Problems and Solutions

Albert Natian, Section Editor

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This section of the Journal offers readers an opportunity to exchange interesting mathematical problems and solutions. Please email them to Prof. Albert Natian at Department of Mathematics, Los Angeles Valley College. Please make sure every proposed problem or proposed solution is provided in both ***LaTeX*** and pdf documents. Please make sure your proposals adhere to **Formats, Styles and Requirements** noted below. Thank you!

To propose problems, email them to: problems4ssma@gmail.com

To propose solutions, email them to: solutions4ssma@gmail.com

Solutions to the problems published in this issue should be submitted *before* October 1, 2026.

• **5840** *Proposed by Paolo Perfetti, dipartimento di matematica Università di "Tor Vergata", Rome, Italy.*

For $x > 0$, evaluate

$$\sum_{n=0}^{N}(-1)^n\binom{N}{n}\sum_{k=0}^{2n}(-1)^k\,\frac{(k+1)\,\binom{2n}{k}2^k}{x+k}.$$

• **5841** *Proposed by Ovidiu Furdui and Alina Sîntămărian, Technical University of Cluj-Napoca, Cluj-Napoca, Romania.*

Calculate

(a) $$L=\lim_{n\to\infty}\sqrt[n]{\int_0^1\frac{1}{x^n+(1-x)^n}\mathrm{d}x}$$

(b) $$M=\lim_{n\to\infty}\frac{n}{\ln n}\left(L-\sqrt[n]{\int_0^1\frac{1}{x^n+(1-x)^n}\mathrm{d}x}\right).$$

• **5842** *Proposed by Michel Bataille, Rouen, France.*

Let z_1, z_2, z_3 be complex numbers such that

$$z_1^2-z_2^2=z_3z_1,\quad z_2^2-z_3^2=z_1z_2,\quad z_3^2-z_1^2=z_2z_3.$$

Prove that $z_1^5z_2^4+z_2^5z_3^4+z_3^5z_1^4+8z_1^3z_2^3z_3^3=0$.

• **5843** *Proposed by Michael Brozinsky, Central Islip, New York.*

Let $a, b \in \mathbb{N}$ with $a, b \geqslant 2$. Set $N = ab$. Suppose N people attend a party with a *knowing* relation that is symmetric and transitive, and where each person is considered to know himself/herself. Prove that there exist either a mutual acquaintances or b mutual strangers.

• **5844** *Proposed by Daniel Sitaru, National Economic College "Theodor Costescu," Drobeta Turnu - Severin, Romania.*

Suppose $x, y, z \in (0, 1)$ and $xy + yz + zx = 1$. Show that

$$8\sqrt{3}xyz \leqslant (1 + x^2)(1 + y^2)(1 + z^2) + (1 - x^2)(1 - y^2)(1 - z^2).$$

Solutions

To Formerly Published Problems

• **5820** *Proposed by Michel Bataille, Rouen, France.*

Let r, s be integers with $r \geqslant s \geqslant 0$ and let the real number x satisfy $|x| < 1$. Show that

$$\sum_{n=0}^{\infty} \binom{r+n}{s} x^n = \sum_{j=0}^{s} \binom{r}{s-j} \frac{x^j}{(1-x)^{j+1}}.$$

Solution 1 by Ulrich Abel, Technische Hochschule Mittelhessen, Friedberg, Germany.

We have

$$\sum_{j=0}^{s} \binom{r}{s-j} \frac{x^j}{(1-x)^{j+1}} = \sum_{j=0}^{s} \binom{r}{s-j} x^j \sum_{i=0}^{\infty} \binom{i+j}{j} x^i = \sum_{n=0}^{\infty} c_n x^n$$

with

$$c_n = \sum_{j=0}^{s} \binom{r}{s-j} \binom{n}{j} = \binom{r+n}{s},$$

where the last equation follows by the Vandermonde convolution formula.

Solution 2 by Prakash Pant, The University of Vermont, Bardiya, Nepal.

We encode the binomial coefficient as follows:

$$\binom{r+n}{s} = [t^s](1+t)^{r+n}$$

where $[t^s]$ means the coefficient of t^s. Then,

$$\text{LHS } = \sum_{n=0}^{\infty}[t^s](1+t)^{r+n}x^n$$

Since sum of coefficients is coefficient of sum,

$$[t^s](1+t)^r\sum_{n=0}^{\infty}(x+xt)^n$$

Since $|x| < 1$ and t is an arbitrary variable which can be chosen to be small enough such that $|x||1+t| < 1$. Thus, using infinite geometric series,

$$[t^s](1+t)^r\frac{1}{1-x-xt}$$

$$[t^s](1+t)^r\left(\frac{1}{1-x}\right)\left(\frac{1}{1-\frac{x}{1-x}t}\right).$$

Again, if t is small enough and since $|x| < 1$, $\left|\dfrac{xt}{1-x}\right| < 1$. Thus,

$$[t^s](1+t)^r\left(\frac{1}{1-x}\right)\sum_{j=0}^{\infty}\frac{x^j}{(1-x)^j}t^j$$

$$[t^s]\sum_{j=0}^{\infty}\frac{x^j}{(1-x)^{j+1}}(1+t)^r t^j$$

Again, since coefficient of sum is sum of coefficients,

$$\sum_{j=0}^{\infty}\frac{x^j}{(1-x)^{j+1}}[t^s](1+t)^r t^j = \sum_{j=0}^{\infty}\frac{x^j}{(1-x)^{j+1}}[t^{s-j}](1+t)^r$$

$$\sum_{j=0}^{\infty}\frac{x^j}{(1-x)^{j+1}}\binom{r}{s-j}.$$

Since $s-j$ needs to be positive for non-zero binomial coefficient, $s-j \geqslant 0 \implies s \geqslant j \implies j \leqslant s$. Thus,

$$\sum_{j=0}^{s}\binom{r}{s-j}\frac{x^j}{(1-x)^{j+1}} = \text{ RHS}$$

which concludes the proof.

Solution 3 by Péter Fülöp, Gyömrő, Hungary.

