Problems and Solutions

Albert Natian, Section Editor

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This section of the Journal offers readers an opportunity to exchange interesting mathematical problems and solutions. Please email them to Prof. Albert Natian at Department of Mathematics, Los Angeles Valley College. Please make sure every proposed problem or proposed solution is provided in both ***LaTeX*** and pdf documents. Please make sure your proposals adhere to **Formats, Styles and Requirements** noted below. Thank you!

To propose problems, email them to: problems4ssma@gmail.com

To propose solutions, email them to: solutions4ssma@gmail.com

Solutions to the problems published in this issue should be submitted *before* December 1, 2026.

• **5845** *Proposed by Vasile Mircea Popa, Lucian Blaga University, Sibiu, Romania.*

Calculate the integral:

$$I := \int_0^\infty \frac{\sqrt{x}\,\arctan(x)}{\sqrt{x^6+1}}\,dx.$$

• **5846** *Proposed by Vasile Cîrtoaje, Petroleum-Gas University of Ploiesti, Romania.*

Let x_1, x_2, x_3, x_4, x_5 be nonngative real numbers such that at most one of them is larger than 1 and $x_1x_2 + x_2x_3 + x_3x_4 + x_4x_5 + x_5x_1 = 5$. Prove that

$$\frac{1}{x_1+2} + \frac{1}{x_2+2} + \frac{1}{x_3+2} + \frac{1}{x_4+2} + \frac{1}{x_5+2} \geqslant \frac{5}{3}.$$

• **5847** *Proposed by Daniel Sitaru, National Economic College "Theodor Costescu," Drobeta Turnu - Severin, Romania.*

Suppose $0 < a \leqslant b$. Show that

$$\int_a^b \int_a^b \frac{x}{y+\sqrt{xy}}dxdy + 2\int_a^b \int_a^b \frac{y}{3x+y}dxdy \geqslant (b-a)^2.$$

• **5848** *Proposed by Shivam Sharma, Delhi University, New Delhi, India.*

Where $\llbracket . \rrbracket$ denotes the Greatest Integer Function, evaluate:

$$J := \int_0^\infty \left(\frac{\llbracket x \rrbracket}{\llbracket x+1 \rrbracket} + \frac{x}{x+1} \right) \left(\frac{\llbracket x \rrbracket}{\llbracket x+1 \rrbracket} - \frac{x}{x+1} \right) dx.$$

• **5849** *Proposed by Toyesh Prakash Sharma, Agra College, Agra, India.*

Suppose $0 < \theta < \pi/2$. Show that

$$(\sin\theta)^{\cos\theta}(\cos\theta)^{\tan\theta}(\tan\theta)^{\sin\theta}(\sin\theta+\cos\theta+\tan\theta)^{\sin\theta+\cos\theta+\tan\theta} \leqslant \left(\sec^2\theta\right)^{\sin\theta+\cos\theta+\tan\theta}.$$

Solutions

To Formerly Published Problems

• **5825** *Proposed by Daniel Sitaru, National Economic College "Theodor Costescu" Drobeta Turnu - Severin, Romania.*

Let $(f_n)_{n=0}^{\infty}$ be the Fibonacci sequence; i.e., $f_0 = 0, f_1 = 1, f_{n+2} = f_{n+1} + f_n$ for $n \geqslant 0$. Prove that for all $n \geqslant 1$:

$$8\left(\sum_{k=1}^{n} f_k \sin\frac{1}{k}\right)\left(\sum_{k=1}^{n} f_k \cos\frac{1}{k}\right) < n f_{n+2}^2.$$

Solution 1 by Ángel Plaza, Universidad de Las Palmas de Gran Canaria, Spain.

By the AM-GM inequality,

$$2\left(\sum_{k=1}^{n} f_k \sin\frac{1}{k}\right)\left(\sum_{k=1}^{n} f_k \cos\frac{1}{k}\right) \leqslant \left(\sum_{k=1}^{n} f_k \sin\frac{1}{k}\right)^2 + \left(\sum_{k=1}^{n} f_k \cos\frac{1}{k}\right)^2.$$

Now, by the Cauchy-Schwartz inequality,

$$\begin{aligned}\left(\sum_{k=1}^{n} f_k \sin\frac{1}{k}\right)^2 &\leqslant \left(\sum_{k=1}^{n} f_k^2\right)\cdot\left(\sum_{k=1}^{n}\sin^2\frac{1}{k}\right),\\ \left(\sum_{k=1}^{n} f_k \cos\frac{1}{k}\right)^2 &\leqslant \left(\sum_{k=1}^{n} f_k^2\right)\cdot\left(\sum_{k=1}^{n}\cos^2\frac{1}{k}\right).\end{aligned}$$

From where, the left-hand side of the proposed inequality, say LHS verifies

$$LHS \leqslant 4\left(\sum_{k=1}^{n} f_k^2\right)\cdot\left(\sum_{k=1}^{n}\sin^2\frac{1}{k}+\sum_{k=1}^{n}\cos^2\frac{1}{k}\right) = 4n f_n f_{n+1}$$

because $\sin^2 \dfrac{1}{k} + \cos^2 \dfrac{1}{k} = 1$ and $\sum_{k=1}^{n} f_k^2 = f_n f_{n+1}$. Since $4 f_n f_{n+1} < (f_n + f_{n+1})^2 = f_{n+2}^2$ by the AM-GM inequality, the conclusion follows.

Solution 2 by Brian D. Beasley, Simpsonville, SC.

