Problems Ted Eisenberg, Section Editor

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This section of the Journal offers readers an opportunity to exchange interesting mathematical problems and solutions. Please send them to Ted Eisenberg, Department of Mathematics, Ben-Gurion University, Beer-Sheva, Israel or fax to: 972-86-477-648. Questions concerning proposals and/or solutions can be sent e-mail to <eisenbt@013.net>. Solutions to previously stated problems can be seen at <http://www.ssma.org/publications>.

Solutions to the problems stated in this issue should be posted before
January 15, 2020

- **5559:** *Proposed by Kenneth Korbin, New York, NY*

  For every positive integer N there are two Pythagorean triangles with area $(N)(N+1)(2N+1)(2N-1)(4N+1)(4N^2+2N+1)$. Find the sides of the triangles if $N=4$.

- **5560:** *Proposed by Michael Brozinsky, Central Islip, NY*

  Square ABCD (in clockwise order) with all sides equal to x has point E as the midpoint of side AB. The right triangle EBC is folded along segment EC so that what was previously corner B is now at point B' which is at a distance d from side AD. Find d and also the distance of B' from AB.

- **5561:** *Proposed by Pedro Pantoja, Natal/RN, Brazil*

  Calculate the *exact value* of:

  $$\cos\frac{5\pi}{28}+\cos\frac{13\pi}{28}-\cos\frac{17\pi}{28}.$$

- **5562:** *Proposed by Daniel Sitaru, National Economic College "Theodor Costescu," Mehedinti, Romania*

  Prove: If $a,b,c\geq 1$, then

  $$e^{ab}+e^{bc}+e^{ca}>3+\frac{c}{a}+\frac{b}{c}+\frac{a}{b}.$$

- **5563:** *Proposed by José Luis Díaz-Barrero, Barcelona Tech, Barcelona, Spain*

  Without the aid of a computer, find the value of

  $$\sum_{n=1}^{+\infty}\frac{15}{25n^2+45n-36}.$$

- **5564:** *Proposed by Ovidiu Furdui and Alina Sîntămărian, Technical University of Cluj-Napoca, Cluj-Napoca, Romania*

  Let $a > 0$ and let $f : [0, a] \to \Re$ be a Riemann integrable function. Calculate

$$\lim_{n\to\infty} \int_0^a \frac{f(x)}{1+nx^n}\mathrm{d}x.$$

Solutions

5541: *Proposed by Kenneth Korbin, New York, NY*

A convex cyclic quadrilateral has inradius r and circumradius R. The distance from the incenter to the circumcenter is 169. Find positive integers r and R.

Solution 1 by Ioannis D. Sfikas, National and Kapodistrian University of Athens, Greece

Fuss' theorem gives a relation between the inradius r, the circumradius R, and the distance d between the incenter I and the circumcenter O, for any bicentric quadrilateral. The relation is:

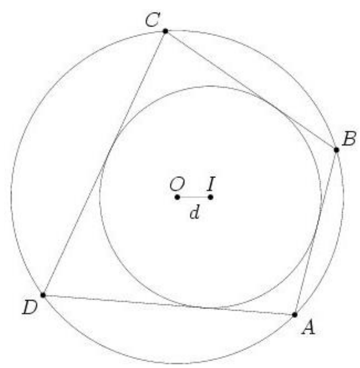
$$\frac{1}{(R+d)^2} + \frac{1}{(R-d)^2} = \frac{1}{r^2}, \qquad (1)$$

or equivalently:

$$2r^2(R^2+d^2) = (R^2-d^2)^2.$$

It was derived by Nicolaus Fuss (1755-1826) in 1792. Solving for d yields:

$$d = \sqrt{R^2 + h^2 - r\sqrt{4R^2+h^2}}.$$



Since $d = 169 = 13^2$, then we may assume the relation (1) as a Diophantine equation:

$$\frac{1}{(R+13^2)^2} + \frac{1}{(R-13^2)^2} = \frac{1}{r^2},$$

with $r, R > 0$ or:

$$r^2 = \frac{(R^2 - 13^4)^2}{2(R^2 + 13^4)}$$

We may assume the Diophantine equation:

$$2(R^2 + 13^4) = y^2,$$

and:

$$R = -\frac{169}{2}\left[(\sqrt{2}+1)(3-2\sqrt{2})^n - (\sqrt{2}-1)(3+2\sqrt{2})^n\right],$$

$$R = \frac{169}{2}\left[(\sqrt{2}+2)(3-2\sqrt{2})^n - (\sqrt{2}-2)(3+2\sqrt{2})^n\right],$$

for $n \geq 1$ and $n \in N$. So, we have: $r = r(n)$ which must be an integer. By calculations, we have:

$$r = 28560 \text{ and } R = 40391.$$

Solution 2 by Albert Stadler, Herrliberg, Switzerland

By Fuss' theorem (https://en.wikipedia.org/wiki/Bicentric-quadrilateral),

$$\frac{1}{(R-x)^2}+\frac{1}{(R+x)^2}=\frac{1}{r^2},$$

or equivalently

$$r=\frac{R^2-x^2}{\sqrt{2(R^2+x^2)}},$$

where r is the inradius, R the circumradius and x the distance between the incenter and the circumcenter of the bicentric quadrilateral.

By assumption, $x=169$ and r is an integer. Therefore $\sqrt{2(R^2+x^2)}$ is a (rational) integer. We note that

$$2r=\frac{2R^2+2x^2-4^2}{\sqrt{2(R^2+x^2)}}=\sqrt{2(R^2+x^2)}-\frac{4x^2}{\sqrt{2(R^2+x^2)}},$$

which implies that $\sqrt{2(R^2+x^2)}$ divides $2^2 13^4$. We conclude that $2(R^2+x^2)\in \{4,16,676,2704,114244,456976,19307236,77228944,3262922884,13051691536\}$.

The only feasible value for R is $R=40391$ which leads to $r=28560$.

Solution 3 by Ed Gray of Highland Beach, FL

Editor′s comment: I am taking the liberty of jumping into the middle of Ed's solution. Like those above, his solution started off using Fuss' Formula, and immediately substituted $d=169$ into it. After some algebra he obtained that

$$R^4-(2r^2+57122)R^2=815730721-57122r^2,$$

that he solved as a quadratic in R^2. Solving this he obtained that $R^2=r^2+28561\pm r\sqrt{r^2+114244}$. Letting $e^2=r^2+114244$, he continued on as follows:

Then $114244=2\cdot 2\cdot 13^4=e^2-r^2=(e-r)(e+r)$. The sum of the factors $e-r$ and $e+r$ is even and equals $2e$. Therefore the factors are both even or both odd. Since their product is even, they both must be even. There are only 2 possibilities:

(i) $e-r=2$ and $e+r=2\cdot 28561=57122$. Then $2e=57124, e=28562$, and $r=28560$. From the equation $R^2=r^2+28561\pm -r\sqrt{r^2+114244})$,

$$R^2=815673600+28561\pm(28560)(28562)=815702161\pm 815730720.$$

Clearly the negative sign is not viable.

So $R^2=815702161+815730720=1631432881$ and $R=40391$. The solution pair (r,R) is $(28560,40391)$, and they satisfy Fuss' Theorem.

(ii) $e-r=2\cdot 13=26$ and $e+r=2\cdot(13^3)=4394$. Then $2e=4420, e=2210$, and $r=2184$. From the equation $R^2=r^2+28561\pm -r\sqrt{r^2+114244})$ we see that $R^2=4769856+28561\pm(2184)(2210)$. However, in this is the case, then R^2 will end in 7, and so R cannot be an integer.