Start from the integral representation of the binomial coefficient:

$$\binom{n}{k} = \frac{1}{2\pi}\int\limits_{-\pi}^{\pi}(1+e^{it})^n e^{-ikt}dt$$

Applying this to both sides of the equation.

$$LHS = \frac{1}{2\pi}\sum_{n=0}^{\infty}\int\limits_{-\pi}^{\pi}(1+e^{it})^{n+r}e^{-ist}dtx^n = \frac{1}{2\pi}\int\limits_{-\pi}^{\pi}(1+e^{it})^r e^{-ist}\sum_{n=0}^{\infty}(1+e^{it})^n x^n dt$$

After performing the summation at LHS, we get:

$$LHS = \frac{1}{2\pi}\int\limits_{-\pi}^{\pi}\frac{(1+e^{it})^r e^{-ist}}{1-x(1+e^{it})}dt$$

Let's split the sum at RSH into two parts in the following way:

$$\sum_{j=0}^{s}\binom{r}{s-j}\frac{x^j}{(1-x)^{j+1}} = \sum_{j=0}^{\infty}\binom{r}{s-j}\frac{x^j}{(1-x)^{j+1}} - \sum_{j=s+1}^{\infty}\binom{r}{s-j}\frac{x^j}{(1-x)^{j+1}}$$

When $j > s$ the second term is vanish, the summation can be performed from $j = 0$ to ∞.

$$RHS = \sum_{j=0}^{\infty}\frac{1}{2\pi}\int\limits_{-\pi}^{\pi}e^{-ist}e^{ijt}(1+e^{it})^r\frac{x^j}{(1-x)^{j+1}}dt = \frac{1}{2\pi}\int\limits_{-\pi}^{\pi}\frac{e^{-ist}(1+e^{it})^r}{1-x}\sum_{j=0}^{\infty}\left(\frac{e^{it}x}{(1-x)}\right)^j dt$$

After performing the summation:

$$RHS = \frac{1}{2\pi}\int\limits_{-\pi}^{\pi}\frac{e^{-ist}(1+e^{it})^r}{1-x}\frac{1}{1-\frac{xe^{it}}{1-x}}dt$$

$$RHS = \frac{1}{2\pi}\int\limits_{-\pi}^{\pi}\frac{e^{-ist}(1+e^{it})^r}{1-x(1+e^{it})}dt.$$

It means that LHS=RHS, so the statement is proved.

Solution 4 by Albert Stadler, Herrliberg, Switzerland.

We start with the binomial series expansion for $\frac{1}{(1-x)^{j+1}}$:

$$\frac{1}{(1-x)^{j+1}} = \sum_{k=0}^{\infty} \binom{-j-1}{k} (-x)^k.$$

We need to prove that

$$\binom{r+n}{s} = [x^n] \left(\sum_{j=0}^{s} \binom{r}{s-j} x^j \sum_{k=0}^{\infty} \binom{-j-1}{k} (-x)^k \right).$$

Expanding this, we obtain

$$\binom{r+n}{s} = \sum_{\substack{0 \leqslant j \leqslant s, k \geqslant 0 \\ j+k=n}} \binom{r}{s-j} \binom{-j-1}{k} (-1)^k.$$

This simplifies to

$$\binom{r+n}{s} = \sum_{0 \leqslant j \leqslant s} \binom{r}{s-j} \binom{-j-1}{n-j} (-1)^{n-j}.$$

Using the identity

$$\binom{-j-1}{n-j} (-1)^{n-j} = \binom{n}{n-j} = \binom{n}{j},$$

we get

$$\binom{r+n}{s} = \sum_{0 \leqslant j \leqslant s} \binom{r}{s-j} \binom{n}{j}.$$

This is precisely the Chu-Vandermonde identity (see, for instance, https://en.wikipedia.org /wiki/ Binomial_coefficient).

Also solved by the problem proposer.

• **5821** *Proposed by D.M. Bătineţu-Giurgiu, Matei Basarab National College, Bucharest, Romania and Neculai Stanciu, George Emil Palade Secondary School, Buzău, Romania.*

Where $a > 0$, $x > 0$, and $\mathcal{E}(x) := \left(1 + \frac{1}{x}\right)^x$, find

$$\lambda := \lim_{n\to\infty} \left[\lim_{x\to\infty} \left(\frac{x}{n}\right)^2 \left(-n\mathcal{E}(x) + \sum_{k=1}^{n} \mathcal{E}(x+ka) \right) \right].$$

Solution 1 by David A. Huckaby, Angelo State University, San Angelo, TX.

Since $\ln\left(1+\frac{1}{w}\right) = \frac{1}{w} - \frac{1}{2w^2} + \frac{1}{3w^3} + O\left(\frac{1}{w^4}\right)$, $w\ln\left(1+\frac{1}{w}\right) = 1 - \frac{1}{2w} + \frac{1}{3w^2} + O\left(\frac{1}{w^3}\right)$, so that

$$\begin{aligned}\mathcal{E}(w) = \left(1+\frac{1}{w}\right)^w &= e^{w\ln\left(1+\frac{1}{w}\right)} = e^1 e^{-\frac{1}{2w}} e^{\frac{1}{3w^2}} \cdots \\ &= e\left(1 - \frac{1}{2w} + \frac{1}{8w^2} + O\left(\frac{1}{w^3}\right)\right)\left(1 + \frac{1}{3w^2} + O\left(\frac{1}{w^3}\right)\right) \\ &= e\left(1 - \frac{1}{2w} + \frac{11}{24w^2} + O\left(\frac{1}{w^3}\right)\right). \quad (1)\end{aligned}$$