For each $n \geqslant 1$, let $S_n = \sum_{k=1}^{n} f_k \sin \dfrac{1}{k}$ and $C_n = \sum_{k=1}^{n} f_k \cos \dfrac{1}{k}$. Since $\sum_{k=1}^{n} f_k = f_{n+2} - 1$, we have

$$S_n < f_{n+2} - 1 \quad \text{and} \quad C_n < f_{n+2} - 1.$$

For $n \in \{1, 2\}$, direct calculations show that

$$8 S_n C_n < n f_{n+2}^2.$$

For $n \in \{3, 4, 5, 6, 7\}$, direct calculations show that

$$8 S_n C_n < 8 S_n (f_{n+2} - 1) < n f_{n+2}^2.$$

For $n \geqslant 8$, we have

$$8 S_n C_n < 8 (f_{n+2} - 1)^2 < 8 f_{n+2}^2 \leqslant n f_{n+2}^2.$$

Addendum. Calculations:

n	$8S_nC_n$	$8S_n(f_{n+2}-1)$	nf_{n+2}^2
1	3.637	(6.732)	4
2	14.983	(21.134)	18
3	52.271	63.209	75
4	135.104	152.180	256
5	329.964	356.241	845
6	765.940	806.082	2646
7	1756.468	1818.655	8092

Solution 3 by Jose Luis Diaz-Barrero, Barcelona, Spain. Let $A = \sum_{k=1}^{n} f_k \sin \dfrac{1}{k}$ and $B = \sum_{k=1}^{n} f_k \cos \dfrac{1}{k}$. From $(A - B)^2 \geqslant 0$ we obtain $8AB \leqslant 4(A^2 + B^2)$ and

$$\begin{aligned} A^2 + B^2 &= \sum_{k=1}^{n} f_k^2 \sin^2 \frac{1}{k} + \sum_{k=1}^{n} f_k^2 \cos^2 \frac{1}{k} \\ &+ 2 \sum_{1 \leqslant i < j \leqslant n} f_i f_j \left(\sin \frac{1}{i} \sin \frac{1}{j} + \cos \frac{1}{i} \cos \frac{1}{j} \right) \end{aligned}$$

$$= \sum_{k=1}^{n} f_k^2 (\sin^2 \frac{1}{k} + \cos^2 \frac{1}{k}) + 2 \sum_{1 \leqslant i < j \leqslant n} f_i f_j \left(\sin \frac{1}{i} \sin \frac{1}{j} + \cos \frac{1}{i} \cos \frac{1}{j} \right).$$

Using the well-known formulae $\sin^2 x + \cos^2 x = 1$ and $\sin a \sin b + \cos a \cos b = \cos(a-b)$, we get

$$A^2 + B^2 = \sum_{k=1}^{n} f_k^2 + 2\sum_{1\leqslant i<j\leqslant n} f_i f_j \cos\left(\frac{1}{i} - \frac{1}{j}\right) \leqslant \sum_{k=1}^{n} f_k^2 + 2\sum_{1\leqslant i<j\leqslant n} f_i f_j.$$

Thus

$$A^2 + B^2 \leqslant \left(\sum_{k=1}^{n} f_k\right)^2, \text{ and therefore } \quad 8AB \leqslant 4\left(\sum_{k=1}^{n} f_k\right)^2.$$

Since $\sum_{k=1}^{n} f_k = f_{n+2} - 1$, as it is well-known, then $8AB \leqslant 4(f_{n+2}-1)^2$. So it suffices to show

$$4(f_{n+2}-1)^2 < n f_{n+2}^2 \quad \Longleftrightarrow \quad 4\left(1 - \frac{1}{f_{n+2}}\right)^2 < n.$$

To do that we will distinguish the following cases:

- For $n = 1$ we have $8\sin 1\cos 1 = 4\sin 2 < 4 = f_3^2$.
- For $n = 2$ we have $8(\sin 1 + \sin\frac{1}{2})(\cos 1 + \cos\frac{1}{2}) < 18 = 2f_4^2$.
- For $n = 3$ we have $8(\sin 1 + \sin\frac{1}{2} + 2\sin\frac{1}{3})(\cos 1 + \cos\frac{1}{2} + 2\cos\frac{1}{3}) < 75 = 3f_5^2$.
- For $n \geqslant 4$, we have $f_{n+2} \geqslant 3$, so

$$0 < 1 - \frac{1}{f_{n+2}} < 1 \quad \Rightarrow \quad 4\left(1 - \frac{1}{f_{n+2}}\right)^2 < 4 \leqslant n.$$

  Thus the inequality holds for all $n \geqslant 4$.

Finally, we conclude that the inequality holds for every $n \geqslant 1$.

Solution 4 by Michel Bataille, Rouen, France.

Let $S_n = \sum_{k=1}^{n} f_k \sin\frac{1}{k}$ and $C_n = \sum_{k=1}^{n} f_k \cos\frac{1}{k}$. The general inequality $4xy \leqslant (x+y)^2$ yields

$$4S_nC_n \leqslant \left(\sum_{k=1}^{n} f_k\left(\sin\frac{1}{k} + \cos\frac{1}{k}\right)\right)^2 \tag{1}$$

and the Cauchy-Schwarz inequality gives

$$\left(\sum_{k=1}^{n} f_k\left(\sin\frac{1}{k} + \cos\frac{1}{k}\right)\right)^2 \leqslant \left(\sum_{k=1}^{n} f_k^2\right)\left(\sum_{k=1}^{n}\left(\sin\frac{1}{k} + \cos\frac{1}{k}\right)^2\right). \tag{2}$$

Now, we have $\sum_{k=1}^{n} f_k^2 = f_n f_{n+1}$ (by an easy induction) and

$$\sum_{k=1}^{n}\left(\sin\frac{1}{k}+\cos\frac{1}{k}\right)^2=\sum_{k=1}^{n}\left(1+\sin\frac{2}{k}\right)<2n$$

(since $\sin\frac{2}{k}<1$ for $k=1,2,\ldots,n$). From (1) and (2), we now deduce

$$8S_nC_n<\frac{1}{2}\cdot 4f_nf_{n+1}\cdot 2n\leqslant n(f_n+f_{n+1})^2=nf_{n+2}^2,$$

and the desired inequality $8S_nC_n<nf_{n+2}^2$ follows.