Also solved by Kee-Wai Lau, Hong Kong, China; David Stone and John Hawkins, Georgia Southern University, Statesboro, GA, and the proposer.

5542: *Proposed by Michel Bataille, Rouen, France*

Evaluate in closed form: $\cos\dfrac{\pi}{13} + \cos\dfrac{3\pi}{13} - \cos\dfrac{4\pi}{13}$.
(Closed form means that the answer should not be expressed as a decimal equivalent.)

Solution 1 by David E. Manes, Oneonta, NY

We will show that $\cos\dfrac{\pi}{13} + \cos\dfrac{3\pi}{13} - \cos\dfrac{4\pi}{13} = \sqrt{\dfrac{7+\sqrt{13}}{8}} = \dfrac{1+\sqrt{13}}{4}$. To do so, we assume the following identities:

$$\cos^2\left(\frac{\pi}{13}\right) + \cos^2\left(\frac{3\pi}{13}\right) + \cos^2\left(\frac{4\pi}{13}\right) = \frac{11+\sqrt{13}}{8},$$

$$\sum_{k=1}^{n} \cos\left(\frac{2k\pi}{2n+1}\right) = \cos\left(\frac{2\pi}{2n+1}\right) + \cos\left(\frac{4\pi}{2n+1}\right) + \cdots + \cos\left(\frac{2n\pi}{2n+1}\right) = -\frac{1}{2},$$

where n is a positive integer. Let $C = \cos\dfrac{\pi}{13} + \cos\dfrac{3\pi}{13} - \cos\dfrac{4\pi}{13}$. Then

$$\begin{aligned} C^2 &= \cos^2\left(\frac{\pi}{13}\right) + \cos^2\left(\frac{3\pi}{13}\right) + \cos^2\left(\frac{4\pi}{13}\right) + 2\cos\frac{\pi}{13}\cos\frac{3\pi}{13} - 2\cos\frac{\pi}{13}\cos\frac{4\pi}{13} - 2\cos\frac{3\pi}{13}\cos\frac{4\pi}{13} \\ &= \frac{11+\sqrt{13}}{8} + 2\cos\frac{\pi}{13}\cos\frac{3\pi}{13} - 2\cos\frac{\pi}{13}\cos\frac{4\pi}{13} - 2\cos\frac{3\pi}{13}\cos\frac{4\pi}{13}. \end{aligned}$$

By the product-to-sum formulas, one finds

$$\begin{aligned} 2\cos\frac{\pi}{13}\cos\frac{3\pi}{13} &= \cos\frac{4\pi}{13} + \cos\frac{2\pi}{13} \\ -2\cos\frac{\pi}{13}\cos\frac{4\pi}{13} &= -\cos\frac{5\pi}{13} - \cos\frac{3\pi}{13} \\ -2\cos\frac{3\pi}{13}\cos\frac{4\pi}{13} &= -\cos\frac{7\pi}{13} - \cos\frac{\pi}{13}. \end{aligned}$$

Using the addition formula for $\cos(\pi - x) = -\cos x$, we get

$$\begin{aligned} -\cos\frac{5\pi}{13} &= \cos\left(\pi - \frac{5\pi}{13}\right) = \cos\frac{8\pi}{13}, -\cos\frac{3\pi}{13} = \cos\frac{10\pi}{13}, \\ -\cos\frac{7\pi}{13} &= \cos\left(\pi - \frac{7\pi}{13}\right) = \cos\frac{6\pi}{13}, -\cos\frac{\pi}{13} = \cos\frac{12\pi}{13}. \end{aligned}$$

Therefore, rearranging the terms, one obtains,

$$2\cos\frac{\pi}{13}\cos\frac{3\pi}{13} - 2\cos\frac{\pi}{13}\cos\frac{4\pi}{13} - 2\cos\frac{3\pi}{13}\cos\frac{4\pi}{13} = \sum_{k=1}^{6}\cos\left(\frac{2k\pi}{13}\right) = -\frac{1}{2}.$$

Therefore,

$$C^2 = \frac{11+\sqrt{13}}{8} - \frac{1}{2} = \frac{7+\sqrt{13}}{8}, \text{ whence } C = \sqrt{\frac{7+\sqrt{13}}{8}}.$$

Note that $\sqrt{\dfrac{7+\sqrt{13}}{8}} = \dfrac{1+\sqrt{13}}{4}$ since $\left(\dfrac{1+\sqrt{13}}{4}\right)^2 = \dfrac{7+\sqrt{13}}{8}$.

Solution 2 by Ángel Plaza, University of Las Palmas de Gran Canaria, Spain

Let $x = \cos\dfrac{\pi}{13} + \cos\dfrac{3\pi}{13} - \cos\dfrac{4\pi}{13}$, $\theta = \dfrac{\pi}{13}$ and $c = \cos\theta$. If $c_k = \cos k\theta$, then $c_{2k} = 2c_k^2 - 1$, $2c_pc_q = c_{p+q} + c_{p-q}$, and $c_{13+k} = c_{13-k}$. Notice that $x > 0$. Therefore

$$\begin{aligned}
x^2 &= c_1^2 + c_3^2 + c_4^2 + 2c_1c_3 - 2c_1c_4 - 2c_3c_4 \\
&= \frac{c_2+1}{2} + \frac{c_6+1}{2} + \frac{c_8+1}{2} + c_4 + c_2 - c_5 - c_3 - c_7 - c_1, \\
x^2 + x &= \frac{1}{2}(3 + 3c_2 - 2c_5 + c_6 - 2c_7 + c_8) \\
&= \frac{1}{2}(3 + 3c_2 + 3c_6 + 3c_8).
\end{aligned}$$

Now, if $y = c_2 + c_6 + c_8$, then

$$\begin{aligned}
y^2 &= \frac{c_4+1}{2} + \frac{c_{12}+1}{2} + \frac{c_{16}+1}{2} + c_8 + c_4 + c_{10} + c_6 + c_{14} + c_2 \\
2y^2 &= 3c_4 + 3c_{10} + 3c_{12} + 2y + 3 \\
2y^2 &= 3(c_4 + c_{10} + c_{12}) + 2y + 3.
\end{aligned}$$

Now, since $c_2 + c_4 + c_6 + c_8 + c_{10} + c_{10} + c_{12} = -\frac{1}{2}$ because $c_2 + c_4 + c_6 + c_8 + c_{10} + c_{10} + c_{12} = \Re\left(\sum_{k=1}^{6} e^{(2k\pi i)/13}\right)$ and applying the sum of a geometric series, we get $-\dfrac{1}{2}$. Then, $c_4 + c_{10} + c_{12} = -\dfrac{1}{2} - (c_2 + c_6 + c_8) = -\dfrac{1}{2} - y$ and so, $2y^2 = 3(-\frac{1}{2} - y) + 2y + 3$ from where, since $y > 0$, $y = \dfrac{-1+\sqrt{13}}{4}$.
Finally, by solving $x^2 + x = \dfrac{1}{2}(3 + 3y)$ and, since $x > 0$ it is obtained $x = \dfrac{1+\sqrt{13}}{4}$.

