluate

$$\int_0^\infty \left(\arctan\frac{1}{1+z^2}\right)^2 dz.$$

Solution 1 by Albert Stadler, Herrliberg, Switzerland.

We will prove that

$$\int_0^\infty \left(\arctan\left(\frac{1}{1+z^2}\right)\right)^2 dz=\frac{\sqrt{\sqrt{2}-1}}{4\sqrt{2}}\pi^2+\sqrt{\frac{\sqrt{2}+1}{2}}\log\left(\frac{\sqrt{2}+1}{2\sqrt{2}}\right)\pi.$$

The Taylor expansion of $(\arctan x\ )^2$ is

$$(\arctan x\ )^2=\sum_{n=1}^{\infty}\left((-1)^{n-1}\frac{1}{n}\sum_{k=0}^{n-1}\frac{1}{2k+1}\right)x^{2n}.$$

(See for instance https://math.stackexchange.com/questions/1459045/how-do-you-write-the-taylor-series-for-arctanx2 for a complete derivation of this formula). We use

$$\sum_{k=0}^{n-1}\frac{1}{2k+1}=\sum_{k=0}^{n-1}\int_0^1 t^{2k}dt=\int_0^1\frac{1-t^{2n}}{1-t^2}dt$$

to derive

$$(\arctan x\ )^2=\sum_{n=1}^{\infty}\left((-1)^{n-1}\frac{1}{n}\int_0^1\frac{1-t^{2n}}{1-t^2}dt\right)x^{2n}=\int_0^1\frac{\log\left(1+x^2\right)\ -\log\left(1+x^2t^2\right)}{1-t^2}dt.$$

We next evaluate the integral $\int_0^\infty \log\left(1+\frac{x^2}{\left(1+z^2\right)^2}\right)dz$, where $0\leqslant x\leqslant 1$. Integration by parts gives

$$\int_0^\infty \log\left(1+\frac{x^2}{\left(1+z^2\right)^2}\right)dz=\int_0^\infty\frac{4x^2z^2}{\left(1+z^2\right)\left(1+x^2+2z^2+z^4\right)}dz$$

$$= \int_0^\infty \left(-\frac{4}{1+z^2} + \frac{4\left(1+x^2+z^2\right)}{1+x^2+2z^2+z^4} \right) dz = -2\pi + 2\int_{-\infty}^\infty \frac{x^2+1+z^2}{x^2+\left(1+z^2\right)^2} dz$$

The last integral can be evaluated by means of the residue calculus. Indeed, the integral equals 2πi times the sum of the residues at the poles of the integrand in the upper half plane. The poles are located at $\pm\sqrt{\pm ix-1}$, where the main branch of the square root is taken which is defined by

$$\sqrt{z} = \sqrt{|z|e^{\frac{1}{2}\arg z}}\ ,\ -\pi < \arg z\ < \pi.$$

The ones in the upper half plane are $\left\{-\sqrt{-ix-1},\ \sqrt{ix-1}\right\}$. So

$$\int_0^\infty \log\left(1+\frac{x^2}{\left(1+z^2\right)^2}\right) dz = -2\pi + 4\pi i \sum_{z\in\left\{-\sqrt{-ix-1},\ \sqrt{ix-1}\right\}} \frac{x^2+1+z^2}{4z\left(1+z^2\right)}$$

$$= -2\pi + 4\pi i\left(\frac{x^2-ix}{-4\sqrt{-ix-1}\,(-ix)} + \frac{x^2+ix}{4\sqrt{ix-1}\,(ix)}\right) = -2\pi + \pi\left(\frac{x-i}{\sqrt{-ix-1}} + \frac{x+i}{\sqrt{ix-1}}\right)$$

$$= -2\pi + 2\pi\left(1+x^2\right)^{\frac{1}{4}}\left(\cos\left(-\frac{\pi}{2} + \arctan x\ + \frac{1}{2}\left(\pi - \arctan x\ \right)\right)\right)$$

$$= -2\pi + 2\pi\left(1+x^2\right)^{\frac{1}{4}}\left(\cos\left(\frac{1}{2}\arctan x\right)\right) = -2\pi + 2\pi\left(1+x^2\right)^{\frac{1}{4}}\sqrt{\frac{1+\frac{1}{\sqrt{1+x^2}}}{2}}$$

$$= -2\pi + \pi\sqrt{2+2\sqrt{1+x^2}},$$

based on the facts that $\cos\left(\frac{x}{2}\right) = \sqrt{\frac{1+\cos x}{2}}$ and $\cos\left(\arctan x\ \right) = \frac{1}{\sqrt{1+x^2}}$. Therefore