Solution 5 by Ulrich Abel, Technische Hochschule Mittelhessen, Friedberg, Germany.

Taking advantage of the well-known formula $\sum_{k=1}^{n} f_k = f_{n+2}-1$ and the obvious relations $0\leqslant \sin\frac{1}{k}\leqslant \sin 1\approx 0.84$ and $0\leqslant\cos\frac{1}{k}\leqslant 1$ for all $k\geqslant 1$, we conclude that

$$\left(\sum_{k=1}^{n} f_k\sin\frac{1}{k}\right)\left(\sum_{k=1}^{n} f_k\cos\frac{1}{k}\right)\leqslant(\sin 1)\left(\sum_{k=1}^{n} f_k\right)\left(\sum_{k=1}^{n} f_k\right)=(\sin 1)\left(f_{n+2}-1\right)^2\leqslant(\sin 1)\,f_{n+2}^2.$$

Observing that $8\sin 1\approx 6.73$ we see that this implies the inequality of Problem #5825, for all $n\geqslant 7$. The finitely many cases $n\leqslant 6$ follow by direct computation.

Remark: The quantity

$$S_n:=\left(\sum_{k=1}^{n} f_k\sin\frac{1}{k}\right)\left(\sum_{k=1}^{n} f_k\cos\frac{1}{k}\right)$$

satisfies the (rough) asymptotic relation

$$\log S_n^{1/n}\sim 2\log\phi \qquad (n\to\infty)\,,$$

where $\phi=\left(1+\sqrt{5}\right)/2$ is the Golden Ratio. From

$$\begin{aligned}\left(\sin\frac{1}{n}\right)(\cos 1)\left(1-\frac{1}{f_{n+2}}\right)^2 f_{n+2}^2 &\leqslant \left(\sin\frac{1}{n}\right)(\cos 1)\left(\sum_{k=1}^{n} f_k\right)^2\leqslant S_n\\ &\leqslant (\sin 1)\left(\cos\frac{1}{n}\right)\left(\sum_{k=1}^{n} f_k\right)^2\leqslant f_{n+2}^2,\end{aligned}$$

it follows that

$$a\,(n)+\frac{1}{n}\log f_{n+2}^2\leqslant\frac{1}{n}\log S_n\leqslant\frac{1}{n}\log f_{n+2}^2,$$

where

$$a(n)=\frac{1}{n}\log\left(\sin\frac{1}{n}\right)+\frac{1}{n}\log(\cos 1)+\frac{1}{n}\log\left(1-\frac{1}{f_{n+2}}\right)^2=o(1)$$

as $n\to\infty$. For the first term we used that $\left|\log\left(\sin\frac{1}{n}\right)\right|\sim\log n=o(n)$ as $n\to\infty$. Therefore,

$$\frac{1}{n}\log S_n=\frac{1}{n}\log f_{n+2}^2+o(1)\qquad(n\to\infty).$$

It is a consequence of the Moivre–Binet representation that $f_n\sim\phi^n/\sqrt{5}$ as $n\to\infty$. We infer that

$$\frac{1}{n}\log S_n=\frac{1}{n}\log\phi^{2n+2}-\frac{1}{n}\log\sqrt{5}+o(1)=\frac{2n+2}{n}\log\phi+o(1)=2\log\phi+o(1)$$

as $n\to\infty$, which proves our claim.

Also solved by Albert Stadler, Herrliberg, Switzerland and the problem proposer.

• **5826** *Proposed by Jose Luis Diaz-Barrero, Barcelona, Spain.*

Let $A(z)=\sum_{k=0}^{n}a_kz^k$ $(a_k\neq 0\ \forall k)$ be a non-constant polynomial with complex coefficients. Show that all its zeros lie in the annulus $C=\{z\in\mathbb{C}:\ r_1\leqslant|z|\leqslant r_2\}$, where

$$r_1=\min_{1\leqslant k\leqslant n}\left\{\frac{5^k\binom{n}{k}}{6^n-1}\left|\frac{a_0}{a_k}\right|\right\}^{1/k}\quad\text{and}\quad r_2=\max_{1\leqslant k\leqslant n}\left\{\frac{6^n-1}{5^k\binom{n}{k}}\left|\frac{a_{n-k}}{a_n}\right|\right\}^{1/k}.$$

Solution 1 by Albert Stadler, Herrliberg, Switzerland.

Assume, for contradiction, that z is a zero of $A(z)$ such that $|z|<r_1$. Since $A(z)=0$, we can write

$$a_0=-\sum_{k=1}^{n}a_kz^k.$$

Taking absolute values and applying the triangle inequality gives

$$|a_0|=\left|\sum_{k=1}^{n}a_kz^k\right|\leqslant\sum_{k=1}^{n}|a_k|\,|z|^k.$$

Because $|z|<r_1$, we obtain

$$|a_0|<\sum_{k=1}^{n}|a_k|\,r_1^k.$$

By the definition of r_1, for each $k = 1, \ldots, n$,

$$r_1^k \leqslant \frac{5^k \binom{n}{k}}{6^n - 1} \left| \frac{a_0}{a_k} \right|.$$

Multiplying both sides by $|a_k|$, we get

$$|a_k| \, r_1^k \leqslant \frac{5^k \binom{n}{k}}{6^n - 1} \, |a_0| \, .$$

Summing over $k = 1$ to n, it follows that

$$\sum_{k=1}^{n} |a_k| \, r_1^k \leqslant \frac{|a_0|}{6^n - 1} \sum_{k=1}^{n} 5^k \binom{n}{k}.$$

