Problems and Solutions

Albert Natian, Section Editor

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This section of the Journal offers readers an opportunity to exchange interesting mathematical problems and solutions. Please email them to Prof. Albert Natian at Department of Mathematics, Los Angeles Valley College. Please make sure every proposed problem or proposed solution is provided in both ***LaTeX*** and pdf documents. Please make sure your proposals adhere to **Formats, Styles and Requirements** noted below. Thank you!

To propose problems, email them to: problems4ssma@gmail.com

To propose solutions, email them to: solutions4ssma@gmail.com

Solutions to previously published problems can be seen at <www.ssma.org/publications>.

Solutions to the problems published in this issue should be submitted *before* April 1, 2027.

• **5855** *Proposed by Paolo Perfetti, dipartimento di matematica Universitą di "Tor Vergata",Rome, Italy.*

Calculate

$$S := \sum_{n=1}^{\infty}(-1)^n \left[\left(\frac{1}{2n-1} - \frac{1}{2n+1} + \frac{1}{2n+3} - \cdots \right) - \frac{1}{4n} \right]^2 .$$

• **5856** *Proposed by Mohammed Aassila, Strasbourg, France.*

Let $r \geqslant 1$, and let $a_1, a_2, \ldots, a_r$ be arbitrary positive integers. Prove that, for every integer $n \geqslant 1$, the set of integers $m \geqslant 1$ satisfying

$$n \mid \left(m + a_1^m + a_2^m + \cdots + a_r^m \right)$$

contains an infinite arithmetic progression. Note: The notation "$x \mid y$" means "x divides y".

• **5857** *Proposed by Goran Conar, Varaždin, Croatia.*

Let n be in $\mathbb{N}$ and suppose α, β, k are positive reals with $\alpha > 1$. Further suppose the real numbers $x_1, x_2, \ldots, x_n$ satisfy the conditions

$$\sum_{j=1}^{n} \frac{1}{x_j} = 1 \quad \text{and} \quad \forall j \in \mathbb{N}: \quad x_j > \left(\frac{k(\alpha-1)}{1+\frac{\alpha}{\beta}} \right)^{1/\alpha} .$$

Prove the following inequality and then tell when the equality occurs:

$$(k+n^{\alpha})^{-1/\beta} \leqslant \sum_{j=1}^{n} \frac{1}{x_j} \cdot \left(k + x_j^{\alpha} \right)^{-1/\beta} .$$

• **5858** *Proposed by Shivam Sharma, Delhi University, New Delhi, India.*

Evaluate

$$I = \int_0^1 \int_0^1 \frac{\ln^2(x)}{(1+y)(1+xy)}\, dx\, dy.$$

• **5859** *Proposed by Michael Brozinsky, Central Islip, New York.*

At a party attended by 91 people, each person was given a list of people (possibly empty) in the party to whom she or he would give chocolate. If person A would give chocolate to person B, then person B would not give chocolate to person A. Furthermore, if person A would give chocolate to person B, then person A would also give chocolate to every person on B's list. Show that among any 50 people chosen from the 91 people in the party, either there exists a group of eight people none of whom would give chocolate to anyone else in the group, or there exists a gourp of eight people such that for every pair of individuals in the group, one would give chocolate to the other.

Solutions

To Formerly Published Problems

• **5835** *Proposed by D.M. Bătineţu-Giurgiu, Bucharest, Romania and Neculai Stanciu, "George Emil Palade" School, Buzău, Romania.*

Find the limit $L = \lim\limits_{n\to\infty} \left(\sqrt[n]{a} - 1\right) \sqrt[n]{b_n F_n}$ where $a > 0$, F_n is the n-th Fibonacci number and where $(b_n)_{n\geqslant 1}$ is a positive real sequence with $\lim\limits_{n\to\infty} \dfrac{b_{n+1}}{nb_n} = \pi$.

Solution 1 by Michel Bataille, Rouen, France.

Let $u_n = \dfrac{n^n}{b_n F_n}$. Then, we have

$$\frac{u_{n+1}}{u_n} = \frac{b_n F_n}{b_{n+1} F_{n+1}} \cdot \left(\frac{n+1}{n}\right)^n (n+1) = \left(1+\frac{1}{n}\right)^n \frac{F_n}{F_{n+1}} \cdot \frac{nb_n}{b_{n+1}} \left(1+\frac{1}{n}\right),$$

hence

$$\lim_{n\to\infty} \frac{u_{n+1}}{u_n} = \frac{e}{\alpha\pi}$$

(where $\alpha = \dfrac{1+\sqrt{5}}{2}$ and because $F_n = \dfrac{\alpha^n - (-1/\alpha)^n}{\sqrt{5}} \sim \dfrac{\alpha^n}{\sqrt{5}}$ as $n \to \infty$).

From a well-known result, we also have $\lim\limits_{n\to\infty} \sqrt[n]{u_n} = \dfrac{e}{\alpha\pi}$ and therefore

$$\sqrt[n]{b_n F_n} \sim \frac{\alpha\pi}{e} \cdot n \qquad (n \to \infty).$$

In addition, we have

$$\sqrt[n]{a} - 1 = e^{\frac{\ln(a)}{n}} - 1 \sim \frac{\ln(a)}{n} \qquad (n \to \infty).$$

Thus,

$$\left(\sqrt[n]{a} - 1\right) \sqrt[n]{b_n F_n} \sim \frac{\ln(a)}{n} \cdot \frac{\alpha\pi}{e} \cdot n \qquad (n \to \infty)$$

and

$$L = \frac{\alpha\pi \ln(a)}{e}.$$

Solution 2 by David A. Huckaby, Angelo State University, San Angelo, TX.

Note that the limit $\lim_{n\to\infty} n\left(\sqrt[n]{a} - 1\right)$ yields the indeterminate form $\infty \cdot 0$. Rearranging to obtain the indeterminate form $\frac{0}{0}$ and then applying L'Hôpital's Rule gives

$$\begin{aligned}\lim_{n\to\infty} n\left(\sqrt[n]{a} - 1\right) &= \lim_{n\to\infty} \frac{\sqrt[n]{a} - 1}{\frac{1}{n}} \\ &= \lim_{n\to\infty} \frac{-\frac{\ln a \sqrt[n]{a}}{n^2}}{-\frac{1}{n^2}} = \lim_{n\to\infty} \ln a \sqrt[n]{a} = \ln a. \end{aligned} \tag{1}$$

The following result follows from, for example, the Stolz-Cesàro Theorem: If for a sequence $(y_n)_{n\geqslant 1}$ of positive real numbers, the limit $\lim_{n\to\infty} \frac{y_{n+1}}{y_n}$ exists, then $\lim_{n\to\infty} \sqrt[n]{y_n} = \lim_{n\to\infty} \frac{y_{n+1}}{y_n}$. (See `https://en.wikipedia.org/wiki/StolzâĂŞCesaro_theorem`.) Define the sequence $(y_n)_{n\geqslant 1}$ by $y_n = \frac{b_n F_n}{n^n}$. Then

$$\begin{aligned}\lim_{n\to\infty} \frac{y_{n+1}}{y_n} &= \lim_{n\to\infty} \frac{\frac{b_{n+1}F_{n+1}}{(n+1)^{n+1}}}{\frac{b_n F_n}{n^n}} = \lim_{n\to\infty} \left[\frac{b_{n+1}}{nb_n} \cdot \frac{F_{n+1}}{F_n} \cdot \left(\frac{n}{n+1}\right)^n \left(\frac{n}{n+1}\right)\right] \\ = \lim_{n\to\infty} \frac{b_{n+1}}{nb_n} \cdot \lim_{n\to\infty} \frac{F_{n+1}}{F_n} \cdot \lim_{n\to\infty} \frac{1}{\left(1+\frac{1}{n}\right)^n} \cdot \lim_{n\to\infty} \frac{n}{n+1} &= \pi \cdot \varphi \cdot \frac{1}{e} \cdot 1 = \frac{\pi\varphi}{e},\end{aligned}$$

where φ is the golden ratio. So $\lim_{n\to\infty} \frac{\sqrt[n]{b_n F_n}}{n} = \lim_{n\to\infty} \sqrt[n]{\frac{b_n F_n}{n^n}} = \lim_{n\to\infty} \sqrt[n]{y_n} = \lim_{n\to\infty} \frac{y_{n+1}}{y_n} = \frac{\pi\varphi}{e}$. Combining this with the result from (1), we have

$$\begin{aligned}\lim_{n\to\infty} \left(\sqrt[n]{a} - 1\right) \sqrt[n]{b_n F_n} &= \lim_{n\to\infty} n\left(\sqrt[n]{a} - 1\right) \cdot \lim_{n\to\infty} \frac{\sqrt[n]{b_n F_n}}{n} \\ &= \ln a \cdot \lim_{n\to\infty} \sqrt[n]{\frac{b_n F_n}{n^n}} = \ln a \cdot \lim_{n\to\infty} \sqrt[n]{y_n} = \ln a \cdot \frac{\pi\varphi}{e}.\end{aligned}$$

Solution 3 by Ángel Plaza, Universidad de Las Palmas de Gran Canaria, Spain.

