Problems and Solutions

Albert Natian, Section Editor

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This section of the Journal offers readers an opportunity to exchange interesting mathematical problems and solutions. Please email them to Prof. Albert Natian at Department of Mathematics, Los Angeles Valley College. Please make sure every proposed problem or proposed solution is provided in both ***LaTeX*** and pdf documents. Please make sure your proposals adhere to **Formats, Styles and Requirements** noted below. Thank you!

To propose problems, email them to: problems4ssma@gmail.com

To propose solutions, email them to: solutions4ssma@gmail.com

Solutions to previously published problems can be seen at <www.ssma.org/publications>.

Solutions to the problems published in this issue should be submitted *before* February 1, 2027.

• **5850** *Proposed by Daniel Sitaru, National Economic College "Theodor Costescu" Drobeta Turnu - Severin, Romania.*

Show that if $0 \leqslant a, b \leqslant \dfrac{\pi}{2}$, then

$$4(b-a)^3(\sin b - \sin a) \leqslant 3(b-a)^4 + (\sin b - \sin a)^4.$$

• **5851** *Proposed by Jose Luis Diaz-Barrero, Barcelona, Spain.*

Determine all continuous functions $f : \mathbb{R} \to \mathbb{R}$, such that $f(0) > 0$ and for all $x, y \in \mathbb{R}$:

$$f(x+y) = f(x)\,f(3-y) + f(y)\,f(3-x).$$

• **5852** *Proposed by Yao Mawugna Dzokotoe, Ecole Nationale Supérieure de Statistique et d'Economie Appliquée, Abidjan, Côte d'Ivoire.*

Prove

$$\sum_{n=1}^{\infty} \frac{1}{4^n\, n\,(n+\frac{1}{2})} \cdot \frac{\binom{4n}{2n}}{\binom{2n}{n}} = \sum_{n=0}^{\infty} \frac{(-1)^n \binom{2n}{n}}{4^n (n+1)^2} = -4 + 4\sqrt{2} - 4\ln\left(\frac{1+\sqrt{2}}{2}\right).$$

• **5853** *Proposed by Michel Bataille, Rouen, France.*

Let $n \in \mathbb{N}$, $m \in \{0\} \cup \mathbb{N}$ and define $P_m(x) := \displaystyle\sum_{j=0}^{m} \binom{2m}{2j} x^j$. Show that the equation

$$P_{2n}(x) = (4x)^n P_n(x)$$

has a unique solution in $(0, \infty)$. Are there any solutions in $(-\infty, 0]$?

• **5854** *Proposed by Problem Section Editor, Albert Natian.*

Solve the following system of equations for real x and y:

$$\left\{ \begin{array}{r} 16x^2 - 12x + 12y^2 + 12x\cos(2y) + \displaystyle\int_0^1 \left(16x^2\cos\left(2y\xi^2\right) + 9y\sin\left(2y\xi^2\right)\right) d\xi = 9y\sin(2y) \\ 3 + xe^{1-\tan y} + ye^{1-\tan x} = e^x + e^y + \cos(2xy) \end{array} \right\}.$$

Solutions

To Formerly Published Problems

• **5830** *Proposed by Vasile Mircea Popa, Lucian Blaga University Sibiu, Romania.*

Prove the equality

$$\int_0^\infty \frac{x\sqrt{x}\ln(x)}{x^4+1}dx = \frac{1}{2}\pi^2\sin^3\left(\frac{\pi}{8}\right).$$

Solution 1 by Albert Stadler, Herrliberg, Switzerland.

We begin with the well-known integral formula (see, for example, [1], 3.222.2):

$$\int_0^\infty \frac{x^{\mu-1}}{x+1}dx = \frac{\pi}{\sin(\pi\mu)}, \quad 0 < \mu < 1.$$

Differentiating both sides with respect to μ, we obtain

$$\int_0^\infty \frac{x^{\mu-1}\ln(x)}{x+1}dx = -\frac{\pi^2\cos(\pi\mu)}{\sin^2(\pi\mu)}.$$

Next, we perform the change of variables $x \mapsto x^4$. A straightforward computation yields

$$\int_0^\infty \frac{x^{\mu-1}\ln(x)}{x+1}dx = 16\int_0^\infty \frac{x^{4\mu-1}\ln(x)}{x^4+1}dx.$$

Now set $\mu = \dfrac{5}{8}$. Then $4\mu - 1 = \dfrac{3}{2}$, and therefore

$$\int_0^\infty \frac{x\sqrt{x}\ln(x)}{x^4+1}dx = -\frac{\pi^2\cos\left(\frac{5\pi}{8}\right)}{16\sin^2\left(\frac{5\pi}{8}\right)} = \frac{1}{2}\pi^2\sin^3\left(\frac{\pi}{8}\right),$$

noting that

$$8\sin^3\left(\frac{\pi}{8}\right)\sin^2\left(\frac{5\pi}{8}\right) + \cos\left(\frac{5\pi}{8}\right) = 8\left(\frac{e^{\frac{i\pi}{8}} - e^{-\frac{i\pi}{8}}}{2i}\right)^3\left(\frac{e^{\frac{5i\pi}{8}} - e^{-\frac{5i\pi}{8}}}{2i}\right)^2 + \frac{e^{\frac{5i\pi}{8}} + e^{-\frac{5i\pi}{8}}}{2} = 0.$$

References [1] I.S. Gradshteyn / I.M. Ryzhik, Table of Integrals, Series, and Products, corrected and enlarged edition, Academic Press, 1980.

Solution 2 by Devis Alvarado, UNAH and UPNFM, Tegucigalpa, Hoduras.

Let I be the value of the integral and let us define the integral $I(\alpha) = \int_0^\infty \frac{x^\alpha}{x^4+1}\,dx$. By making the change of variable $x^2 = \tan(\theta)$, we obtain

$$\begin{aligned}
I(\alpha) &= \int_0^\infty \frac{x^\alpha}{x^4+1}\,dx \\
&= \int_0^{\frac{\pi}{2}} \frac{\left(\tan(\theta)\right)^{\frac{\alpha}{2}}}{\tan^2(\theta)+1}\frac{1}{2}\left(\tan(\theta)\right)^{-\frac{1}{2}}\sec^2(\theta)\,d\theta \\
&= \frac{1}{2}\int_0^{\frac{\pi}{2}} \left(\sin(\theta)\right)^{\frac{\alpha-1}{2}}\left(\cos(\theta)\right)^{-\frac{\alpha-1}{2}}\,d\theta \\
&= \frac{1}{2}\int_0^{\frac{\pi}{2}} \left(\sin(\theta)\right)^{2\left(\frac{\alpha+1}{4}\right)-1}\left(\cos(\theta)\right)^{2\left(\frac{-\alpha+3}{4}\right)-1}\,d\theta \\
&= \frac{1}{4}B\left(\frac{\alpha+1}{4},\frac{-\alpha+3}{4}\right) = \frac{1}{4}B\left(\frac{\alpha+1}{4},1-\frac{\alpha+1}{4}\right) \\
&= \frac{1}{4}\frac{\pi}{\sin\left(\pi\frac{\alpha+1}{4}\right)}.
\end{aligned}$$

Using this expression, we obtain:

$$\begin{aligned}
I &= \left.\frac{dI(\alpha)}{d\alpha}\right|_{\alpha=\frac{3}{2}} = -\frac{\pi^2}{16}\left.\frac{\cos\left(\pi\frac{\alpha+1}{4}\right)}{\sin^2\left(\pi\frac{\alpha+1}{4}\right)}\right|_{\alpha=\frac{3}{2}} \\
&= -\frac{\pi^2}{16}\frac{\cos\left(\frac{5\pi}{8}\right)}{\sin^2\left(\frac{5\pi}{8}\right)} = \frac{\pi^2}{16}\frac{\sin\left(\frac{\pi}{8}\right)}{\cos^2\left(\frac{\pi}{8}\right)} \\
&\ \frac{\pi^2}{16}\frac{\sin^3\left(\frac{\pi}{8}\right)}{\cos^2\left(\frac{\pi}{8}\right)\sin^2\left(\frac{\pi}{8}\right)} = \frac{\pi^2}{16}\frac{\sin^3\left(\frac{\pi}{8}\right)}{\frac{1}{4}\sin^2\left(\frac{\pi}{4}\right)} \\
&= \frac{\pi^2}{2}\sin^3\left(\frac{\pi}{8}\right).
\end{aligned}$$

Here we use the properties of the Gamma and Beta functions:

$$B(\alpha,\beta) = 2\int_0^{\frac{\pi}{2}} \sin^{2\alpha-1}(\theta)\cos^{2\beta-1}(\theta)\,d\theta.$$

$$B(\alpha, 1-\alpha) = \Gamma(\alpha)\Gamma(1-\alpha) = \frac{\pi}{\sin(\pi\alpha)}.$$

Solution 3 by Jose Luis Diaz-Barrero, Barcelona, Spain.

First, we define the following integral

$$I(a) = \int_0^\infty \frac{x^{a-1}}{x^4+1}\,dx.$$

To determine for which values of a it converges, we examine the behavior of the integrand near 0 and ∞. For x close to 0, we have $x^4 + 1 \approx 1$, so $\dfrac{x^{a-1}}{x^4+1} \sim x^{a-1}$. Thus we require the integral

$$\int_0^1 x^{a-1}\,dx$$

to converge. This happens iff $a-1 > -1$ or $a > 0$.

For large x, the denominator behaves like x^4, so $\dfrac{x^{a-1}}{x^4+1} \sim x^{a-5}$. Thus we require the integral

$$\int_1^\infty x^{a-5}\,dx$$

to converge. This happens iff $a-5 < -1$ or $a < 4$. Both conditions must hold simultaneously, so the integral converges exactly for $0 < a < 4$.