Using the binomial identity,

$$\sum_{k=0}^{n} 5^k \binom{n}{k} = (1+5)^n = 6^n,$$

we obtain

$$\sum_{k=1}^{n} 5^k \binom{n}{k} = 6^n - 1.$$

Hence,

$$\sum_{k=1}^{n} |a_k| \, r_1^k \leqslant |a_0| \, .$$

Combining the inequalities, we arrive at

$$|a_0| < |a_0| \, ,$$

a contradiction. Therefore, every zero z of $A(z)$ satisfies

$$|z| \geqslant r_1.$$

Now suppose that z is a zero of $A(z)$ such that

$$|z| > r_2.$$

Define the reversed polynomial

$$A^* \left(z\right) = z^n A \left(\frac{1}{z}\right) = \sum_{k=0}^{n} a_{n-k} z^k.$$

Then A^* is also a polynomial of degree n, and since $A(z) = 0$, it follows that

$$A^* \left(\frac{1}{z}\right) = 0,$$

so $\dfrac{1}{z}$ is a zero of A^*. Applying the first part of the proof to A^*, we obtain

$$\frac{1}{|z|} \geqslant \min_{1\leqslant k\leqslant n} \left\{ \frac{5^k\binom{n}{k}}{6^n-1}\left|\frac{a_n}{a_{n-k}}\right| \right\}^{1/k}$$

This inequality is equivalent to

$$|z| \leqslant \max_{1\leqslant k\leqslant n} \left\{ \frac{6^n-1}{5^k\binom{n}{k}}\left|\frac{a_{n-k}}{a_n}\right| \right\}^{1/k} = r_2.$$

Solution 2 by Michel Bataille, Rouen, France. We remark that $6^n-1=(1+5)^n-1=\sum_{k=1}^{n}\binom{n}{k}5^k$. Let z be a complex number with $|z|<r_1$. Then $|z|^k < \dfrac{5^k\binom{n}{k}}{6^n-1}\left|\dfrac{a_0}{a_k}\right|$ for $k=1,2,\ldots,n$, hence

$$|A(z)| \geqslant |a_0| - \sum_{k=1}^{n}|a_k||z|^k > |a_0|\left(1-\sum_{k=1}^{n}\frac{|a_k|}{|a_0|}\cdot\frac{5^k\binom{n}{k}}{6^n-1}\left|\frac{a_0}{a_k}\right|\right) = |a_0|\left(1-\frac{\sum_{k=1}^{n}5^k\binom{n}{k}}{6^n-1}\right) = 0.$$

Therefore any zero z_0 of $A(z)$ must satisfy $|z_0|\geqslant r_1$.

Now, consider the polynomial $B(z)=|a_n|z^n-\sum_{k=1}^{n}|a_{n-k}|z^{n-k}$ and let z be such that $|z|>r_2$. Then $\dfrac{1}{|z|^k} < \dfrac{5^k\binom{n}{k}}{6^n-1}\left|\dfrac{a_n}{a_{n-k}}\right|$ for $k=1,2,\ldots,n$ so that

$$B(|z|) = |a_n||z|^n\left(1-\sum_{k=1}^{n}\frac{|a_n|}{|a_{n-k}|}\cdot\frac{1}{|z|^k}\right) > |a_n||z|^n\left(1-\frac{\sum_{k=1}^{n}5^k\binom{n}{k}}{6^n-1}\right) = 0.$$

This said, the function f defined by $f(x)=\dfrac{B(x)}{x^n}=|a_n|-\sum_{k=1}^{n}\dfrac{|a_{n-k}|}{x^k}$ is continuous and increasing on $(0,\infty)$ with $\lim_{x\to 0^+} f(x)=-\infty$ and $\lim_{x\to\infty} f(x)=|a_n|>0$. It follows that for some $r>0$, we have $B(x)<0$ if $0<x<r$, $B(r)=0$ and $B(x)>0$ if $x>r$. We deduce that if $|z|>r_2$ then we also have $|z|>r$ (since $B(|z|)>0$).

But, since $|A(z)| \geqslant |a_nz^n| - \left|\sum_{k=1}^{n}a_{n-k}z^{n-k}\right| \geqslant B(|z|)$, any zero z_0 of $A(z)$ satisfies $B(|z_0|)\leqslant 0$, hence $|z_0|\leqslant r$. Thus, we must have $A(z)\neq 0$ whenever $|z|>r_2$. We conclude that any zero z_0 of $A(z)$ must satisfy $r_1\leqslant|z_0|\leqslant r_2$.

Also solved by Bruno Salgueiro Fanego, Viveiro, Lugo and the problem proposer.

• **5827** *Proposed by Shivam Sharma, Delhi University, New Delhi, India.*

Find the limits L and M:

$$L = \lim_{n\to\infty} \left(\sqrt[n]{\prod_{k=2}^{n+1} \left(\frac{k-1}{k+1}\right)^{k+1}} \right),$$

$$M = \lim_{n\to\infty} \frac{n}{\ln(n)} \left(L - \sqrt[n]{\prod_{k=2}^{n+1} \left(\frac{k-1}{k+1}\right)^{k+1}} \right).$$

Solution 1 by Sri Hari, Bhupala Haribhakta, Aumovio, Bengaluru, India.

1. Evaluation of Limit L: Let x_n denote the argument of the above limit; i.e.,

$$x_n = \left(\prod_{k=2}^{n+1} \left(\frac{k-1}{k+1}\right)^{k+1} \right)^{\frac{1}{n}}.$$

We take the natural logarithm to convert the product into a sum:

$$\ln(x_n) = \frac{1}{n} \sum_{k=2}^{n+1} (k+1) \ln\left(\frac{k-1}{k+1}\right).$$

We rewrite the argument of the logarithm as $1 - \dfrac{2}{k+1}$. Apply the Taylor expansion

$$\ln(1-u) = -u - \frac{u^2}{2} - O(u^3)$$

for small u:

$$\ln\left(1 - \frac{2}{k+1}\right) = -\frac{2}{k+1} - \frac{2}{(k+1)^2} - O\left(\frac{1}{k^3}\right).$$

Multiply the latter by $(k+1)$:

$$(k+1)\ln\left(\frac{k-1}{k+1}\right) = -2 - \frac{2}{k+1} - O\left(\frac{1}{k^2}\right).$$