We have $\lim\limits_{n\to\infty}\sqrt[n]{a}=1 \Rightarrow \lim\limits_{n\to\infty}\dfrac{\sqrt[n]{a}-1}{\ln\sqrt[n]{a}}=1$. Also, $\lim\limits_{n\to\infty}\sqrt[n]{F_n}=\alpha$ because $F_n=\dfrac{\alpha^n-\beta^n}{\sqrt{5}}$ with $\alpha=\dfrac{1+\sqrt{5}}{2}$ and $\beta=\dfrac{1-\sqrt{5}}{2}$. Then,

$$\begin{aligned}
L &= \lim_{n\to\infty}\ln\sqrt[n]{a}\sqrt[n]{b_n}\,\alpha \\
&= \alpha\ln a\lim_{n\to\infty}\frac{1}{n}\sqrt[n]{b_n} \\
&= \alpha\ln a\lim_{n\to\infty}\sqrt[n]{\frac{b_n}{n^n}} \\
&= \alpha\ln a\lim_{n\to\infty}\frac{b_{n+1}}{b_n}\cdot\frac{n^n}{(n+1)^{n+1}} \\
&= \alpha\ln a\lim_{n\to\infty}\frac{b_{n+1}}{(n+1)b_n}\cdot\frac{n^n}{(n+1)^n} \\
&= \alpha\ln a\,\frac{\pi}{e}.
\end{aligned}$$

Solution 4 by Daniel Văcaru, National Economic College, Pitești, Romania. One has

$$\left(\sqrt[n]{a}-1\right)\sqrt[n]{b_nF_n}=\frac{\sqrt[n]{a}-1}{\frac{1}{n}}\sqrt[n]{\frac{b_nF_n}{n^n}}. \tag{2}$$

One has

$$\lim_{n\to\infty}\frac{\sqrt[n]{a}-1}{\frac{1}{n}}=\ln a \tag{3}$$

and

$$\lim_{n\to\infty}\sqrt[n]{\frac{b_nF_n}{n^n}}=\lim_{n\to\infty}\frac{b_{n+1}F_{n+1}}{(n+1)^{n+1}}\cdot\frac{n^n}{b_nF_n} \tag{4}$$

One has

$$\frac{b_{n+1}F_{n+1}}{(n+1)^{n+1}}\cdot\frac{n^n}{b_nF_n}=\frac{F_{n+1}}{F_n}\cdot\left(\frac{n}{n+1}\right)^n\cdot\frac{b_{n+1}}{(n+1)\,b_n} \tag{5}$$

But $\lim\limits_{n\to\infty}\dfrac{F_{n+1}}{F_n}=\dfrac{1+\sqrt{5}}{2}$, $\lim\limits_{n\to\infty}\left(\dfrac{n}{n+1}\right)^n=\dfrac{1}{e}$, and $\lim\limits_{n\to\infty}\dfrac{b_{n+1}}{(n+1)\,b_n}=\pi$. It follows that

$$L=\lim_{n\to\infty}\left(\sqrt[n]{a}-1\right)\sqrt[n]{b_nF_n}=\ln a\cdot\frac{1+\sqrt{5}}{2}\cdot\frac{1}{e}\cdot\pi \tag{6}$$

Also solved by Albert Stadler, Herrliberg, Switzerland and the problem proposer.

• **5836** *Proposed by Daniel Sitaru, National Economic College "Theodor Costescu" Drobeta Turnu - Severin, Romania.*

Show that in any $\triangle ABC$ the following inequality holds:

$$\sin^4A+\sin^4B+\sin^4C+\sin^4\left(\frac{\pi}{3}+A\right)+\sin^4\left(\frac{\pi}{3}+B\right)+\sin^4\left(\frac{\pi}{3}+C\right)\leqslant\frac{27}{8}.$$

Solution 1 by Albert Stadler, Herrliberg, Switzerland.

We may assume without loss of generality that $A \leqslant B \leqslant C$. Then $C \geqslant \pi/3$. Indeed, if $C < \pi/3$, then $A < \frac{\pi}{3},\ B < \frac{\pi}{3},\ C < \frac{\pi}{3}$, which would imply $A + B + C < \pi$, contradicting the fact that the angles of a triangle satisfy $A + B + C = \pi$. Hence

$$A + B = \pi - C \leqslant \frac{2\pi}{3}.$$

Define

$$f(x) := \sin^4 x \ + \sin^4\left(\frac{\pi}{3} + x\right).$$

We claim that f is concave on the interval $\left[0, \frac{2\pi}{3}\right]$. Indeed

$$f''(x) = 12\cos^2 x \sin^2 x \ - 4\sin^4 x \ + 12\cos^2\left(\frac{\pi}{3} + x\right)\sin^2\left(\frac{\pi}{3} + x\right) \ - 4\sin^4\left(\frac{\pi}{3} + x\right).$$

Observe that

$$12\cos^2 x \sin^2 x \ - 4\sin^4 x \ = 4\left(3\cos^2 x \sin^2 x \ - \sin^4 x\right) = 4\sin x \sin(3x).$$

Similarly,

$$12\cos^2\left(\frac{\pi}{3} + x\right)\sin^2\left(\frac{\pi}{3} + x\right) \ - 4\sin^4\left(\frac{\pi}{3} + x\right) \ = -4\sin\left(\frac{\pi}{3} + x\right)\sin(3x),$$

and therefore

$$f''(x) = 4\sin x \sin(3x) \ - -4\sin\left(\frac{\pi}{3} + x\right)\sin(3x).$$

Using

$$\sin\left(+\frac{\pi}{3}\,\middle|\, x\right) \ = \frac{\sqrt{3}}{2}\cos x \ + \frac{1}{2}\sin x,$$

we obtain

$$f''(x) = 2\left(-\sqrt{3}\cos x \ + \sin x\right)\sin(3x).$$

Now, $\sin(3x) \ > 0$ for $0 < x < \pi/3$, while $-\sqrt{3}\cot x \ + 1 < 0$ on the same interval. Hence $f''(x) \leqslant 0$ for $0 < x < \pi/3$.

Likewise, $\sin(3x) \ < 0$ for $\pi/3 < x < 2\pi/3$, while $-\sqrt{3}\cot x \ + 1 > 0$ there. Again $f''(x) \leqslant 0$.

Consequently, $f''(x) \leqslant 0$ for all $x \in \left[0, \frac{2\pi}{3}\right]$, so f is concave on $\left[0, \frac{2\pi}{3}\right]$.

By Jensen's inequality,

$$f(A) + f(B) \leqslant 2f\left(\frac{A + B}{2}\right).$$

Equivalently,

$$\sin^4 A \ + \sin^4 B \ + \sin^4\left(\frac{\pi}{3}+A\right) \ + \sin^4\left(\frac{\pi}{3}+B\right) \ \leqslant 2\sin^4\left(\frac{A+B}{2}\right) \ + 2\sin^4\left(\frac{\pi}{3}+\frac{A+B}{2}\right).$$

Thus we may replace (A,B) by $\left(\frac{A+B}{2},\frac{A+B}{2}\right)$. Let

$$x=\frac{A+B}{2}.$$

Since $A+B\leqslant 2\pi/3$, we have $0\leqslant x\leqslant \pi/3$, and it remains to prove

$$2\sin^4 x \ + 2\sin^4\left(\frac{\pi}{3}+x\right) \ + \sin^4(\pi-2x) \ + \sin^4\left(\frac{\pi}{3}+\pi-2x\right) \ \leqslant \frac{27}{8}.$$

Define

$$\omega=\frac{1}{2}+i\frac{\sqrt{3}}{2}=e^{i\pi/3}, z=e^{ix}.$$

Consider

$$\begin{aligned} g(x) &:= \frac{27}{8}-2\sin^4 x \ -2\sin^4\left(\frac{\pi}{3}+x\right) \ -\sin^4(\pi-2x) \ -\sin^4\left(\frac{\pi}{3}+\pi-2x\right) \\ &= \frac{27}{8}-2\sin^4 x \ -2\sin^4\left(x+\frac{\pi}{3}\right) \ -\sin^4(2x) \ -\sin^4\left(2x-\frac{\pi}{3}\right). \end{aligned}$$

Using the standard complex-exponential representation of the sine function, we obtain

$$g(x) := \frac{27}{8}-\frac{2}{16}\left(z-\frac{1}{z}\right)^4-\frac{2}{16}\left(\omega z-\frac{1}{\omega z}\right)^4-\frac{1}{16}\left(z^2-\frac{1}{z^2}\right)^4-\frac{1}{16}\left(\frac{z^2}{\omega}-\frac{\omega}{z^2}\right)^4.$$

After simplification,

$$g(x)=-\frac{1}{16\left(\frac{z}{\omega}\right)^8}\left(-1+2\left(\frac{z}{\omega}\right)^4+8\left(\frac{z}{\omega}\right)^6-18\left(\frac{z}{\omega}\right)^8+8\left(\frac{z}{\omega}\right)^{10}+2\left(\frac{z}{\omega}\right)^{12}-\left(\frac{z}{\omega}\right)^{16}\right).$$

Factoring the polynomial gives

$$g(x)=\frac{1}{16\left(\frac{z}{\omega}\right)^8}\left(-1+\left(\frac{z}{\omega}\right)^2\right)^4\left(\left(1+\left(\frac{z}{\omega}\right)^2\right)^4+2\left(\frac{z}{\omega}\right)^4\right).$$

This can be rewritten as

$$g(x)=\frac{1}{16}\left(\left(\left(\frac{z}{\omega}\right)^2-\left(\frac{\omega}{z}\right)^2\right)^4+2\left(\frac{z}{\omega}-\frac{\omega}{z}\right)^4\right).$$

Returning to trigonometric form,

$$g(x)=\sin^4\left(2x-\frac{2\pi}{3}\right) \ +2\sin^4\left(x-\frac{\pi}{3}\right).$$

Since both terms on the right-hand side are non-negative, $g(x) \geqslant 0$ for all real x.

Therefore,

$$2\sin^4 x \ + 2\sin^4\left(\frac{\pi}{3}+x\right) \ + \sin^4(2x) \ + \sin^4\left(2x-\frac{\pi}{3}\right) \ \leqslant \frac{27}{8},$$

which proves

$$\sin^4 A \ + \sin^4 B \ + \sin^4 C \ + \sin^4\left(\frac{\pi}{3}+A\right) \ + \sin^4\left(\frac{\pi}{3}+B\right) \ + \sin^4\left(\frac{\pi}{3}+C\right) \ \leqslant \frac{27}{8}.$$

The proof is complete.

Solution 2 by Michel Bataille, Rouen, France.

We will show this stronger result:

$$\forall x \in \mathbb{R}: \quad \sin^4 x + \sin^4\left(x+\frac{\pi}{3}\right) \leqslant \frac{9}{8}.$$

We shall use the following lemma:

$$\forall x \in \mathbb{R}: \quad \cos^2 x + \cos^2\left(x+\frac{\pi}{3}\right) + \cos^2\left(x+\frac{2\pi}{3}\right) = \frac{3}{2}$$

which can be verified via a simple calculation using

$$\cos(a+b) = \cos a \cos b - \sin a \sin b$$

and

$$\cos\frac{\pi}{3} = \frac{1}{2} = -\cos\frac{2\pi}{3} \ \text{ and } \ \sin\frac{\pi}{3} = \sin\frac{2\pi}{3} = \frac{\sqrt{3}}{2}.$$

We now work on the expression $\sin^4 x + \sin^4\left(x+\frac{\pi}{3}\right)$:

We know that $\sin^3 x = \frac{3}{4}\sin x - \frac{1}{4}\sin 3x$. It follows that

$$\sin^4 x = \frac{3}{4}\sin^2 x - \frac{1}{4}(\sin x \sin 3x) = \frac{3}{4}(1-\cos^2 x) - \frac{1}{8}(\cos 2x - \cos 4x) = \frac{3}{4} - \cos^2 x + \frac{1}{4}\cos^2 2x$$

(via the formula $\cos 2a = 2\cos^2 a - 1$).

