Problems

Ted Eisenberg, Section Editor

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This section of the Journal offers readers an opportunity to exchange interesting mathematical problems and solutions. Please send them to Ted Eisenberg, Department of Mathematics, Ben-Gurion University, Beer-Sheva, Israel or fax to: 972-86-477-648. Questions concerning proposals and/or solutions can be sent e-mail to <eisenbt@013.net>. Solutions to previously stated problems can be seen at <http://www.ssma.org/publications>.

Solutions to the problems stated in this issue should be posted before June 15, 2012

- **5206:** *Proposed by Kenneth Korbin, New York, NY*

  The distances from the vertices of an equilateral triangle to an interior point P are $\sqrt{a}$, $\sqrt{b}$, and $\sqrt{c}$ respectively, where $a, b,$ and c are positive integers.

  Find the minimum and the maximum possible values of the sum $a+b+c$ if the side of the triangle is 13.

- **5207:** *Proposed by Roger Izard, Dallas, TX*

  Consider the following four algebraic terms:

$$\begin{aligned} T_1 &= a^2(b+c)+b^2(a+c)+c^2(a+b) \\ T_2 &= (a+b)(a+c)(b+c) \\ T_3 &= abc \\ T_4 &= \frac{b+c-a}{a}+\frac{a+c-b}{b}+\frac{a+b-c}{c} \end{aligned}$$

  Suppose that $\dfrac{T_1 \cdot T_2}{(T_3)^2} = \dfrac{616}{9}$. What values would then be possible for T_4?

- **5208:** *Proposed by D. M. Bătinetu-Giurgiu, Bucharest and Neculai Stanciu, Buzău, Romania*

  Let the sequence of positive real numbers $\{a_n\}_{n\geq 1}$, $N \in Z^+$ be such that $\lim\limits_{n\to\infty} \dfrac{a_{n+1}}{n^2 \cdot a_n} = b$. Calculate:

$$\lim_{n\to\infty}\left(\frac{\sqrt[n+1]{a_{n+1}}}{n+1} - \frac{\sqrt[n]{a_n}}{n}\right).$$

- **5209:** *Proposed by Tom Moore, Bridgewater, MA*

  We noticed that 27 is a cube and 28 is an even perfect number. Find all pairs of consecutive integers such that one is cube and the other is an even perfect number.

- **5210:** *Proposed by José Luis Díaz-Barrero, Barcelona, Spain*

  Let a, b, c, d be four positive real numbers. Prove that

$$1+\frac{1}{8}\left(\frac{a}{b+c}+\frac{b}{c+d}+\frac{c}{d+a}+\frac{d}{a+b}\right)>\frac{2\sqrt{3}}{3}.$$

- **5211:** *Proposed by Ovidiu Furdui, Cluj-Napoca, Romania*

  Let $n \geq 1$ be a natural number and let

$$f_n(x)=x^{x^{\cdot^{\cdot^{\cdot^{x}}}}},$$

where the number of x's in the definition of f_n is n. For example

$$f_1(x)=x, \quad f_2(x)=x^x, \quad f_3(x)=x^{x^x}, \ldots .$$

Calculate the limit

$$\lim_{x\to 1}\frac{f_n(x)-f_{n-1}(x)}{(1-x)^n}.$$

Solutions

- **5188:** *Proposed by Kenneth Korbin, New York, NY*

  Given $\triangle ABC$ with coordinates $A(-5,0), B(0,12)$ and $C(9,0)$. The triangle has an interior point P such that $\angle APB = \angle APC = 120^\circ$. Find the coordinates of point P.

Solution 1 by Ercole Suppa, Teramo, Italy

Let us construct equilateral triangles $\triangle ABD$, $\triangle AEC$ externally on the sides AB, AC of triangle $\triangle BAC$ and denote by ω_1, ω_2 the circumcircles of $\triangle ABD$, $\triangle AEC$. The point P is the intersection point of ω_1, ω_2 different from O. In order to find the coordinates of D, E we use complex numbers. If we denote respectively by $a=-5$, $b=12i$, $c=9$ the affixes of A, B, C we get:

$$\begin{aligned} d &= a+(b-a)e^{\frac{\pi}{3}i}=\frac{-5-12\sqrt{3}}{2}+\frac{12+5\sqrt{3}}{2}i \\ e &= a+(c-a)e^{\frac{\pi}{3}i}=2-7\sqrt{3}i \end{aligned}$$

so the coordinates of D, E are $D\left(\frac{-5-12\sqrt{3}}{2}, \frac{12+5\sqrt{3}}{2}\right)$ and $E\left(2,-7\sqrt{3}\right)$.