Coming back, we have

$$\int_0^\infty \frac{x\sqrt{x}\,\ln x}{x^4+1}\,dx = I'(5/2).$$

Now we evaluate $I(a)$, after we compute its derivative. A standard Beta–Gamma integral gives

$$\int_0^\infty \frac{x^{s-1}}{1+x^n}\,dx = \frac{\pi}{n}\csc\left(\frac{\pi s}{n}\right), \quad 0 < s < n.$$

Putting $n = 4$ and $s = a$, we get $I(a) = \dfrac{\pi}{4}\csc\left(\dfrac{\pi a}{4}\right)$,

$$I'(a) = \int_0^\infty \frac{x^{a-1}\ln x}{x^4+1}\,dx = -\frac{\pi^2}{16}\csc\left(\frac{\pi a}{4}\right)\cot\left(\frac{\pi a}{4}\right).$$

Thus

$$\int_0^\infty \frac{x\sqrt{x}\,\ln x}{x^4+1}\,dx = I'\left(\frac{5}{2}\right) = -\frac{\pi^2}{16}\csc\left(\frac{5\pi}{8}\right)\cot\left(\frac{5\pi}{8}\right).$$

To finish, it remains a trigonometric simplification. Indeed, using

$$\sin\left(\frac{5\pi}{8}\right) = \sin\left(\frac{3\pi}{8}\right), \quad \cos\left(\frac{5\pi}{8}\right) = -\cos\left(\frac{3\pi}{8}\right),$$

we have

$$-\csc\left(\frac{5\pi}{8}\right)\cot\left(\frac{5\pi}{8}\right) = \frac{\cos(3\pi/8)}{\sin^2(3\pi/8)}.$$

Now using

$$\cos\left(\frac{3\pi}{8}\right) = \sin\left(\frac{\pi}{8}\right), \quad \sin\left(\frac{3\pi}{8}\right) = \cos\left(\frac{\pi}{8}\right),$$

we get

$$\frac{\cos(3\pi/8)}{\sin^2(3\pi/8)} = \frac{\sin(\pi/8)}{\cos^2(\pi/8)} = \sin(\pi/8)\sec^2(\pi/8).$$

Finally, using

$$\sec^2\theta = \frac{1}{\cos^2\theta}, \quad \cos\left(\frac{\pi}{8}\right) = \frac{\sqrt{2}}{4\sin(\pi/8)},$$

we obtain

$$\frac{\sin(\pi/8)}{\cos^2(\pi/8)} = 8\sin^3\left(\frac{\pi}{8}\right),$$

and the final result is

$$\int_0^\infty \frac{x\sqrt{x}\,\ln x}{x^4+1}\,dx = \frac{\pi^2}{16}\cdot 8\sin^3\left(\frac{\pi}{8}\right) = \frac{\pi^2}{2}\sin^3\left(\frac{\pi}{8}\right),$$

as required.

Solution 4 by Michel Bataille, Rouen, France.

Let I denote the given integral. The change of variables $x = \dfrac{1}{u}$ gives $I = -J$ where

$$J = \int_0^\infty \frac{\sqrt{u}\ln(u)}{u^4+1}\,du = J_1 + J_2$$

with

$$J_1 = \int_0^1 \frac{\sqrt{u}\ln(u)}{u^4+1}\,du = \int_0^1 \frac{v(2\ln(v))(2v\,dv)}{1+v^8} = 4\int_0^1 \frac{v^2\ln(v)}{1+v^8}\,dv$$

and

$$J_2 = \int_1^\infty \frac{\sqrt{u}\ln(u)}{u^4+1}\,du = \int_1^0 \frac{v^{-1}(-2\ln(v))(-2v^{-3}dv)}{v^{-8}+1} = -4\int_0^1 \frac{v^4\ln(v)}{1+v^8}\,dv.$$

We recall that $\displaystyle\int_0^1 x^k\ln(x)\,dx = \frac{-1}{(k+1)^2}$ $\quad(k = 0, 1, \ldots)$. It follows that

$$J_1 = 4\int_0^1\left(\sum_{n=0}^\infty (-1)^n v^{8n+2}\ln(v)\right)dv = 4\sum_{n=0}^\infty\int_0^1 (-1)^n v^{8n+2}\ln(v)\,dv = 4\sum_{n=0}^\infty \frac{(-1)^{n+1}}{(8n+3)^2}.$$

(The interchange $\sum/\int$ is possible as $\displaystyle\sum_{n=0}^\infty\int_0^1\left|(-1)^n v^{8n+2}\ln(v)\right|dv = \sum_{n=0}^\infty\frac{1}{(8n+3)^2} < \infty$.)

Similarly, $J_2 = -4\displaystyle\sum_{n=0}^\infty \frac{(-1)^{n+1}}{(8n+5)^2}$ and therefore

$$I = -J = -J_1 - J_2 = \frac{1}{16}\left(\sum_{n=0}^\infty \frac{(-1)^n}{(n+3/8)^2} - \sum_{n=0}^\infty \frac{(-1)^n}{(n+5/8)^2}\right).$$

Now, we introduce the digamma function ψ defined by $\psi(x) = \dfrac{\Gamma'(x)}{\Gamma(x)}$ for positive x. We recall two known results about this function:

$$\psi'(x) = \sum_{n=0}^{\infty} \frac{1}{(n+x)^2} \quad (x > 0)$$

and

$$\psi(1-x) = \psi(x) + \pi \cot(\pi x) \quad (0 < x < 1).$$

We first deduce that

$$\begin{aligned}\sum_{n=0}^{\infty} \frac{(-1)^n}{(n+x)^2} &= \sum_{n=0}^{\infty} \frac{1}{(2n+x)^2} - \sum_{n=0}^{\infty} \frac{1}{(2n+1+x)^2} \\ &= \frac{1}{4}\big(\psi'(x/2) - \psi'((x+1)/2)\big) \quad (x > 0)\end{aligned}$$

and also that $\psi'(x) + \psi'(1-x) = \dfrac{\pi^2}{\sin^2(\pi x)} \quad (0 < x < 1)$.
As a result,

$$I = \frac{1}{64}\left(\psi'\left(\frac{3}{16}\right) - \psi'\left(\frac{11}{16}\right) - \psi'\left(\frac{5}{16}\right) + \psi'\left(\frac{13}{16}\right)\right) = \frac{\pi^2}{64}\left(\frac{1}{\sin^2(3\pi/16)} - \frac{1}{\sin^2(5\pi/16)}\right).$$

Using the trig formulas $\sin^2 a - \sin^2 b = \sin(a+b)\sin(a-b)$ and $2\sin a \sin b = \cos(a-b) - \cos(a+b)$, easy calculations lead to

$$I = \frac{\pi^2}{64} \cdot \frac{\sin(\pi/8)}{\frac{\cos^2(\pi/8)}{4}} = \frac{\pi^2}{16} \cdot \frac{\sin^3(\pi/8)}{\frac{\sin^2(\pi/4)}{4}} = \frac{\pi^2}{2}\sin^3\left(\frac{\pi}{8}\right).$$

Solution 5 by Perfetti Paolo, dipartimento di matematica Universitą di "Tor Vergata", Roma, Italy.

The integral is $\dfrac{d}{da}\displaystyle\int_0^{\infty} \frac{x^a\,dx}{x^4+1}\Big|_{a=3/2}$. By passing to the complex variable z it follows

$$\begin{aligned}\int_0^{\infty} \frac{x^a\,dx}{x^4+1} &= \frac{2\pi i}{1-e^{2i\pi a}}\sum_{k=0}^{3} \mathrm{Res}\frac{z^a}{z^4+1}\Big|_{z_k = e^{\frac{i(1+2k)\pi}{4}}} = \frac{2\pi i}{1-e^{2i\pi a}}\sum_{k=0}^{3}\frac{1}{4}e^{\frac{i\pi(1+2k)(a-3)}{4}} = \\ &= \frac{\pi i}{2(1-e^{2i\pi a})}e^{\frac{i\pi(a-3)}{4}}\sum_{k=0}^{3} e^{\frac{i\pi k(a-3)}{2}} = \frac{\pi i}{2(1-e^{2i\pi a})}e^{\frac{i\pi(a-3)}{4}}\frac{1-e^{i\pi 2k(a-3)}}{1-e^{\frac{i\pi(a-3)}{2}}} = \\ &= \frac{\pi i}{2}\frac{e^{\frac{i\pi(a-3)}{4}}}{1-e^{\frac{i\pi(a-3)}{2}}} = \frac{\pi i}{2}\frac{1}{-2i\sin\frac{\pi(a-3)}{4}}\end{aligned}$$

Doing the derivative and evaluating at $a = 3/2$ we get

$$\frac{\pi}{4}\frac{\cos\frac{\pi(a-3)}{4}}{\sin^2\frac{\pi(a-3)}{4}}\Bigg|_{a=\frac{3}{2}} = \frac{\pi^2}{16}\frac{\cos\frac{3\pi}{8}}{\sin^2\frac{3\pi}{8}} = \frac{\pi^2}{16}\frac{\sin\frac{\pi}{8}}{\cos^2\frac{\pi}{8}} = \frac{\pi^2}{16}\frac{\frac{\sqrt{2-\sqrt{2}}}{2}}{\frac{2+\sqrt{2}}{4}} = \frac{\pi^2}{2}\frac{(2-\sqrt{2})^{3/2}}{8} = \frac{\pi^2}{2}\sin^3\frac{\pi}{8}$$

Solution 6 by Péter Fülöp, Gyömrő, Hungary.

Use $x = t^{\frac{1}{4}}$ substitution and write

$$I = \int_0^\infty \frac{x\sqrt{x}\ln(x)}{x^4+1}dx = \frac{1}{16}\int_0^\infty \frac{t^{-\frac{3}{8}}\ln(t)}{1+t}dt.$$

Let's split the integral into two domains: $[0,1]$ and $[1,\infty]$:

$$I = \frac{1}{16}\int_0^1 \frac{t^{-\frac{3}{8}}\ln(t)}{1+t} + \frac{1}{16}\int_1^\infty \frac{t^{-\frac{3}{8}}\ln(t)}{1+t}dt.$$

Use $z = \dfrac{1}{t}$ substitution on the second integral:

$$I = \frac{1}{16}\int_0^1 \frac{t^{-\frac{3}{8}}\ln(t)}{1+t} + \frac{1}{16}\int_0^1 \frac{z^{\frac{3}{8}}\ln(\frac{1}{z})}{1+\frac{1}{z}}.\frac{1}{z^2}dz$$

After merging the two integrals and recalling the fact $\dfrac{dt^a}{da}_{|a=0} = \ln(t)$, we have:

$$I = \frac{1}{16}\frac{d}{da}_{|a=0}\int_0^1 \frac{t^{a-\frac{3}{8}} - t^{a-\frac{5}{8}}}{1+t}dt.$$

Since $\dfrac{1}{1+t} = \sum_{k=0}^\infty (-1)^k t^k$, replace the order of the integration and summation we get the followings:

$$I = \frac{1}{16}\frac{d}{da}_{|a=0}\sum_{k=0}^\infty (-1)^k \int_0^1 t^{k+a-\frac{3}{8}} - t^{k+a-\frac{5}{8}}dt.$$

Let's perform the integration and then the derivation:

$$I = \frac{1}{16}\sum_{k=0}^\infty \Big[\frac{(-1)^k}{(k+\frac{3}{8})^2} - \frac{(-1)^k}{(k+\frac{5}{8})^2}\Big].$$

Decomposing into even and odd parts of the sum, we write:

$$I = \frac{1}{64}\sum_{n=0}^\infty \Big[\frac{1}{(n+\frac{3}{16})^2} - \frac{1}{(n+\frac{11}{16})^2} + \frac{1}{(n+\frac{13}{16})^2} - \frac{1}{(n+\frac{5}{16})^2}\Big].$$

Avail of the trigamma function $\psi^{(1)}(z) = \sum_{k=0}^\infty \dfrac{1}{(z+k)^2}$ and write:

$$I = \frac{1}{64}\Big(\psi^{(1)}(\frac{3}{16}) - \psi^{(1)}(\frac{5}{16}) + \psi^{(1)}(1-\frac{3}{16}) - \psi^{(1)}(1-\frac{5}{16})\Big).$$

Here we can use the reflection relation of polygamma function:

$$(-1)^{(m)})\psi^{(m)}(1-z)-\psi^{(m)}(z)=\pi\frac{d^m}{dz^m}\Big(\cot(\pi z)\Big).$$

In this case $m=1$, and $z=\dfrac{3}{16}$ then $z=\dfrac{5}{16}$ so

$$-\psi^{(1)}(1-z)-\psi^{(1)}(z)=\pi\frac{d}{dz}\Big(\cot(\pi z)\Big)$$

$$I=\frac{\pi^2}{64}\Big(\frac{1}{\sin^2(\frac{3\pi}{16})}-\frac{1}{\sin^2(\frac{5\pi}{16})}\Big)=\frac{\pi^2}{32}\Big(\frac{1}{1-\cos(\frac{3\pi}{8})}-\frac{1}{1-\cos(\frac{5\pi}{8})}\Big)$$

.