From the lemma, we deduce that

$$\begin{aligned}\sin^4 x + \sin^4\left(x+\frac{\pi}{3}\right) &= \frac{3}{2} - \left(\frac{3}{2} - \cos^2\left(x+\frac{2\pi}{3}\right)\right) + \frac{1}{4}\left(\frac{3}{2} - \cos^2\left(2x+\frac{\pi}{3}\right)\right)\\ &= \frac{3}{8} + \frac{1}{2}\left(1+\cos\left(2x+\frac{4\pi}{3}\right)\right) - \frac{1}{4}\cos^2\left(2x+\frac{\pi}{3}\right)\\ &= \frac{3}{8} + \frac{1}{4}\left(2+2\cos u - \cos^2 u\right)\end{aligned}$$

where $u = 2x + \dfrac{4\pi}{3}$.

Since $2 + 2\cos u - \cos^2 u = 3 - (1 - \cos u)^2 \leqslant 3$, we see that $\sin^4 x + \sin^4\left(x + \dfrac{\pi}{3}\right) \leqslant \dfrac{3}{8} + \dfrac{3}{4} = \dfrac{9}{8}$.

Also solved by Daniel Văcaru, National Economic College, Pitești, Romania; Jose Luis Diaz-Barrero, Barcelona, Spain and the problem proposer.

• **5837** *Proposed by Jose Luis Diaz-Barrero, Barcelona, Spain.*

Find all triples (x, y, z) of real numbers that are solutions to the equation

$$\sqrt{11^x(13^y + 17^z)} + \sqrt{13^y(17^z + 11^x)} + \sqrt{17^z(11^x + 13^y)} = \sqrt{2}(11^x + 13^y + 17^z).$$

Solution 1 by Devis Alvarado, UNAH and UPNFM, Tegucigalpa, Honduras.

Introduce the substitutions

$$a = 13^y + 17^z, \qquad b = 17^z + 11^x, \qquad c = 11^x + 13^y.$$

These yield the identities

$$11^x + 13^y + 17^z = \frac{a+b+c}{2},$$

$$11^x = \frac{-a+b+c}{2},$$

$$13^y = \frac{a-b+c}{2},$$

and

$$17^z = \frac{a+b-c}{2}.$$

Substituting these expressions into the given equation and applying the Cauchy–Schwarz inequality, we obtain

$$\begin{aligned}
& \sqrt{11^x\,(13^y + 17^z)} + \sqrt{13^y\,(17^z + 11^x)} + \sqrt{17^z\,(11^x + 13^y)} \\
= \;& \sqrt{a\left(\frac{-a+b+c}{2}\right)} + \sqrt{b\left(\frac{a-b+c}{2}\right)} + \sqrt{c\left(\frac{a+b-c}{2}\right)} \\
\leqslant \;& \sqrt{a+b+c}\,\sqrt{\frac{-a+b+c}{2} + \frac{a-b+c}{2} + \frac{a+b-c}{2}} \\
= \;& \sqrt{2}\left(\frac{a+b+c}{2}\right) \\
= \;& \sqrt{2}\,(11^x + 13^y + 17^z)\,.
\end{aligned}$$

Equality holds if and only if

$$\frac{-a+b+c}{2a} = \frac{a-b+c}{2b} = \frac{a+b-c}{2c}.$$

Since $a, b, c > 0$, these equalities are satisfied if and only if

$$a = b = c.$$

Substituting back into the original definitions of a, b, and c, we conclude that

$$11^x = 13^y = 17^z.$$

Therefore, the complete family of solutions is

$$(x, y, z) = \left(t,\ \frac{\ln 11}{\ln 13}\, t,\ \frac{\ln 11}{\ln 17}\, t\right), \qquad t \in \mathbb{R}.$$

Solution 2 by Mohammed Aassila, Strasbourg, France.

Introduce the positive variables

$$a = 11^x, \qquad b = 13^y, \qquad c = 17^z.$$

Then the equation of of interest becomes

$$\sqrt{a(b+c)} + \sqrt{b(c+a)} + \sqrt{c(a+b)} \ = \ \sqrt{2}(a+b+c). \tag{1}$$

We first establish that

$$\sqrt{a(b+c)} + \sqrt{b(c+a)} + \sqrt{c(a+b)} \ \leqslant \ \sqrt{2}(a+b+c).$$

Indeed, by the Cauchy–Schwarz inequality,

$$\left(\sqrt{a(b+c)} + \sqrt{b(c+a)} + \sqrt{c(a+b)}\right)^2 \ \leqslant \ (a+b+c)\big((b+c)+(c+a)+(a+b)\big).$$

Since

$$(b+c)+(c+a)+(a+b) \ = \ 2(a+b+c),$$

we obtain

$$\left(\sqrt{a(b+c)} + \sqrt{b(c+a)} + \sqrt{c(a+b)}\right)^2 \ \leqslant \ 2(a+b+c)^2,$$

and therefore

$$\sqrt{a(b+c)} + \sqrt{b(c+a)} + \sqrt{c(a+b)} \ \leqslant \ \sqrt{2}(a+b+c).$$

Since equality holds in (1), equality must also hold in the Cauchy–Schwarz inequality.
The equality condition for Cauchy–Schwarz gives

$$\frac{\sqrt{a}}{\sqrt{b+c}} \ = \ \frac{\sqrt{b}}{\sqrt{c+a}} \ = \ \frac{\sqrt{c}}{\sqrt{a+b}},$$

or equivalently,

$$\frac{a}{b+c} \;=\; \frac{b}{c+a} \;=\; \frac{c}{a+b}.$$

Let the common value be $k > 0$. Then

$$a = k(b+c), \qquad b = k(c+a), \qquad c = k(a+b).$$

Subtracting the first two equations yields $a - b = k(b-a) = -k(a-b)$, so that

$$(1+k)(a-b) \;=\; 0.$$

Since $k > 0$, we conclude that $a = b$.
Similarly,

$$(1+k)(b-c) \;=\; 0,$$

hence

$$a \;=\; b \;=\; c.$$

Let $a = b = c = \lambda > 0$. Then $11^x = 13^y = 17^z = \lambda$. Writing $\lambda = e^t$, $t \in \mathbb{R}$, we obtain

$$x = \log_{11}\lambda = t\log_{11} e, \qquad y = \log_{13}\lambda = t\log_{13} e, \qquad z = \log_{17}\lambda = t\log_{17} e.$$

Conversely, every such triple satisfies the original equation, since both sides are equal to $3\lambda\sqrt{2}$. Therefore all solutions are

$$(x,y,z) \;=\; \left(t\log_{11} e,\, t\log_{13} e,\, t\log_{17} e\right), \quad t \in \mathbb{R}.$$

Solution 3 by Ángel Plaza, Universidad de Las Palmas de Gran Canaria, Spain.

We change variables by doing $a = 11^x$, $b = 13^x$, and $c = 17^z$, so the equation becomes

$$\sqrt{a(b+c)} + \sqrt{b(c+a)} + \sqrt{c(a+b)} = \sqrt{2}(a+b+c). \tag{7}$$

By the AM-GM inequality, $\sqrt{a(b+c)} = \dfrac{1}{\sqrt{2}}\sqrt{2a(b+c)} \leqslant \dfrac{2a+b+c}{2\sqrt{2}}$.

Therefore, if LHS denotes the left-hand side of equation (7), then

$$LHS \leqslant \frac{2a+b+c}{2\sqrt{2}} + \frac{a+2b+c}{2\sqrt{2}} + \frac{a+b+2c}{2\sqrt{2}} = \frac{4(a+b+c)}{2\sqrt{2}} = \sqrt{2}(a+b+c).$$

The equality holds if and only if the equality holds for every AM-GM application, that is if $2a = b+c$, $2b = c+a$ and $2c = a+b$. This implies that $a = b = c$, which means that $11^x = 13^y = 17^z$. Or, $(x,y,z) = \left(x, x\dfrac{\ln 11}{\ln 13}, x\dfrac{\ln 11}{\ln 17}\right)$.

Solution 4 by Brian D. Beasley, Simpsonville, SC.

Let $a = 11^x$, $b = 13^y$, and $c = 17^z$, so that a, b, and c are positive real numbers. Let

$$L = \sqrt{a(b+c)} + \sqrt{b(c+a)} + \sqrt{c(a+b)} \quad \text{and} \quad R = \sqrt{2}(a+b+c).$$

Then by the Cauchy-Bunyakovsky-Schwarz inequality, we have

$$L \leqslant \sqrt{(a+b+c)[(b+c)+(c+a)+(a+b)]} = R,$$

with equality if and only if $k(a,b,c) = (b+c, c+a, a+b)$ for some positive constant k, which implies $k = 2$ and hence $a = b = c$. Thus we obtain

$$11^x = 13^y = 17^z,$$

so $x = \log_{11} w$, $y = \log_{13} w$, and $z = \log_{17} w$ for some $w > 0$. This yields the following triples as all real solutions of the original equation:

$$(x,y,z) = \ln w \left(\frac{1}{\ln 11}, \frac{1}{\ln 13}, \frac{1}{\ln 17} \right) \quad \text{for any positive real number } w.$$

Equivalently, we may write the solutions in the form

$$(x,y,z) = v \left(\frac{1}{\ln 11}, \frac{1}{\ln 13}, \frac{1}{\ln 17} \right) \quad \text{for any real number } v.$$

Solution 5 by David A. Huckaby, Angelo State University, San Angelo, TX.