Solution 3 by Andrea Fanchini, Cantú, Italy

Let p be an odd prime number. Then we know that

$$g_p = \sum_{k=0}^{p-1} \exp(2\pi k^2/p)$$

is a *quadratic Gaussian sum*, where $g_p = \sqrt{p}$ or $i\sqrt{p}$ according to whether $p \equiv 1$ or $p \equiv 3 \pmod 4$. So $g_{13} = \sqrt{13}$. Therefore,

$$\begin{aligned}
\sqrt{13} = 1 + e^{2\pi i/13} &+ e^{8\pi i/13} + e^{-8\pi i/13} + e^{6\pi i/13} + e^{-2\pi i/13} + e^{-6\pi i/13} \\
&+ e^{-6\pi i/13} + e^{-2\pi i/13} + e^{6\pi i/13} + e^{-8\pi i/13} + e^{8\pi i/13} + e^{2\pi i/13}.
\end{aligned}$$

Recalling that $e^{ix} + e^{-ix} = 2\cos x$ we then have:

$$\cos\frac{2\pi}{13} + \cos\frac{6\pi}{13} + \cos\frac{8\pi}{13} = \frac{\sqrt{13}-1}{4}, \qquad (1)$$

Now we consider the sum of cosines with arguments in arithmetic progression.

$$\sum_{k=0}^{n-1} \cos(a+kd) = \frac{\sin(nd/2)}{\sin(d/2)} \cos\left(a + \frac{(n-1)d}{2}\right)$$

where $a, d \in R, d \neq 0$, and that n is a positive integer.

In our case, we set $a = d = \frac{2\pi}{13}$ and $n = 6$, then

$$\begin{aligned}
\cos\frac{2\pi}{13} + \cos\frac{4\pi}{13} + \cos\frac{6\pi}{13} + \cos\frac{8\pi}{13} + \cos\frac{10\pi}{13} + \cos\frac{12\pi}{13} &= \frac{\sin\frac{6\pi}{13}\cos\frac{7\pi}{13}}{\sin\frac{\pi}{13}} \\
&= \frac{-2\sin\frac{6\pi}{13}\cos\frac{6\pi}{13}}{2\sin\frac{\pi}{13}} \\
&= -\frac{\sin\frac{12\pi}{13}}{2\sin\frac{\pi}{13}} \\
&= -\frac{\sin\frac{\pi}{13}}{2\sin\frac{\pi}{13}} \\
&= -\frac{1}{2}.
\end{aligned}$$

Substituting the sum in (1) into this last expression we obtain;

$$\cos\frac{4\pi}{13} + \cos\frac{10\pi}{13} + \cos\frac{12\pi}{13} = -\frac{1}{2} - \frac{\sqrt{13}-1}{4} = -\frac{\sqrt{13}+1}{4}.$$

So finally we have:

$$-\cos\frac{12\pi}{13} - \cos\frac{10\pi}{13} - \cos\frac{4\pi}{13} = \cos\frac{\pi}{13} + \cos\frac{3\pi}{13} - \cos\frac{4\pi}{13} = \frac{\sqrt{13}+1}{4}.$$

Solution 4 by Kee-Wai Lau, Hong Kong, China

We show that

$$\cos\frac{\pi}{13} + \cos\frac{3\pi}{13} - \cos\frac{3\pi}{13} = \frac{1+\sqrt{13}}{4}. \qquad (1)$$

Denote the left side of (1) by x, which is clearly positive. So (1) will follow from

$$4x^2 - 2x - 3 = 0. \qquad (2)$$

Let $\theta = \dfrac{\pi}{13}$ and $i = \sqrt{-1}$. Since $\cos^2\theta = \dfrac{1+\cos 2\theta}{2}$, $\cos^2 3\theta = \dfrac{1+\cos 6\theta}{2}$,

$\cos^2 4\theta = \dfrac{1-\cos 5\theta}{2}$, $2\cos\theta\cos 3\theta = \cos 2\theta + \cos 4\theta$, $2\cos\theta\cos 4\theta = \cos 3\theta + \cos 5\theta$, and

$2\cos 3\theta \cos 4\theta = \cos\theta - \cos 6\theta$, so

$$4x^2 - 2x - 3 = 3 + 6\sum_{k=1}^{6}(-1)^k\cos(k\theta) = 3\sum_{k=0}^{12}(-1)^k\cos(k\theta)$$

$$= 3Re\sum_{k=0}^{12}(-1)^k e^{ik\theta} = 3Re\left(\frac{1+e^{13i\theta}}{1+e^{i\theta}}\right) = 0.$$

This proves (2) and completes the solution.

Solution 5 by Brian D. Beasley, Presbyterian College, Clinton, SC

We show that the given expression equals $\dfrac{1+\sqrt{13}}{4}$.

Let $a = \cos(\pi/13)$, $b = \cos(3\pi/13)$, and $c = \cos(4\pi/13)$. Using the multiple-angle formulas for cosine, we have $b = 4a^3 - 3a$ and $c = 8a^4 - 8a^2 + 1$. Then $a + b - c = (1 + \sqrt{13})/4$ if and only if

$$[4(-8a^4 + 4a^3 + 8a^2 - 2a - 1) - 1]^2 = 13.$$

This in turn holds if and only if $f(a)(16a^2 - 8a - 12) = 0$, where

$$f(x) = 64x^6 - 32x^5 - 80x^4 + 32x^3 + 24x^2 - 6x - 1.$$

Using another multiple-angle formula for cosine, namely
$\cos(13\theta) = 4096r^{13} - 13312r^{11} + 16640r^9 - 9984r^7 + 2912r^5 - 364r^3 + 13r = (r+1)[f(r)]^2 - 1$
with $r = \cos\theta$, we have

$$(a + 1)[f(a)]^2 = 0.$$

Since $a \neq -1$, we conclude $f(a) = 0$, completing the proof.

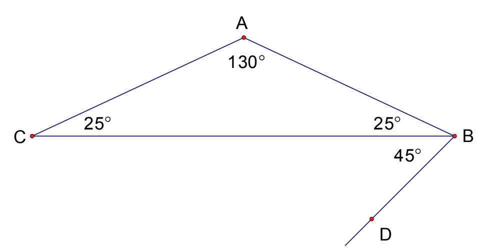
Also solved by Brian Bradie, Christopher Newport University, Newport News,VA; Ed Gray, Highland Beach, FL; Ioannis D. Sfikas, National and Kapodistrian University of Athens, Greece; Albert Stadler, Herrliberg, Switzerland, and the proposer.

5543: *Proposed by Titu Zvonaru, Comănesti, Romania*

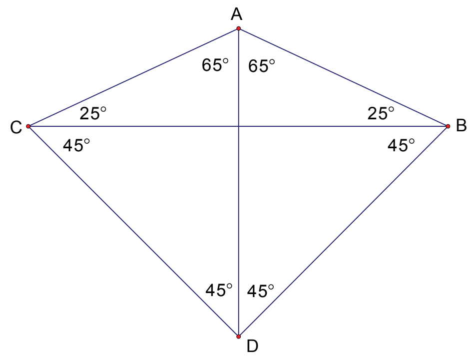
Let $ABDC$ be a convex quadrilateral such that
$\angle ABC = \angle BCA = 25°, \angle CBD = \angle ADC = 45°$. Compute the value of $\angle DAC$. (Note the order of the vertices.)

Solution 1 by David A. Huckaby, Angelo State University, San Angelo, TX

From the given facts $\angle ABC = \angle BCA = 25°$ and $\angle CBD = 45°$, we know that $\angle CAB = 130°$ and that D lies on the ray emanating from point B at a $45°$ angle from BC, as shown in the figure below.