Since $\cos(\dfrac{3\pi}{8})=\sin(\dfrac{\pi}{8})$ and $\cos(\dfrac{5\pi}{8})=-\sin(\dfrac{\pi}{8})$, then:

$$I=\frac{\pi^2}{32}\Big(\frac{1}{1-\sin(\frac{\pi}{8})}-\frac{1}{1+\sin(\frac{\pi}{8})}\Big)=\frac{\pi^2}{16}\frac{\sin(\frac{\pi}{8})}{\cos^2(\frac{\pi}{8})}.$$

Finally, expand the fraction by $\sin^2(\dfrac{\pi}{8})$ to conclude:

$$I=\frac{\pi^2}{2}\sin^3(\frac{\pi}{8}).$$

Solution 7 by Prakash Pant, The University of Vermont, Bardiya, Nepal.

We have

$$\int_0^\infty \frac{x^{\frac{3}{2}}\ln(x)}{x^4+1}dx$$

Use the substitution $y=x^4$

$$\frac{1}{16}\int_0^\infty \frac{y^{\frac{3}{8}}\ln(y)}{1+y}y^{-\frac{3}{4}}dy=\frac{1}{16}\int_0^\infty \frac{y^{-\frac{3}{8}}\ln(y)}{1+y}dy=\frac{1}{16}\frac{\partial}{\partial a}\int_0^\infty \frac{y^{a-1}}{1+y}dy\Big|_{a=\frac{5}{8}}$$

$$\frac{1}{16}\frac{\partial}{\partial a}\beta(a,1-a)\Big|_{a=\frac{5}{8}}=\frac{1}{16}\frac{\partial}{\partial a}\left(\frac{\pi}{\sin(\pi a)}\right)\Big|_{a=\frac{5}{8}}=\frac{\pi}{16}\frac{-1}{\sin^2(\pi a)}\cos(\pi a)\pi\Big|_{a=\frac{5}{8}}$$

$$-\frac{\pi^2}{16}\frac{\cos(\frac{5\pi}{8})}{\sin^2(\frac{5\pi}{8})}=\frac{\pi^2}{16}\frac{\cos(\frac{3\pi}{8})}{\sin^2(\frac{3\pi}{8})}=\frac{\pi^2}{16}\frac{\sin^3(\frac{\pi}{8})}{\sin^2(\frac{\pi}{8})\sin^2(\frac{3\pi}{8})}=\frac{\pi^2}{4}\frac{\sin^3(\frac{\pi}{8})}{\Big(\cos(\frac{\pi}{4})-\cos(\frac{\pi}{2})\Big)^2}$$

$$=\frac{1}{2}\pi^2\sin^3(\frac{\pi}{8})$$

This completes the proof.

Also solved by Bruno Salgueiro Fanego, Viveiro, Lugo, Spain and the problem proposer.

• **5831** *Proposed by Vasile Cirtoaje, Petroleum-Gas University of Ploiesti, Romania.*

Suppose $0 \leqslant a \leqslant b \leqslant c \leqslant d \leqslant e$ and $ab + bc + cd + de + ea = 5$. Prove that

$$(2a + 1)^2 + (2b + 1)^2 + (2c + 1)^2 \leqslant 27.$$

Solution 1 by Michel Bataille, Rouen, France.

Clearly, we have to prove that

$$a^2 + b^2 + c^2 + a + b + c \leqslant 6. \tag{1}$$

If $c \leqslant 1$, then $a, b, c \leqslant 1$, hence $a^2, b^2, c^2 \leqslant 1$ as well and (1) obviously holds.
Suppose now that $c > 1$. We cannot have $a > 1$ (otherwise $a, b, c, d, e > 1$ and the constraint $ab + bc + cd + de + ea = 5$ would be violated). Therefore we have $0 \leqslant a \leqslant 1$, $a \leqslant b$ and $1 < c \leqslant d \leqslant e$ and it follows that $5 = ab + bc + cd + de + ea > ab + b + c + c^2 + a$ so that

$$a^2 + b^2 + c^2 + a + b + c < a^2 + b^2 + 5 - ab = 5 + b^2 - a(b - a) \leqslant 5 + b^2$$

and (1) holds if $b \leqslant 1$.
Lastly, if $0 \leqslant a \leqslant 1 < b \leqslant c \leqslant d \leqslant e$, then $5 > a + b^2 + c^2 + c^2 + a^2 = a^2 + b^2 + c^2 + a + c^2$. Since $c - 1 \geqslant b - 1 > 0$ and $c > 1$, we have $c(c - 1) \geqslant b - 1$ and consequently $c^2 \geqslant b + c - 1$. We deduce $5 > a^2 + b^2 + c^2 + a + b + c - 1$ and therefore (1) holds.
Thus, (1) holds under the given constraints in any event.

Solution 2 by Albert Stadler, Herrliberg, Switzerland.

The expression to be maximized does not involve d or e. Hence, under the constraint

$$ab + bc + cd + de + ea = 5,$$

we may minimize d and e, so $d = e = c$. Thus it suffices to prove

$$\big(2a + 1)^2 + (2b + 1)^2 + (2c + 1)^2 \leqslant 27$$

under

$$0 \leqslant a \leqslant b \leqslant c, \;\; ab + bc + ca + 2c^2 = 5.$$

Since

$$5 = ab + bc + ca + 2c^2 \leqslant c^2 + c^2 + c^2 + 2c^2 = 5c^2,$$

we have $c \geqslant 1$. Similarly,

$$5 = ab + bc + ca + 2c^2 \geqslant a^2 + a^2 + a^2 + 2a^2 = 5a^2,$$

so $a \leqslant 1$.
The desired inequality is equivalent to

$$a^2 + a + b^2 + b + c^2 + c \leqslant 6.$$

Set
$$u = a + b, \quad v = ab.$$
Then
$$u^2 \geqslant 4v, \quad v + uc + 2c^2 = 5.$$
Also,
$$a^2 + b^2 = u^2 - 2v,$$
hence
$$a^2 + a + b^2 + b + c^2 + c = u^2 + u - 2v + c^2 + c.$$
Using $v = 5 - uc - 2c^2$,
$$a^2 + a + b^2 + b + c^2 + c = u^2 + (2c + 1)u + 5c^2 + c - 10. \qquad (1)$$
From
$$u^2 \geqslant 4v = 4(5 - uc - 2c^2),$$
we obtain
$$(u + 2c)^2 \geqslant 4(5 - c^2),$$
so
$$u \geqslant -2c + 2\sqrt{5 - c^2}. \qquad (2)$$
On the other hand,
$$u = a + b \leqslant 2c. \qquad (3)$$
For fixed c, the right-hand side of (**??**) is a convex quadratic in u. Therefore its maximum on the interval determined by (**??**) and (**??**) occurs at an endpoint.
If $u = 2c$, then $a = b = c$, and the constraint gives $5c^2 = 5$, so $c = 1$. Substituting into (**??**) yields
$$13c^2 + 3c - 10 = 6.$$
Now let
$$u = -2c + 2\sqrt{5 - c^2}.$$
Substituting into (**??**) gives
$$f(c) = 10 + c^2 - c + 2(1 - 2c)\sqrt{5 - c^2}.$$
Thus we must prove
$$f(c) \leqslant 6,$$
equivalently,
$$2(2c - 1)\sqrt{5 - c^2} \geqslant c^2 - c + 4. \qquad (4)$$
Since $u \geqslant 0$, relation (**??**) implies
$$-2c + 2\sqrt{5 - c^2} \geqslant 0,$$
hence
$$1 \leqslant c \leqslant \sqrt{\frac{5}{2}}. \qquad (5)$$
On this interval both sides of (**??**) are nonnegative, so squaring is valid:
$$4(2c - 1)^2(5 - c^2) \geqslant (c^2 - c + 4)^2.$$

After simplification,
$$(c-2)(c-1)(c+2)(17c-1) \leqslant 0.$$
This holds because of (**??**).
Therefore $f(c) \leqslant 6$, and consequently
$$a^2+a+b^2+b+c^2+c \leqslant 6.$$
Finally,
$$(2a+1)^2+(2b+1)^2+(2c+1)^2 = 4(a^2+a+b^2+b+c^2+c)+3 \leqslant 27.$$
Hence
$$(2a+1)^2+(2b+1)^2+(2c+1)^2 \leqslant 27.$$

Also solved by the problem proposer.

• **5832** *Proposed by Toyesh Prakash Sharma, Agra College, Agra, India.*

If $z = z(x,y)$, then solve the differential equation
$$\left(y^2+1\right)\left(x^4+1\right)\frac{\partial z}{\partial x} = \left(x^2-1\right)\left(y^4+1\right)\frac{\partial z}{\partial y}.$$

Solution 1 by Prakash Pant, The University of Vermont, Bardiya, Nepal.