Now, summing from $k = 2$ to $n+1$, we get:

$$\sum_{k=2}^{n+1} (k+1)\ln\left(\frac{k-1}{k+1}\right) = \sum_{k=2}^{n+1} (-2) - \sum_{k=2}^{n+1} \frac{2}{k+1} - \sum O\left(\frac{1}{k^2}\right),$$

where the first term sums to $-2n$. The second term is a harmonic sum, which behaves asymptotically as $2\ln n$. The remaining terms converge to a constant. Thus:

$$n\ln(x_n) = -2n - 2\ln n + O(1).$$

Dividing by n, we get:

$$\ln(x_n) = -2 - \frac{2\ln n}{n} + O\left(\frac{1}{n}\right).$$

Taking the limit as $n \to \infty$, we get:

$$\lim_{n\to\infty} \ln(x_n) = -2 \implies \boxed{L = e^{-2}}.$$

2. Evaluation of Limit M: We see that

$$M = \lim_{n\to\infty} \frac{n}{\ln n}(L - x_n).$$

From our previous expansion, we have $\ln(x_n) = -2 - \epsilon_n$, where $\epsilon_n = \dfrac{2\ln n}{n} - O(\dfrac{1}{n})$. Exponentiating gives:

$$x_n = e^{-2-\epsilon_n} = e^{-2}e^{-\epsilon_n}$$

Using the approximation $e^{-u} \approx 1 - u$ as $u \to 0$:

$$x_n \approx e^{-2}(1 - \epsilon_n) = e^{-2} - e^{-2}\left(\frac{2\ln n}{n}\right).$$

Rearranging for $L - x_n$ (where $L = e^{-2}$):

$$L - x_n \approx e^{-2}\frac{2\ln n}{n}.$$

Substituting this into the expression for M, we get:

$$M = \lim_{n\to\infty} \frac{n}{\ln n}\left(e^{-2}\frac{2\ln n}{n}\right) = 2e^{-2}.$$

Solution 2 by Michel Bataille, Rouen, France.

We show that $L = e^{-2}$ and $M = 2e^{-2}$.

Let $P_n = \displaystyle\prod_{k=2}^{n+1}\left(\frac{k-1}{k+1}\right)^{k+1} = \frac{\prod_{k=1}^{n} k^{k+2}}{\prod_{k=3}^{n+2} k^k}$. We readily obtain

$$P_n = \frac{4(n!)^2}{(n+1)^{n+1}(n+2)^{n+2}}.$$

Recall that $n! \sim \sqrt{2\pi n}e^{-n}n^n$ as $n \to \infty$. Since for $a > 0$,

$$\lim_{n\to\infty} a^{1/n} = \lim_{n\to\infty} \frac{n^2}{(n+1)(n+2)} = \lim_{n\to\infty} n^{1/n} = \lim_{n\to\infty} \left(1 + \frac{a}{n}\right)^{a/n} = 1$$

we see that

$$P_n^{1/n} = \frac{4^{1/n}(n!)^{2/n}}{(n+1)(n+2)n^{3/n}\left(1+\frac{1}{n}\right)^{1/n}\left(1+\frac{2}{n}\right)^{2/n}} \sim e^{-2}$$

as $n \to \infty$. Thus $L = e^{-2}$.

We have

$$\ln\left(P_n^{1/n}\right) = \frac{2\ln(2)}{n} + \frac{2}{n}\ln(n!) - \ln(n+1) - \ln(n+2) - \frac{3\ln(n)}{n} - \frac{\ln(1+1/n)}{n} - \frac{2\ln(1+2/n)}{n}$$

Using the well-known $\ln(n!) = n\ln(n) - n + \dfrac{\ln(n)}{2} + o(\ln(n))$ as $n \to \infty$, we obtain

$$\ln\left(P_n^{1/n}\right) = -2 - 2\cdot\frac{\ln(n)}{n} + o((\ln(n))/n).$$

(Note that $\ln(1 + a/n) \sim \dfrac{a}{n}$, $\dfrac{1}{n} = o((\ln(n))/n)$ and $\ln(n+1) + \ln(n+2) = 2\ln(n) + o(1/n)$ as $n \to \infty$.) We deduce that

$$e^{-2} - P_n^{1/n} = e^{-2}\left(1 - e^{-2\cdot\frac{\ln(n)}{n} + o((\ln(n))/n)}\right) \sim e^{-2}\cdot 2\cdot\frac{\ln(n)}{n}$$

as $n \to \infty$. The result $M = 2e^{-2}$ follows at once.

Solution 3 by Ángel Plaza, Universidad de Las Palmas de Gran Canaria, Spain.

$\ln L = \lim_{n\to\infty} \dfrac{\sum_{k=2}^{n+1} \ln\left(\frac{k-1}{k+1}\right)^{k+1}}{n}$. Now, by the Stolz-Cezaro Lemma $\lim_{n\to\infty} \ln\left(\dfrac{n}{n+2}\right)^{n+2} = -2$, so $L = e^{-2}$.

If P_n denotes the product $\prod_{k=2}^{n+1}\left(\dfrac{k-1}{k+1}\right)^{k+1}$. Then, $P_n = \dfrac{1\cdot 2^4\cdot 3^5\cdot\ldots\cdot n^{n+2}}{3^3\cdot 4^4\cdot\ldots\cdot(n+1)^{n+1}(n+2)^{n+2}} = \dfrac{1\cdot 2^4\cdot 3^2\cdot\ldots\cdot n^2}{(n+1)^{n+1}(n+2)^{n+2}} = \dfrac{4(n!)^2}{(n+1)^{n+1}(n+2)^{n+2}}$.