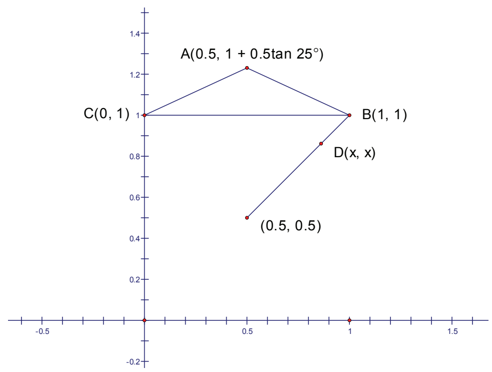


From the additional given fact $\angle ADC = 45°$, by inspection one solution is the kite shown in the figure below, in which $\angle DAC = 65°$.



It is clear that if D is farther from B (on the ray emanating from point B at a $45°$ angle from BC), then $\angle ADC < 45°$, and that as D moves closer to B that $\angle ADC > 45°$ and increases, reaches a maximum, and then decreases to approach $\angle ABC = 25°$ as D approaches B. This implies that there is one more location for point D such that $\angle ADC = 45°$.

To find it, let us place the points on a coordinate grid. See the figure below.



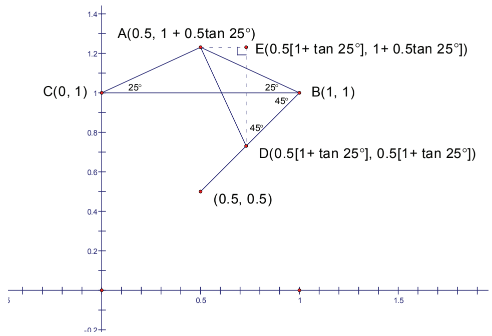
Consider $\angle ADC = 45°$ to be an inscribed angle of the circle passing through the points A, C, and the first location for D, namely $(\frac{1}{2}, \frac{1}{2})$. The other point where this circle intersects the line $y = x$ is the second location for D.

If the circle has center (h, k) and radius r, we have $(0-h)^2 + (1-k)^2 = (\frac{1}{2} - h)^2 + (1 + \frac{1}{2}\tan 25° - k)^2 = (\frac{1}{2} - h)^2 + (\frac{1}{2} - k)^2$. The second equation gives $1 + \frac{1}{2}\tan 25° - k = k - \frac{1}{2}$, whence $k = \frac{1}{4}(3 + \tan 25°)$. Using this value for k in the first equation gives $(0-h)^2 + (1 - \frac{1}{4}[3 + \tan 25°])^2 = (\frac{1}{2} - h)^2 + (1 + \frac{1}{2}\tan 25° - \frac{1}{4}[3 + \tan 25°])^2$. Expanding terms and solving for h yields $h = \frac{1}{4}(1 + \tan 25°)$.

To find r^2, we substitute the point $C(0, 1)$ into the equation of the circle $(x - \frac{1}{4}[1 + \tan 25°])^2 + (y - \frac{1}{4}[3 + \tan 25°])^2 = r^2$. Doing this and expanding terms yields $r^2 = \frac{1}{8}(1 + \tan^2 25°)$. So the equation of the circle is $(x - \frac{1}{4}[1 + \tan 25°])^2 + (y - \frac{1}{4}[3 + \tan 25°])^2 = \frac{1}{8}(1 + \tan^2 25°)$.

The circle intersects the line $y = x$ when $(x - \frac{1}{4}[1 + \tan 25°])^2 + (x - \frac{1}{4}[3 + \tan 25°])^2 = \frac{1}{8}(1 + \tan^2 25°)$. Expanding this yields the quadratic equation $2x^2 - (2 + \tan 25°)x + \frac{1}{2}(1 + \tan°) = 0$. The quadratic formula yields the two solutions $x = \frac{1}{2}$, which we already know, and $x = \frac{1}{2}(1 + \tan 25°)$.

Referring to the figure below, we see that
$\tan \angle ADE = \frac{AE}{DE} = \frac{\frac{1}{2}(1+\tan 25^\circ)-\frac{1}{2}}{1+\frac{1}{2}\tan 25^\circ - \frac{1}{2}(1+\tan 25^\circ)} = \tan 25^\circ$ so that $\angle ADE = 25^\circ$. So $\angle ADB = 25^\circ + 45^\circ = 70^\circ$, whence $\angle DAB = 180^\circ - 70^\circ - 70^\circ = 40^\circ$, so that $\angle DAC = 130^\circ - 40^\circ = 90^\circ$.



So the two possible values of $\angle DAC$ are 65° and 90°.

Solution 2 by Ioannis D. Sfikas, National and Kapodistrian University of Athens, Greece

The triangle ABC is isosceles and BCD is right isosceles, making AD the angle bisector of $\angle BAC$. So $\angle DCA = 65^\circ$. Furthermore, there are two possible answers since there are two positions for D.

Let F lie on BD, such that CF and BD are perpendicular. Let the circumcircle of ACF intersect BD at E. Now, $\angle EAC = 90^\circ$, or $\angle BAE = 40^\circ$, or $\angle AEB = \angle ABE$. So, $AB = AE$. But $AB = AC$. So, $AE = AC$ and ∠AEC=∠AFC=45°. So, E and F both satisfy the conditions imposed on D and in each case, we have $\angle DAC = 90^\circ, 65^\circ$, respectively (as D=E,F).

Solution 3 by Ed Gray, Highland Beach, FL

From the information given, triangle ABC is isosceles, with $AB = AC$. To enhance the lucidity of the calculations, we assign the value of 2.0 to each of these sides. We define $\angle DAC = x$, $\angle BAD = a$, $\angle BCD = c$, and $\angle BDA = b$.

(**1**) In triangle ABC, by the Law of Sines, $\dfrac{BC}{\sin(130^\circ)} = \dfrac{2}{\sin(c)}$; $BC = 3.625231148$

(**2**) In triangle CBD, $c + 45^\circ + b + 45^\circ = 180^\circ$, so $b + c = 90^\circ$.

(**3**) In triangle CBD, by the Law of Sines, $\dfrac{BC}{\sin(b+45^\circ)} = \dfrac{BD}{\sin(c)}$, or

(**4**) $\dfrac{BC}{\sin(b+45^\circ)} = \dfrac{BD}{\cos(b)}$.

(**5**) In triangle ABD, $\dfrac{BD}{\sin(a)} = \dfrac{2}{\sin(b)}$, $BD = \dfrac{2\sin(a)}{\sin(b)}$

(**6**) In triangle $ABD, a + 70^\circ + b = 180^\circ, a + b = 110^\circ, a = 110^\circ - b$.

From Equation **(4),**

(**7**) $\dfrac{BC}{\sin(b+45^\circ)} = \dfrac{2\cdot\sin(a)}{\sin(b)\cdot\cos(b)}$,

(**8**) $\dfrac{BC}{\sin(b+45^\circ)} = \dfrac{2\cdot\sin(110^\circ - b)}{\sin(b)\cdot\cos(b)}$

Substituting BC from Equation (1), we have a trigonometric equation for b.

(**9**) $1.812615574 \cdot \sin(b) \cdot \cos(b) =$
$[\sin(110°) \cdot \cos(b) - \cos(110°) \cdot \sin(b)] \cdot [\sin(b) \cdot \cos(45°) + \cos(b) \cdot \sin(45°)]$.

Since $\cos(45°) = \sin(45°) = \dfrac{\sqrt{2}}{2}$, we divide both sides by 0.707106781

(**10**) $2.563425529 \cdot \sin(b) \cdot \cos(b) =$
$\sin(110) \cdot \sin(b) \cdot \cos(b) + \sin(110°) \cdot \cos^2(b) - \cos(110°) \cdot \sin^2(b) - \cos(110°) \cdot \sin(b) \cdot \cos(b)$

(**11**) $2.563425529 \cdot \sin(b) \cdot \cos(b) = 0.939692621 \cdot \sin(b) \cdot \cos(b) + 0.939692621 \cdot \cos^2(b) +$
$0.342020143 \cdot \sin^2(b) + 0.342020143 \cdot \sin(b) \cdot \cos(b)$

(**12**) $1.281712765 \cdot \sin(b) \cdot \cos(b) = 0.342020143 \cdot \sin^2(b) + 0.939692621 \cdot \cos^2(b)$.