The equation can be written as:
$$(y^2+1)(x^4+1)\frac{\partial z}{\partial x} - (x^2-1)(y^4+1)\frac{\partial z}{\partial y} = 0$$

For a first order pde of the form $a(x,y)\dfrac{\partial z}{\partial x} + b(x,y)\dfrac{\partial z}{\partial y} = c(x,y)$, the characteristic equation is given by
$$\frac{dx}{a(x,y)} = \frac{dy}{b(x,y)} = \frac{dz}{c(x,y)}$$

In our case, $a(x,y) = (y^2+1)(x^4+1)$, $b(x,y) = -(x^2-1)(y^4+1)$, $c(x,y) = 0$. Here, the last equality $\dfrac{dz}{0}$ just means that z is constant along the characteristic curve. Next, we get the characteristic curve by solving
$$\frac{dx}{(y^2+1)(x^4+1)} = -\frac{dy}{(x^2-1)(y^4+1)}$$
$$-\frac{x^2-1}{x^4+1}dx = \frac{y^2+1}{y^4+1}dy$$
Integrate both sides
$$-\int \frac{x^2-1}{x^4+1}dx = \int \frac{y^2+1}{y^4+1}dy$$

$$-\int \frac{1-\frac{1}{x^2}}{x^2+\frac{1}{x^2}}dx = \int \frac{1+\frac{1}{y^2}}{y^2+\frac{1}{y^2}}dy$$

$$-\int \frac{1-\frac{1}{x^2}}{(x+\frac{1}{x})^2-2}dx = \int \frac{1+\frac{1}{y^2}}{(y-\frac{1}{y})^2+2}dy$$

$$-\frac{1}{2\sqrt{2}}\ln\left(\frac{x+\frac{1}{x}-\sqrt{2}}{x+\frac{1}{x}+\sqrt{2}}\right)+C = \frac{1}{\sqrt{2}}\arctan\left(\frac{y-\frac{1}{y}}{\sqrt{2}}\right)$$

$$C = \frac{1}{2\sqrt{2}}\ln\left(\frac{x+\frac{1}{x}-\sqrt{2}}{x+\frac{1}{x}+\sqrt{2}}\right)+\frac{1}{\sqrt{2}}\arctan\left(\frac{y-\frac{1}{y}}{\sqrt{2}}\right)$$

which is the required characteristic curve. Since z is constant along each characteristic curve, z is an arbitrary function of this constant.

$$z = F(C) = F\left(\frac{1}{2\sqrt{2}}\ln\left(\frac{x+\frac{1}{x}-\sqrt{2}}{x+\frac{1}{x}+\sqrt{2}}\right)+\frac{1}{\sqrt{2}}\arctan\left(\frac{y-\frac{1}{y}}{\sqrt{2}}\right)\right)$$

where F is an arbitrary differentiable function.This concludes the solution.

Solution 2 by Jose Luis Diaz-Barrero, Barcelona, Spain.

We begin by rewriting the PDE in the standard form $Pz_x + Qz_y = 0$:

$$(y^2+1)(x^4+1)z_x - (x^2-1)(y^4+1)z_y = 0.$$

The associated Lagrange characteristic equations are:

$$\frac{dx}{(y^2+1)(x^4+1)} = \frac{dy}{-(x^2-1)(y^4+1)}.$$

Rearranging the terms to separate variables x and y, we obtain

$$\frac{y^2+1}{y^4+1}\,dy = -\frac{x^2-1}{x^4+1}\,dx \tag{1}$$

We integrate both sides:

$$\int \frac{y^2+1}{y^4+1}\,dy + \int \frac{x^2-1}{x^4+1}\,dx = C.$$

Integral I: For the y-term, divide the numerator and denominator by y^2:

$$\int \frac{1+\frac{1}{y^2}}{y^2+\frac{1}{y^2}}\,dy$$

Let $u = y - \frac{1}{y}$, then $du = (1 + \frac{1}{y^2})\, dy$ and $y^2 + \frac{1}{y^2} = u^2 + 2$.

$$\int \frac{du}{u^2 + 2} = \frac{1}{\sqrt{2}} \arctan\left(\frac{u}{\sqrt{2}}\right) = \frac{1}{\sqrt{2}} \arctan\left(\frac{y - \frac{1}{y}}{\sqrt{2}}\right).$$

Integral II: For the x-term, divide the numerator and denominator by x^2:

$$\int \frac{1 - \frac{1}{x^2}}{x^2 + \frac{1}{x^2}}\, dx$$

Let $v = x + \frac{1}{x}$, then $dv = (1 - \frac{1}{x^2})\, dx$ and $x^2 + \frac{1}{x^2} = v^2 - 2$.

$$\int \frac{dv}{v^2 - 2} = \frac{1}{2\sqrt{2}} \ln\left|\frac{v - \sqrt{2}}{v + \sqrt{2}}\right| = \frac{1}{2\sqrt{2}} \ln\left|\frac{x + \frac{1}{x} - \sqrt{2}}{x + \frac{1}{x} + \sqrt{2}}\right|.$$

Combining these results, the characteristic invariant is:

$$\frac{1}{\sqrt{2}} \arctan\left(\frac{y - \frac{1}{y}}{\sqrt{2}}\right) + \frac{1}{2\sqrt{2}} \ln\left|\frac{x + \frac{1}{x} - \sqrt{2}}{x + \frac{1}{x} + \sqrt{2}}\right| = C.$$

Multiplying by $2\sqrt{2}$ to simplify, the general solution is:

$$z(x, y) = F\left(2 \arctan\left(\frac{y - \frac{1}{y}}{\sqrt{2}}\right) + \ln\left|\frac{x + \frac{1}{x} - \sqrt{2}}{x + \frac{1}{x} + \sqrt{2}}\right|\right),$$

where F is an arbitrary differentiable function.

Solution 3 by Devis Alvarado, UNAH and UPNFM, Tegucigalpa, Hoduras.

Using the method of separation of variables, there exist functions $u(x)$ and $v(y)$ such that $z(x, y) = u(x)v(y)$. Then,

$$\begin{aligned}
& \left(y^2 + 1\right)\left(x^4 + 1\right) \frac{\partial z}{\partial x} = \left(x^2 - 1\right)\left(y^4 + 1\right) \frac{\partial z}{\partial y} \\
\Longrightarrow \quad & \left(y^2 + 1\right)\left(x^4 + 1\right) v(y) \frac{du}{dx} = \left(x^2 - 1\right)\left(y^4 + 1\right) u(x) \frac{dv}{dy} \\
\Longrightarrow \quad & \frac{x^4 + 1}{x^2 - 1} \frac{1}{u(x)} \frac{du}{dx} = \frac{y^4 + 1}{y^2 + 1} \frac{1}{v(y)} \frac{dv}{dy}.
\end{aligned}$$

Since each side of the equation depends on different variables, they must be equal to a constant. Thus,

$$\frac{x^4 + 1}{x^2 - 1} \frac{1}{u(x)} \frac{du}{dx} = \frac{y^4 + 1}{y^2 + 1} \frac{1}{v(y)} \frac{dv}{dy} = \lambda$$

Let us solve each equation separately:

$$\begin{aligned}
& \frac{x^4+1}{x^2-1}\frac{1}{u(x)}\frac{du}{dx} = \lambda \\
\implies \quad & \frac{1}{u(x)}\frac{du}{dx} = \lambda\frac{x^2-1}{x^4+1} \\
\implies \quad & \ln(u(x)) = \frac{\lambda}{2\sqrt{2}}\ln\left(\frac{x^2-\sqrt{2}x+1}{x^2+\sqrt{2}x+1}\right) \\
\implies \quad & u(x) = \left(\frac{x^2-\sqrt{2}x+1}{x^2+\sqrt{2}x+1}\right)^{\frac{\lambda}{2\sqrt{2}}}
\end{aligned}$$

Similarly for the second one:

$$\begin{aligned}
& \frac{y^4+1}{y^2+1}\frac{1}{v(y)}\frac{dv}{dy} = \lambda \\
\implies \quad & \frac{1}{v(y)}\frac{dv}{dy} = \lambda\frac{y^2+1}{y^4+1} \\
\implies \quad & \ln(v(y)) = \frac{\lambda}{\sqrt{2}}\left(\arctan(1+\sqrt{2}y) - \arctan(1-\sqrt{2}y)\right) \\
\implies \quad & v(y) = e^{\frac{\lambda}{\sqrt{2}}\left(\arctan(1+\sqrt{2}y)-\arctan(1-\sqrt{2}y)\right)} \\
\implies \quad & v(y) = e^{\frac{\lambda}{\sqrt{2}}\arctan\left(\frac{\sqrt{2}y}{1-y^2}\right)}
\end{aligned}$$

Therefore, the solution is:

$$z(x,y) = \left(\frac{x^2-\sqrt{2}x+1}{x^2+\sqrt{2}x+1}\right)^{\frac{\lambda}{2\sqrt{2}}} e^{\frac{\lambda}{\sqrt{2}}\arctan\left(\frac{\sqrt{2}y}{1-y^2}\right)}.$$

Also solved by David A. Huckaby, Angelo State University, San Angelo, TX; Albert Stadler, Herrliberg, Switzerland and the problem proposer.

• **5833** *Proposed by Paolo Perfetti, dipartimento di matematica Universitą di “Tor Vergata", Rome, Italy.*

Let $(a_k)_{k\geqslant 1}$ be a sequence of positive real numbers with $a_{k+1} \geqslant a_k(1+c/k)^{-1}$ for some real $c > 0$. Let $S_n = \sum_{k=1}^{n} a_k$. Prove that the series $\sum_{k=1}^{\infty} \frac{S_k}{(S_{2k}-S_k)(k+1)^{1+c}(\ln(k+1))^2}$ converges.

Solution 1 by Albert Stadler, Herrliberg, Switzerland.

We will show that

$$\frac{S_k}{S_{2k}-S_k} \ll \begin{cases} 1, & 0 < c < 1, \\ \ln k, & c = 1, \\ k^{c-1}, & c > 1. \end{cases}$$

Once this is established, we obtain

$$\frac{S_k}{\left(S_{2k} - S_k\right)(k+1)^{1+c}\left(\ln(k+1)\right)^2} \ll \begin{cases} \dfrac{1}{(k+1)^{1+c}\left(\ln(k+1)\right)^2}, & 0 < c < 1, \\ \dfrac{1}{(k+1)^2 \ln(k+1)}, & c = 1, \\ \dfrac{1}{(k+1)^2\left(\ln(k+1)\right)^2}, & c > 1. \end{cases}$$

Each of these expressions is bounded by a constant multiple of

$$\frac{1}{k(\ln k)^2},$$

and since the series

$$\sum_{k=2}^{\infty} \frac{1}{k(\ln k)^2}$$

converges, it follows by comparison that

$$\sum_{k=1}^{\infty} \frac{S_k}{\left(S_{2k} - S_k\right)(k+1)^{1+c}\left(\ln(k+1)\right)^2}$$

also converges. It therefore remains to prove the estimate for

$$\frac{S_k}{S_{2k} - S_k}.$$

From the hypothesis,

$$a_{m+1} \geqslant a_m \left(+1 \,\middle|\, \frac{c}{m}\right)^{-1} = a_m \frac{m}{m+c}.$$

Iterating this inequality from $m = j$ to $m = k-1$ (with $j < k$) gives

$$a_k \geqslant a_j \prod_{m=j}^{k-1} \frac{m}{m+c}.$$

Equivalently,

$$a_j \leqslant a_k \prod_{m=j}^{k-1} \left(1 + \frac{c}{m}\right).$$

Using the elementary inequality

$$\ln(1+x) \leqslant x,$$

we obtain

$$\prod_{m=j}^{k-1} \left(1 + \frac{c}{m}\right) \leqslant \exp\left(c \sum_{m=j}^{k-1} \frac{1}{m}\right).$$

Since

$$\sum_{m=j}^{k-1} \frac{1}{m} = H_{k-1} - H_{j-1} = \ln\left(\frac{k}{j}\right) + O(1),$$

it follows that

$$a_j \ll a_k \left(\frac{k}{j}\right)^c. \qquad (1)$$

Now,

$$S_k = \sum_{j=1}^{k} a_j.$$

Applying (**??**), we get

$$S_k \ll a_k k^c \sum_{j=1}^{k} j^{-c}. \qquad (2)$$

Next, we use the standard estimates

$$\sum_{j=1}^{k} j^{-c} = \begin{cases} O(k^{1-c}), & 0 < c < 1, \\ O(\ln k), & c = 1, \\ O(1), & c > 1. \end{cases}$$

Substituting these into (**??**) yields

$$S_k \ll \begin{cases} k a_k, & 0 < c < 1, \\ k(\ln k) a_k, & c = 1, \\ k^c a_k, & c > 1. \end{cases} \qquad (3)$$

On the other hand, for $k < j \leqslant 2k$, relation (**??**) gives

$$a_k \ll a_j \left(\frac{j}{k}\right)^c.$$

Since $j \leqslant 2k$, the factor $\left(j/k\right)^c$ is bounded above by a constant depending only on c. Hence

$$a_j \gg a_k.$$

Therefore,

$$S_{2k} - S_k = \sum_{j=k+1}^{2k} a_j \gg k a_k. \qquad (4)$$

Finally, combining (**??**) and (**??**), we obtain

$$\frac{S_k}{S_{2k} - S_k} \ll \begin{cases} 1, & 0 < c < 1, \\ \ln k, & c = 1, \\ k^{c-1}, & c > 1, \end{cases}$$

which completes the proof.