$$I := \int_0^\infty \left(\arctan\left(\frac{1}{1+z^2}\right)\right)^2 dz = \int_0^\infty\int_0^1 \frac{\log\left(1+\frac{1}{\left(1+z^2\right)^2}\right) - \log\left(1+\frac{t^2}{\left(1+z^2\right)^2}\right)}{1-t^2} dt dz$$

$$= \int_0^1\int_0^\infty \frac{\log\left(1+\frac{1}{\left(1+z^2\right)^2}\right) - \log\left(1+\frac{t^2}{\left(1+z^2\right)^2}\right)}{1-t^2} dz dt$$

$$= \pi\sqrt{2}\int_0^1 \frac{1}{1-t^2}\left(\sqrt{1+\sqrt{2}} - \sqrt{1+\sqrt{1+t^2}}\right) dt.$$

We note that

$$\int \frac{1}{1-t^2}\left(\sqrt{\sqrt{2}+1} - \sqrt{1+\sqrt{1+t^2}}\right) dt =$$

$$= \sqrt{\sqrt{2}-1}\arctan\left(\frac{\sqrt{\sqrt{2}-1}\,t}{\sqrt{1+\sqrt{1+t^2}}}\right) + \frac{1}{2}\sqrt{\sqrt{2}+1}\log\left(\frac{1+t}{1-t}\right) -$$

$$-\frac{1}{2}\sqrt{\sqrt{2}+1}\log\left(\frac{\sqrt{1+\sqrt{1+t^2}}+\sqrt{\sqrt{2}+1}\,t}{\sqrt{1+\sqrt{1+t^2}}-\sqrt{\sqrt{2}+1}\,t}\right) + C,$$

which is easily verified by differentiation. It follows that

$$I = \frac{\sqrt{\sqrt{2}-1}}{4\sqrt{2}}\pi^2 + \sqrt{\frac{\sqrt{2}+1}{2}}\log\left(\frac{\sqrt{2}+1}{2\sqrt{2}}\right)\pi.$$

Also solved by the problem proposer.

• **5819** *Proposed by Toyesh Prakash Sharma, Agra College, Agra, India.*

For $a, b, c \geqslant 0$, show that

$$\frac{a+b}{a^2+b^2}\sqrt{ab} + \frac{b+c}{b^2+c^2}\sqrt{bc} + \frac{c+a}{c^2+a^2}\sqrt{ca} \leqslant 3.$$

Solution 1 by Henry Ricardo, Westchester Area Math Circle, Purchase, NY.

First Solution: Observe that $\sqrt{ab} \leqslant \left(\frac{a^2+b^2}{2}\right)^{1/2}$, or $\frac{ab}{a^2+b^2} \leqslant \frac{1}{2}$. Now, noting that

$$\frac{a+b}{a^2+b^2}\sqrt{ab} = \frac{\frac{a+b}{2}}{\frac{a^2+b^2}{2}}\sqrt{ab} \leqslant \frac{\left(\frac{a+b}{2}\right)^2}{\frac{a^2+b^2}{2}} = \frac{1}{2} + \frac{ab}{a^2+b^2},$$

we have

$$\sum_{cyclic}\frac{a+b}{a^2+b^2}\sqrt{ab} \leqslant \sum_{cyclic}\left(\frac{1}{2}+\frac{ab}{a^2+b^2}\right) \leqslant \sum_{cyclic}\left(\frac{1}{2}+\frac{1}{2}\right) = 3.$$

Second Solution: Applying familiar inequalities between the arithmetic and geometric means and the root mean square, we have

$$\frac{a+b}{a^2+b^2}\sqrt{ab} \leqslant \frac{a+b}{a^2+b^2}\sqrt{\frac{a^2+b^2}{2}} = \frac{\frac{a+b}{2}}{\sqrt{\frac{a^2+b^2}{2}}} \leqslant 1.$$

Therefore,

$$\sum_{cyclic}\frac{a+b}{a^2+b^2}\sqrt{ab} \leqslant \sum_{cyclic}\frac{\frac{a+b}{2}}{\sqrt{\frac{a^2+b^2}{2}}} \leqslant \sum_{cyclic} 1 = 3.$$

Solution 2 by Michel Bataille, Rouen, France.