Squaring,

(**13**) $1.6427876 \cdot \sin^2(b) \cdot \cos^2(b) =$
$0.116977778 \cdot \sin^4(b) + 0.642787609 \cdot \sin^2(b) \cdot \cos^2(b) + 0.8883022222 \cdot \cos^4(b)$,

(**14**) $\cos^2(b) = 1 - \sin^2(b)$

(**15**) $\cos^4(b) = 1 - 2 \cdot \sin^2(b) + \sin^4(b)$

(**16**) $0.116977778 \cdot \sin^4(\mathrm{b}) - \sin^2(\mathrm{b}) \cdot \cos^2(\mathrm{b}) + 0.883022222(1 - 2 \cdot \sin^2(\mathrm{b}) + \sin^4(\mathrm{b})) = 0$

(**17**) $0.116977778 \cdot \sin^4(b) - \sin^2(b)(1 - \sin^2(b)) + 0.883022222 - 1.766044444 \cdot \sin^2(b) +$
$0.883022222 \cdot \sin^4(b) = 0$

(**18**) $2 \cdot \sin^4(b) - 2.766044444 \cdot \sin^2(b) + 0.883022222 = 0$

This is a quadratic equation in $\sin^2(b)$, with solutions:

(**19**) $4 \cdot \sin^2(b) = 2.766044444 \pm \sqrt{7.651001866 - 7.064177776)}$, or

(**20**) $4 \cdot \sin^2(b) = 2.766044444 \pm 7.66044444$

(**21**) So $\sin^2(b_1) = \dfrac{2}{4},\ \sin(b_1) = 0.707106781, b_1 = 45°$.

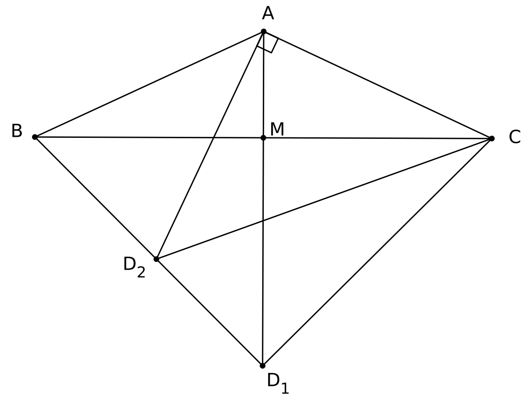
(**22**) $\sin^2(b_2) = \dfrac{3.532088888}{4} = 0.883022222,\ \sin(b_2) = 0.939692621, b_2 = 70°$.

When $b = 45°, a = 65°, x = 65°$.

When $b = 70°, a = 40°, x = 90°$.

Editor's comment: The following remark followed this solution: "I must admit having 2 answers is a surprise,..., however both solutions satisfy Equation **(12),** which is a good sign, because that is the fundamental equation and no extraneous root was introduced by squaring."

Solution 4 by Michel Bataille, Rouen, France



We consider ΔABC, which we suppose positively oriented, and let M be the midpoint of BC (see figure). Since $\angle ABC = \angle BCA$, AM is the perpendicular bisector of BC.

First, let D_1 be the image of B under the rotation with centre M and angle $+90°$. Then, ABD_1C is a convex quadrilateral and $\angle CBD_1 = \angle AD_1C = 45°$.

Second, let D_2 on BD_1 be such that $\angle CAD_2 = 90°$. Since $\angle BAC = 130°$, we have $\angle BAD_2 = 40°$. Also, $\angle ABD_2 = 25° + 45° = 70°$ and so
$\angle AD_2B = 180° - 40° - 70° = 70° = \angle ABD_2$. It follows that $AD_2 = AB = AC$ and the triangle CAD_2 is right-angled at A and isosceles. As a result the quadrilateral ABD_2C is convex with $\angle AD_2C = 45° = \angle CBD_2$.

Thus, we have found two candidates D_1, D_2 for the vertex D. There cannot be more: indeed, because of the convexity of $ABDC$, D must be on the ray BD_1 (to ensure that $\angle CBD = 45°$) and on the arc of circle, locus of the points P such that
$\angle(\overrightarrow{PC}, \overrightarrow{PA}) = +45°$ (to ensure that $\angle ADC = 45°$). We conclude that the answer to the problem is twofold: if $D = D_1$, then $\angle DAC = \frac{1}{2}\angle BAC = 65°$; if $D = D_2$, then $\angle DAC = 90°$.

Also solved by Andrea Fanchini, Cantú, Italy; Kee-Wai Lau, Hong Kong, China; Raquel Rosado, Hallie Kaiser, Mitch DeJong, and Caleb Edington, students at Taylor University, Upland, IN; David Stone and John Hawkins, Georgia Southern University, Statesboro, GA, and the proposer.

5544: *Proposed by Seyran Brahimov, Baku State University, Masalli, Azerbaijan*

Solve in $\Re$:

$$\begin{cases} \tan^{-1} x = & \tan y + \tan z \\ \tan^{-1} y = & \tan x + \tan z \\ \tan^{-1} z = & \tan x + \tan y \end{cases}$$

Solution by Ioannis D. Sfikas, National and Kapodistrian University of Athens, Greece

Adding the equations we have:

$$\sum_{cyc} (2\tan x - \tan^{-1} x) = 0.$$

Let $f(x) = \left|2\tan x - \tan^{-1} x\right|$, for ever $\left\{x \in \Re : k\pi - \frac{\pi}{2} < x < k\pi + \frac{\pi}{2} \text{ and } k \in Z\right\}$.

Then $f'(x) = \dfrac{2}{\cos^2 x} - \dfrac{1}{x^2+1} > 0$ for every $x \in \Re$. So, $f(x)$ is an increasing monotonic function and $f(x) \geq f(0) = 0$, since equality holds if $x = 0$.

Similarly, $f(y) \geq f(0) = 0$ and $f(z) \geq f(0) = 0$, since equality holds if $y = z = 0$.

Then, the only real solution is $x = y = z = 0$.

Also solved by Ed Gray, Highland Beach, FL and the proposer.

5545: *Proposed by José Luis Díaz-Barrero, Barcelona Tech, Barcelona, Spain*

Let p, q be two twin primes. Show that

$$1 + 4\left(\sum_{j=1}^{\frac{p-1}{2}} \left\lfloor \frac{jq}{p} \right\rfloor + \sum_{k=1}^{\frac{q-1}{2}} \left\lfloor \frac{kp}{q} \right\rfloor\right)$$

is a perfect square and determine it. (Here $\lfloor x \rfloor$ represents the integer part of x).

Solution 1 by Albert Stadler, Herrliberg, Switzerland

The integers p and q are odd (since they are twin primes) and so their difference is two. Let $x = (p+q)/2$. Then $\min(p,q) = x-1$, $\max(p,q) = x+1$.