Solution 2 by Michel Bataille, Rouen, France.

Let
$$u_k = \frac{(1+c)(2+c)\cdots(k+c)}{k!}.$$
We first prove that $\dfrac{S_k}{S_{2k}-S_k} \leqslant u_k$ for $k = 1, 2, \ldots$.

For any positive integer n, we have $a_n \leqslant \dfrac{n+c}{n} a_{n+1}$ (from the hypothesis). By an easy induction (on m for any fixed n), we deduce that $a_n \leqslant b_{n,m} a_{n+m+1}$ for any integers n, m with $n \geqslant 1$ and $m \geqslant 0$ where $b_{n,m} = \dfrac{(n+c)(n+c+1)\cdots(n+c+m)}{n(n+1)\cdots(n+m)}$.

Let $k \geqslant 1$. From $a_1 \leqslant b_{1,k-1}a_{k+1}, \quad a_2 \leqslant b_{2,k-1}a_{k+2}, \quad \ldots \quad a_k \leqslant b_{k,k-1}a_{2k}$, we deduce that
$$\frac{S_k}{S_{2k}-S_k} = \frac{a_1 + a_2 + \cdots + a_k}{a_{k+1} + a_{k+2} + \cdots + a_{2k}} \leqslant \frac{b_{1,k-1}a_{k+1} + b_{2,k-1}a_{k+2} + \cdots + b_{k,k-1}a_{2k}}{a_{k+1} + a_{k+2} + \cdots + a_{2k}}. \tag{1}$$
Now, we have $b_{n,0} - b_{n+1,0} = \dfrac{c}{n(n+1)} > 0$ and for $m \geqslant 1$:
$$\begin{aligned} b_{n,m} - b_{n+1,m} &= \frac{(n+c+1)\cdots(n+c+m)}{(n+1)\cdots(n+m)} \left(\frac{n+c}{n} - \frac{n+c+m+1}{n+m+1} \right) \\ &= \frac{(n+c+1)\cdots(n+c+m)}{(n+1)\cdots(n+m)} \cdot \frac{c(m+1)}{n(n+m+1)} > 0, \end{aligned}$$
hence $b_{n,m} > b_{n+1,m}$ (and therefore $b_{n,m} \leqslant b_{1,m}$) for any $n \geqslant 1, m \geqslant 0$.

(1) now yields $\dfrac{S_k}{S_{2k}-S_k} \leqslant b_{1,k-1} = u_k$. Thus, for all $k \geqslant 1$:
$$0 < \frac{S_k}{(S_{2k}-S_k)(k+1)^{1+c}(\ln(k+1))^2} \leqslant \frac{u_k}{(k+1)^{1+c}(\ln(k+1))^2}$$
and it suffices to show that the series $\displaystyle\sum_{k=1}^{\infty} \frac{u_k}{(k+1)^{1+c}(\ln(k+1))^2}$ converges.

We know that $\displaystyle\lim_{n\to\infty} \frac{n!n^c}{c(c+1)(c+2)\cdots(c+n)} = \Gamma(c)$. It follows that $u_k \sim \dfrac{k^c}{c\Gamma(c)}$ as $k \to \infty$. Thus, $\dfrac{u_k}{(k+1)^{1+c}(\ln(k+1))^2} \sim \dfrac{1}{c\Gamma(c)} \cdot \dfrac{1}{(k+1)(\ln(k+1))^2}$ and the sought convergence follows from the well-known results about Bertrand series.

Also solved by the problem proposer.

• **5834** *Proposed by Daniel Sitaru, National Economic College "Theodor Costescu" Drobeta Turnu - Severin, Romania.*

Evaluate
$$\Omega = \int_0^{\pi} \frac{(x+2)\sin x}{1+3\cos^2 x} dx.$$

Solution 1 by Zuming Bi, Academy of Mathematics and Science, Hays, KS; and Paul Flesher, Fort Hays State University, Hays, KS; and Jonathan Rehmert, Fort Hays State University, Hays, KS.

Method 1: Partial Fraction Decomposition

$$\begin{aligned}\Omega &= \int_0^\pi \frac{(x+2)\sin x}{1+3\cos^2 x}\,dx = \int_0^\pi \frac{(x+2)\sin x}{4-3\sin^2 x}\,dx \\ &= \int_0^\pi \frac{(x+2)\sin x}{(2+\sqrt{3}\sin x)(2-\sqrt{3}\sin x)}\,dx\end{aligned}$$

which we re-express as

$$\begin{aligned}\Omega &= \int_0^\pi \frac{1}{2\sqrt{3}}\left(\frac{-(x+2)(2-\sqrt{3}\sin x)+(x+2)(2+\sqrt{3}\sin x)}{4-3\sin^2 x}\right)dx \\ &= \int_0^\pi -\frac{(x+2)(2-\sqrt{3}\sin x)}{2\sqrt{3}(4-3\sin^2 x)} + \frac{(x+2)(2+\sqrt{3}\sin x)}{2\sqrt{3}(4-3\sin^2 x)}\,dx \\ &= \int_0^\pi -\frac{x+2}{2\sqrt{3}(2+\sqrt{3}\sin x)} + \frac{x+2}{2\sqrt{3}(2-\sqrt{3}\sin x)}\,dx \\ &= \int_0^\pi -\frac{x+2}{4\sqrt{3}+6\sin x} + \frac{x+2}{4\sqrt{3}-6\sin x}\,dx. \qquad (1)\end{aligned}$$

Look at the latter separately, we have

$$\int_0^\pi \frac{x+2}{4\sqrt{3}-6\sin x}\,dx = \int_0^\pi \frac{x}{4\sqrt{3}-6\sin x}\,dx + \int_0^\pi \frac{2}{4\sqrt{3}-6\sin x}\,dx.$$

Let

$$\begin{aligned}k &= \int_0^\pi \frac{x}{4\sqrt{3}-6\sin x}\,dx \\ p &= \int_0^\pi \frac{2}{4\sqrt{3}-6\sin x}\,dx.\end{aligned}$$

Do a u substitution:

$$u = \pi - x$$

$$k = \int_\pi^0 \frac{\pi-u}{4\sqrt{3}-6\sin(\pi-u)}\,(-du) = \int_0^\pi \frac{\pi-u}{4\sqrt{3}-6\sin(u)}\,du$$

Note the trig identity

$$\sin(\pi-u) = \sin(u).$$

Change to equivalent integral

$$k = \int_0^\pi \frac{\pi-u}{4\sqrt{3}-6\sin(u)}\,du = \int_0^\pi \frac{\pi-x}{4\sqrt{3}-6\sin(x)}\,dx$$

$$2k = \int_0^\pi \frac{\pi - x}{4\sqrt{3} - 6\sin(x)}\,dx + \int_0^\pi \frac{x}{4\sqrt{3} - 6\sin x}\,dx = \int_0^\pi \frac{\pi}{4\sqrt{3} - 6\sin x}\,dx$$

$$2k = \pi \int_0^\pi \frac{1}{4\sqrt{3} - 6\sin x}\,dx$$

$$k = \frac{\pi}{2}\int_0^\pi \frac{1}{4\sqrt{3} - 6\sin x}\,dx = \frac{\pi}{2} \times \frac{p}{2} = \frac{\pi}{4}p$$

$$k = \frac{\pi}{4}p.$$

So

$$\int_0^\pi \frac{x+2}{4\sqrt{3} - 6\sin x}\,dx = \frac{\pi}{4}p + p = \frac{\pi + 4}{4}p.$$

Let

$$m = \int_0^\pi \frac{x}{4\sqrt{3} + 6\sin x}\,dx$$
$$n = \int_0^\pi \frac{2}{4\sqrt{3} + 6\sin x}\,dx.$$

Do a u substitution:

$$u = \pi - x$$

$$k = \int_\pi^0 \frac{\pi - u}{4\sqrt{3} + 6\sin(\pi - u)}\,(-du) = \int_0^\pi \frac{\pi - u}{4\sqrt{3} + 6\sin(u)}\,du.$$

Change to equivalent integral

$$m = \int_0^\pi \frac{\pi - u}{4\sqrt{3} + 6\sin(u)}\,du = \int_0^\pi \frac{\pi - x}{4\sqrt{3} + 6\sin(x)}\,dx$$

$$2m = \int_0^\pi \frac{\pi - x}{4\sqrt{3} + 6\sin(x)}\,dx + \int_0^\pi \frac{x}{4\sqrt{3} + 6\sin x}\,dx = \int_0^\pi \frac{\pi}{4\sqrt{3} + 6\sin x}\,dx$$

$$2m = \pi \int_0^\pi \frac{1}{4\sqrt{3} + 6\sin x}\,dx$$

$$m = \frac{\pi}{2}\int_0^\pi \frac{1}{4\sqrt{3} - 6\sin x}\,dx = \frac{\pi}{2} \times \frac{n}{2} = \frac{\pi}{4}n$$

$$m = \frac{\pi}{4} n.$$

Then

$$\int_0^\pi \frac{x+2}{4\sqrt{3} + 6\sin x}\, dx = \frac{\pi + 4}{4} \int_0^\pi \frac{2}{4\sqrt{3} + 6\sin x}\, dx.$$

Deal with the other term in (1):

$$\begin{aligned}
\Omega &= \frac{\pi + 4}{4}\Big(\int_0^\pi \frac{2}{4\sqrt{3} - 6\sin x}\, dx - \int_0^\pi \frac{2}{4\sqrt{3} + 6\sin x}\, dx\Big) \\
&= \frac{\pi + 4}{2}\Big(\int_0^\pi \frac{1}{4\sqrt{3} - 6\sin x}\, dx - \int_0^\pi \frac{1}{4\sqrt{3} + 6\sin x}\, dx\Big).
\end{aligned}$$