We assume that at most one of a, b, c is zero.
For nonnegative x, y with $x^2 + y^2 \neq 0$, we have $\sqrt{xy} \leqslant \frac{x+y}{2}$ and $2(x^2 + y^2) \geqslant (x + y)^2$ (since $2(x^2 + y^2) - (x + y)^2 = (x - y)^2 \geqslant 0$), hence

$$\frac{x+y}{x^2+y^2}\sqrt{xy} \leqslant \frac{(x+y)^2}{2(x^2+y^2)} \leqslant 1.$$

It follows that

$$\frac{a+b}{a^2+b^2}\sqrt{ab} + \frac{b+c}{b^2+c^2}\sqrt{bc} + \frac{c+a}{c^2+a^2}\sqrt{ca} \leqslant 1 + 1 + 1 = 3.$$

Solution 3 by Péter Fülöp, Gyömrő, Hungary.

Let's take the AM-GM inequality: $\sqrt{ab} \leqslant \frac{a+b}{2}$ and multiplie by $(a + b)$ ($a, b \geqslant 0$).

We get the following $2(a + b)\sqrt{ab} - 2ab \leqslant a^2 + b^2$ inequality.

Using this fact at the first term of the LHS of the statement we get:

$$\frac{a+b}{a^2+b^2}\sqrt{ab} \leqslant \frac{a+b}{2(a+b)\sqrt{ab} - 2ab}\sqrt{ab} = \frac{1}{2}\frac{1}{1 - \frac{\sqrt{ab}}{a+b}} \leqslant \frac{1}{2}\frac{1}{1 - \frac{1}{2}} = 1$$

because of the AM-GM inequality $\frac{\sqrt{ab}}{a+b} \leqslant \frac{1}{2}$.

Perform the same method at the other two terms of the LHS, the statement is proved.

Solution 4 by Prakash Pant, The University of Vermont, Bardiya, Nepal.

Notice that proving the following statement is enough

$$\frac{a+b}{a^2+b^2}\sqrt{ab} \leqslant 1 \tag{2}$$

because other two terms in L.H.S are just the same expressions with (a,b) replaced by with (b,c) or (c,a) each of which is just a generic dummy variable $\geqslant 0$. Thus, the total of the expression would be $\leqslant 1 + 1 + 1 = 3$

To prove equation (2), since $a^2 + b^2$ is positive, we can cross-mulitply:

$$a^{\frac{3}{2}}b^{\frac{1}{2}} + b^{\frac{3}{2}}a^{\frac{1}{2}} \leqslant a^2 + b^2$$

which can be written as symmetric sums

$$\sum_{sym} a^{\frac{3}{2}} b^{\frac{1}{2}} \leqslant \sum_{sym} a^2 b^0 \tag{3}$$

To prove this statement, we use Muirhead Inequality. Since the sequence $\left(\frac{3}{2}, \frac{1}{2}\right) \prec (2,0)$, by Muirhead inequality, equation (3) is true. This completes the proof.

Solution 5 by Saurab Banstola, Gandaki Boarding School, Pokhara, Nepal.

Observe first that the expression is cyclic in the variables a, b, c. Thus it is enough to verify the inequality

$$\frac{a+b}{a^2+b^2}\sqrt{ab} \leqslant 1, \tag{1}$$

because the two remaining terms have exactly the same form, obtained by rotating the variables (a, b, c). If (1) holds, then the entire sum is bounded by

$$1+1+1=3.$$

To prove (1), note that the denominator a^2+b^2 is positive for $a, b \geqslant 0$, so multiplying both sides of (1) by it yields the equivalent inequality

$$a^{3/2}b^{1/2} + b^{3/2}a^{1/2} \leqslant a^2 + b^2.$$

This can be written in terms of symmetric sums as

$$\sum_{\text{sym}} a^{3/2}b^{1/2} \ \leqslant \ \sum_{\text{sym}} a^2 b^0. \tag{2}$$

Now, compare the exponent pairs. The sequence $(2,0)$ majorizes $(3/2,\ 1/2)$, that is,

$$(3/2,\ 1/2) \prec (2,\ 0).$$

Since the function describing the symmetric sum is Schur-convex, Muirhead's inequality directly applies, ensuring that (2) is valid. Thus inequality (1) is established, and consequently the original inequality follows.

Solution 6 by Albert Stadler, Herrliberg, Switzerland.