The Cauchy-Schwarz inequality applied to the vectors

$$\mathbf{u} = \left\langle \sqrt{11^x}, \sqrt{13^y}, \sqrt{17^z} \right\rangle$$

and

$$\mathbf{v} = \left\langle \sqrt{13^y + 17^z}, \sqrt{17^z + 11^x}, \sqrt{11^x + 13^y} \right\rangle$$

implies

$$\left(\sqrt{11^x(13^y + 17^z)} + \sqrt{13^y(17^z + 11^x)} + \sqrt{17^z(11^x + 13^y)} \right)^2 \leqslant (11^x + 13^y + 17^z)(13^y + 17^z + 17^z + 11^x + 11^x + 13^y).$$

That is,

$$\left(\sqrt{11^x(13^y + 17^z)} + \sqrt{13^y(17^z + 11^x)} + \sqrt{17^z(11^x + 13^y)} \right)^2 \leqslant 2(11^x + 13^y + 17^z)^2.$$

which, in turn, by taking the square roots of both sides, yields

$$\sqrt{11^x(13^y + 17^z)} + \sqrt{13^y(17^z + 11^x)} + \sqrt{17^z(11^x + 13^y)} \leqslant \sqrt{2}(11^x + 13^y + 17^z).$$

Equality—which is the original problem statement—occurs when the vectors $\mathbf{u}$ and $\mathbf{v}$ are dependent, that is, multiples of each other. So

$$\frac{\sqrt{13^y + 17^z}}{\sqrt{11^x}} = \frac{\sqrt{17^z + 11^x}}{\sqrt{13^y}} = \frac{\sqrt{11^x + 13^y}}{\sqrt{17^z}}. \tag{8}$$

The first and second equations give, respectively,

$$11^x(17^z + 11^x) = 13^y(13^y + 17^z)$$
$$17^z(17^z + 11^x) = 13^y(11^x + 13^y),$$

that is,

$$11^x 17^z + 11^{2x} = 13^{2y} + 13^y 17^z$$
$$17^{2z} + 17^z 11^x = 13^y 11^x + 13^{2y}.$$

Subtracting the second equation from the first gives $11^{2x} - 17^{2z} = 13^y 17^z - 13^y 11^x$, that is, $(11^x + 17^z)(11^x - 17^z) = 13^y(17^z - 11^x)$. Assuming that $11^x - 17^z \neq 0$ gives $13^y = -(11^x + 17^z)$, which is impossible since $13^y > 0$ and $-(11^x + 17^z) < 0$. So $11^x = 17^z$. By symmetry (or by using the equality of the first and third terms in (8)), $11^x = 13^y = 17^z$. Setting $a = 11^x = 13^y = 17^z$, we have $x = \dfrac{\ln a}{\ln 11}$, $y = \dfrac{\ln a}{\ln 13}$, and $z = \dfrac{\ln a}{\ln 17}$. So the set of solutions to the original equation is all $(x, y, z) = \left(\dfrac{\ln a}{\ln 11}, \dfrac{\ln a}{\ln 13}, \dfrac{\ln a}{\ln 17}\right)$ where $a > 0$.

Solution 6 by Michel Bataille, Rouen, France.

We show that the solutions are the triples $\left(\dfrac{a}{\ln(11)}, \dfrac{a}{\ln(13)}, \dfrac{a}{\ln(17)}\right)$ where a is a real number.

Let (x, y, z) be a solution and let $\mathcal{L}$ denote the left-hand side of the equation. From the Cauchy-Schwarz inequality, we have

$$\mathcal{L} \leqslant (11^x + 13^y + 17^z)^{1/2}\left((13^y + 17^z) + (17^z + 11^x) + (11^x + 13^y)\right)^{1/2} = \sqrt{2}(11^x + 13^y + 17^z).$$

Since $\mathcal{L} = \sqrt{2}(11^x + 13^y + 17^z)$, we are in the case of equality for the Cauchy-Schwarz inequality, meaning that

$$\frac{13^y + 17^z}{11^x} = \frac{17^z + 11^x}{13^y} = \frac{11^x + 13^y}{17^z}.$$

Since the common value of these ratios is

$$\frac{(13^y + 17^z) + (17^z + 11^x) + (11^x + 13^y)}{11^x + 13^y + 17^z} = 2,$$

it quickly follows that $11^x = 13^y = 17^z$, that is, $x\ln(11) = y\ln(13) = z\ln(17)$. Thus, $(x, y, z) = \left(\dfrac{a}{\ln(11)}, \dfrac{a}{\ln(13)}, \dfrac{a}{\ln(17)}\right)$ for some real a.

Conversely, if $(x, y, z) = \left(\dfrac{a}{\ln(11)}, \dfrac{a}{\ln(13)}, \dfrac{a}{\ln(17)}\right)$, then $11^x = e^a = 13^y = 17^z$, hence

$$\mathcal{L} = 3\sqrt{2e^{2a}} = 3\sqrt{2}e^a = \sqrt{2}(11^x + 13^y + 17^z)$$

and (x, y, z) is a solution. This completes the proof.

Also solved by Albert Stadler, Herrliberg, Switzerland and the problem proposer.

• **5838** *Proposed by Michel Bataille, Rouen, France.*

Let n be a positive integer. Prove that

$$\sum_{k=1}^{n} \frac{1}{\sin^2\left(\frac{(2k-1)\pi}{4n+2}\right)} = 2n(n+1).$$

Solution 1 by Ángel Plaza, Universidad de Las Palmas de Gran Canaria, Spain.

Using that $1 = \sin^2\alpha + \cos^2\alpha$, we have

$$\begin{aligned}\sum_{k=1}^{n} \frac{1}{\sin^2\left(\frac{(2k+1)\pi}{4n+2}\right)} &= n + \sum_{k=1}^{n} \cot^2\left(\frac{(2k+1)\pi}{4n+2}\right) \\ &= n + \sum_{k=1}^{n} \cot^2\left(\frac{2k+1}{2n+1}\cdot\frac{\pi}{2}\right) \\ &= n + \sum_{k=1}^{n} \tan^2\left(\frac{k}{2n+1}\cdot\pi\right)\end{aligned}$$

because by changing k to $n+1-k$, we get

$$\sum_{k=1}^{n} \cot^2\left(\frac{2k+1}{2n+1}\cdot\frac{\pi}{2}\right) = \sum_{k=1}^{n} \tan^2\left(\frac{2n-2k+2}{2n+1}\cdot\frac{\pi}{2}\right) = \sum_{k=1}^{n} \tan^2\left(\frac{k}{2n+1}\cdot\pi\right).$$

So, the problem is now to prove that $\displaystyle\sum_{k=1}^{n} \tan^2\left(\frac{k}{2n+1}\,\pi\right) = 2n^2+n$.

Let $\theta = \dfrac{k}{2n+1}$, and $z = \cos\theta + i\sin\theta$. Since $\sin((2n+1)\theta) = \operatorname{Im}\left((\cos\theta + i\sin\theta)^{2n+1}\right)$. Let $x = \tan\theta$, so $\cos\theta = \dfrac{1}{\sqrt{1+x^2}}$, $\sin\theta = \dfrac{x}{\sqrt{1+x^2}}$.

$$\sin((2n+1)\theta) = \operatorname{Im}\left(\frac{(1+ix)^{2n+1}}{\sqrt{1+x^2}^{2n+1}}\right).$$

The roots of $\sin((2n+1)\theta) = 0$ are $(2n+1)\theta = k\pi$, so $\theta = \dfrac{k\pi}{2n+1}$. The roots $x_k = \tan\left(\dfrac{k\pi}{2n+1}\right)$ for $k = 1,\ldots,n$ are the imaginary part of $(1+ix)^{2n+1} = 0$.

$$(1+ix)^{2n+1} = \sum_{j=0}^{2n+1} \binom{2n+1}{j} (ix)^j.$$

$$\operatorname{Im}\left((1+ix)^{2n+1}\right) = \sum_{j=0}^{n} \binom{2n+1}{2j+1} (-1)^j x^{2j+1}$$

Set this to zero and divide by $x \neq 0$,

$$0 = \binom{2n+1}{1} - \binom{2n+1}{3}x^2 + \binom{2n+1}{5}x^4 - \ldots + (-1)^n \binom{2n+1}{2j+1}x^{2n}$$

By doing $y = x^2 = \tan^2\left(\dfrac{k}{2n+1}\,\pi\right)$, the equation is a polynomial in y of degree n, with roots $y_k = \tan^2\left(\dfrac{k}{2n+1}\,\pi\right)$, for $k = 1, \ldots, n$. Finally, by Vieta's formulas, the sum of the roots is

$$\sum_{k=1}^{n} y_k = -\frac{a_{n-1}}{a_n} = \frac{\binom{2n+1}{2n-1}}{\binom{2n+1}{2n+1}} = \frac{(2n+1)!}{(2n-1)!2!} = \frac{(2n+1)2n}{2} = n(2n+1).$$

Solution 2 by Jose Luis Diaz-Barrero, Barcelona, Spain.

We will prove this identity by constructing a polynomial whose roots are related to the terms in the summation, and then applying Vieta's formulas. Indeed, we begin considering the trigonometric equation:

$$\cos((2n+1)\theta) = 0$$

which roots occur when the angle is an odd multiple of $\dfrac{\pi}{2}$. That is,

$$(2n+1)\theta = \frac{(2k-1)\pi}{2} \implies \theta_k = \frac{(2k-1)\pi}{2(2n+1)} = \frac{(2k-1)\pi}{4n+2}$$

For $k = 1, 2, \ldots, n$, we obtain exactly the n distinct angles present in the given summation. Note that all these angles lie in the first quadrant, meaning $\sin\theta_k \neq 0$ and $\cos\theta_k \neq 0$.

Using Euler's formula and the binomial expansion, we can express $\cos((2n+1)\theta)$ as:

$$\cos((2n+1)\theta) = \sum_{j=0}^{n} (-1)^j \binom{2n+1}{2j} \cos^{2n+1-2j}\theta \sin^{2j}\theta$$

Setting this expression to 0 and dividing the entire equation by $\cos^{2n+1}\theta$ (which is valid since $\cos\theta_k \neq 0$), we get:

$$\sum_{j=0}^{n} (-1)^j \binom{2n+1}{2j} \tan^{2j}\theta = 0$$

Letting $t = \tan^2\theta$, this transforms into a polynomial equation in t of degree n:

$$\binom{2n+1}{0} - \binom{2n+1}{2}t + \binom{2n+1}{4}t^2 - \cdots + (-1)^n \binom{2n+1}{2n}t^n = 0$$

The roots of this polynomial are precisely $t_k = \tan^2\left(\dfrac{(2k-1)\pi}{4n+2}\right)$ for $k = 1, 2, \ldots, n$.