By the Stirling's approximation, $\ln n! = n\ln n - n + \dfrac{1}{2}\ln(2\pi n) + O(1)$, so
$2\ln n! = 2n\ln n - 2n + \ln(2\pi) + \ln n + O(1)$,
$(n+1)\ln(n+1) = (n+1)(\ln n + \ln(1+1/n)) = n\ln n + \ln n + 1 + O(1/n)$,
$(n+2)\ln(n+2) = (n+2)(\ln n + \ln(1+2/n)) = n\ln n + 2\ln n + 2 + O(1/n)$,
so, $\ln\sqrt[n]{P_n} = \dfrac{1}{n}\left(-2n - 2\ln n + O(1)\right)$. Therefore,

$$\sqrt[n]{P_n} = e^{-2-\frac{2\ln n}{n}+\frac{C}{n}} = e^{-2}\cdot e^{-\frac{2\ln n}{n}+\frac{C}{n}} \sim e^{-2}\left(1 - \frac{2\ln n}{n} + \frac{C}{n}\right).$$

Then

$$\begin{aligned}
M &= \lim_{n\to\infty} \frac{n}{\ln n}\left(e^{-2} - \left(e^{-2} - \frac{2e^{-2}\ln n}{n} + \frac{Ce^{-2}}{n}\right)\right) \\
&= \lim_{n\to\infty} \frac{n}{\ln n}\left(\frac{2e^{-2}\ln n}{n} - \frac{Ce^{-2}}{n}\right) \\
&= \lim_{n\to\infty} \left(2e^{-2} - \frac{Ce^{-2}}{\ln n}\right) \\
&= 2e^{-2}.
\end{aligned}$$

Also solved by Albert Stadler, Herrliberg, Switzerland and the problem proposer.

• **5828** *Proposed by Toyesh Prakash Sharma, Agra College, Agra, India.*

If $a, b, c \geqslant 5$ and $a + b + c = 18$ then show that

$$a^{1/a}b^{1/b} + b^{1/b}c^{1/c} + c^{1/c}a^{1/a} \geqslant 3\sqrt[3]{6}.$$

Solution 1 by Albert Stadler, Herrliberg, Switzerland.

By the AM–GM inequality,

$$a^{1/a}b^{1/b} + b^{1/b}c^{1/c} + c^{1/c}a^{1/a} \geqslant 3(a^{1/a}b^{1/b}c^{1/c})^{2/3}.$$

Thus it is enough to prove that

$$a^{1/a}b^{1/b}c^{1/c} \geqslant \sqrt{6}.$$

To establish this, note that the function

$$f(x) = \frac{\ln x}{x}$$

is convex for all $x \geqslant 5$. Indeed,

$$f''(x) = \frac{-3 + 2\ln x}{x^3} > 0 \qquad (x \geqslant 5).$$

Applying Jensen's inequality to the convex function f, and using $a + b + c = 18$, we obtain

$$\frac{\ln a}{a} + \frac{\ln b}{b} + \frac{\ln c}{c} \geqslant 3 \cdot \frac{\ln\left(\frac{a+b+c}{3}\right)}{\left(\frac{a+b+c}{3}\right)} = \ln\left(\sqrt{6}\right).$$

Exponentiating both sides gives

$$a^{1/a}b^{1/b}c^{1/c} \geqslant \sqrt{6}.$$

Equality holds if and only if $a = b = c = 6$.

Solution 2 by Ángel Plaza, Universidad de Las Palmas de Gran Canaria, Spain.

Notice that function $a^{1/a}$ is a positive decreasing and convex for $a \in [5, 8]$, and so the left-hand side of the proposed inequality is convex. Therefore, by Jensen's inequality,

$$a^{1/a}b^{1/b} + b^{1/b}c^{1/c} + c^{1/c}a^{1/a} \geqslant 3 \cdot 6^{1/6} \cdot 6^{1/6} = 3\sqrt[3]{6}.$$

Solution 3 by David A. Huckaby, Angelo State University, San Angelo, TX.

By the AM-GM inequality, $a^{1/a}b^{1/b} + b^{1/b}c^{1/c} + c^{1/c}a^{1/a} \geqslant 3\sqrt[3]{a^{2/a}b^{2/b}c^{2/c}}$, so it suffices to show that $a^{1/a}b^{1/b}c^{1/c} \geqslant \sqrt{6}$, or equivalently, that

$$\frac{1}{a}\ln a + \frac{1}{b}\ln b + \frac{1}{c}\ln c \geqslant \frac{1}{2}\ln 6. \tag{1}$$

Define $f(x) = \dfrac{\ln x}{x}$. Then $f''(x) = \dfrac{2\ln x - 3}{x^3}$, which is positive for $x > e^{3/2} \approx 4.48$.

So $f(x)$ is convex for $x \geqslant 5$. So by Jensen's inequality,

$$\frac{1}{3}f(a) + \frac{1}{3}f(b) + \frac{1}{3}f(c) \geqslant f\left(\frac{1}{3}(a+b+c)\right) = f(\frac{1}{3} \cdot 18) = f(6).$$

So $f(a) + f(b) + f(c) \geqslant 3f(6)$; that is, $\dfrac{1}{a}\ln a + \dfrac{1}{b}\ln b + \dfrac{1}{c}\ln c \geqslant 3 \cdot \dfrac{1}{6}\ln 6 = \dfrac{1}{2}\ln 6$, which is inequality (1).

Solution 4 by Michel Bataille, Rouen, France.