The equations of ω_1, ω_2 are:

$$\begin{aligned} \omega_1 &: \quad 169\sqrt{3}x^2+169\sqrt{3}y^2+\left(2028+845\sqrt{3}\right)x+\left(-845-2028\sqrt{3}\right)y+10140=0 \\ \omega_2 &: \quad 196\sqrt{3}x^2+196\sqrt{3}y^2-784\sqrt{3}x+2744y-8820\sqrt{3}=0 \end{aligned}$$

and, after some calculations, we obtain

$$P = \omega_1 \cap \omega_2 = \left(-\frac{2\left(-981+112\sqrt{3}\right)}{2353}, -\frac{21\left(-896+263\sqrt{3}\right)}{2353} \right).$$

Solution 2 by Ángel Plaza, University of Las Palmas de Gran Canaria, Spain

The isogonic center P is the point from which we see the sides of ABC under equal angles (that is 120° mod 180°). The isogonic center is the common intersection point of the three circumcircles to the equilateral triangles constructed on the sides of $\triangle ABC$ (Napoleon's theorem).

Let $\triangle AC'B$, $\triangle BA'C$, $\triangle ACB'$ be equilateral triangles constructed on the outside on the edges of $\triangle ABC$, then AA', BB', CC' intersect in the isogonic center P. Of course, it is enough to find two of these lines. In our case,

$$\begin{aligned} B' &= -5 + (14,0)\cdot 1_{-\pi/3} = 2 - 7\sqrt{3}i \\ C' &= -5 + (5,12)\cdot 1_{\pi/3} = -5/2 - 6\sqrt{3} + (6 + 5\sqrt{3}/2)i. \end{aligned}$$

The intersection of lines BB' and CC' gives the solution

$$P = \left(\frac{6(56+33\sqrt{3})}{504+295\sqrt{3}}, \frac{21(93+56\sqrt{3})}{504+295\sqrt{3}} \right) = \left(\frac{1962-224\sqrt{3}}{2353}, \frac{18816-5523\sqrt{3}}{2353} \right).$$

Solution 3 by Michael Brozinsky, Central Islip, NY

Clearly $AC = 14$, and $AB = 13$. Consider the circumscribed circle of $\triangle APC$. If we denote its center by O_1 and the midpoint of AC by $M(2,0)$, then, since an inscribed angle is measured by one half of its intercepted arc, and a radius perpendicular to a chord bisects the chord and its arc, it readily follows that $\triangle AO_1M$ is a 30, 60, 90 degree right triangle and so since $MA = 7$, the radius is $14\frac{\sqrt{3}}{3}$, $O_1M = 7\frac{\sqrt{3}}{3}$, and P lies on the circle

$$(x-2)^2 + \left(y + \frac{7}{3}\sqrt{3}\right)^2 = \frac{196}{3}. \qquad (1)$$

(Note that since the segment of this circle having minor arc AC contains P, the center O_1 is below AC.)

Similarly, consider the circumscribed circle of $\triangle APB$. If we denote its center by O_2 and the midpoint of AB by $N\left(-\frac{5}{2}, 6\right)$, then $\triangle AO_2N$ is a 30, 60, 90 degree right triangle and so since $NA = \frac{13}{2}$ the radius is $13\frac{\sqrt{3}}{3}$ and $O_2N = 13\frac{\sqrt{3}}{6}$.

The perpendicular bisector of AB is

$$y - 6 = -\frac{5}{12}\cdot\left(x + \frac{5}{2}\right) \Longrightarrow y = -\frac{5}{12}x + \frac{119}{24}. \qquad (2)$$

If (X, Y) are the coordinates of O_2 we have

$$\left(X + \frac{5}{2}\right)^2 + \left(-\frac{5}{12}X + \frac{119}{24} - 6\right)^2 = \left(13\frac{\sqrt{3}}{6}\right)^2,$$

and thus, $X = -\frac{5}{2} \pm 2\sqrt{3}$. We choose $X = -\frac{5}{2} - 2\sqrt{3}$ since O_2 is to the left of AB, and from (2), $Y = 6 + \frac{5}{6}\sqrt{3}$.

Thus P also lies on the circle

$$\left(x + \frac{5}{2} + 2\sqrt{3}\right)^2 + \left(y - 6 - \frac{5}{6}\sqrt{3}\right)^2 = \frac{169}{3}. \qquad (3)$$

If we subtract equation (3) from equation (1) we obtain the line

$$-9x - 36 + \frac{19}{3}y\sqrt{3} - 4x\sqrt{3} - 20\sqrt{3} + 12y = 9 \qquad (4)$$

or equivalently,

$$y = -\frac{3}{71}\left(-32 + 9\sqrt{3}\right)(x + 5). \qquad (5)$$

This line just found passes through the points of intersections of these two circles and thus P is that point that is interior to $\triangle ABC$. Substituting (1) into (5), solving the resulting quadratic equation and rejecting $x = -5$, gives the (x, y) coordinates of P.