We want to evaluate a sum involving $\dfrac{1}{\sin^2\theta}$. We use the fundamental trigonometric identity:

$$\frac{1}{\sin^2\theta} = 1 + \cot^2\theta = 1 + \frac{1}{\tan^2\theta}$$

To find the sum of $\cot^2\theta_k$, we look for the polynomial whose roots are $x = 1/t = \cot^2\theta$. We can obtain this by reversing the coefficients of our original polynomial:

$$\binom{2n+1}{0}x^n - \binom{2n+1}{2}x^{n-1} + \cdots + (-1)^n\binom{2n+1}{2n} = 0$$

Evaluating the coefficients, we have

$$x^n - n(2n+1)x^{n-1} + \cdots + (-1)^n(2n+1) = 0$$

By Vieta's formulas, the sum of the roots $x_k = \cot^2\theta_k$ is equal to the negative of the coefficient of x^{n-1} divided by the coefficient of x^n:

$$\sum_{k=1}^{n}\cot^2\left(\frac{(2k-1)\pi}{4n+2}\right) = \frac{-(-n(2n+1))}{1} = n(2n+1)$$

Now, we substitute the identity $\dfrac{1}{\sin^2\theta} = 1 + \cot^2\theta$ back into our target summation:

$$\begin{aligned}\sum_{k=1}^{n}\frac{1}{\sin^2\left(\frac{(2k-1)\pi}{4n+2}\right)} &= \sum_{k=1}^{n}\left(1+\cot^2\left(\frac{(2k-1)\pi}{4n+2}\right)\right)\\ &= \left(\sum_{k=1}^{n}1\right) + \sum_{k=1}^{n}\cot^2\left(\frac{(2k-1)\pi}{4n+2}\right)\\ &= n + n(2n+1) = 2n(n+1),\end{aligned}$$

as we wanted to prove.

Solution 3 by Mohammed Aassila, Strasbourg, France.

Lemma: For every positive integer m and every real number x such that none of the denominators vanish,

$$\sum_{k=0}^{m-1}\csc^2\left(x+\frac{k\pi}{m}\right) \;=\; m^2\csc^2(mx).$$

Proof: We start from the classical identity

$$\sin(mx) \;=\; 2^{m-1}\prod_{k=0}^{m-1}\sin\left(x+\frac{k\pi}{m}\right).$$

Taking logarithmic derivatives gives

$$m\cot(mx) \;=\; \sum_{k=0}^{m-1}\cot\left(x+\frac{k\pi}{m}\right).$$

Differentiating once more, and using $\dfrac{d}{dx}\cot x = -\csc^2 x$, we obtain

$$-m^2\csc^2(mx) \;=\; -\sum_{k=0}^{m-1}\csc^2\left(x+\frac{k\pi}{m}\right).$$

Therefore,

$$\sum_{k=0}^{m-1} \csc^2\left(x + \frac{k\pi}{m}\right) \;=\; m^2 \csc^2(mx),$$

as required. Now, apply the lemma with $m = 2n+1$ and choose $x = \dfrac{\pi}{2(2n+1)}$. Then $mx = \dfrac{\pi}{2}$, so that $\csc^2(mx) = 1$. Hence the lemma gives

$$\sum_{k=0}^{2n} \csc^2\left(\frac{\pi}{2(2n+1)} + \frac{k\pi}{2n+1}\right) \;=\; (2n+1)^2.$$

The arguments of the sine functions are $\dfrac{(2k+1)\pi}{2(2n+1)}$, therefore

$$\sum_{k=0}^{2n} \csc^2\left(\frac{(2k+1)\pi}{4n+2}\right) \;=\; (2n+1)^2.$$

Among these $2n+1$ terms, the middle term corresponding to $k = n$ is

$$\csc^2\left(\frac{(2n+1)\pi}{4n+2}\right) \;=\; \csc^2\left(\frac{\pi}{2}\right) \;=\; 1.$$

The remaining terms occur in equal pairs because $\sin(\pi-\theta) = \sin\theta$. The positive representatives of these pairs are precisely $\dfrac{(2k-1)\pi}{4n+2}$, $k = 1, \ldots, n$. Consequently,

$$(2n+1)^2 \;=\; 1 + 2\sum_{k=1}^{n} \csc^2\left(\frac{(2k-1)\pi}{4n+2}\right).$$

Solving for the desired sum gives

$$2\sum_{k=1}^{n} \csc^2\left(\frac{(2k-1)\pi}{4n+2}\right) \;=\; (2n+1)^2 - 1.$$

Since $(2n+1)^2 - 1 = 4n^2 + 4n = 4n(n+1)$, we finally obtain

$$\sum_{k=1}^{n} \frac{1}{\sin^2\left(\frac{(2k-1)\pi}{4n+2}\right)} \;=\; 2n(n+1).$$

Also solved by Albert Stadler, Herrliberg, Switzerland and the problem proposer.

• **5839** *Proposed by Prakash Pant, The University of Vermont, Bardiya, Nepal.*

Find all 2×2 matrices M with real entries that satisfy the equation:

$$M^2 - 5M = \begin{pmatrix} 0 & 9 \\ 4 & 0 \end{pmatrix}.$$

Solution 1 by Albert Stadler, Herrliberg, Switzerland.

First solution (Linear Algebra Approach: Let

$$A = \begin{pmatrix} 0 & 9 \\ 4 & 0 \end{pmatrix}.$$

We wish to determine all real 2×2 matrices M satisfying

$$M^2 - 5M = A.$$

The characteristic polynomial of A is

$$\det(\lambda I - A) = \begin{vmatrix} \lambda & -9 \\ -4 & \lambda \end{vmatrix} = \lambda^2 - 36.$$

Therefore, the eigenvalues of A are

$$\lambda = 6, \;\; \lambda = -6.$$

Since the eigenvalues are distinct, A is diagonalizable over $\mathbb{R}$. Hence there exists an invertible matrix

$$P = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

such that

$$P^{-1}AP = \begin{pmatrix} 6 & 0 \\ 0 & -6 \end{pmatrix}.$$

Equivalently,

$$P \begin{pmatrix} 6 & 0 \\ 0 & -6 \end{pmatrix} = AP.$$

Computing both sides gives

$$P \begin{pmatrix} 6 & 0 \\ 0 & -6 \end{pmatrix} = \begin{pmatrix} 6a & -6b \\ 6c & -6d \end{pmatrix},$$

and

$$AP = \begin{pmatrix} 0 & 9 \\ 4 & 0 \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 9c & 9d \\ 4a & 4b \end{pmatrix}.$$

Comparing corresponding entries yields

$$6a = 9c, -6b = 9d, 6c = 4a, -6d = 4b.$$

These equations imply

$$c = \frac{2}{3}a, d = -\frac{2}{3}b.$$

Choosing $a = 3$ and $b = -3$, we obtain

$$c = 2, d = 2.$$

Thus we may take

$$P = \begin{pmatrix} 3 & -3 \\ 2 & 2 \end{pmatrix},$$

for which

$$P^{-1}AP = \begin{pmatrix} 6 & 0 \\ 0 & -6 \end{pmatrix}.$$

Define

$$N = P^{-1}MP.$$

Since $M^2 - 5M = A$, multiplying on the left by P^{-1} and on the right by P gives

$$N^2 - 5N = \begin{pmatrix} 6 & 0 \\ 0 & -6 \end{pmatrix}.$$

Let

$$D = \begin{pmatrix} 6 & 0 \\ 0 & -6 \end{pmatrix}.$$

Then

$$N^2 - 5N = D.$$

Since D is a polynomial in N, namely

$$D = N^2 - 5N,$$

it follows that N commutes with D:

$$ND = DN.$$

We now use the fact that any matrix commuting with a diagonal matrix having distinct diagonal entries must itself be diagonal. Let

$$N = (n_{ij}), D = (d_i\delta_{ij}),$$

and suppose $i \neq k$. Then

$$d_k n_{ik} = \sum_j n_{ij} d_j \delta_{jk} = (ND)_{ik} = (DN)_{ik} = \sum_j d_i \delta_{ij} n_{jk} = d_i n_{ik}.$$

Hence

$$(d_k - d_i)n_{ik} = 0.$$

Since $d_1 = 6$ and $d_2 = -6$ are distinct, we obtain

$$n_{ik} = 0 \quad (i \neq k).$$

Therefore N is diagonal:

$$N = \begin{pmatrix} x & 0 \\ 0 & y \end{pmatrix}.$$

Substituting this form of N into

$$N^2 - 5N = D$$

gives

$$x^2 - 5x = 6, \;\; y^2 - 5y = -6.$$

Thus

$$x^2 - 5x - 6 = 0,$$

and

$$y^2 - 5y + 6 = 0.$$

Factoring,

$$(x-6)(x+1) = 0, \;\; (y-2)(y-3) = 0.$$

Hence

$$x \in \{6, -1\}, y \in \{2, 3\}.$$

Therefore there are four possibilities:

$$N = \begin{pmatrix} 6 & 0 \\ 0 & 2 \end{pmatrix}, \begin{pmatrix} 6 & 0 \\ 0 & 3 \end{pmatrix}, \begin{pmatrix} -1 & 0 \\ 0 & 2 \end{pmatrix}, \begin{pmatrix} -1 & 0 \\ 0 & 3 \end{pmatrix}.$$

Since

$$M = PNP^{-1},$$

we compute M for each of the four choices of N. This yields the four matrices

$$\begin{pmatrix} \frac{9}{2} & \frac{9}{4} \\ 1 & \frac{9}{2} \end{pmatrix}, \begin{pmatrix} 4 & 3 \\ \frac{4}{3} & 4 \end{pmatrix}, \begin{pmatrix} \frac{1}{2} & -\frac{9}{4} \\ -1 & \frac{1}{2} \end{pmatrix}, \begin{pmatrix} 1 & -3 \\ -\frac{4}{3} & 1 \end{pmatrix}.$$

Therefore, the complete set of real 2×2 matrices satisfying

$$M^2 - 5M = \begin{pmatrix} 0 & 9 \\ 4 & 0 \end{pmatrix}$$

is

$$\begin{pmatrix} \frac{9}{2} & \frac{9}{4} \\ 1 & \frac{9}{2} \end{pmatrix}, \begin{pmatrix} 4 & 3 \\ \frac{4}{3} & 4 \end{pmatrix}, \begin{pmatrix} \frac{1}{2} & -\frac{9}{4} \\ -1 & \frac{1}{2} \end{pmatrix}, \begin{pmatrix} 1 & -3 \\ -\frac{4}{3} & 1 \end{pmatrix}.$$

These are exactly all real solutions.