For positive x, $x^{1/x} = e^{\frac{\ln x}{x}}$, hence the left-hand side of the inequality $\mathcal{L}$ satisfies

$$\mathcal{L} = e^{\frac{\ln a}{a} + \frac{\ln b}{b}} + e^{\frac{\ln b}{b} + \frac{\ln c}{c}} + e^{\frac{\ln c}{c} + \frac{\ln a}{a}}.$$

From the convexity of the exponential function, we deduce

$$\mathcal{L} \geqslant 3e^{\frac{2}{3}\left(\frac{\ln a}{a} + \frac{\ln b}{b} + \frac{\ln c}{c}\right)}$$

and therefore it is sufficient to prove that

$$\frac{\ln a}{a} + \frac{\ln b}{b} + \frac{\ln c}{c} \geqslant \frac{\ln 6}{2}. \tag{1}$$

Now, let $f(x) = \dfrac{\ln x}{x}$. An easy calculation gives

$$f'(x) = x^{-2} - x^{-2}\ln x, \qquad f''(x) = 2x^{-3}\left(\ln x - \frac{3}{2}\right)$$

so that $f''(x) > 0$ when $x \geqslant 5$ and f is convex on the interval $[5, \infty)$. It follows that

$$\frac{\ln a}{a} + \frac{\ln b}{b} + \frac{\ln c}{c} = f(a) + f(b) + f(c) \geqslant 3f\left(\frac{a+b+c}{3}\right) = 3 \cdot \frac{\ln((a+b+c)/3)}{(a+b+c)/3} = \frac{\ln 6}{2}$$

(using the hypothesis $a + b + c = 18$). Thus, (1) holds and we are done.

Also solved by the problem proposer.

• **5829** *Proposed by Vasile Cîrtoaje, Petroleum-Gas University of Ploiesti, Romania.*

Find the largest positive value of the constant k such that the inequality

$$\frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2} \geqslant a^k + b^k + c^k$$

holds for any positive numbers a, b, c, with $ab + bc + ca = 3$, such that at most one of the numbers a, b, c is less than 1.

Solution by the problem proposer.

For $b = 1$, $c \in (0, 1]$ and $a = \dfrac{3-c}{1+c}$, the constraints are satisfied and the inequality is equivalent to $E(c) \geqslant 0$, where

$$E(c) = \frac{1}{a^2} + \frac{1}{c^2} - a^k - c^k.$$

Note that $c = 1$ implies $a = 1$. We have

$$a'(c) = \frac{-4}{(1+c)^2}, \qquad a'(1) = -1,$$

$$a''(c) = \frac{8}{(1+c)^2}, \qquad a''(1) = 1,$$

and

$$E'(c) = -\left(\frac{2}{a^3} + ka^{k-1}\right)a' - \frac{2}{c^3} - kc^{k-1},$$

$$E''(c) = \left(\frac{6}{a^4} - k(k-1)a^{k-2}\right)(a')^2 - \left(\frac{2}{a^3} + ka^{k-1}\right)a'' + \frac{6}{c^4} - k(k-1)c^{k-2}.$$

Since $c = 1$ implies $a = 1$, we have $E(1) = 0$, $E'(1) = 0$ and $E''(1) = 10+k-2k^2 = (5-2k)(2+k)$. Since $E(1) = E'(1) = 0$, the condition $E''(1) \geqslant 0$ is necessary to have $E(c) \geqslant 0$ for $c \in (0, 1]$. This condition implies $k \leqslant \dfrac{5}{2}$. To prove that $\dfrac{5}{2}$ is the largest positive value of k, we need to show that $F \geqslant 0$ for $x \geqslant y \geqslant 1 \geqslant z > 0$ such that $x^2y^2 + y^2z^2 + z^2x^2 = 3$, where

$$F = \frac{1}{x^4} + \frac{1}{y^4} + \frac{1}{z^4} - x^5 - y^5 - z^5.$$

For fixed x, assume that z and F are functions of y. We have

$$(x^2+y^2)zz' + (x^2+z^2)y = 0, \qquad z' = \frac{-(x^2+z^2)y}{(x^2+y^2)z}$$

and

$$F'(y) = \left(\frac{-4}{z^5} - 5z^4\right)z' - \frac{4}{y^5} - 5y^4 = \frac{(5z^9+4)(x^2+z^2)y}{(x^2+y^2)z^6} - \frac{5y^9+4}{y^5}.$$

We will show that $F'(y) \geqslant 0$, that is

$$\frac{(5z^9+4)y^6}{(5y^9+4)z^6} \geqslant \frac{x^2+y^2}{x^2+z^2}, \qquad \frac{(5z^9+4)y^6}{(5y^9+4)z^6} - 1 \geqslant \frac{x^2+y^2}{x^2+z^2} - 1,$$

$$\frac{4(y^6-z^6) - 5y^6z^6(y^3-z^3)}{(5y^9+4)z^6} \geqslant \frac{y^2-z^2}{x^2+z^2}.$$

This is true if

$$\frac{(y^2+yz+z^2)(4y^3+4z^3-5y^6z^6)}{(5y^9+4)z^6} \geqslant \frac{y+z}{x^2+z^2}.$$

Since $2(y^2+yz+z^2) \geqslant 3z(y+z)$ and $x^2+z^2 \geqslant 2z^2$, it suffices to show that

$$\frac{3z(y+z)(4y^3+4z^3-5y^6z^6)}{2(5y^9+4)z^6} \geqslant \frac{y+z}{2z^2},$$

that is

$$\frac{3(4y^3+4z^3-5y^6z^6)}{(5y^9+4)z^3} \geqslant 1, \qquad 12y^3+8z^3 \geqslant 15y^6z^6 + 5y^9z^3.$$

From

$$3 = x^2y^2 + y^2z^2 + z^2x^2 \geqslant 3(xyz)^{4/3},$$

we obtain $xyz \leqslant 1$, hence $y^2z \leqslant xyz \leqslant 1$. Thus, we have

$$12y^3+8z^3 - (15y^6z^6+5y^9z^3) \geqslant (y^2z)^3(12y^3+8z^3) - (15y^6z^6+5y^9z^3) = 7y^6z^3(y^3-z^3) \geqslant 0.$$