$$(x, y) = \left(\frac{1962}{2353} - \frac{224}{2353}\sqrt{3},\ \frac{18816}{2353} - \frac{5523}{2353}\sqrt{3}\right).$$

Also solved by Brian D. Beasley, Clinton, SC; Michael Brozinsky (two solutions), Central Islip, NY; Paul M. Harms, North Newton, KS; Kee-Wai Lau, Hong Kong, China; David E. Manes, Oneonta, NY; Charles McCracken, Dayton, OH; John Nord, Spokane, WA; Titu Zvonaru, Comănesti, Romania jointly with Neculai Stanciu, Buzău, Romania; Albert Stadler, Herrliberg, Switzerland; David Stone and John Hawkins (jointly), Statesboro, GA, and the proposer.

- **5189:** *Proposed by Kenneth Korbin, New York, NY*

Given triangle ABC with integer length sides and with $\angle A = 60^\circ$ and with $(a, b, c) = 1$.

Find the lengths of b and c if

$$\begin{aligned} i)\ a &= 13, \text{ and if} \\ ii)\ a &= 13^2 = 169, \text{ and if} \\ iii)\ a &= 13^4 = 28561. \end{aligned}$$

Solution 1 by Kee-Wai Lau, Hong Kong, China

By the cosine formula formula, we have $a^2 = b^2 + c^2 - bc$ so that $c = \dfrac{b \pm \sqrt{4a^2 - 3b^2}}{2}$ with $1 \leq b < \dfrac{2\sqrt{3}a}{3}$. A computer search yields the following solutions with $(a, b, c) = 1$:

$$i)\ (b, c) = (7, 15), (8, 15), (15, 7), (15, 8).$$

$$ii)\ (b,c) = (15,176),(161,176),(176,15),(176,161).$$

$$iii)\ (b,c) = (5055,30751),(25696,30751),(30751,5055),(30751,25696).$$

Solution 2 by Albert Stadler, Herrliberg, Switzerland

(*Editor:* Albert gave two solutions to this problem; in the first solution he used the above computer-aided approach. But in his second solution he used the complex roots of unity.)

A more inspired approach is based on Eisenstein integers (see e.g., http : //en.wikipedia.org/wiki/Eisenstein integer).

Let $\omega = \dfrac{-1+i\sqrt{3}}{2}$. The set of Eisenstein integers $Z[\omega] = \{a + b\omega | a, b \in Z\}$ has the following properties:

- (i) $Z[\omega]$ forms a commutative ring of algebraic integers in the real number field $Q(\omega)$
- (ii) $Z[\omega]$ is an Euclidean domain whose norm N is given by $N(a+b\omega) = a^2 - ab + b^2$. As a result of this $Z[\omega]$ is a factorial ring.
- (iii) The group of units in $Z[\omega]$ is the cyclic group formed by the sixth root of unity in the complex plane. Specifically, they are $\{\pm 1, \pm\omega, \pm\omega^2\}$. These are just the Eisenstein integers of norm one.
- (iv) An ordinary prime number (or rational prime) which is 3 or congruent to 1 mod 3 is of the form $x^2 - xy + y^2$ for some integers x, y and may therefore be factored into $(x + y\omega)(x + y\omega^2)$ and because of that it is not prime in the Eisenstein integers. Ordinary primes congruent to 2 mod 3 cannot be factored in this way and they are primes in the Eisenstein integers as well.

Based on the above we find the factorization $13 = (4+\omega)(4+\omega^2)$, where $4+\omega$ and $4+\omega^2$ are two Eisenstein primes that are conjugate to each other . So $13^n = (4+\omega)^n(4+\omega^2)^n$, and this is (up to units) the only factorization into two factors of the form $b + c\omega$ with b and c coprime. We find

$$(4+\omega)^2 = 16 + 8\omega + \omega^2 = 15 + 7\omega,$$

$$(4+\omega)^2(-\omega^2) = -15\omega^2 - 7 = 8 + 15\omega,$$

$$(4+\omega)^4 = (15+7\omega)^2 = 225 + 210\omega + 49\omega^2 = 176 + 161\omega,$$

$$(4+\omega)^4(-\omega^2) = (176+161\omega)(-\omega^2) = -176\omega^2 - 161 = 15 + 176\omega,$$

$$(4+\omega)^8 = (176+161\omega)^2 = 30976 + 56672\omega + 25921\omega^2 = 5055 + 30751\omega$$

$$(4+\omega)^8(-\omega) = (5055+30751\omega)(-\omega) = -5055\omega - 30751\omega^2 = 30751 + 25696\omega.$$

We note that

$$\begin{aligned}
N((4+\omega)^2) &= N((4+\omega)^2(-\omega^2)) = 13^2 \\
N((4+\omega)^4) &= N((4+\omega)^4(-\omega^2)) = 13^4 \\
N((4+\omega)^8) &= N((4+\omega)^8(-\omega^2)) = 13^8 \\
N(x+y\omega) &= x^2 - xy + y^2,
\end{aligned}$$

and we get the same solutions as with the exhaustive computer search.