We have

$$\int_0^\pi \frac{1}{4\sqrt{3} - 6\sin x}\, dx - \int_0^\pi \frac{1}{4\sqrt{3} + 6\sin x}\, dx = \int_0^\pi \frac{12\sin x}{48 - 36\sin^2 x}\, dx.$$

Let

$$u = \cos x.$$

Then

$$\int_0^\pi \frac{12\sin x}{48 - 36\sin^2 x}\, dx = \int_0^\pi \frac{\sin x}{4 - 3\sin^2 x}\, dx = \int_1^{-1} \frac{-du}{4 - 3(1 - u^2)} = \int_{-1}^1 \frac{1}{1 + 3u^2}\, du$$

$$= \frac{1}{\sqrt{3}} \arctan(\sqrt{3}u)\Big|_{-1}^1 = \frac{1}{\sqrt{3}}(\arctan(\sqrt{3}) - \arctan(-\sqrt{3})) = \frac{1}{\sqrt{3}}\Big(\frac{\pi}{3} - \Big(-\frac{\pi}{3}\Big)\Big)$$

$$= \frac{2\pi}{3\sqrt{3}}.$$

Then

$$\Omega = \frac{\pi + 4}{2} \times \frac{2\pi}{3\sqrt{3}} = \frac{\pi(\pi + 4)}{3\sqrt{3}}.$$

<u>Method 2: Direct u Substitution</u>

Let

$$u = \frac{\pi}{2} - x.$$

Then

$$\Omega = \int_{\frac{\pi}{2}}^{-\frac{\pi}{2}} \frac{(\frac{\pi}{2} - u + 2)\sin(\frac{\pi}{2} - u)}{1 + 3\cos^2(\frac{\pi}{2} - u)} (-du) = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{(\frac{\pi}{2} - u + 2)\sin(\frac{\pi}{2} - u)}{1 + 3\cos^2(\frac{\pi}{2} - u)}\, du.$$

Note

$$\sin(\frac{\pi}{2} - u) = \cos(u).$$

$$\Omega = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{(\frac{\pi}{2} - u + 2)\cos u}{1 + 3\sin^2 u}\, du = (\frac{\pi}{2} + 2)\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{\cos u}{1 + 3\sin^2 u}\, du - \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{u\cos u}{1 + 3\sin^2 u}\, du.$$

Notice since $\cos u$ is an even function, then $\dfrac{\cos u}{1 + 3\sin^2 u}$ is an even function and $\dfrac{u\cos u}{1 + 3\sin^2 u}$ is an odd function. Hence

$$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{u\cos u}{1 + 3\sin^2 u}\, du = 0.$$

Therefore

$$\begin{aligned}\Omega = (\frac{\pi}{2} + 2)\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{\cos u}{1 + 3\sin^2 u}\, du &= (\frac{\pi}{2} + 2)\frac{1}{\sqrt{3}} \arctan(\sqrt{3}\sin u)|_{-\pi/2}^{\pi/2} \\ &= (\frac{\pi}{2} + 2)\frac{2\pi}{3\sqrt{3}} = \frac{\pi(\pi + 4)}{3\sqrt{3}}.\end{aligned}$$

<u>Method 3: Integration by Parts</u>

Let

$$u = x + 2,\ du = dx.$$

So

$$dv = \frac{\sin x}{1 + 3\cos^2 x}\, dx,\ v = -\frac{1}{\sqrt{3}} \arctan(\sqrt{3}\cos x).$$

Then

$$\Omega = -(x + 2)\frac{1}{\sqrt{3}} \arctan(\sqrt{3}\cos x)|_0^{\pi} + \frac{1}{\sqrt{3}}\int_0^{\pi} \arctan(\sqrt{3}\cos x)\, dx.$$

Calculate the second part:

$$\begin{aligned} u &= \frac{\pi}{2} - x, \\ du &= -dx. \end{aligned}$$

Note

$$\cos(\frac{\pi}{2} - u) = \sin(u).$$

$$\frac{1}{\sqrt{3}}\int_0^{\pi} \arctan(\sqrt{3}\cos x)\, dx = \frac{1}{\sqrt{3}}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \arctan(\sqrt{3}\cos(\frac{\pi}{2} - u))\,(-du) = -\frac{1}{\sqrt{3}}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \arctan(\sqrt{3}\sin u)\, du.$$

Since

$$f(x) = \arctan(\sqrt{3}\sin u),\ f(-x) = -f(x)$$

$$f(x)\text{is an odd function, then } -\frac{1}{\sqrt{3}}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \arctan(\sqrt{3}\sin u)\,du = 0,$$

Then

$$\Omega = -(x+2)\frac{1}{\sqrt{3}}\arctan(\sqrt{3}\cos x)|_0^\pi = (\pi+2)(-\frac{1}{\sqrt{3}}\arctan(-\sqrt{3})) - 2(-\frac{1}{\sqrt{3}}\arctan(\sqrt{3}))$$
$$= \frac{\pi+2}{\sqrt{3}} \times \frac{\pi}{3} + \frac{2}{\sqrt{3}} \times \frac{\pi}{3} = \frac{\pi(\pi+4)}{3\sqrt{3}}$$

In conclusion,

$$\Omega = \frac{\pi(\pi+4)}{3\sqrt{3}}.$$

Solution 2 by Prakash Pant, The University of Vermont, Bardiya, Nepal.

Use the substitution $x \mapsto \pi - x$,

$$\Omega = \int_0^\pi \frac{(x+2)\sin(x)}{1+3\cos^2(x)}dx = \int_0^\pi \frac{(\pi-x+2)\sin(\pi-x)}{1+3\cos^2(\pi-x)}dx = \int_0^\pi \frac{(\pi-x+2)\sin(x)}{1+3\cos^2(x)}dx$$

Adding the first and third expression expressions,

$$2\Omega = \int_0^\pi \frac{(\pi+4)\sin(x)}{1+3\cos^2(x)}dx$$

Use the substitution $\cos(x) \mapsto x$,

$$2\Omega = \int_1^{-1} \frac{(\pi+4)(-dx)}{1+3x^2} = \int_{-1}^1 \frac{(\pi+4)(dx)}{1+3x^2}$$

Since the integrand is even,

$$\Omega = \frac{\pi+4}{3}\int_0^1 \frac{1}{x^2+\frac{1}{3}} = \frac{\pi+4}{3}\sqrt{3}\arctan(\sqrt{3}x)\Big|_0^1 = \frac{\pi(\pi+4)\sqrt{3}}{9}$$

which solves the given problem.

Solution 3 by Péter Fülöp, Gyömrő, Hungary.

Integration by parts:

$$u = x + 2 \qquad v' = \frac{\sin(x)}{1 + \cos^2(x)}$$

$$u' = 1 \qquad v = \int \frac{\sin(x)}{1 + \cos^2(x)} dx.$$

Calculation of $v(x)$:

Substitution: $t = \cos(x)$

$$x = \cos^{-1}(t) \qquad \frac{dx}{dt} = -\frac{1}{\sqrt{1 - t^2}}$$

$$v(x) = \int -\frac{1}{\sqrt{1 - t^2}} \frac{\sqrt{1 - t^2}}{\sqrt{1 - 3t^2}} dt = -\frac{\tan^{-1}(\sqrt{3}t)}{\sqrt{3}}\Big|_{t=\cos(x)} .$$

Calculation of Ω:

$$\Omega = \Big[-(x + 2)\frac{\tan^{-1}(\sqrt{3}\cos(x))}{\sqrt{3}} \Big]_0^\pi + \int\limits_0^\pi \frac{\tan^{-1}(\sqrt{3}\cos(x))}{\sqrt{3}} dx .$$

First term:

$$\Omega_1 = 2\frac{\tan^{-1}(\sqrt{3})}{\sqrt{3}} - (\pi + 2)\frac{\tan^{-1}(-\sqrt{3})}{\sqrt{3}} = (\pi + 4)\frac{\tan^{-1}(\sqrt{3})}{\sqrt{3}} = \frac{\pi\sqrt{3}(\pi + 4)}{9}.$$

Substitution at the second term: $z = \sqrt{3}\cos(x)$:

$$x = \cos^{-1}(\frac{z}{\sqrt{3}}) \qquad \frac{dx}{dz} = -\frac{\frac{1}{\sqrt{3}}}{\sqrt{1 - \frac{t^2}{3}}}$$

$$\Omega_2 = \int\limits_0^\pi \frac{\tan^{-1}(\sqrt{3}\cos(x))}{\sqrt{3}} dx = \frac{1}{3} \int\limits_{-\sqrt{3}}^{+\sqrt{3}} \frac{\tan^{-1}(z)}{\sqrt{1 - \frac{z^2}{3}}} dz = 0 .$$

As the function to be integrated is odd, and its integral in a symmetric domain is zero. So $\Omega = \Omega_1$.

$$\Omega = \frac{\pi(\pi + 4)}{3\sqrt{3}}.$$

Solution 4 by Albert Stadler, Herrliberg, Switzerland.

Apply the change of variable $x \mapsto \pi - x$. Since $\sin(\pi - x) = \sin x$ and $\cos(\pi - x) = -\cos x$, we obtain

$$\Omega = \int_0^\pi \frac{(\pi - x + 2)\sin x}{1 + 3\cos^2 x} dx.$$

Adding the original and transformed expressions yields

$$2\Omega = \int_0^\pi \frac{(\pi + 4)\sin x}{1 + 3\cos^2 x} dx.$$

Since the integrand is symmetric about $\pi/2$, we write

$$\int_0^\pi \frac{(\pi + 4)\sin x}{1 + 3\cos^2 x} dx = 2\int_0^{\frac{\pi}{2}} \frac{(\pi + 4)\sin x}{1 + 3\cos^2 x} dx.$$

Hence,

$$2\Omega = 2\int_0^{\frac{\pi}{2}} \frac{(\pi + 4)\sin x}{1 + 3\cos^2 x} dx.$$

Let $y = \cos x$. Then $dy = -\sin x\, dx$, and as x goes from 0 to $\pi/2$, y goes from 1 to 0. Therefore,

$$\int_0^{\frac{\pi}{2}} \frac{\sin x}{1 + 3\cos^2 x} dx = \int_0^1 \frac{1}{1 + 3y^2} dy = \frac{1}{\sqrt{3}} \arctan(\sqrt{3}y)\Big|_0^1 = \frac{1}{\sqrt{3}} \arctan(\sqrt{3}) = \frac{\pi}{3\sqrt{3}}.$$

Thus,

$$\Omega = \frac{(\pi + 4)\pi}{3\sqrt{3}}.$$

Solution 5 by Ángel Plaza, Universidad de Las Palmas de Gran Canaria, Spain.