From $F'(y) \geqslant 0$, it follows that $F(y)$ is increasing and has the minimum value when y is minimum, hence when $y = 1$. So, it suffices to prove the required inequality for $y = 1$. We need to show that

$$\frac{1}{x^4} + \frac{1}{z^4} \geqslant x^5 + z^5$$

for $x \geqslant 1 \geqslant z > 0$ such that $x^2+z^2+x^2z^2 = 3$. Letting $S = x+z$ and $p = xz$, we need to show that

$$\frac{p^4-8p^2+9}{p^4} \geqslant S\,(p^4+p^3-7p^2-3p+9)$$

for $S^2 = 3+2p-p^2$. From $S^2 = 4-(1-p)^2 \leqslant 4$, we get $S \leqslant 2$ and $p \leqslant S^2/4 = 1$. So, it suffices to show that

$$p^4-8p^2+9 \geqslant 2p^4(p^4+p^3-7p^2-3p+9)$$

for $p \leqslant 1$. Since

$$p^4 - 8p^2 + 9 = 9(p^2 - 1)^2 + 2p^2(5 - 4p^2) \geqslant 2p^2(5 - 4p^2),$$

we have

$$p^4 - 8p^2 + 9 - 2p^4(p^4 + p^3 - 7p^2 - 3p + 9) \geqslant 2p^2(5 - 4p^2) - 2p^4(p^4 + p^3 - 7p^2 - 3p + 9)$$
$$= 2p^2(5 - 13p^2 + 3p^3 + 7p^4 - p^5 - p^6) = 2p^2(1 - p)^2(5 + 10p + 2p^2 - 3p^3 - p^4) \geqslant 0.$$

The proof is finished. If $k = \dfrac{5}{2}$, then the equality occurs for $a = b = c = 1$.

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Editor's Statement: It goes without saying that the problem proposers, as well as the solution proposers, are the *élan vital* of the Problems/Solutions Section of SSMJ. As the editor of this Section of the Journal, I consider myself fortunate to be in a position to receive, compile and organize a wealth of proposed ingenious problems and solutions intended for online publication. My unwavering gratitude goes to all the amazingly creative contributors. We come together from across continents because we find intellectual value, joy and satisfaction in mathematical problems, both in their creation as well as their solution. So that our collective efforts serve us well, I kindly ask all contributors to adhere to the following guidelines. As you peruse below, you may construe that the guidelines amount to a lot of work. But, as the samples show, there's not much to do. Your cooperation is much appreciated!

Keep in mind that the examples given below are your best guide!

Formats, Styles and Requirements

When submitting proposed problem(s) or solution(s), please send both **LaTeX** document and **pdf** document of your proposed problem(s) or solution(s). There are ways (discoverable from the internet) to convert from Word to proper LaTeX code. Porposals without a *proper* **LaTeX** document will not be published regrettably.

Regarding Proposed Solutions:

Below is the FILENAME format for all the documents of your proposed solution(s).

#ProblemNumber_FirstName_LastName_Solution_SSMJ

- FirstName stands for YOUR first name.
- LastName stands for YOUR last name.

Examples:

#1234_Max_Planck_Solution_SSMJ

#9876_Charles_Darwin_Solution_SSMJ

Please note that every problem number is *preceded* by the sign # .

All you have to do is copy the FILENAME format (or an example below it), paste it and then modify portions of it to your specs.

Please adopt the following structure, in the order shown, for the presentation of your solution:

1. On top of the first page of your solution, begin with the phrase:

"Proposed Solution to #**** SSMJ"

where the string of four astrisks represents the problem number.

2. On the second line, write

"Solution proposed by [your First Name, your Last Name]",

followed by your affiliation, city, country, all on the same linear string of words. Please see the example below. Make sure you do the same for your collaborator(s).

3. On a new line, state the problem proposer's name, affiliation, city and country, just as it appears published in the Problems/Solutions section.

4. On a new line below the above, write in bold type: "**Statement of the Problem**".

5. Below the latter, state the problem. Please make sure the statement of your problem (unlike the preceding item) is not in bold type.

6. Below the statement of the problem, write in bold type: "**Solution of the Problem**".

7. Below the latter, show the entire solution of the problem.

Here is a sample for the above-stated format for proposed solutions:

Proposed solution to #1234 SSMJ

Solution proposed by Emmy Noether, University of Göttingen, Lower Saxony, Germany.

Problem proposed by Isaac Newton, Trinity College, Cambridge, England.

Statement of the problem:

Compute $\sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}$.

Solution of the problem: . . . . . .

Regarding Proposed Problems:

For all your proposed problems, please adopt for all documents the following FILENAME format:

FirstName_LastName_ProposedProblem_SSMJ_YourGivenNumber_ProblemTitle

If you do not have a ProblemTitle, then leave that component as it already is (i.e., ProblemTitle).

The component YourGivenNumber is any UNIQUE 3-digit (or longer) number you like to give to your problem.

Examples:

Max_Planck_ProposedProblem_SSMJ_314_HarmonicPatterns

Charles_Darwin_ProposedProblem_SSMJ_358_ProblemTitle

Please adopt the following structure, in the order shown, for the presentation of your proposal:

1. On the top of first page of your proposal, begin with the phrase:

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"Problem proposed by [your First Name, your Last Name]", followed by your affiliation, city, country all on the same linear string of words. Please see the example below. Make sure you do the same for your collaborator(s) if any.

3. On a new line state the title of the problem, if any.

4. On a new line below the above, write in bold type: "**Statement of the Problem**".

5. Below the latter, state the problem. Please make sure the statement of your problem (unlike the preceding item) is not in bold type.

6. Below the statement of the problem, write in bold type: "**Solution of the Problem**".

7. Below the latter, show the entire solution of your problem.

Here is a sample for the above-stated format for proposed problems:

Problem proposed to SSMJ

Problem proposed by Isaac Newton, Trinity College, Cambridge, England.

Principia Mathematica (⟵ You may choose to not include a title.)

Statement of the problem:

Compute $\sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}$.

Solution of the problem: . . . . . .

♣ ♣ ♣ Thank You! ♣ ♣ ♣