Now $\frac{1}{x+ka} = \frac{1}{x}\left[\frac{1}{1+\frac{ka}{x}}\right] = \frac{1}{x}\left[1 - \frac{ka}{x} + \frac{(ka)^2}{x^2} + O\left(\frac{1}{x^3}\right)\right] = \frac{1}{x} - \frac{ka}{x^2} + O\left(\frac{1}{x^3}\right)$, and $\frac{1}{(x+ka)^2} = \frac{1}{x^2}\left[\frac{1}{\left(1+\frac{ka}{x}\right)^2}\right] = \frac{1}{x^2}\left[1 - \frac{ka}{x} + O\left(\frac{1}{x^2}\right)\right]^2 = \frac{1}{x^2}\left[1 - \frac{2ka}{x} + O\left(\frac{1}{x^3}\right)\right] = \frac{1}{x^2} + O\left(\frac{1}{x^3}\right)$. So from equation (1),

$$\begin{aligned}\mathcal{E}(x+ka) &= e\left\{1 - \frac{1}{2}\left[\frac{1}{x} - \frac{ka}{x^2} + O\left(\frac{1}{x^3}\right)\right] + \frac{11}{24}\left[\frac{1}{x^2} + O\left(\frac{1}{x^3}\right)\right] + O\left(\frac{1}{x^3}\right)\right\} \\ &= e\left(1 - \frac{1}{2x} + \frac{12ka+11}{24x^2} + O\left(\frac{1}{x^3}\right)\right).\end{aligned}$$

Thus

$$\begin{aligned}\sum_{k=1}^{n}\mathcal{E}(x+ka) &= e\sum_{k=1}^{n}\left(1 - \frac{1}{2x} + \frac{12ka+11}{24x^2} + O\left(\frac{1}{x^3}\right)\right) \\ &= e\left(n - \frac{n}{2x} + \frac{12n(n+1)a}{48x^2} + \frac{11n}{24x^2} + O\left(\frac{n}{x^3}\right)\right).\end{aligned}$$

So

$$
\begin{aligned}
&-n\mathcal{E}(x)+\sum_{k=1}^{n}\mathcal{E}(x+ka)\\
&\qquad=-n\left(e\left[1-\frac{1}{2x}+O\left(\frac{1}{x^3}\right)\right]\right)+e\left(n-\frac{n}{2x}+\frac{12n(n+1)a}{48x^2}+\frac{11n}{24x^2}+O\left(\frac{n}{x^3}\right)\right)\\
&\qquad=e\left(\frac{12n(n+1)a}{48x^2}+\frac{11n}{24x^2}+O\left(\frac{n}{x^3}\right)\right).
\end{aligned}
$$

Thus

$$
\begin{aligned}
\lambda&=\lim_{n\to\infty}\left[\lim_{x\to\infty}\left(\frac{x}{n}\right)^2\left(-n\mathcal{E}(x)+\sum_{k=1}^{n}\mathcal{E}(x+ka)\right)\right]\\
&=\lim_{n\to\infty}\left[\lim_{x\to\infty}\left(\frac{x}{n}\right)^2\left\{e\left(\frac{12n(n+1)a}{48x^2}+\frac{11n}{24x^2}+O\left(\frac{n}{x^3}\right)\right)\right\}\right]\\
&=e\lim_{n\to\infty}\left[\lim_{x\to\infty}\left\{\frac{a}{4}+\frac{a}{4n}+\frac{11}{24n}+O\left(\frac{1}{xn}\right)\right\}\right]=\frac{ea}{4}.
\end{aligned}
$$

Solution 2 by Michel Bataille, Rouen, France.

As $x\to\infty$, we have

$$
\ln\mathcal{E}(x)=x\ln\left(1+\frac{1}{x}\right)=x\left(\frac{1}{x}-\frac{1}{2x^2}+\frac{1}{3x^3}+o(1/x^3)\right)=1-\frac{1}{2x}+\frac{1}{3x^2}+o(1/x^2),
$$

hence

$$
\begin{aligned}
\mathcal{E}(x)&=e\left(e^{-\frac{1}{2x}+\frac{1}{3x^2}+o(1/x^2)}\right)\\
&=e\left(1-\frac{1}{2x}+\frac{1}{3x^2}+\frac{1}{2}\left(-\frac{1}{2x}+\frac{1}{3x^2}\right)^2+o(1/x^2)\right)\\
&=e-\frac{e}{2x}+\frac{11e}{24x^2}+o(1/x^2)
\end{aligned}
$$

and

$$
\begin{aligned}
\mathcal{E}(x+ka)&=e-\frac{e}{2x}\left(1+\frac{ka}{x}\right)^{-1}+\frac{11e}{24x^2}\left(1+\frac{ka}{x}\right)^{-2}+o(1/x^2)\\
&=e-\frac{e}{2x}\left(1-\frac{ka}{x}+o(1/x)\right)+\frac{11e}{24x^2}+o(1/x^2)\\
&=e-\frac{e}{2x}+\frac{e(11+12ka)}{24x^2}+o(1/x^2).
\end{aligned}
$$

We deduce that

$$\sum_{k=1}^{n} \mathcal{E}(x+ka) = ne - \frac{ne}{2x} + \frac{11ne}{24x^2} + \frac{ea}{2x^2} \cdot \frac{n(n+1)}{2} + o(1/x^2).$$

so that

$$\left(\frac{x}{n}\right)^2 \left(-n\mathcal{E}(x) + \sum_{k=1}^{n} \mathcal{E}(x+ka)\right) = \frac{x^2}{n^2}\left(\frac{ea}{2x^2} \cdot \frac{n(n+1)}{2} + o(1/x^2)\right) = \frac{ea(n+1)}{4n} + o(1).$$

Thus, $\lim_{x\to\infty} \left(\frac{x}{n}\right)^2 \left(-n\mathcal{E}(x) + \sum_{k=1}^{n} \mathcal{E}(x+ka)\right) = \frac{ea(n+1)}{4n}$
and consequently

$$\lambda = \frac{ea}{4}.$$

Solution 3 by Ulrich Abel, Technische Hochschule Mittelhessen, Friedberg, Germany.