We consider the rectangle R with vertices $A(0,0), B(p/2,0), C(p/2,q/2), D(0,q/2)$ in the Euclidean plane. The number of lattice points that are strictly inside R equals

$$L = \frac{p-1}{2} \cdot \frac{q-1}{2}.$$

There are no lattice points on the diagonal AC, since p and q are relatively prime. Clearly L equals the number of lattice points strictly inside the triangle ABC plus the number of lattice points strictly inside the triangle CDA. Therefore

$$L = \sum_{j=1}^{\frac{p-1}{2}} \left\lfloor \frac{jq}{p} \right\rfloor + \sum_{k=1}^{\frac{q-1}{2}} \left\lfloor \frac{kq}{p} \right\rfloor.$$

We conclude that

$$\begin{aligned} 1 + 4\left(\sum_{j=1}^{\frac{p-1}{2}} \left\lfloor \frac{jq}{p} \right\rfloor + \sum_{k=1}^{\frac{q-1}{2}} \left\lfloor \frac{kq}{p} \right\rfloor\right) &= 1 + 4L = 1 + (p-1)(q-1) = 1 + (x-2)x = (x-1)^2 = \\ &= (\min(p,q))^2. \end{aligned}$$

Solution 2 by Charles Diminnie and Simon Pfeil, Angelo State University, San Angelo, TX

We will assume only that p is odd, $p \geq 3$, and $q = p+2$. It is unnecessary to restrict p and/or q to be prime. To begin, if $j = 1, 2, \ldots, \dfrac{p-1}{2}$, then

$$
\begin{aligned}
j &< \frac{jq}{p} \\
&= \frac{j\,(p+2)}{p} \\
&= j + \frac{2j}{p} \\
&\le j + \left(\frac{2}{p}\right)\left(\frac{p-1}{2}\right) \\
&= j + \frac{p-1}{p} \\
&< j + 1.
\end{aligned}
$$

Hence, $\left\lfloor \frac{jq}{p} \right\rfloor = j$ for $j = 1, 2, \ldots, \frac{p-1}{2}$.

Further, for $k = 1, 2, \ldots, \frac{q-1}{2}$,

$$
\begin{aligned}
k &> \frac{kp}{q} \\
&= \frac{k\,(q-2)}{q} \\
&= k - \frac{2k}{q} \\
&\ge k - \left(\frac{2}{q}\right)\left(\frac{q-1}{2}\right) \\
&= k - \frac{q-1}{q} \\
&> k - 1.
\end{aligned}
$$

Therefore, $\left\lfloor \frac{kp}{q} \right\rfloor = k - 1$ for $k = 1, 2, \ldots, \frac{q-1}{2} = \frac{p+1}{2}$.

Using the known result that

$$
\sum_{i=1}^{n} i = \frac{n\,(n+1)}{2}
$$

for $n \ge 1$, we obtain

$$
\begin{aligned}
\sum_{j=1}^{\frac{p-1}{2}} \left\lfloor \frac{jq}{p} \right\rfloor &= \sum_{j=1}^{\frac{p-1}{2}} j \\
&= \left(\frac{1}{2}\right)\left(\frac{p-1}{2}\right)\left(\frac{p+1}{2}\right) \\
&= \frac{p^2-1}{8}
\end{aligned}
$$

and

$$\begin{aligned}\sum_{k=1}^{\frac{q-1}{2}} \left\lfloor \frac{kp}{q} \right\rfloor &= \sum_{k=1}^{\frac{p+1}{2}} (k-1) \\ &= \sum_{k=2}^{\frac{p+1}{2}} (k-1) \\ &= \sum_{i=1}^{\frac{p-1}{2}} i \\ &= \frac{p^2-1}{8},\end{aligned}$$

(substituting $i = k-1$ in the last sum.)
As a result,

$$\begin{aligned}&1+4\left(\sum_{j=1}^{\frac{p-1}{2}} \left\lfloor \frac{jq}{p} \right\rfloor + \sum_{k=1}^{\frac{q-1}{2}} \left\lfloor \frac{kp}{q} \right\rfloor\right) \\ &= 1+4\left(\frac{p^2-1}{8} + \frac{p^2-1}{8}\right) \\ &= 1+\left(p^2-1\right) \\ &= p^2.\end{aligned}$$

Solution 3 by Anthony J. Bevelacqua, University of North Dakota, Grand Forks, ND

For any relatively prime odd integers $p, q \geq 3$ we have

$$\left(\sum_{j=1}^{\frac{p-1}{2}} \left\lfloor \frac{jq}{p} \right\rfloor + \sum_{k=1}^{\frac{q-1}{2}} \left\lfloor \frac{kp}{q} \right\rfloor\right) = \frac{p-1}{2} \cdot \frac{q-1}{2}$$

by, for example, Theorem 86 of Nagell's *Number Theory.* (The proof is standard and elementary: Consider the set of integer points (j,k) with $1 \leq j \leq (p-1)/2$ and $1 \leq k \leq (q-1)/2$. There are $\frac{p-1}{2}\frac{q-1}{2}$ such points. None of these are on the line $py = qx$. The number of points below the line is $\sum_{j=1}^{\frac{p-1}{2}} \left\lfloor \frac{jq}{p} \right\rfloor$ while the number of points above is $\sum_{k=1}^{\frac{q-1}{2}} \left\lfloor \frac{kp}{q} \right\rfloor$.)
Now suppose p and q are twin primes with $p < q$. Then p and q are relatively prime odd integers ≥ 3 with $q = p+2$. So

$$\begin{aligned}1+4\left(\sum_{j=1}^{\frac{p-1}{2}} \left\lfloor \frac{jq}{p} \right\rfloor + \sum_{k=1}^{\frac{q-1}{2}} \left\lfloor \frac{kp}{q} \right\rfloor\right) &= 1+4\cdot\frac{p-1}{2}\cdot\frac{q-1}{2} \\ &= 1+4\cdot\frac{p-1}{2}\cdot\frac{p+1}{2} \\ &= p^2.\end{aligned}$$

(Note that we only need p and q to be consecutive odd integers ≥ 3 in this argument.)

Solution 4 by Brian Bradie, Christopher Newport University, Newport News,VA

Without loss of generality, suppose p is the smaller of the two primes. Then $p \geq 3$, and $p+2=q$. Therefore,

$$\begin{aligned}
\sum_{j=1}^{\frac{p-1}{2}} \left\lfloor \frac{jq}{p} \right\rfloor &= \sum_{j=1}^{\frac{p-1}{2}} \left\lfloor j\left(1+\frac{2}{p}\right) \right\rfloor = \sum_{j=1}^{\frac{p-1}{2}} j \\
&= \frac{\frac{p-1}{2}\cdot\frac{p+1}{2}}{2} = \frac{p^2-1}{8}, \\
\sum_{k=1}^{\frac{q-1}{2}} \left\lfloor \frac{kp}{q} \right\rfloor &= \sum_{k=1}^{\frac{q-1}{2}} \left\lfloor k\left(1-\frac{2}{q}\right) \right\rfloor = \sum_{k=1}^{\frac{q-1}{2}} (k-1) \\
&= \frac{\frac{q-3}{2}\cdot\frac{q-1}{2}}{2} = \frac{\frac{p-1}{2}\cdot\frac{p+1}{2}}{2} = \frac{p^2-1}{8},
\end{aligned}$$

and

$$\begin{aligned}
1+4\left(\sum_{j=1}^{\frac{p-1}{2}} \left\lfloor \frac{jq}{p} \right\rfloor + \sum_{k=1}^{\frac{q-1}{2}} \left\lfloor \frac{kp}{q} \right\rfloor\right) &= 1+4\left(\frac{p^2-1}{8}+\frac{p^2-1}{8}\right) \\
&= p^2.
\end{aligned}$$