Using the identity $\displaystyle\int_a^b f(x)dx = \int_a^b f(a + b - x)dx$, we assert

$$\Omega = \int_0^\pi \frac{(\pi - x + 2)\sin(\pi - x)}{1 + 3\cos^2(\pi - x)}\, dx.$$

Since $\sin(\pi - x) = \sin x$ and $\cos(\pi - x) = -\cos x$, then

$$\Omega = \int_0^\pi \frac{(\pi - x + 2)\sin x}{1 + 3\cos^2 x}\, dx.$$

So

$$
\begin{aligned}
2\,\Omega &= \int_0^{\pi} \frac{(x+2+\pi-x+2)\sin x}{1+3\cos^2 x}\,dx \\
&= \int_0^{\pi} \frac{(\pi+4)\sin x}{1+3\cos^2 x}\,dx \\
&= -\frac{\pi+4}{\sqrt{3}} \int_0^{\pi} \frac{-\sqrt{3}\sin x}{1+\left(\sqrt{3}\cos x\right)^2}\,dx \\
&= \frac{\pi+4}{\sqrt{3}} \int_{-\sqrt{3}}^{\sqrt{3}} \frac{dt}{1+t^2} \\
&= \frac{\pi+4}{\sqrt{3}} \left[\arctan t\right]_{-\sqrt{3}}^{\sqrt{3}} \\
&= \frac{\pi+4}{\sqrt{3}} \left(\frac{\pi}{3} - \left(\frac{-\pi}{3}\right)\right) \\
&= \frac{2(\pi+4)\pi}{3\sqrt{3}}.
\end{aligned}
$$

In short,

$$
\Omega = \frac{\pi^2+4\pi}{3\sqrt{3}} \sim 4.3178\ldots.
$$

Solution 6 by Brian D. Beasley, Simpsonville, SC.

Using the substitution $w = \cos x$, we note that

$$
\int \frac{\sin x}{1+3\cos^2 x} dx = -\int \frac{dw}{1+3w^2} = f(x) + C,
$$

where $f(x) = (-1/\sqrt{3})\tan^{-1}(\sqrt{3}\cos x)$. This result allows us in turn to apply integration by parts with $u = x+2$, $du = dx$, $v = f(x)$, and $dv = \sin x\, dx/(1+3\cos^2 x)$ to yield

$$
\Omega = uv]_0^{\pi} - \int_0^{\pi} v du = (x+2)f(x)]_0^{\pi} - \int_0^{\pi} f(x)dx.
$$

We claim that the definite integral in this expression equals zero, since $f(\pi/2 - x) = -f(\pi/2 + x)$:

$$
f\left(\frac{\pi}{2} - x\right) = -\frac{1}{\sqrt{3}} \tan^{-1}\left(\sqrt{3}\cos\left(\frac{\pi}{2} - x\right)\right) = -\frac{1}{\sqrt{3}} \tan^{-1}(\sqrt{3}\sin x)
$$

and

$$
f\left(\frac{\pi}{2} + x\right) = -\frac{1}{\sqrt{3}} \tan^{-1}\left(\sqrt{3}\cos\left(\frac{\pi}{2} + x\right)\right) = -\frac{1}{\sqrt{3}} \tan^{-1}(-\sqrt{3}\sin x) = \frac{1}{\sqrt{3}} \tan^{-1}(\sqrt{3}\sin x)
$$

In particular, we obtain

$$\int_0^{\pi/2} f(x)dx = -\int_{\pi/2}^{0} f\left(\frac{\pi}{2}-u\right)du = \int_0^{\pi/2} f\left(\frac{\pi}{2}-u\right)du$$

and

$$\int_{\pi/2}^{\pi} f(x)dx = \int_0^{\pi/2} f\left(\frac{\pi}{2}+u\right)du = -\int_0^{\pi/2} f\left(\frac{\pi}{2}-u\right)du.$$

Hence $\int_0^{\pi} f(x)dx = 0$, so we conclude

$$\Omega = (\pi+2)f(\pi) - 2f(0) = -\frac{(\pi+2)}{\sqrt{3}}\left(-\frac{\pi}{3}\right) + \frac{2}{\sqrt{3}}\left(\frac{\pi}{3}\right) = \frac{\pi(\pi+4)}{3\sqrt{3}}.$$

Solution 7 by Devis Alvarado, UNAH and UPNFM, Tegucigalpa, Hoduras.

$$\begin{aligned}
\Omega &= \int_0^{\pi} \frac{(x+2)\sin(x)}{1+3\cos^2(x)}\,dx \\
&= \int_0^{\pi} \frac{(\pi-y+2)\sin(\pi-y)}{1+3\cos^2(\pi-y)}\,dy, \quad y=\pi-x \\
&= \int_0^{\pi} \frac{(\pi+4-2-y)\sin(y)}{1+3\cos^2(y)}\,dy \\
&= (\pi+4)\int_0^{\pi} \frac{\sin(y)}{1+3\cos^2(y)}\,dy - \int_0^{\pi} \frac{(2+y)\sin(y)}{1+3\cos^2(y)}\,dy \\
&= (\pi+4)\int_0^{\pi} \frac{\sin(y)}{1+3\cos^2(y)}\,dy - \Omega \\
\Longrightarrow \Omega &= \frac{\pi+4}{2}\int_{-\sqrt{3}}^{\sqrt{3}} \frac{1}{1+u^2}\frac{du}{\sqrt{3}}, \quad u = \sqrt{3}\cos(y) \\
&= \frac{\pi+4}{2\sqrt{3}}\arctan(u)\Big|_{-\sqrt{3}}^{\sqrt{3}} = \frac{\pi+4}{2\sqrt{3}}\frac{2\pi}{3} = \frac{\pi(\pi+4)}{3\sqrt{3}}.
\end{aligned}$$

Therefore the solution is:

$$\Omega = \frac{\pi(\pi+4)}{3\sqrt{3}}.$$

Solution 8 by Jose Luis Diaz-Barrero, Barcelona, Spain.

To evaluate

$$\Omega = \int_0^{\pi} \frac{(x+2)\sin x}{1+3\cos^2 x}\,dx.$$

we make the substitution $x \mapsto \pi - x$. Then

$$\Omega = \int_0^{\pi} \frac{(\pi-x+2)\sin x}{1+3\cos^2 x}\,dx.$$

Add the two expressions for Ω yields

$$2\Omega = \int_0^{\pi} \frac{(x+2)+(\pi - x + 2)}{1+3\cos^2 x} \sin x \, dx = \int_0^{\pi} \frac{\pi+4}{1+3\cos^2 x} \sin x \, dx.$$

Thus $\Omega = \dfrac{\pi+4}{2} \displaystyle\int_0^{\pi} \frac{\sin x}{1+3\cos^2 x}\, dx$. Now, we evaluate

$$I = \int_0^{\pi} \frac{\sin x}{1+3\cos^2 x}\, dx.$$

Using the substitution $u = \cos x$, $du = -\sin x\, dx$, we have

$$I = \int_1^{-1} \frac{-1}{1+3u^2}\, du = \int_{-1}^{1} \frac{1}{1+3u^2}\, du.$$

Since the integrand is even, then

$$I = 2\int_0^{1} \frac{1}{1+3u^2}\, du.$$

Since $\displaystyle\int \frac{1}{1+3u^2}\, du = \frac{1}{\sqrt{3}} \arctan(\sqrt{3}u)$, then

$$I = \frac{2}{\sqrt{3}} \arctan(\sqrt{3}) = \frac{2\pi}{3\sqrt{3}}$$

on account that $\arctan(\sqrt{3}) = \dfrac{\pi}{3}$. Finally,

$$\Omega = \frac{\pi+4}{2} I = \frac{\pi+4}{2} \cdot \frac{2\pi}{3\sqrt{3}} = \frac{\pi(\pi+4)}{3\sqrt{3}}.$$

Solution 9 by Luke Paluso, Blinn College, Brenham, TX.

Using the substitution method (U) with $u = \sqrt{3}\cos x$ and properties of definite integrals one has

$$\begin{aligned}
\Omega &= \int_0^{\pi} \frac{(x+2)\sin x}{1+(\sqrt{3}\cos x)^2} dx \\
&\stackrel{\mathrm{U}}{=} \frac{1}{\sqrt{3}} \int_{-\sqrt{3}}^{\sqrt{3}} \frac{\arccos\left(\frac{u}{\sqrt{3}}\right)+2}{1+u^2} du \\
&= \frac{1}{\sqrt{3}} \left[\int_{-\sqrt{3}}^{\sqrt{3}} \frac{\arccos\left(\frac{u}{\sqrt{3}}\right)}{1+u^2} du + \int_{-\sqrt{3}}^{\sqrt{3}} \frac{2}{1+u^2} du \right].
\end{aligned}$$

Do the substitution $u = \tan\theta$ (A), using the identity $\arccos(y) + \arcsin(y) = \pi/2$ (B), the Maclaurin series for arcsine (C), and the fact that $\tan^{2n+1}\theta$ is odd for $n \in \mathbb{N}$ (D), one evaluates the 1st integral:

$$\begin{aligned}
\int_{-\sqrt{3}}^{\sqrt{3}} \frac{\arccos\left(\frac{u}{\sqrt{3}}\right)}{1+u^2} du &=^{\mathrm{A}} \int_{-\frac{\pi}{3}}^{\frac{\pi}{3}} \arccos\left(\frac{1}{\sqrt{3}}\tan\theta\right) d\theta \\
&=^{\mathrm{B}} \frac{\pi}{2}\int_{-\frac{\pi}{3}}^{\frac{\pi}{3}} d\theta - \int_{-\frac{\pi}{3}}^{\frac{\pi}{3}} \arcsin\left(\frac{1}{\sqrt{3}}\tan\theta\right) d\theta \\
&=^{\mathrm{C}} \frac{\pi}{2}\cdot\frac{2\pi}{3} - \int_{-\frac{\pi}{3}}^{\frac{\pi}{3}} \sum_{n=0}^{\infty} \frac{(2n)!}{(2^n n!)^2(2n+1)} \left(\frac{1}{\sqrt{3}}\tan\theta\right)^{2n+1} d\theta \\
&= \frac{\pi^2}{3} - \sum_{n=0}^{\infty} \frac{(2n)!}{(2^n n!)^2(2n+1)\sqrt{3}^{2n+1}} \int_{-\frac{\pi}{3}}^{\frac{\pi}{3}} \tan^{2n+1}\theta d\theta \\
&=^{\mathrm{D}} \frac{\pi^2}{3}.
\end{aligned}$$

The 2nd integral is straight forward

$$\int_{-\sqrt{3}}^{\sqrt{3}} \frac{2}{1+u^2} du = 2\arctan u\Big|_{-\sqrt{3}}^{\sqrt{3}} = 2\cdot\left(\arctan\sqrt{3} - \arctan(-\sqrt{3})\right) = \frac{4\pi}{3}.$$

Substituting these into the original chain of equalities yields the answer:

$$\Omega = \frac{1}{\sqrt{3}}\left(\frac{\pi^2}{3} + \frac{4\pi}{3}\right) = \frac{\pi^2+4\pi}{3\sqrt{3}} = \frac{\sqrt{3}\pi^2 + 4\sqrt{3}\pi}{9}.$$

Solution 10 by Michel Bataille, Rouen, France.