Taking advantage of the asymptotic expansion

$$\left(1+\frac{1}{x}\right)^x = e\left(1 - \frac{1}{2x} + \frac{11}{24x^2} + O\left(\frac{1}{x^3}\right)\right) \qquad (x \to \infty)$$

we obtain, for fixed n,

$$\begin{aligned} -n\mathcal{E}(x) + \sum_{k=1}^{n} \mathcal{E}(x+ka) &= \sum_{k=1}^{n} \left(\mathcal{E}(x+ka) - \mathcal{E}(x)\right) \\ &= e\sum_{k=1}^{n}\left(\frac{1}{2x} - \frac{1}{2(x+ka)} - \frac{11}{24}\left(\frac{1}{x^2} - \frac{1}{(x+ka)^2}\right)\right) + O\left(\frac{1}{x^3}\right) \\ &= e\sum_{k=1}^{n} \frac{ka}{2x(x+ka)} + O\left(\frac{1}{x^3}\right) \qquad (x\to\infty), \end{aligned}$$

since

$$\frac{1}{x^2} - \frac{1}{(x+ka)^2} = \frac{(x+ka)^2 - x^2}{x^2(x+ka)^2} = \frac{2kax + (ka)^2}{x^2(x+ka)^2} = O\left(\frac{1}{x^3}\right) \qquad (x\to\infty).$$

We conclude that

$$\begin{aligned} &\lim_{x\to\infty}\left(\frac{x}{n}\right)^2\left(-n\mathcal{E}(x) + \sum_{k=1}^{n}\mathcal{E}(x+ka)\right) \\ = &\ \frac{e}{n^2}\lim_{x\to\infty}\sum_{k=1}^{n}\frac{kax^2}{2x(x+ka)} = \frac{e}{n^2}\sum_{k=1}^{n}\frac{ka}{2} = \frac{ae}{2n^2}\cdot\frac{n(n+1)}{2}. \end{aligned}$$

Hence,

$$\lambda = \lim_{n\to\infty}\left(\frac{ae}{2n^2}\cdot\frac{n(n+1)}{2}\right) = \frac{ae}{4}.$$

Also solved by Albert Stadler, Herrliberg, Switzerland and the problem proposer.

• **5822** *Proposed by Paolo Perfetti, dipartimento di matematica Universitą di "Tor Vergata", Rome, Italy.*

Find the value of

$$S := \sum_{n=0}^{\infty}\left(\frac{1}{n+1}-\frac{1}{n+2}+\frac{1}{n+3}-\dots\right)^3.$$

Solution 1 by Albert Stadler, Herrliberg, Switzerland.

We will show that

$$S = \frac{5}{16}\zeta(3).$$

Recall that

$$\frac{1}{k} = \int_0^1 x^{k-1}dx.$$

Using this representation, the alternating harmonic tail becomes

$$\frac{1}{n+1}-\frac{1}{n+2}+\frac{1}{n+3}-\dots = \sum_{k=0}^{\infty}(-1)^k\int_0^1 x^{n+k}dx = \int_0^1\frac{x^n}{1+x}dx.$$

Interchanging the sum and integrals (which is justified by the positivity of all terms) gives

$$S = \sum_{n=0}^{\infty}\int_0^1\frac{x^n}{1+x}dx\int_0^1\frac{y^n}{1+y}dy\int_0^1\frac{z^n}{1+z}dz = \int_0^1\int_0^1\int_0^1\frac{1}{(1+x)(1+y)(1+z)}\sum_{n=0}^{\infty}(xyz)^n dzdydx$$

$$= \int_0^1\int_0^1\int_0^1\frac{1}{(1-xyz)(1+x)(1+y)(1+z)}dzdydx.$$

Make the substitution

$$x = \frac{1-u}{1+u},\quad y = \frac{1-v}{1+v},\quad z = \frac{1-w}{1+w}.$$

The triple integral becomes

$$\int_0^1\int_0^1\int_0^1\frac{8}{\left(1-\frac{1-u}{1+u}\frac{1-v}{1+v}\frac{1-w}{1+w}\right)\left(1+\frac{1-u}{1+u}\right)\left(1+\frac{1-v}{1+v}\right)\left(1+\frac{1-w}{1+w}\right)}\frac{1}{(1+u)^2(1+v)^2(1+w)^2}dudvdw$$

$$= \int_0^1\int_0^1\int_0^1\frac{1}{(1+u)(1+v)(1+w)-(1-u)(1-v)(1-w)}dudvdw$$

$$= \int_0^1 \int_0^1 \int_0^1 \frac{1}{2\left(u+v+w+uvw\right)} dudvdw.$$

Since

$$\int \frac{1}{2\left(u+v+w+uvw\right)} du = \frac{\ln\left(u+v+w+uvw\right)}{2\left(1+vw\right)} + C,$$

we obtain

$$S = \int_0^1 \int_0^1 \frac{\ln\left(1+v\right) + \ln\left(1+w\right) - \ln\left(v+w\right)}{2\left(1+vw\right)} dvdw.$$

Splitting terms:

$$S = \int_0^1 \int_0^1 \frac{\ln\left(1+v\right)}{\left(1+vw\right)} dvdw - \int_0^1 \int_0^1 \frac{\ln\left(v+w\right)}{2\left(1+vw\right)} dvdw.$$

Since

$$\int_0^1 \frac{1}{\left(1+vw\right)} dw = \frac{\ln\left(1+vw\right)}{v}\bigg|_{w=0}^{w=1} = \frac{\ln\left(1+v\right)}{v},$$

the first double integral evaluates to

$$\int_0^1 \frac{\ln^2\left(1+v\right)}{v} dv.$$

A classical result (see [1], formula (3.18)) is:

$$\int_0^1 \frac{\ln^2\left(1+v\right)}{v} dv = \frac{1}{4}\zeta\left(3\right).$$

Thus it remains to show that

$$\int_0^1 \int_0^1 \frac{\ln\left(v+w\right)}{1+vw} dvdw = -\frac{1}{8}\zeta\left(3\right),$$

since then

$$S = \int_0^1 \frac{\ln^2\left(1+v\right)}{v} dv - \int_0^1 \int_0^1 \frac{\ln\left(v+w\right)}{2\left(1+vw\right)} dvdw = \frac{1}{4}\zeta\left(3\right) + \frac{1}{16}\zeta\left(3\right) = \frac{5}{16}\zeta\left(3\right).$$

By symmetry,

$$\int_0^1 \int_0^1 \frac{\ln\left(v+w\right)}{\left(1+vw\right)} dvdw = 2\int_0^1 \int_0^w \frac{\ln\left(v+w\right)}{\left(1+vw\right)} dvdw.$$