We have $\frac{x+y}{x^2+y^2}\sqrt{xy} \leqslant 1$, since this inequality is equivalent to each of the following lines

$$(x+y)\sqrt{xy} \leqslant x^2+y^2$$

$$(x+y)^2 xy \leqslant \left(x^2+y^2\right)^2$$

$$(x-y)^2(x^2+xy+y^2) \geqslant 0$$

$$(x-y)^2\left(\left(x+\frac{1}{2}y\right)^2+\frac{3}{4}y^2\right)\geqslant 0.$$

So

$$\frac{a+b}{a^2+b^2}\sqrt{ab}+\frac{b+c}{b^2+c^2}\sqrt{bc}+\frac{c+a}{c^2+a^2}\sqrt{ca}\leqslant 1+1+1\leqslant 3.$$

Solution 7 by Ángel Plaza, Universidad de Las Palmas de Gran Canaria, Spain.

It is enough to show that for $a,b\geqslant 0$, $\frac{a+b}{a^2+b^2}\sqrt{ab}\leqslant 1$. This last inequality follows by the AM-GM inequality. Indeed, by the the AM-GM inequality

$$\sqrt{ab}\leqslant\frac{a+b}{2},$$

so

$$\frac{a+b}{a^2+b^2}\sqrt{ab}\leqslant\frac{a+b}{a^2+b^2}\cdot\frac{a+b}{2}=\frac{a^2/2+ab+b^2/2}{a^2+b^2}\leqslant 1,$$

which it is equivalent to

$$a^2/2+ab+b^2/2\leqslant a^2+b^2,$$

that is,

$$a^2/2-ab+b^2/2=\frac{(a-b)^2}{2}\geqslant 1,$$

which it is true.

Solution 8 by Brian D. Beasley, Simpsonville, SC.

We note that at most one of a, b, or c may equal zero. With that assumption, it suffices to prove that for any $x,y\geqslant 0$ with at least one of x or y not zero,

$$\frac{x+y}{x^2+y^2}\sqrt{xy}\leqslant 1.$$

Without loss of generality, we assume $y\geqslant x\geqslant 0$ with $y\neq 0$. Since $x^2+y^2>0$, this inequality in turn is equivalent to $(x+y)\sqrt{xy}\leqslant x^2+y^2$. Then squaring both sides (which are non-negative) yields another equivalent inequality, namely $x^3y+2x^2y^2+xy^3\leqslant x^4+2x^2y^2+y^4$. To establish this inequality, we observe

$$x^3y+xy^3\leqslant x^4+y^4\iff 0\leqslant(y^3-x^3)(y-x),$$

which completes the proof and also shows that equality holds in the original result if and only if $a=b=c\neq 0$.

Solution 9 by Daniel Văcaru, National Economic College „Maria Teiuleanu", Pitești, Romania.

One has

$$a^2+b^2\geqslant 2ab\Rightarrow\frac{a+b}{a^2+b^2}\sqrt{ab}\leqslant\frac{a+b}{2\sqrt{ab}}\leqslant 1.$$

In the same manner, one has

$$\frac{b+c}{b^2+c^2}\sqrt{bc} \leqslant 1$$

and

$$\frac{c+a}{c^2+a^2}\sqrt{ca} \leqslant 1.$$

The relationship

$$\frac{a+b}{a^2+b^2}\sqrt{ab} + \frac{b+c}{b^2+c^2}\sqrt{bc} + \frac{c+a}{c^2+a^2}\sqrt{ca} \leqslant 3$$

follows.

Solution 10 by David A. Huckaby, Angelo State University, San Angelo, TX.

First note that if two or more of the variables a, b, and c equal 0, then the left-hand side is undefined. So we assume that at most one of the variables is equal to 0.

By the AGM inequality,

$$\frac{a+b}{a^2+b^2}\sqrt{ab} \quad \leqslant \quad \frac{a+b}{a^2+b^2} \cdot \frac{a+b}{2} \quad = \quad \frac{1}{2} \cdot \frac{a^2+b^2+2ab}{a^2+b^2} \quad = \quad \frac{1}{2} + \frac{ab}{a^2+b^2}. \quad (4)$$

Since $a^2+b^2 \geqslant 2ab$ (from $(a-b)^2 \geqslant 0$), $\dfrac{ab}{a^2+b^2} \leqslant \dfrac{1}{2}$. So from (4),

$$\frac{a+b}{a^2+b^2}\sqrt{ab} \leqslant \frac{1}{2} + \frac{1}{2} = 1.$$

So each of the three terms on the left-hand side of the inequality is less than or equal to 1, thus the inequality holds.

Also solved by Bruno Salgueiro Fanego, Viveiro, Lugo and the problem proposer.