Solution 5 by Moti Levy, Rehovot, Israel

Without loss of generality, suppose p is the smaller of the two primes. Then $p \geq 3$, and $p+2=q$. Therefore,

$$\begin{aligned}
\sum_{j=1}^{\frac{p-1}{2}} \left\lfloor \frac{jq}{p} \right\rfloor &= \sum_{j=1}^{\frac{p-1}{2}} \left\lfloor j\left(1+\frac{2}{p}\right) \right\rfloor = \sum_{j=1}^{\frac{p-1}{2}} j \\
&= \frac{\frac{p-1}{2}\cdot\frac{p+1}{2}}{2} = \frac{p^2-1}{8}, \\
\sum_{k=1}^{\frac{q-1}{2}} \left\lfloor \frac{kp}{q} \right\rfloor &= \sum_{k=1}^{\frac{q-1}{2}} \left\lfloor k\left(1-\frac{2}{q}\right) \right\rfloor = \sum_{k=1}^{\frac{q-1}{2}} (k-1) \\
&= \frac{\frac{q-3}{2}\cdot\frac{q-1}{2}}{2} = \frac{\frac{p-1}{2}\cdot\frac{p+1}{2}}{2} = \frac{p^2-1}{8},
\end{aligned}$$

and

$$\begin{aligned}
1+4\left(\sum_{j=1}^{\frac{p-1}{2}} \left\lfloor \frac{jq}{p} \right\rfloor + \sum_{k=1}^{\frac{q-1}{2}} \left\lfloor \frac{kp}{q} \right\rfloor\right) &= 1+4\left(\frac{p^2-1}{8}+\frac{p^2-1}{8}\right) \\
&= p^2.
\end{aligned}$$

Solution 6 by Ulrich Abel, Technische Hochschule Mittelhessen, Germany

We show the slightly more general formula

$$1+4\left(\sum_{j=1}^{(n-1)/2}\left\lfloor j\frac{n+2}{n}\right\rfloor+\sum_{k=1}^{(n+1)/2}\left\lfloor k\frac{n}{n+2}\right\rfloor\right)=n^2 \qquad (n=3,5,7,9,\ldots).$$

Proof: Let $n\geq 3$ be an odd integer. Since $j<j\frac{n+2}{n}=j+\frac{2j}{n}<j+1$, for $1\leq j\leq (n-1)/2$, and $k-1<k-\frac{2k}{n+2}=k\frac{n}{n+2}<k$, for $1\leq k\leq (n+1)/2$, we conclude that

$$\begin{aligned} & 1+4\left(\sum_{j=1}^{(n-1)/2}\left\lfloor j\frac{n+2}{n}\right\rfloor+\sum_{k=1}^{(n+1)/2}\left\lfloor k\frac{n}{n+2}\right\rfloor\right) \\ = & \ 1+4\left(\sum_{j=1}^{(n-1)/2} j+\sum_{k=1}^{(n+1)/2}(k-1)\right) \\ = & \ 1+4\frac{n-1}{2}\frac{n+1}{2}=n^2. \end{aligned}$$

Also solved by Michel Bataille, Rouen, France; Brian D. Beasley, Presbyterian College, Clinton, SC; Ed Gray, Highland Beach, FL; Kee-Wai Lau, Hong Kong, China; David E. Manes, Oneonta, NY; Ángel Plaza, University of Las Palmas de Gran Canaria, Spain; Henry Ricardo, Westchester Area Math Circle, Purchase, NY; Ioannis D. Sfikas, National and Kapodistrian University of Athens, Greece; David Stone and John Hawkins, Georgia Southern University, Statesboro, GA; Nicusor Zlota, "Traian Vuia" Technical College, Focsani, Romania, and the proposer.

5546: *Proposed by Ovidiu Furdui and Alina Sîntămărian, Technical University of Cluj-Napoca, Cluj-Napoca, Romania*

Calculate

$$\sum_{n=1}^{\infty}(-1)^{\left\lfloor \frac{n}{2}\right\rfloor}\left(\mathrm{e}^x-1-\frac{x}{1!}-\frac{x^2}{2!}-\cdots-\frac{x^n}{n!}\right).$$

Solution 1 by Ángel Plaza, University of Las Palmas de Gran Canaria, Spain

Since $e^x-1-\dfrac{x}{1!}-\dfrac{x^2}{2!}-\ldots-\dfrac{x^n}{n!}=\displaystyle\sum_{k=n+1}^{\infty}\frac{x^k}{k!}$, the proposed series, say S, is absolutely convergent, and

$$\begin{aligned}
S &= \sum_{n=1}^{\infty}(-1)^{\lfloor \frac{n}{2} \rfloor} \sum_{k=n+1}^{\infty} \frac{x^k}{k!} \\
&= \sum_{k=2}^{\infty} \frac{x^k}{k!} \sum_{n=1}^{k-1}(-1)^{\lfloor \frac{n}{2} \rfloor} \\
&= \sum_{k=1}^{\infty} \frac{x^k}{k!} \cos\left(\frac{(k-2)\pi}{2}\right) \\
&= \sum_{k=1}^{\infty}(-1)^{n+1} \frac{x^{2k}}{(2k)!} \\
&= 1 - \cos x.
\end{aligned}$$

Solution 2 by Bruno Salgueiro Fanego, Viveiro, Spain

$$\sum_{n=1}^{\infty} (-1)^{\lfloor \frac{n}{2} \rfloor} \left(e^x - 1 - \frac{x}{1!} - \frac{x^2}{2!} - \ldots - \frac{x^n}{n!}\right) = \sum_{n=1}^{\infty}((-1)^{\lfloor \frac{n}{2} \rfloor} \left(\sum_{k=1}^{\infty} \frac{x^k}{k!} - 1 - \frac{x}{1!} - \frac{x^2}{2!} - \ldots - \frac{x^n}{n!}\right)$$

$$= \sum_{n=1}^{\infty} (-1)^{\lfloor \frac{n}{2} \rfloor} \left(\sum_{k=n+1}^{\infty} \frac{x^k}{k!}\right) = \sum_{n=1}^{\infty} \left(\sum_{k=n+1}^{\infty} (-1)^{\lfloor \frac{n}{2} \rfloor} \frac{x^k}{k!}\right) = \sum_{k=2}^{\infty} \left(\sum_{n=1}^{k-1} (-1)^{\lfloor \frac{n}{2} \rfloor} \frac{x^k}{k!}\right)$$

$$= 1\frac{x^2}{2!} + 0\frac{x^3}{3!} - 1\frac{x^4}{4!} + 0\frac{x^5}{5!} + 1\frac{x^6}{6!} + \ldots = -\sum_{i=1}^{\infty}(-1)^i \frac{x^{2i}}{i!} = 1 - \sum_{i=0}^{\infty} (-1)^i \frac{x^{2i}}{i!} = 1 - \cos x.$$

Solution 3 by Kee-Wai Lau, Hong Kong, China

We show that the given sum equals $1 - \cos x$.

Let $f(x) = \sin x + \cos x$, so that

$$f^{(n)}(x) = \begin{cases} \sin x + \cos x & n \equiv 0 (\text{mod } 4) \\ \cos x - \sin x & n \equiv 1 (\text{mod } 4) \\ -\sin x - \cos x & n \equiv 2 (\text{mod } 4) \\ -\cos x + \sin x & n \equiv 3 (\text{mod } 4) \end{cases}$$

It follows that the given sum $\sum_{n=1}^{\infty} f^{(n)}(0) \left(e^x - 1 - \frac{x}{1!} - \frac{x^2}{2!} - \cdots - \frac{x^n}{n!}\right)$.