Letting $v \to wv$

$$= 2\int_0^1 w \int_0^1 \frac{\ln\left(vw+w\right)}{\left(1+vw^2\right)} dvdw = 2\int_0^1 w \int_0^1 \frac{\ln w + \ln\left(1+v\right)}{\left(1+vw^2\right)} dvdw.$$

Setting $u = w^2$, we obtain

$$\int_0^1 \int_0^1 \frac{\ln\left(v+w\right)}{\left(1+vw\right)} dvdw = \int_0^1 \int_0^1 \frac{\frac{1}{2}\ln u + \ln\left(1+v\right)}{\left(1+uv\right)} dvdu$$

$$= \int_0^1 \frac{\ln u \ln(1+u)}{2u} du + \int_0^1 \frac{\ln^2(1+v)}{v} dv,$$

since

$$\int_0^1 \frac{\frac{1}{2}\ln u}{(1+uv)} dv = \frac{\ln u \ln(1+u)}{2u} \text{ and } \int_0^1 \frac{\ln(1+v)}{(1+uv)} du = \frac{\ln^2(1+v)}{v}.$$

Here

$$\int_0^1 \frac{\ln u \ln(1+u)}{2u} du = \frac{1}{2}\sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k} \int_0^1 u^{k-1} \ln u \, du = \frac{1}{2}\sum_{k=1}^{\infty} \frac{(-1)^k}{k^3}$$

$$= -\frac{1}{2}\sum_{k=1}^{\infty} \frac{1}{k^3} + \sum_{k=1}^{\infty} \frac{1}{(2k)^3} = -\frac{3}{8}\zeta(3).$$

Thus

$$\int_0^1 \int_0^1 \frac{\log(v+w)}{(1+vw)} dv dw = -\frac{3}{8}\zeta(3) + \frac{1}{4}\zeta(3) = -\frac{1}{8}\zeta(3),$$

and we are done.

Solution 2 by Ángel Plaza, Universidad de Las Palmas de Gran Canaria, Spain.

This sum appears with a different indexation in [1], from where we have adapted the solution. Let S denote the proposed series. We calculate the series by shifting the index of summation. We have

$$\begin{aligned}
S &= \left(1 - \frac{1}{2} + \frac{1}{3} - \dots\right)^3 + \sum_{n=1}^{\infty} \left(\frac{1}{n+1} - \frac{1}{n+2} + \frac{1}{n+3} - \dots\right)^3 \\
&= \ln^3 2 + \sum_{i=1}^{\infty} \left(\frac{1}{i+1} - \frac{1}{i+2} + \frac{1}{i+3} - \dots\right)^3 \\
&= \ln^3 2 + \sum_{i=1}^{\infty} \left[\frac{1}{i} - \left(\frac{1}{i} - \frac{1}{i+1} + \frac{1}{i+2} - \dots\right)\right]^3 \\
&= \ln^3 2 + \sum_{i=1}^{\infty} \left[\frac{1}{i^3} - \frac{3}{i^2}\left(\frac{1}{i} - \frac{1}{i+1} + \frac{1}{i+2} - \dots\right)\right. \\
&+ \left.\frac{3}{i}\left(\frac{1}{i} - \frac{1}{i+1} + \frac{1}{i+2} - \dots\right)^2 - \left(\frac{1}{i} - \frac{1}{i+1} + \frac{1}{i+2} - \dots\right)^3\right] \\
&= \ln^3 2 + \zeta(3) - 3\sum_{i=1}^{\infty} \frac{1}{i^2}\left(\frac{1}{i} - \frac{1}{i+1} + \frac{1}{i+2} - \dots\right) \\
&+ 3\sum_{i=1}^{\infty} \frac{1}{i}\left(\frac{1}{i} - \frac{1}{i+1} + \frac{1}{i+2} - \dots\right)^2 - S.
\end{aligned}$$

It follows, based on parts (d) and (h) of Lemma 1.3 [1], that

$$\begin{aligned} 2S &= \ln^3 2 + \zeta(3) - 3\sum_{i=1}^{\infty}\frac{1}{i^2}\left(\frac{1}{i} - \frac{1}{i+1} + \frac{1}{i+2} - \dots\right) \\ &+ 3\sum_{i=1}^{\infty}\frac{1}{i}\left(\frac{1}{i} - \frac{1}{i+1} + \frac{1}{i+2} - \dots\right)^2 \\ &= \ln^3 2 + \zeta(3) - 3\left(\frac{13}{8}\zeta(3) - \zeta(2)\ln 2\right) + 3\left(\frac{3}{2}\zeta(3) - \zeta(2)\ln 2 - \frac{\ln^3 2}{3}\right) \\ &= \frac{5}{2}\zeta(3). \end{aligned}$$

From the latter, we get $S = \dfrac{5}{16}\zeta(3)$.

References

[1] Furdui, O., Sîntămărian, A., *Cubic and quartic series with the tail of* ln 2, CUBO, A Mathematical J., Vol. 25, no. 1, pp. 89–101, 2003.

Also solved by the problem proposer.

• **5823** *Proposed by Daniel Sitaru, National Economic College "Theodor Costescu" Drobeta Turnu - Severin, Romania.*

Prove that if $0 < a \leqslant b$, then:

$$\left(\int_a^b e^{-x^2}dx\right)^2 \geqslant \left(\int_{\frac{a+b}{2}}^b e^{-x^2}dx + \int_{\sqrt{ab}}^b e^{-x^2}dx\right)\left(\int_a^{\frac{a+b}{2}} e^{-x^2}dx + \int_a^{\sqrt{ab}} e^{-x^2}dx\right).$$

Solution 1 by Sri Hari Bhupala Haribhakta, Bengaluru, India.

Let $f(x) = e^{-x^2}$ and set

$$m = \frac{a+b}{2}, \qquad g = \sqrt{ab}.$$

Although the integrand is Gaussian, the inequality holds for any integrable function f; it is a consequence of splitting $\displaystyle\int_a^b f(x)\,dx$ at the two points m and g.