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Editor's Statement: It goes without saying that the problem proposers, as well as the solution proposers, are the *élan vital* of the Problems/Solutions Section of SSMJ. As the editor of this Section of the Journal, I consider myself fortunate to be in a position to receive, compile and organize a wealth of proposed ingenious problems and solutions intended for online publication. My unwavering gratitude goes to all the amazingly creative contributors. We come together from across continents because we find intellectual value, joy and satisfaction in mathematical problems, both in their creation as well as their solution. So that our collective efforts serve us well, I kindly ask all contributors to adhere to the following guidelines. As you peruse below, you may construe that the guidelines amount to a lot of work. But, as the samples show, there's not much to do. Your cooperation is much appreciated!

Keep in mind that the examples given below are your best guide!

Formats, Styles and Requirements

When submitting proposed problem(s) or solution(s), please send both **LaTeX** document and **pdf** document of your proposed problem(s) or solution(s). There are ways (discoverable from the internet) to convert from Word to proper LaTeX code. Porposals without a *proper* **LaTeX** document will not be published regrettably.

Regarding Proposed Solutions:

Below is the FILENAME format for all the documents of your proposed solution(s).

#ProblemNumber_FirstName_LastName_Solution_SSMJ

- FirstName stands for YOUR first name.
- LastName stands for YOUR last name.

Examples:

#1234_Max_Planck_Solution_SSMJ

#9876_Charles_Darwin_Solution_SSMJ

Please note that every problem number is *preceded* by the sign # .

All you have to do is copy the FILENAME format (or an example below it), paste it and then modify portions of it to your specs.

Please adopt the following structure, in the order shown, for the presentation of your solution:

1. On top of the first page of your solution, begin with the phrase:

"Proposed Solution to #**** SSMJ"

where the string of four astrisks represents the problem number.

2. On the second line, write

"Solution proposed by [your First Name, your Last Name]", followed by your affiliation, city, country, all on the same linear string of words. Please see the example below. Make sure you do the same for your collaborator(s).

3. On a new line, state the problem proposer's name, affiliation, city and country, just as it appears published in the Problems/Solutions section.

4. On a new line below the above, write in bold type: "**Statement of the Problem**".

5. Below the latter, state the problem. Please make sure the statement of your problem (unlike the preceding item) is not in bold type.

6. Below the statement of the problem, write in bold type: "**Solution of the Problem**".

7. Below the latter, show the entire solution of the problem.

Here is a sample for the above-stated format for proposed solutions:

> *Proposed solution to #1234 SSMJ*
>
> *Solution proposed by Emmy Noether, University of Göttingen, Lower Saxony, Germany.*
>
> *Problem proposed by Isaac Newton, Trinity College, Cambridge, England.*
>
> **Statement of the problem:**
>
> Compute $\sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}$.
>
> **Solution of the problem:** . . . . . .

Regarding Proposed Problems:

For all your proposed problems, please adopt for all documents the following FILENAME format:

FirstName_LastName_ProposedProblem_SSMJ_YourGivenNumber_ProblemTitle

If you do not have a ProblemTitle, then leave that component as it already is (i.e., ProblemTitle).

The component YourGivenNumber is any UNIQUE 3-digit (or longer) number you like to give to your problem.

Examples:

Max_Planck_ProposedProblem_SSMJ_314_HarmonicPatterns

Charles_Darwin_ProposedProblem_SSMJ_358_ProblemTitle

Please adopt the following structure, in the order shown, for the presentation of your proposal:

1. On the top of first page of your proposal, begin with the phrase:

"Problem proposed to SSMJ"

2. On the second line, write

"Problem proposed by [your First Name, your Last Name]",

followed by your affiliation, city, country all on the same linear string of words. Please see the example below. Make sure you do the same for your collaborator(s) if any.

3. On a new line state the title of the problem, if any.

4. On a new line below the above, write in bold type: "**Statement of the Problem**".

5. Below the latter, state the problem. Please make sure the statement of your problem (unlike the preceding item) is not in bold type.

6. Below the statement of the problem, write in bold type: "**Solution of the Problem**".

7. Below the latter, show the entire solution of your problem.

Here is a sample for the above-stated format for proposed problems:

> *Problem proposed to SSMJ*
>
> *Problem proposed by Isaac Newton, Trinity College, Cambridge, England.*
>
> **Principia Mathematica** (⟵ You may choose to not include a title.)
>
> **Statement of the problem:**
>
> Compute $\sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}$.
>
> **Solution of the problem:** . . . . . .

♣ ♣ ♣ Thank You! ♣ ♣ ♣