Second solution - Direct Approach: Let

$$M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

Then

$$M^2 = \begin{pmatrix} a^2 + bc & b(a+d) \\ c(a+d) & d^2 + bc \end{pmatrix}.$$

This yields the system of equations

$$a^2 + bc - 5a = 0, \;\; b\,(a + d - 5) = 9, \;\; c\,(a + d - 5) = 4, \;\; d^2 + bc - 5d = 0.$$

From the second or third equation, we see that

$$a + d - 5 \neq 0.$$

Dividing the third equation by the second gives

$$c = \frac{4b}{9}.$$

Subtracting the fourth equation from the first, we obtain

$$(a - d)(a + d - 5) = 0.$$

Since $a + d - 5 \neq 0$, it follows that

$$a = d.$$

The system therefore reduces to

$$a^2 + \frac{4}{9}b^2 - 5a = 0, \;\; b(2a - 5) = 9.$$

Substituting

$$b = \frac{9}{2a - 5}$$

into the first equation yields

$$a^2 + \frac{36}{(2a - 5)^2} - 5a = 0.$$

Equivalently,

$$\frac{4a^4 - 40a^3 + 125a^2 - 125a + 36}{(2a - 5)^2} = 0,$$

which factors as

$$\frac{\big(a - 4)(a - 1)(2a - 9)(2a - 1\big)}{(2a - 5)^2} = 0.$$

Hence,

$$a \in 1 \;\Big|\; \frac{1}{2} \;\Big|\; \frac{9}{2} \;\Big|\; 4 \Big\}.$$

Substituting these values back into the system gives the four solutions:

$$(a, b, c, d) \in \left\{ \left(1, -3, -\frac{4}{3}, 1\right), \left(\frac{1}{2}, -\frac{9}{4}, -1, \frac{1}{2}\right), \left(\frac{9}{2}, \frac{9}{4}, 1, \frac{9}{2}\right), \left(4, 3, \frac{4}{3}, 4\right) \right\}.$$

Solution 2 by Ángel Plaza, Universidad de Las Palmas de Gran Canaria, Spain.

Since matrices $5I$ and $\begin{pmatrix} 0 & 9 \\ 4 & 0 \end{pmatrix}$ commute, the given equation may be written as

$$\left(M - \frac{5}{2}I\right)^2 = \frac{25}{4}I^2 + \begin{pmatrix} 0 & 9 \\ 4 & 0 \end{pmatrix}.$$

Let $Y = M - \frac{5}{2}I$ and $N = \frac{25}{4}I^2 + \begin{pmatrix} 0 & 9 \\ 4 & 0 \end{pmatrix} = \begin{pmatrix} \frac{25}{4} & 9 \\ 4 & \frac{25}{4} \end{pmatrix}$.

The decomposition $P^{-1}DP = N$ for the matrix N is $D = \begin{pmatrix} \frac{49}{4} & 0 \\ 0 & \frac{1}{4} \end{pmatrix}$, $P = \begin{pmatrix} 3 & -3 \\ 2 & 2 \end{pmatrix}$.

$$\det(N - \lambda I) = \left(\frac{25}{4} - \lambda\right)^2 - 36 = 0$$

from where $\lambda = \frac{25}{4} \mp 6$, that is $\lambda = \frac{49}{4}$, and $\lambda = \frac{1}{4}$, so $D = \begin{pmatrix} \frac{49}{4} & 0 \\ 0 & \frac{1}{4} \end{pmatrix}$.

For $\lambda_1 = \frac{49}{4}$, the associated eigenvector v_1 verifies $\left(N - \frac{49}{4}I\right)v_1 = 0$, that is

$$\begin{pmatrix} -6 & 9 \\ 4 & -6 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = 0 \;\Rightarrow\; -6x + 9y = 0 \;\Rightarrow\; 2x = 3y \;\Rightarrow\; v_1 = \begin{pmatrix} 3 \\ 2 \end{pmatrix}$$

For $\lambda_2 = \frac{1}{4}$, the associated eigenvector $\mathbf{v_2}$ verifies $\left(N - \frac{1}{4}I\right)v_2 = 0$, that is

$$\begin{pmatrix} 6 & 9 \\ 4 & 6 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = 0 \;\Rightarrow\; 6x + 9y = 0 \;\Rightarrow\; 2x = -3y \;\Rightarrow\; v_2 = \begin{pmatrix} -3 \\ 2 \end{pmatrix}.$$

Therefore matrix $P = \begin{pmatrix} 3 & -3 \\ 2 & 2 \end{pmatrix}$ and $P^{-1} = \begin{pmatrix} 1/6 & 1/4 \\ -1/6 & 1/4 \end{pmatrix}$. The equation to be solved is

$$\left(M - \frac{5}{2}I\right)^2 = \begin{pmatrix} 1/6 & 1/4 \\ -1/6 & 1/4 \end{pmatrix}\begin{pmatrix} \frac{49}{4} & 0 \\ 0 & \frac{1}{4} \end{pmatrix}\begin{pmatrix} 3 & -3 \\ 2 & 2 \end{pmatrix}$$

$$M - \frac{5}{2}I = \begin{pmatrix} 1/6 & 1/4 \\ -1/6 & 1/4 \end{pmatrix}\begin{pmatrix} \pm\frac{7}{2} & 0 \\ 0 & \pm\frac{1}{2} \end{pmatrix}\begin{pmatrix} 3 & -3 \\ 2 & 2 \end{pmatrix}.$$

From where it is obtained:

$$M \in \left\{ \begin{pmatrix} \frac{1}{2} & -\frac{9}{4} \\ -1 & \frac{1}{2} \end{pmatrix}, \begin{pmatrix} 1 & -3 \\ -\frac{4}{3} & 1 \end{pmatrix}, \begin{pmatrix} 4 & 3 \\ \frac{4}{3} & 4 \end{pmatrix}, \begin{pmatrix} \frac{9}{2} & \frac{9}{4} \\ 1 & \frac{9}{2} \end{pmatrix}, \right\}.$$

Solution 3 by Brian D. Beasley, Simpsonville, SC.

Let $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ with a, b, c, d in $\mathbb{R}$. Then

$$M^2 - 5M = \begin{pmatrix} a^2 + bc - 5a & ab + bd - 5b \\ ac + cd - 5c & bc + d^2 - 5d \end{pmatrix} = \begin{pmatrix} 0 & 9 \\ 4 & 0 \end{pmatrix},$$

so $a^2 - d^2 - 5a + 5d = (a - d)(a + d - 5) = 0$, $b(a + d - 5) = 9$, and $c(a + d - 5) = 4$. Since $a + d - 5 \neq 0$, we have $a = d$ and $4b = 9c$. This yields

$$a^2 + \frac{4}{9}b^2 - 5a = 0 \text{ and } b(2a - 5) = 9,$$

where $2a - 5 \neq 0$ and hence $b = 9/(2a - 5)$. Thus

$$a^2 + \frac{4}{9}\left(\frac{9}{2a - 5}\right)^2 - 5a = 0,$$

or equivalently

$$4a^4 - 40a^3 + 125a^2 - 125a + 36 = (a - 1)(a - 4)(2a - 1)(2a - 9) = 0.$$

Then $a \in \{1, 4, 1/2, 9/2\}$, so the four solutions for M are

$$\begin{pmatrix} 1 & -3 \\ -\frac{4}{3} & 1 \end{pmatrix}, \begin{pmatrix} 4 & 3 \\ \frac{4}{3} & 4 \end{pmatrix}, \begin{pmatrix} \frac{1}{2} & -\frac{9}{4} \\ -1 & \frac{1}{2} \end{pmatrix}, \begin{pmatrix} \frac{9}{2} & \frac{9}{4} \\ 1 & \frac{9}{2} \end{pmatrix}.$$

Solution 4 by Devis Alvarado, UNAH and UPNFM, Tegucigalpa, Honduras.

We first rewrite the given equation as

$$\begin{aligned} & M^2 - 5M = \begin{pmatrix} 0 & 9 \\ 4 & 0 \end{pmatrix} \\ \iff \quad & \left(M - \frac{5}{2}I\right)^2 = \begin{pmatrix} 0 & 9 \\ 4 & 0 \end{pmatrix} + \frac{25}{4}I \\ \iff \quad & \left(M - \frac{5}{2}I\right)^2 = \begin{pmatrix} \frac{25}{4} & 9 \\ 4 & \frac{25}{4} \end{pmatrix}. \end{aligned}$$

Next, let

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

be a matrix satisfying

$$A^2 = \begin{pmatrix} \frac{25}{4} & 9 \\ 4 & \frac{25}{4} \end{pmatrix}.$$

Then the entries of A satisfy the system

$$\begin{cases} a^2 + bc &= \frac{25}{4}, \\ ab + bd &= 9, \\ ac + cd &= 4, \\ bc + d^2 &= \frac{25}{4}. \end{cases}$$

From the first and fourth equations we obtain $a^2 = d^2$, so that $d = \pm a$. Substituting into either the second or the third equation shows that the case $d = -a$ is impossible. Hence $d = a$. The second and third equations then become

$$ab = \frac{9}{2}, \qquad ac = 2,$$

which imply

$$b = \frac{9}{2a}, \qquad c = \frac{2}{a}.$$

Substituting these expressions into the first equation yields

$$a^2 + \frac{9}{a^2} = \frac{25}{4}.$$

Equivalently, $a + \frac{3}{a} = \pm\frac{7}{2}$, from which we obtain

$$a = \pm\frac{3}{2}, \qquad a = \pm 2.$$

Finally, since

$$M = \frac{5}{2}I + A,$$

the four solutions are

$$\begin{pmatrix} \frac{1}{2} & -\frac{9}{4} \\ -1 & \frac{1}{2} \end{pmatrix}, \qquad \begin{pmatrix} 1 & -3 \\ -\frac{4}{3} & 1 \end{pmatrix}, \qquad \begin{pmatrix} 4 & 3 \\ \frac{4}{3} & 4 \end{pmatrix}, \qquad \begin{pmatrix} \frac{9}{2} & \frac{9}{4} \\ 1 & \frac{9}{2} \end{pmatrix}.$$

Solution 5 by Eagle Problem Solvers, Georgia Southern University, Savannah, GA and Statesboro, GA.