We calculate

$$\Omega_1 = \int_0^{\pi} \frac{\sin x}{1+3\cos^2 x} dx \quad \text{and} \quad \Omega_2 = \int_0^{\pi} \frac{x\sin x}{1+3\cos^2 x} dx.$$

First, the substitution $u = \cos x$ leads to

$$\Omega_1 = \int_{-1}^{1} \frac{du}{1+3u^2} = \frac{1}{\sqrt{3}}\left[\arctan(u\sqrt{3})\right]_{-1}^{1} = \frac{2\pi}{3\sqrt{3}}.$$

Then we have

$$\Omega_2 = \int_0^{\pi} \frac{(\pi-u)\sin u}{1+3\cos^2 u} du = \pi\cdot\Omega_1 - \Omega_2,$$

hence $\Omega_2 = \dfrac{\pi}{2}\cdot\Omega_1 = \dfrac{\pi^2}{3\sqrt{3}}$. Finally, we obtain

$$\Omega = 2\Omega_1 + \Omega_2 = \frac{\pi(\pi+4)}{3\sqrt{3}}.$$

Solution 11 by Perfetti Paolo, dipartimento di matematica Università di "Tor Vergata", Roma, Italy.

The integral is

$$\Omega = \int_0^\pi \frac{x\sin x}{1+3\cos^2 x}dx + \int_0^\pi \frac{2\sin x}{1+3\cos^2 x}dx \doteq \Omega_1 + \Omega_2$$

$$\begin{aligned}
\Omega_2 &= \int_0^\pi \frac{\sin x}{1-i\sqrt{3}\cos x}dx + \int_0^\pi \frac{\sin x}{1+i\sqrt{3}\cos x}dx = \frac{1}{i\sqrt{3}}\mathrm{Ln}\frac{1-i\sqrt{3}\cos x}{1+i\sqrt{3}\cos x}\Big|_0^\pi = \\
&= \frac{1}{i\sqrt{3}}\mathrm{Ln}\frac{1+i\sqrt{3}}{1-i\sqrt{3}} - \frac{1}{i\sqrt{3}}\mathrm{Ln}\frac{1-i\sqrt{3}}{1+i\sqrt{3}} = \frac{2}{i\sqrt{3}}\mathrm{Ln}\frac{1+i\sqrt{3}}{1-i\sqrt{3}} = \\
&= \frac{2}{i\sqrt{3}}\mathrm{Ln}\frac{e^{i\pi/3}}{e^{-i\pi/3}} = \frac{2}{i\sqrt{3}}\frac{2i\pi}{3} = \frac{4\pi}{3\sqrt{3}}
\end{aligned}$$

$$\begin{aligned}
\Omega_1 &= \frac{1}{2}\int_{-\pi}^\pi \frac{x\sin x}{1+3\cos^2 x}dx = \frac{1}{2}\int_{-\pi}^0 \frac{x\sin x}{1+3\cos^2 x}dx + \frac{1}{2}\int_0^\pi \frac{x\sin x}{1+3\cos^2 x}dx = \\
&= \frac{1}{2}\int_0^\pi \frac{-(x-\pi)\sin x}{1+3\cos^2 x}dx + \frac{1}{2}\int_0^\pi \frac{x\sin x}{1+3\cos^2 x}dx = \frac{\pi}{2}\int_0^\pi \frac{\sin x}{1+3\cos^2 x}dx = \\
&= \frac{2\pi}{3\sqrt{3}}\frac{\pi}{2} = \frac{\pi^2}{3\sqrt{3}}.
\end{aligned}$$

Then result is $\dfrac{\pi(4+\pi)}{3\sqrt{3}}$. Now consider this: $\dfrac{d}{da}\displaystyle\int_0^\infty \frac{x^a\,dx}{x^4+1}\Big|_{a=3/2}$. By passing to the complex variable z, it follows that

$$\begin{aligned}
\int_0^\infty \frac{x^a\,dx}{x^4+1} &= \frac{2\pi i}{1-e^{2i\pi a}}\sum_{k=0}^{3}\mathrm{Res}\frac{z^a}{z^4+1}\Big|_{z_k=e^{\frac{i(1+2k)\pi}{4}}} = \frac{2\pi i}{1-e^{2i\pi a}}\sum_{k=0}^{3}\frac{1}{4}e^{\frac{i\pi(1+2k)(a-3)}{4}} = \\
&= \frac{\pi i}{2(1-e^{2i\pi a})}e^{\frac{i\pi(a-3)}{4}}\sum_{k=0}^{3}e^{\frac{i\pi k(a-3)}{2}} = \frac{\pi i}{2(1-e^{2i\pi a})}e^{\frac{i\pi(a-3)}{4}}\frac{1-e^{i\pi 2k(a-3)}}{1-e^{\frac{i\pi(a-3)}{2}}} = \\
&= \frac{\pi i}{2}\frac{e^{\frac{i\pi(a-3)}{4}}}{1-e^{\frac{i\pi(a-3)}{2}}} = \frac{\pi i}{2}\frac{1}{-2i\sin\frac{\pi(a-3)}{4}}.
\end{aligned}$$

Doing the derivative and evaluating at $a = 3/2$ we get

$$\frac{\pi}{4}\frac{\cos\frac{\pi(a-3)}{4}}{\sin^2\frac{\pi(a-3)}{4}}\Bigg|_{a=\frac{3}{2}} = \frac{\pi^2}{16}\frac{\cos\frac{3\pi}{8}}{\sin^2\frac{3\pi}{8}} = \frac{\pi^2}{16}\frac{\sin\frac{\pi}{8}}{\cos^2\frac{\pi}{8}} = \frac{\pi^2}{16}\frac{\frac{\sqrt{2-\sqrt{2}}}{2}}{\frac{2+\sqrt{2}}{4}} = \frac{\pi^2}{2}\frac{(2-\sqrt{2})^{3/2}}{8} = \frac{\pi^2}{2}\sin^3\frac{\pi}{8}.$$

Also solved by Bruno Salgueiro Fanego, Viveiro, Lugo, Spain and the problem proposer.

* * * * *   * * * * *   * * * * *   * * * * *   * * * * *

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Editor's Statement: It goes without saying that the problem proposers, as well as the solution proposers, are the *élan vital* of the Problems/Solutions Section of SSMJ. As the editor of this Section of the Journal, I consider myself fortunate to be in a position to receive, compile and organize a wealth of proposed ingenious problems and solutions intended for online publication. My unwavering gratitude goes to all the amazingly creative contributors. We come together from across continents because we find intellectual value, joy and satisfaction in mathematical problems, both in their creation as well as their solution. So that our collective efforts serve us well, I kindly ask all contributors to adhere to the following guidelines. As you peruse below, you may construe that the guidelines amount to a lot of work. But, as the samples show, there's not much to do. Your cooperation is much appreciated!

Keep in mind that the examples given below are your best guide!

Formats, Styles and Requirements

When submitting proposed problem(s) or solution(s), please send both **LaTeX** document and **pdf** document of your proposed problem(s) or solution(s). There are ways (discoverable from the internet) to convert from Word to proper LaTeX code. Porposals without a *proper* **LaTeX** document will not be published regrettably.

Regarding Proposed Solutions:

Below is the FILENAME format for all the documents of your proposed solution(s).

#ProblemNumber_FirstName_LastName_Solution_SSMJ

- FirstName stands for YOUR first name.
- LastName stands for YOUR last name.

Examples:

#1234_Max_Planck_Solution_SSMJ

#9876_Charles_Darwin_Solution_SSMJ

Please note that every problem number is *preceded* by the sign # .

All you have to do is copy the FILENAME format (or an example below it), paste it and then modify portions of it to your specs.

Please adopt the following structure, in the order shown, for the presentation of your solution:

1. On top of the first page of your solution, begin with the phrase:

"Proposed Solution to #**** SSMJ"

where the string of four astrisks represents the problem number.

2. On the second line, write

"Solution proposed by [your First Name, your Last Name]",

followed by your affiliation, city, country, all on the same linear string of words. Please see the example below. If you have collaborator(s), then, immediately following your address, after inserting the connective "and", provide full name, affiliation and address of your collaborator(s).

3. On a new line, state the problem proposer's name, affiliation, city and country, just as it appears published in the Problems/Solutions section.

4. On a new line below the above, write in bold type: "**Statement of the Problem**".

5. Below the latter, state the problem. Please make sure the statement of your problem (unlike the preceding item) is not in bold type.

6. Below the statement of the problem, write in bold type: "**Solution of the Problem**".

7. Below the latter, show the entire solution of the problem.

Here is a sample for the above-stated format for proposed solutions:

Proposed solution to #1234 SSMJ

Solution proposed by Emmy Noether, University of Göttingen, Lower Saxony, Germany.

Problem proposed by Isaac Newton, Trinity College, Cambridge, England.

Statement of the problem:

Compute $\sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}$.

Solution of the problem: . . . . . .

Regarding Proposed Problems:

For all your proposed problems, please adopt for all documents the following FILENAME format:

FirstName_LastName_ProposedProblem_SSMJ_YourGivenNumber_ProblemTitle

If you do not have a ProblemTitle, then leave that component as it already is (i.e., ProblemTitle).

The component YourGivenNumber is any UNIQUE 3-digit (or longer) number you like to give to your problem.

Examples:

Max_Planck_ProposedProblem_SSMJ_314_HarmonicPatterns

Charles_Darwin_ProposedProblem_SSMJ_358_ProblemTitle

Please adopt the following structure, in the order shown, for the presentation of your proposal:

1. On the top of first page of your proposal, begin with the phrase:

"Problem proposed to SSMJ"

2. On the second line, write

"Problem proposed by [your First Name, your Last Name]",

followed by your affiliation, city, country, all on the same linear string of words. Please see the example below. If you have collaborator(s), then, immediately following your address, after inserting the connective "and", provide full name, affiliation and address of your collaborator(s).

3. On a new line state the title of the problem, if any.

4. On a new line below the above, write in bold type: "**Statement of the Problem**".

5. Below the latter, state the problem. Please make sure the statement of your problem (unlike the preceding item) is not in bold type.

6. Below the statement of the problem, write in bold type: "**Solution of the Problem**".

7. Below the latter, show the entire solution of your problem.

Here is a sample for the above-stated format for proposed problems:

Problem proposed to SSMJ

Problem proposed by Isaac Newton, Trinity College, Cambridge, England.

Principia Mathematica (⟵ You may choose to not include a title.)

Statement of the problem:

Compute $\sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}$.

Solution of the problem: . . . . . .

♣ ♣ ♣ Thank You! ♣ ♣ ♣