Since $0 < a \leqslant b$, we have $a \leqslant g \leqslant b$, hence all integrals below are well-defined.

Define

$$A = \int_a^m f(x)\,dx, \quad B = \int_m^b f(x)\,dx, \quad C = \int_a^g f(x)\,dx, \quad D = \int_g^b f(x)\,dx.$$

Then

$$\int_a^b f(x)\,dx = A + B = C + D.$$

Let

$$I := \int_a^b f(x)\,dx, \qquad S := A + C.$$

Using $B = I - A$ and $D = I - C$, we obtain

$$B + D = (I - A) + (I - C) = 2I - (A + C) = 2I - S.$$

Therefore the right-hand side of the desired inequality equals

$$\left(\int_m^b f(x)\,dx + \int_g^b f(x)\,dx\right)\left(\int_a^m f(x)\,dx + \int_a^g f(x)\,dx\right) = (B + D)(A + C) = (2I - S)S.$$

Hence the inequality becomes

$$I^2 \geqslant (2I - S)S \quad \Longleftrightarrow \quad I^2 - 2IS + S^2 \geqslant 0 \quad \Longleftrightarrow \quad (I - S)^2 \geqslant 0,$$

which is always true. This proves the required inequality.

Solution 2 by Michel Bataille, Rouen, France.

Let

$$X = \int_{\frac{a+b}{2}}^b e^{-x^2}\,dx + \int_{\sqrt{ab}}^b e^{-x^2}\,dx, \qquad Y = \int_a^{\frac{a+b}{2}} e^{-x^2}\,dx + \int_a^{\sqrt{ab}} e^{-x^2}\,dx.$$

Then, we have

$$\begin{aligned} X + Y &= \int_a^{\frac{a+b}{2}} e^{-x^2}\,dx + \int_{\frac{a+b}{2}}^b e^{-x^2}\,dx + \int_a^{\sqrt{ab}} e^{-x^2}\,dx + \int_{\sqrt{ab}}^b e^{-x^2}\,dx \\ &= \int_a^b e^{-x^2}\,dx + \int_a^b e^{-x^2}\,dx = 2\int_a^b e^{-x^2}\,dx \end{aligned}$$

From the obvious inequality $(X+Y)^2 \geqslant 4XY$, we deduce $4\left(\int_a^b e^{-x^2}\,dx\right)^2 \geqslant 4XY$, hence

$$\left(\int_a^b e^{-x^2}\,dx\right)^2 \geqslant XY,$$

as desired.

Solution 3 by Prakash Pant, The University of Vermont, Bardiya, Nepal.

Let F(x) be the real antiderivative of e^{-x^2}. Then the given problem can be restated as:

$$[F(b)-F(a)]^2 \geqslant \left[F(b)-F(\frac{a+b}{2})+F(b)-F(\sqrt{ab})\right]\left[F(\frac{a+b}{2})-F(a)+F(\sqrt{ab})-F(a)\right]$$

$$[F(b)-F(a)]^2 \geqslant \left[2F(b)-F(\frac{a+b}{2})-F(\sqrt{ab})\right]\left[F(\frac{a+b}{2})+F(\sqrt{ab})-2F(a)\right]$$

$$F^2(b)-2F(b)F(a)+F^2(a) \geqslant 2F(b)\left[F(\frac{a+b}{2})+F(\sqrt{ab})\right]-4F(b)F(a)-\left[F(\frac{a+b}{2})+F(\sqrt{ab})\right]^2$$
$$+2F(a)\left[F(\frac{a+b}{2})+F(\sqrt{ab})\right]$$

which can be rewritten as

$$\left[F(b)+F(a)\right]^2+\left[F(\frac{a+b}{2})+F(\sqrt{ab})\right]^2 \geqslant 2\left[F(\frac{a+b}{2})+F(\sqrt{ab})\right]\left[F(b)+F(a)\right]$$

$$\left[F(b)+F(a)-F(\frac{a+b}{2})-F(\sqrt{ab})\right]^2 \geqslant 0$$

Since square of a real number cannot be negative, the given statement is true which proves the original statement we began with, hence concluding the proof.

Solution 4 by Albert Stadler, Herrliberg, Switzerland.

We prove more generally that

$$\left(\int_a^b f(x)\,dx\right)^2 \geqslant \left(\int_u^b f(x)\,dx+\int_v^b f(x)\,dx\right)\left(\int_a^u f(x)\,dx+\int_a^v f(x)\,dx\right)$$

for any integrable function f and any $u, v \in [a, b]$. Let

$$A := \int_a^b f(x)\,dx, B := \int_u^b f(x)\,dx, C := \int_v^b f(x)\,dx$$

Then above inequality reads as

$$A^2 \geqslant (B+C)(A-B+A-C)$$

which is equivalent to $(A-B-C)^2 \geqslant 0$ which is obviously true.

Solution 5 by Ángel Plaza, Universidad de Las Palmas de Gran Canaria, Spain.

Subintegral function, e^{-x^2} is positive. The solution follows by several applications of the GM-AM inequality:

Since
$$\sqrt{ab} \leqslant \frac{a+b}{2},$$
then
$$\int_{\frac{a+b}{2}}^{b} e^{-x^2}\,dx + \int_{\sqrt{ab}}^{b} e^{-x^2}\,dx \leqslant 2\int_{\frac{a+b}{2}}^{b} e^{-x^2}\,dx$$
and
$$\int_{a}^{\frac{a+b}{2}} e^{-x^2}\,dx + \int_{a}^{\sqrt{ab}} e^{-x^2}\,dx \leqslant 2\int_{a}^{\frac{a+b}{2}} e^{-x^2}\,dx.$$
Therefore, for the right-hand side (RHS) of the proposed inequality we have
$$RHS \leqslant 4\left(\int_{\frac{a+b}{2}}^{b} e^{-x^2}\,dx\right)\left(\int_{a}^{\frac{a+b}{2}} e^{-x^2}\,dx\right).$$
Again, by the GM-AM inequality,
$$RHS \leqslant 4\frac{\left(\int_{a}^{\frac{a+b}{2}} e^{-x^2}\,dx + \int_{\frac{a+b}{2}}^{b} e^{-x^2}\,dx\right)^2}{4} = \left(\int_{a}^{b} e^{-x^2}\,dx\right)^2.$$

Also solved by the problem proposer.