$$M \in \left\{ \begin{pmatrix} 4 & 3 \\ \frac{4}{3} & 4 \end{pmatrix}, \begin{pmatrix} 1 & -3 \\ -\frac{4}{3} & 1 \end{pmatrix}, \begin{pmatrix} \frac{9}{2} & \frac{9}{4} \\ 1 & \frac{9}{2} \end{pmatrix}, \begin{pmatrix} \frac{1}{2} & -\frac{9}{4} \\ -1 & \frac{1}{2} \end{pmatrix} \right\}.$$

Let $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. Then

$$M^2 - 5M = \begin{pmatrix} a^2 + bc - 5a & b(a+d-5) \\ c(a+d-5) & bc + d^2 - 5d \end{pmatrix} = \begin{pmatrix} 0 & 9 \\ 4 & 0 \end{pmatrix}.$$

Since $a^2 - 5a + bc = 0 = d^2 - 5d + bc$, then $0 = a^2 - 5a - d^2 + 5d = (a-d)(a+d-5)$. Since $b(a+d-5) = 9$, then $a + d - 5 \neq 0$, so we must have $a = d$. Thus, $2a - 5 = \dfrac{9}{b} = \dfrac{4}{c}$, so $a = \dfrac{5c+4}{2c}$ and $b = \dfrac{9c}{4}$. Since $a^2 - 5a + bc = 0$, we have

$$\frac{(5c+4)^2}{4c^2} - \frac{5(5c+4)}{2c} + \frac{9c^2}{4} = 0.$$

Multiplying by $4c^2$ gives

$$0 = 9c^4 + (25c^2 + 40c + 16) - (50c^2 + 40c) = 9c^4 - 25c^2 + 16 = (9c^2 - 16)(c^2 - 1),$$

so that $c = \pm\dfrac{4}{3}$ or $c = \pm 1$. Since $b = \dfrac{9c}{4}$, then $b = \pm 3$ or $b = \pm\dfrac{9}{4}$. Since $a = \dfrac{5c+4}{2c}$, then $a \in \left\{4, 1, \dfrac{9}{2}, \dfrac{1}{2}\right\}$, and since $d = a$ we have

$$M \in \left\{ \begin{pmatrix} 4 & 3 \\ \frac{4}{3} & 4 \end{pmatrix}, \begin{pmatrix} 1 & -3 \\ -\frac{4}{3} & 1 \end{pmatrix}, \begin{pmatrix} \frac{9}{2} & \frac{9}{4} \\ 1 & \frac{9}{2} \end{pmatrix}, \begin{pmatrix} \frac{1}{2} & -\frac{9}{4} \\ -1 & \frac{1}{2} \end{pmatrix} \right\}.$$

Solution 6 by Michel Bataille, Rouen, France.

We show that the equation has four solutions, namely the matrices

$$\begin{pmatrix} 4 & 3 \\ 4/3 & 4 \end{pmatrix}, \quad \begin{pmatrix} 1/2 & -9/4 \\ -1 & 1/2 \end{pmatrix}, \quad \begin{pmatrix} 9/2 & 9/4 \\ 1 & 9/2 \end{pmatrix}, \quad \begin{pmatrix} 1 & -3 \\ -4/3 & 1 \end{pmatrix}.$$

Simple calculations show that these matrices are solutions.

Conversely, let M be any solution and let λ be an eigenvalue of M. Then, $\lambda^2 - 5\lambda$ is an eigenvalue of $A := \begin{pmatrix} 0 & 9 \\ 4 & 0 \end{pmatrix}$, hence $\lambda^2 - 5\lambda \in \{6, -6\}$ (since $\det(xI_2 - A) = x^2 - 36$). Thus, $\lambda^2 - 5\lambda = -6$ or $\lambda^2 - 5\lambda = 6$ and the possible eigenvalues of M are $2, 3, -1, 6$. It follows that M is similar to a matrix $\begin{pmatrix} \lambda_1 & \alpha \\ 0 & \lambda_2 \end{pmatrix}$ where $\lambda_1, \lambda_2 \in \{2, 3, -1, 6\}$ (possibly $\lambda_1 = \lambda_2$) and $\alpha \in \mathbb{R}$. Therefore $M^2 - 5M$ is similar to

$$\begin{pmatrix} \lambda_1 & \alpha \\ 0 & \lambda_2 \end{pmatrix}^2 - 5\begin{pmatrix} \lambda_1 & \alpha \\ 0 & \lambda_2 \end{pmatrix} = \begin{pmatrix} \lambda_1^2 - 5\lambda_1 & \alpha(\lambda_1 + \lambda_2 - 5) \\ 0 & \lambda_2^2 - 5\lambda_2 \end{pmatrix}.$$

Since similar matrices have the same trace, we must have $\lambda_1^2 + \lambda_2^2 - 5(\lambda_1 + \lambda_2) = \mathrm{tr}(A) = 0$. As a result,we cannot have $\lambda_1 = \lambda_2$; also $\{\lambda_1, \lambda_2\} = \{2, 3\}$ and $\{\lambda_1, \lambda_2\} = \{-1, 6\}$ are excluded. Thus, we are left with four possibilities for $\{\lambda_1, \lambda_2\}$, namely $\{2, 6\}, \{2, -1\}, \{3, 6\}, \{3, -1\}$, that we examine in turn.
If $\{\lambda_1, \lambda_2\} = \{2, 6\}$, then $M^2 - 8M + 12I_2 = O_2$ (from the Hamilton-Cayley theorem), hence $3M - 12I_2 = A$ and therefore $M = \begin{pmatrix} 4 & 3 \\ 4/3 & 4 \end{pmatrix}$.

If $\{\lambda_1, \lambda_2\} = \{2, -1\}$, then $M^2 - M - 2I_2 = O_2$, hence $-4M + 2I_2 = A$ and $M = \begin{pmatrix} 1/2 & -9/4 \\ -1 & 1/2 \end{pmatrix}$.

If $\{\lambda_1, \lambda_2\} = \{3, 6\}$, then $M^2 - 9M + 18I_2 = O_2$, hence $4M - 18I_2 = A$ and $M = \begin{pmatrix} 9/2 & 9/4 \\ 1 & 9/2 \end{pmatrix}$.

If $\{\lambda_1, \lambda_2\} = \{3, -1\}$, then $M^2 - 2M - 3I_2 = O_2$, hence $-3M + 3I_2 = A$ and $M = \begin{pmatrix} 1 & -3 \\ -4/3 & 1 \end{pmatrix}$.

The proof is complete.

Solution 7 by Mohammed Aassila, Strasbourg, France.

From the equation

$$M^2 - 5M = \begin{pmatrix} 0 & 9 \\ 4 & 0 \end{pmatrix} =: A.$$

Multiplying on the right by M gives $AM = (M^2 - 5M)M = M^3 - 5M^2$. Multiplying on the left by M gives $MA = M(M^2 - 5M) = M^3 - 5M^2$. Hence $AM = MA$, so M and A commute.
The characteristic polynomial of A is

$$\det(\lambda I - A) = \det \begin{pmatrix} \lambda & -9 \\ -4 & \lambda \end{pmatrix} = \lambda^2 - 36.$$

Thus the eigenvalues are 6 and -6, which are distinct. Therefore A is diagonalizable over $\mathbb{R}$.

A standard result says that any matrix commuting with a diagonalizable matrix having distinct eigenvalues must be a polynomial in that matrix. Hence there exist real numbers α, β such that $M = \alpha I + \beta A$. We have

$$A^2 = \begin{pmatrix} 0 & 9 \\ 4 & 0 \end{pmatrix}^2 = \begin{pmatrix} 36 & 0 \\ 0 & 36 \end{pmatrix} = 36I.$$

Now compute:

$$M^2 = (\alpha I + \beta A)^2 = \alpha^2 I + 2\alpha\beta A + \beta^2 A^2 = (\alpha^2 + 36\beta^2)I + 2\alpha\beta A.$$

Also, $5M = 5\alpha I + 5\beta A$. Therefore, $M^2 - 5M = (\alpha^2 + 36\beta^2 - 5\alpha)I + (2\alpha\beta - 5\beta)A$. This must equal A. Comparing coefficients of I and A, we get

$$\alpha^2 + 36\beta^2 - 5\alpha = 0, \tag{1}$$

$$2\alpha\beta - 5\beta \;=\; 1. \tag{2}$$

From (2), $\beta(2\alpha-5) \;=\; 1$, so $\beta = \dfrac{1}{2\alpha-5}$, $\alpha \neq \dfrac{5}{2}$. Substitute into (1):

$$\alpha^2 - 5\alpha + \frac{36}{(2\alpha-5)^2} \;=\; 0.$$

Multiply through by $(2\alpha-5)^2$:

$$(\alpha^2 - 5\alpha)(2\alpha-5)^2 + 36 \;=\; 0.$$

Let $t = 2\alpha - 5$. Then $\alpha = \dfrac{t+5}{2}$, and

$$\alpha^2 - 5\alpha = \frac{(t+5)^2}{4} - \frac{5(t+5)}{2} \;=\; \frac{t^2+10t+25-10t-50}{4} \;=\; \frac{t^2-25}{4}.$$

Thus the equation becomes

$$\frac{25-t^2}{4}\cdot t^2 + 36 = 0 \qquad \text{so} \qquad t^2(25-t^2)+144=0.$$

Equivalently,

$$25t^2 - t^4 + 144 = 0 \qquad \text{or} \qquad t^4 - 25t^2 + 144 = 0.$$

Let $u = t^2$. Then $u^2 - 25u + 144 = 0$. Hence

$$u \;=\; \frac{25 \pm \sqrt{625-576}}{2} \;=\; \frac{25 \pm 7}{2}.$$

So $u = 16$ or $u = 9$. Therefore $t = \pm 4$ or $t = \pm 3$. Recall

$$M = \alpha I + \beta A, \qquad \alpha = \frac{t+5}{2}, \qquad \beta = \frac{1}{t}.$$

We compute each case:

Case 1: $t = 4$. Then

$$\alpha = \frac{4+5}{2} = \frac{9}{2}, \qquad \beta = \frac{1}{4}.$$

So

$$M = \frac{9}{2}I + \frac{1}{4}A = \begin{pmatrix} 9/2 & 9/4 \\ 1 & 9/2 \end{pmatrix}.$$

Case 2: $t = -4$. Then

$$\alpha = \frac{-4+5}{2} = \frac{1}{2}, \qquad \beta = -\frac{1}{4}.$$

So

$$M = \frac{1}{2}I - \frac{1}{4}A = \begin{pmatrix} 1/2 & -9/4 \\ -1 & 1/2 \end{pmatrix}.$$

Case 3: $t = 3$. Then

$$\alpha = \frac{3+5}{2} = 4, \qquad \beta = \frac{1}{3}.$$

So

$$M = 4I + \frac{1}{3}A = \begin{pmatrix} 4 & 3 \\ 4/3 & 4 \end{pmatrix}.$$

Case 4: $t = -3$. Then

$$\alpha = \frac{-3+5}{2} = 1, \qquad \beta = -\frac{1}{3}.$$

So

$$M = I - \frac{1}{3}A = \begin{pmatrix} 1 & -3 \\ -4/3 & 1 \end{pmatrix}.$$

In conclusion, all real 2×2 matrices satisfying the equation are

$$\begin{pmatrix} 9/2 & 9/4 \\ 1 & 9/2 \end{pmatrix}, \begin{pmatrix} 1/2 & -9/4 \\ -1 & 1/2 \end{pmatrix}, \begin{pmatrix} 4 & 3 \\ 4/3 & 4 \end{pmatrix}, \begin{pmatrix} 1 & -3 \\ -4/3 & 1 \end{pmatrix}.$$

Solution 8 by Sri Hari Bhupala Haribhakta, Aumovio, Bengaluru, India.