According to entry 3.89 (a) on pp. 154, 227 of [1], we have

$$\sum_{n=1}^{\infty} f^{(n)}(0) \left(e^x - 1 - \frac{x}{1!} - \frac{x^2}{2!} - \cdots - \frac{x^n}{n!}\right) = \int_0^x e^{x-t} f(t)dt$$

which equals $1 - \cos x$, by standard integration. Our claimed result now follows easily.

Reference:

1. O. Furdui: *Limits, Series, and Fractional Part Integrals, Springer, 2013.*

Solution 4 by Michel Bataille, Rouen, France

For every nonnegative integer n and any real number x, let
$R_n(x) = e^x - 1 - \frac{x}{1!} - \frac{x^2}{2!} - \cdots - \frac{x^n}{n!} = \sum\limits_{k=n+1}^{\infty} \frac{x^k}{k!}$ and let $f(x) = \sum\limits_{n=1}^{\infty} (-1)^{\lfloor \frac{n}{2} \rfloor} R_n(x)$ be the required sum. We show that $f(x) = 1 - \cos x$.

Let $A > 0$ and $x \in [-A, A]$. Since $e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \cdots + \frac{x^n}{n!} + \frac{x^{n+1}e^c}{(n+1)!}$ for some c beteween 0 and x (Taylor-Lagrange relation), we see that

$$|R_n(x)| \leq \frac{A^{n+1}}{(n+1)!} \cdot e^A.$$

It follows that the series $\sum\limits_{n=1}^{\infty} (-1)^{\lfloor \frac{n}{2} \rfloor} R_n(x)$ is uniformly convergent on any interval $[-A, A]$ $(A > 0)$. Since the derivative $R_n'(x)$ is equal to R_{n-1} $(n \in N)$, the same is true of the series $\sum\limits_{n=1}^{\infty} (-1)^{\lfloor \frac{n}{2} \rfloor} R_n'(x) = \sum\limits_{n=1}^{\infty} (-1)^{\lfloor \frac{n}{2} \rfloor} R_{n-1}(x)$. As a result, we have

$$f'(x) = \sum_{n=1}^{\infty} (-1)^{\lfloor \frac{n}{2} \rfloor} R_{n-1}(x) = e^x - 1 + \sum_{n=2}^{\infty} (-1)^{\lfloor \frac{n}{2} \rfloor} R_{n-1}(x)$$

for any $x \in R$.

Likewise, f' is differentiable on R and for any real number x,

$$\begin{aligned} f''(x) &= e^x + \sum_{n=2}^{\infty} (-1)^{\lfloor \frac{n}{2} \rfloor} R_{n-2}(x) \\ &= e^x - R_0(x) + \sum_{n=3}^{\infty} (-1)^{\lfloor \frac{n}{2} \rfloor} R_{n-2}(x) \\ &= 1 + \sum_{n=1}^{\infty} (-1)^{\lfloor \frac{n+2}{2} \rfloor} R_n(x) \\ &= 1 + \sum_{n=1}^{\infty} (-1)^{1+\lfloor \frac{n}{2} \rfloor} R_n(x) = 1 - f(x). \end{aligned}$$

Thus, f is the solution to the differential equation $y'' + y = 1$ satisfying $f(0) = 0 = f'(0)$. Solving is classical and we readily obtain $f(x) = 1 - \cos x$.

Solution 5 by David Stone and John Hawkins, Georgia Southern University, Statesboro, GA

The sum of the series is $1 - \cos(x)$.

Recall the Maclaurin series for $\cos(x)$: $\cos(x) = \sum\limits_{n=0}^{\infty} (-1)^{2n} \frac{x^{2n}}{(2n)!} = 1 + \sum\limits_{n=1}^{\infty} (-1)^{2n} \frac{x^{2n}}{(2n)!}$.

As expected, we'll also use the Maclaurin series representation for the exponential function:
$e^x = \sum\limits_{n=0}^{\infty} \frac{x^n}{n!} = E_k + R_k$, for any $k \geq 1$,

where $E_k = \sum_{n=0}^{k} \frac{x^n}{n!}$ is the k^{th} partial sum and $R_k = \sum_{n=k+1}^{\infty} \frac{x^n}{n!}$ is the remainder.

Because the series converges, we know that the sequence $\{R_i\}_{k\geq 1}$ has limit 0.

Note also that $e^x - E_k = R_k$, and $E_{k+1} - E_k = \frac{x^{k+1}}{(k+1)!}$.

Consider the partial sums of our given series:

let $S_m = \sum_{n=1}^{m} (-1)^{\lfloor \frac{n}{2} \rfloor} \left(e^x - 1 = \frac{x}{1!} - \frac{x^2}{2!} - \frac{x^3}{3!} - \ldots - \frac{x^n}{n!} \right) = \sum_{n=1}^{m} (-1)^{\lfloor \frac{n}{2} \rfloor} (e^x - E_n)$.

We compute the first few partial sums.

To simplify the calculations, we first handle the sign term:
its pattern is $1, -1, -1, 1, 1, -1, -1, 1, 1, -1, \ldots$
This "block of four" pattern suggests that it will be productive to consider pairing consecutive terms (although we cannot be content with just carrying out a regrouping of a series without a guarantee of convergence).

$S_1 = e^x - E_1 = e^x - 1 - \frac{x}{1!}$

$S_2 = (e^x - E_1) - (e^x - E_2) = E_2 - E_1 = \frac{x^2}{2!}$

$S_3 = (e^x - E_1) - (e^x - E_2) - (e^x - E_3) = E_2 - E_1 = S_2 - R_2$

$S_4 = (e^x - E_1) - (e^x - E_2) - (e^x - E_3) + (e^x - E_4) = S_2 - (E_4 - E_3) = \frac{x^2}{2!} - \frac{x^4}{4!}$

$S_5 = S_4 + (e^x - E_5) = S_4 + R_5$

$S_6 = S_4 + (e^x - E_5) - (e^x - E_6) = S_4 + (E_6 - E_5) = \frac{x^2}{2!} - \frac{x^4}{4!} + \frac{x^6}{6!}$

$S_7 = S_6 + (e^x - E_7) = S_6 + R_7$

$S_8 = S_6 + (e^x - E_7) - (e^x - E_8) = S_6 + (E_8 - E_7) = \frac{x^2}{2!} - \frac{x^4}{4!} + \frac{x^6}{6!} - \frac{x^8}{8!}$.

Inductively, we can show that, for even subscripts

$S_{4k} = \frac{x^2}{2!} - \frac{x^4}{4!} + \frac{x^6}{6!} - \ldots \frac{x^{4k}}{(4k)!}$

$S_{4k+2} = \frac{x^2}{2!} - \frac{x^4}{4!} + \frac{x^6}{6!} - \ldots + \frac{x^{4k-2}}{(4k-2)!}$

and for odd subscripts

$S_{4k+1} = S_{4k} + R_{4k+1}$

$S_{4k+3} = S_{4k+2} - R_{4k+3}$.

We see that the subsequence $\{S_{2k}\}_{k\geq 1}$ has as its limit the Maclaurin series for $1 - \cos(x)$. If we had a priori knowledge that our given series is convergent, this would guarantee that our series has sum $1 - \cos(x)$.

However, looking at the odd-subscript partial sums will give us enough information to draw that conclusion. The subsequence $\{S_{2k+1}\}_{k\geq 1}$ has as the same limit as $\{S_{2k}\}_{k\geq 1}$ because the sequence $R_n \longrightarrow 0$.

Therefore, the limit of the sequence of partial sums, i.e. the sum of the given series, is $1 - \cos(x)$.

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