• **5824** *Proposed by Toyesh Prakash Sharma, Agra College, Agra, India.*

Calculate
$$I := \int_2^3 \left(\frac{\ln(x+1)\ln x}{x-1} + \frac{\ln x \ln(x-1)}{x+1} + \frac{\ln(x+1)\ln(x-1)}{x}\right) dx.$$

Solution 1 by Albert Stadler, Herrliberg, Switzerland.

Since
$$\frac{d}{dx}\left(\ln(x-1)\ln(x)\ln(x+1)\right) = \frac{\ln(x)\ln(x+1)}{x-1} + \frac{\ln(x-1)\ln(x+1)}{x} + \frac{\ln(x-1)\ln(x)}{x+1},$$
then
$$I = \ln(x-1)\ln(x)\ln(x+1)\Big|_{x=2}^{x=3} = \ln(2)\ln(3)\ln(4) = 2\ln^2(2)\ln(3).$$

Solution 2 by Ángel Plaza, Universidad de Las Palmas de Gran Canaria, Spain.

Since

$$\frac{1}{x-1} = \big(\ln(x-1)\big)',$$
$$\frac{1}{x+1} = \big(\ln(x+1)\big)'$$

and

$$\frac{1}{x} = (\ln x)',$$

it follows that

$$I = \int_2^3 d\left(\ln(x-1)\ln x \ln(x+1)\right) = \ln(x-1)\ln x\ln(x+1)\Big]_2^3 = \ln 2\ln 3\ln 4 = 2(\ln 2)^2\ln 3.$$

Solution 3 by Brian D. Beasley, Simpsonville, SC.

We let $u = \ln(x+1)$, $v = \ln x$, and $w = \ln(x-1)$. Then the given integral becomes

$$I = \int_2^3 \left(uv\cdot\frac{dw}{dx} + vw\cdot\frac{du}{dx} + uw\cdot\frac{dv}{dx}\right)dx = \int_2^3 \left[\frac{d}{dx}(uvw)\right]dx = uvw|_2^3 = \ln(4)\ln(3)\ln(2).$$

Solution 4 by Michael Faleski, Delta College, University Center, MI.

To handle the first integral, we can use integration by parts with

$$u = \ln(x+1)\ln x \to du = \left(\frac{\ln x}{x+1} + \frac{\ln(x+1)}{x}\right)dx$$
$$dv = \frac{1}{x-1}dx \to v = \ln(x-1).$$

Upon substituting this result into the original expression gives

$$\begin{aligned}
I &= \int_2^3 \left(\frac{\ln(x+1)\ln x}{x-1} + \frac{\ln x\ln(x-1)}{x+1} + \frac{\ln(x+1)\ln(x-1)}{x}\right)dx \\
&= \ln(x-1)\ln(x)\ln(x+1)|_2^3 \\
&+ \int_2^3 \left(-\frac{\ln x\ln(x-1)}{x+1} - \frac{\ln(x+1)\ln(x-1)}{x} + \frac{\ln x\ln(x-1)}{x+1} + \frac{\ln(x+1)\ln(x-1)}{x}\right)dx \\
&= \ln(2)\ln(3)\ln(4) - \ln(1)\ln(2)\ln(3) \\
&= \ln(2)\ln(3)\ln(4).
\end{aligned}$$

Solution 5 by Michel Bataille, Rouen, France.

Let $f(x) = \ln(x-1)\ln(x+1)\ln x$ and recall the formula for the derivative of a product: $(uvw)' = u'vw + uv'w + uvw'$. We deduce that

$$f'(x) = \frac{\ln(x+1)\ln x}{x-1} + \frac{\ln x \ln(x-1)}{x+1} + \frac{\ln(x+1)\ln(x-1)}{x}$$

and it follows that

$$I = \int_2^3 f'(x)\,dx = [f(x)]_2^3 = 2(\ln 2)^2 \ln 3.$$

Solution 6 by Patrick Farrell, F-Cubed Math Tutors, West Kingston, RI.

Let $u = \ln(x+1)\ln(x)\ln(x-1)$. Then

$$du = \frac{\ln(x+1)\ln x}{x-1} + \frac{\ln(x)\ln(x-1)}{x+1} + \frac{\ln(x+1)\ln(x-1)}{x}\ dx$$

and I can be rewritten and evaluated as follows:

$$I = \int_0^{\ln 4 \ln 3 \ln 2} du = \ln 4 \ln 3 \ln 2.$$

Solution 7 by Péter Fülöp, Gyömrő, Hungary.

The integral can be devided into three parts (I_1,I_2,I_3):

$$I = \int_2^3 \Big(\frac{\ln(x+1)\ln(x)}{x-1})dx + \int_2^3 \frac{\ln(x)\ln(x-1)}{x+1}dx + \int_2^3 \frac{\ln(x+1)\ln(x-1)}{x}\Big)dx$$

Let's integrate by parts I_1 in the following way:

$$u = \ln(1+x)\ln(x) \qquad \text{and} \qquad v' = \frac{1}{x-1}$$

then

$$u' = \frac{\ln(x)}{x+1} + \frac{\ln(x+1)}{x} \qquad \text{and} \qquad v = \ln(x-1)$$

$$I_1 = \Big[\ln(x+1)\ln(x)\ln(x-1)\Big]_2^3 - \underbrace{\int\limits_2^3 \frac{\ln(x)\ln(x-1)}{x+1} + \frac{\ln(x+1)\ln(x-1)}{x}dx}_{I_2+I_3}.$$

It can be seen that $I_2 + I_3$ term are vanished from I. The the result is

$$I = 2\ln^2(2)\ln(3)$$

Solution 8 by Prakash Pant, The University of Vermont, Bardiya, Nepal.