1. Setting up the system of equations: Let $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. Then

$$M^2 - 5M = \begin{pmatrix} a^2 + bc - 5a & b(a+d-5) \\ c(a+d-5) & bc + d^2 - 5d \end{pmatrix} = \begin{pmatrix} 0 & 9 \\ 4 & 0 \end{pmatrix}.$$

Equating entries gives the system:

$$\text{(i) } a^2 + bc - 5a = 0, \qquad \text{(ii) } b(a+d-5) = 9,$$

$$\text{(iii) } c(a+d-5) = 4, \qquad \text{(iv) } bc + d^2 - 5d = 0.$$

2. Deducing $a = d$: Subtracting (i) from (iv):

$$d^2 - a^2 - 5(d-a) = 0 \quad \Longrightarrow \quad (d-a)(d+a-5) = 0.$$

For $a + d = 5$, (ii) and (iii) require $b \cdot 0 = 9$, a contradiction. Hence $a = d$ is forced.

3. Reducing to a single variable: With $a = d$, dividing (ii) by (iii) gives $b/c = 9/4$, so $b = 9c/4$. Equation (iii) yields $c = \dfrac{4}{2a-5}$, and consequently $b = \dfrac{9}{2a-5}$. Substituting into (i):

$$a^2 + \frac{36}{(2a-5)^2} - 5a = 0.$$

Setting $s = 2a - 5$ (so that $a = (s+5)/2$) and clearing the denominator:

$$\frac{s^2 - 25}{4}\, s^2 + 36 = 0 \quad \Longrightarrow \quad s^4 - 25s^2 + 144 = 0.$$

This factors as

$$(s^2 - 16)(s^2 - 9) = 0,$$

yielding $s \in \{-4,\ -3,\ 3,\ 4\}$ and hence $a \in \left\{\frac{1}{2},\ 1,\ 4,\ \frac{9}{2}\right\}$. 4. The four solutions: Computing b and c for each value of a:

$$M_1 = \begin{pmatrix} \frac{9}{2} & \frac{9}{4} \\ 1 & \frac{9}{2} \end{pmatrix}, \qquad M_2 = \begin{pmatrix} 4 & 3 \\ \frac{4}{3} & 4 \end{pmatrix},$$

$$M_3 = \begin{pmatrix} 1 & -3 \\ -\frac{4}{3} & 1 \end{pmatrix}, \qquad M_4 = \begin{pmatrix} \frac{1}{2} & -\frac{9}{4} \\ -1 & \frac{1}{2} \end{pmatrix}.$$

Solution 9 by Ulrich Abel, Technische Hochschule Mittelhessen, Friedberg, Germany..

Define $A = 2M - 5I$, where I denotes the 2×2 identity matrix. Then

$$A^2 = 4\left(M^2 - 5M\right) + 25I = \begin{pmatrix} 25 & 36 \\ 16 & 25 \end{pmatrix}.$$

We determine all matrices $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ satisfying

$$A^2 = \begin{pmatrix} a^2 + bc & b\,(a+d) \\ c\,(a+d) & d^2 + bc \end{pmatrix} = \begin{pmatrix} 25 & 36 \\ 16 & 25 \end{pmatrix}.$$

$a^2 + bc = d^2 + bc$ implies that $a^2 = d^2$. We conclude that $a = d$, since $b\,(a+d) = 36$ implies that $a \neq -d$. Hence, $2ab = 36$ and $2ac = 16$. It follows that $25 = a^2 + bc = a^2 + \frac{18}{a}\frac{8}{a}$, which can be rewritten in the form $0 = a^4 - 25a^2 + 144 = \left(a^2 - 16\right)\left(a^2 - 9\right)$. Therefore, $a \in \{+4, -4, +3, -3\}$ and $A = \pm \begin{pmatrix} 4 & 9/2 \\ 2 & 4 \end{pmatrix}$ or $A = \pm \begin{pmatrix} 3 & 6 \\ 8/3 & 3 \end{pmatrix}$. Finally, $M = \frac{1}{2}\,(A + 5I)$ implies that the equation is solved if and only if M is equal to one of the four matrices

$$\begin{pmatrix} 9/2 & 9/4 \\ 1 & 9/2 \end{pmatrix}, \quad \begin{pmatrix} 1/2 & -9/4 \\ -1 & 1/2 \end{pmatrix}, \quad \begin{pmatrix} 4 & 3 \\ 4/3 & 4 \end{pmatrix}, \quad \begin{pmatrix} 1 & -3 \\ -4/3 & 1 \end{pmatrix}.$$

Also solved by Bruno Salgueiro Fanego, Viveiro, Lugo, Spain; David Huckaby, Angelo State University, San Angelo, TX; Jose Luis Diaz-Barrero, Barcelona, Spain; Daniel Văcaru, National Economic College "Maria Teiuleanu", Pitești, Romania and the problem proposer.

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Editor's Statement: It goes without saying that the problem proposers, as well as the solution proposers, are the *élan vital* of the Problems/Solutions Section of SSMJ. As the editor of this Section of the Journal, I consider myself fortunate to be in a position to receive, compile and organize a wealth of proposed ingenious problems and solutions intended for online publication. My unwavering gratitude goes to all the amazingly creative contributors. We come together from across continents because we find intellectual value, joy and satisfaction in mathematical problems, both in their creation as well as their solution. So that our collective efforts serve us well, I kindly ask all contributors to adhere to the following guidelines. As you peruse below, you may construe that the guidelines amount to a lot of work. But, as the samples show, there's not much to do. Your cooperation is much appreciated!

Keep in mind that the examples given below are your best guide!

Formats, Styles and Requirements

When submitting proposed problem(s) or solution(s), please send both **LaTeX** document and **pdf** document of your proposed problem(s) or solution(s). There are ways (discoverable from the internet) to convert from Word to proper LaTeX code. Porposals without a *proper* **LaTeX** document will not be published regrettably.

Regarding Proposed Solutions:

Below is the FILENAME format for all the documents of your proposed solution(s).

#ProblemNumber_FirstName_LastName_Solution_SSMJ

- FirstName stands for YOUR first name.
- LastName stands for YOUR last name.

Examples:

#1234_Max_Planck_Solution_SSMJ

#9876_Charles_Darwin_Solution_SSMJ

Please note that every problem number is *preceded* by the sign # .

All you have to do is copy the FILENAME format (or an example below it), paste it and then modify portions of it to your specs.

Please adopt the following structure, in the order shown, for the presentation of your solution:

1. On top of the first page of your solution, begin with the phrase:

"Proposed Solution to #**** SSMJ"

where the string of four astrisks represents the problem number.

2. On the second line, write

"Solution proposed by [your First Name, your Last Name]",

followed by your affiliation, city, country, all on the same linear string of words. Please see the example below. If you have collaborator(s), then, immediately following your address, after inserting the connective "and", provide full name, affiliation and address of your collaborator(s).

3. On a new line, state the problem proposer's name, affiliation, city and country, just as it appears published in the Problems/Solutions section.

4. On a new line below the above, write in bold type: "**Statement of the Problem**".

5. Below the latter, state the problem. Please make sure the statement of your problem (unlike the preceding item) is not in bold type.

6. Below the statement of the problem, write in bold type: "**Solution of the Problem**".

7. Below the latter, show the entire solution of the problem.

Here is a sample for the above-stated format for proposed solutions:

Proposed solution to #1234 SSMJ

Solution proposed by Emmy Noether, University of Göttingen, Lower Saxony, Germany.

Problem proposed by Isaac Newton, Trinity College, Cambridge, England.

Statement of the problem:

Compute $\sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}$.

Solution of the problem: . . . . . .

Regarding Proposed Problems:

For all your proposed problems, please adopt for all documents the following FILENAME format:

FirstName_LastName_ProposedProblem_SSMJ_YourGivenNumber_ProblemTitle

If you do not have a ProblemTitle, then leave that component as it already is (i.e., ProblemTitle).

The component YourGivenNumber is any UNIQUE 3-digit (or longer) number you like to give to your problem.

Examples:

Max_Planck_ProposedProblem_SSMJ_314_HarmonicPatterns

Charles_Darwin_ProposedProblem_SSMJ_358_ProblemTitle

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3. On a new line state the title of the problem, if any.

4. On a new line below the above, write in bold type: "**Statement of the Problem**".

5. Below the latter, state the problem. Please make sure the statement of your problem (unlike the preceding item) is not in bold type.

6. Below the statement of the problem, write in bold type: "**Solution of the Problem**".

7. Below the latter, show the entire solution of your problem.

Here is a sample for the above-stated format for proposed problems:

Problem proposed to SSMJ

Problem proposed by Isaac Newton, Trinity College, Cambridge, England.

Principia Mathematica (⟵ You may choose to not include a title.)

Statement of the problem:

Compute $\displaystyle\sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}$.

Solution of the problem: . . . . . .

♣ ♣ ♣ Thank You! ♣ ♣ ♣