We rewrite the problem as:

$$= \int_2^3 \ln(x)\left(\frac{\ln(x+1)}{x-1} + \frac{\ln(x-1)}{x+1}\right)dx + \int_2^3 \frac{\ln(x+1)\ln(x-1)}{x}dx$$

$$= \int_2^3 \ln(x)\frac{d}{dx}\big[\ln(x+1)\ln(x-1)\big]\,dx + \int_2^3 \frac{\ln(x+1)\ln(x-1)}{x}dx$$

Using integration by parts on the first integral,

$$= \ln(x)\ln(x+1)\ln(x-1)\big|_2^3 - \int_2^3 \frac{1}{x}\ln(x+1)\ln(x-1)dx + \int_2^3 \frac{\ln(x+1)\ln(x-1)}{x}dx$$

Notice the elegant cancellation of two integrals,

$$= \ln(3)\ln(4)\ln(2) - \ln(2)\ln(3)\ln(1) = \ln(2)\ln(3)\ln(4)$$

which is the required answer.

Solution 9 by Sri Hari Bhupala Haribhakta, Bengaluru, India.

Consider the function

$$F(x) := \ln(x-1)\ \ln x\ \ln(x+1), \qquad (x > 1).$$

Differentiating by the product rule for three factors, we obtain

$$F'(x) = \frac{1}{x-1}\ \ln x\ \ln(x+1) + \frac{1}{x}\ \ln(x-1)\ \ln(x+1) + \frac{1}{x+1}\ \ln(x-1)\ \ln x$$
$$= \frac{\ln(x+1)\ln x}{x-1} + \frac{\ln(x+1)\ln(x-1)}{x} + \frac{\ln x\ln(x-1)}{x+1}.$$

Thus the integrand is precisely $F'(x)$, and therefore

$$I = \int_2^3 F'(x)\,dx = F(3) - F(2).$$

Now

$$F(3) = \ln(2)\ \ln(3)\ \ln(4), \qquad F(2) = \ln(1)\ \ln(2)\ \ln(3) = 0,$$

so

$$I = \ln 2\ \ln 3\ \ln 4\,.$$

Also solved by Bruno Salgueiro Fanego, Viveiro, Lugo and the problem proposer.

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Editor's Statement: It goes without saying that the problem proposers, as well as the solution proposers, are the *élan vital* of the Problems/Solutions Section of SSMJ. As the editor of this Section of the Journal, I consider myself fortunate to be in a position to receive, compile and organize a wealth of proposed ingenious problems and solutions intended for online publication. My unwavering gratitude goes to all the amazingly creative contributors. We come together from across continents because we find intellectual value, joy and satisfaction in mathematical problems, both in their creation as well as their solution. So that our collective efforts serve us well, I kindly ask all contributors to adhere to the following guidelines. As you peruse below, you may construe that the guidelines amount to a lot of work. But, as the samples show, there's not much to do. Your cooperation is much appreciated!

Keep in mind that the examples given below are your best guide!

Formats, Styles and Requirements

When submitting proposed problem(s) or solution(s), please send both **LaTeX** document and **pdf** document of your proposed problem(s) or solution(s). There are ways (discoverable from the internet) to convert from Word to proper LaTeX code. Porposals without a *proper* **LaTeX** document will not be published regrettably.

Regarding Proposed Solutions:

Below is the FILENAME format for all the documents of your proposed solution(s).

#ProblemNumber_FirstName_LastName_Solution_SSMJ

- FirstName stands for YOUR first name.
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Examples:

#1234_Max_Planck_Solution_SSMJ

#9876_Charles_Darwin_Solution_SSMJ

Please note that every problem number is *preceded* by the sign # .

All you have to do is copy the FILENAME format (or an example below it), paste it and then modify portions of it to your specs.

Please adopt the following structure, in the order shown, for the presentation of your solution:

1. On top of the first page of your solution, begin with the phrase:

"Proposed Solution to #**** SSMJ"

where the string of four astrisks represents the problem number.

2. On the second line, write

"Solution proposed by [your First Name, your Last Name]",

followed by your affiliation, city, country, all on the same linear string of words. Please see the example below. Make sure you do the same for your collaborator(s).

3. On a new line, state the problem proposer's name, affiliation, city and country, just as it appears published in the Problems/Solutions section.

4. On a new line below the above, write in bold type: "**Statement of the Problem**".

5. Below the latter, state the problem. Please make sure the statement of your problem (unlike the preceding item) is not in bold type.

6. Below the statement of the problem, write in bold type: "**Solution of the Problem**".

7. Below the latter, show the entire solution of the problem.

Here is a sample for the above-stated format for proposed solutions:

Proposed solution to #1234 SSMJ

Solution proposed by Emmy Noether, University of Göttingen, Lower Saxony, Germany.

Problem proposed by Isaac Newton, Trinity College, Cambridge, England.

Statement of the problem:

Compute $\sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}$.

Solution of the problem: . . . . . .

Regarding Proposed Problems:

For all your proposed problems, please adopt for all documents the following FILENAME format:

FirstName_LastName_ProposedProblem_SSMJ_YourGivenNumber_ProblemTitle

If you do not have a ProblemTitle, then leave that component as it already is (i.e., ProblemTitle).

The component YourGivenNumber is any UNIQUE 3-digit (or longer) number you like to give to your problem.

Examples:

Max_Planck_ProposedProblem_SSMJ_314_HarmonicPatterns

Charles_Darwin_ProposedProblem_SSMJ_358_ProblemTitle

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3. On a new line state the title of the problem, if any.

4. On a new line below the above, write in bold type: "**Statement of the Problem**".

5. Below the latter, state the problem. Please make sure the statement of your problem (unlike the preceding item) is not in bold type.

6. Below the statement of the problem, write in bold type: "**Solution of the Problem**".

7. Below the latter, show the entire solution of your problem.

Here is a sample for the above-stated format for proposed problems:

Problem proposed to SSMJ

Problem proposed by Isaac Newton, Trinity College, Cambridge, England.

Principia Mathematica (⟵ You may choose to not include a title.)

Statement of the problem:

Compute $\displaystyle\sum_{k=0}^{n}\binom{n}{k}x^k y^{n-k}$.

Solution of the problem: . . . . . .

♣ ♣ ♣ Thank You! ♣ ♣ ♣