Also solved by Brian D. Beasley, Clinton, SC; Dionne T. Bailey, Elsie M. Campbell, and Charles Diminnie, San Angelo TX; Bruno Salgueiro Fanego, Viveiro Spain; Paul M. Harms, North Newton, KS; David E. Manes, Oneonta, NY; David Stone and John Hawkins (jointly), Statesboro, GA, and the proposer.

- **5190:** *Proposed by Neculai Stanciu, Buzău, Romania*

Prove: If x, y and z are positive integers such that $\dfrac{x(y+1)}{x-1} \in \mathrm{N}$, $\dfrac{y(z+1)}{y-1} \in \mathrm{N}$, and $\dfrac{z(x+1)}{z-1} \in \mathrm{N}$, then $xyz \leq 693$.

Solution by Kee-Wai Lau, Hong Kong, China

Since two consecutive positive integers are relatively prime, so in fact

$$y+1 = a(x-1),\ \ z+1 = b(y-1),\ \ x+1 = c(z-1), \tag{1}$$

where $a, b, c \in N$ and $abc < 1$. Solving (1) for x, y, z we obtain

$$x = \frac{1+2c+2bc+abc}{abc-1},\ \ y = \frac{1+2a+2ac+abc}{abc-1},\ \ \frac{1+2b+2ab+abc}{abc-1}. \tag{2}$$

Also, we have from (1) that

$$abc = \left(\frac{x+1}{x-1}\right)\left(\frac{y+1}{y-1}\right)\left(\frac{z+1}{z-1}\right) \leq (3)(3)(3) = 27. \tag{3}$$

Using (3), we check with a computer that x, y, and z of (2) are positive integers if and only if

$$\begin{aligned}
(a,b,c) \;=\;& (1,1,2),(1,1,3),(1,1,4),(1,1,7),(1,2,1),(1,2,3),(1,3,1),(1,3,5)(1,4,1),\\
& (1,7,1),(2,1,1),(2,2,2,)(2,3,1),(3,1,1),(3,1,2),(3,3,3),(3,5,1),(4,1,1),\\
& (5,1,3),(7,1,1).
\end{aligned}$$

Correspondingly,

$$\begin{aligned}
(x,y,z) \;=\;& (11,9,7),(8,6,4),(7,5,3),(6,4,2),(9,7,11),(5,3,3),(6,4,8),(4,2,2),(5,3,7),\\
& (4,2,6),(7,11,9),(3,3,3),(3,3,5),(4,8,6),(3,5,3),(2,2,2),(2,2,4),(3,7,5),\\
& (2,4,2),(2,6,4).
\end{aligned}$$

Hence, $xyz \leq 693$ as desired.

Also solved by Dionne T. Bailey, Elsie M. Campbell, and Charles Diminnie, San Angelo TX; Paul M. Harms, North Newton, KS; Albert Stadler, Herrilberg, Switzerland; David Stone and John Hawkins (jointly), Statesboro, GA, and the proposer.

- **5191:** *Proposed by José Luis Díaz-Barrero, Barcelona, Spain*

Let a, b, c be positive real numbers such that $ab + bc + ca = 3$. Prove that

$$\frac{a\sqrt{bc} + b\sqrt{ca} + c\sqrt{ab}}{a^4 + b^4 + c^4} \leq 1.$$

Solution 1 by Paul M. Harms, North Newton, KS

From inequalities of the type $\sqrt{bc} \leq \dfrac{b+c}{2}$ we see that

$$a\sqrt{bc} + b\sqrt{ca} + c\sqrt{ab} \ \leq \ a\frac{b+c}{2} + b\frac{c+a}{2} + c\frac{a+b}{2} \ = \ ab + bc + ca.$$

If we can show that the last expression is less than or equal to $\left(a^4 + b^4 + c^4\right)$, then the inequality of the problem is correct.

From the condition in the problem $ab + bc + ca = 3$, so we will prove that
$\dfrac{3}{(a^4 + b^4 + c^4)} \leq 1.$
Let $a \leq b \leq c$ with $b = ta$ and $c = sa$ where $1 \leq t \leq s$. Then

$$3 \leq a^4 + b^4 + c^4 \ = \ \left(a^2\right)^2 \left(1 + t^4 + s^4\right).$$

$$ab + bc + ca \ = \ 3 \Longrightarrow a^2 = \frac{3}{t + ts + s}. \text{ We must prove that}$$

$$3 \ \leq \ \left[\frac{3}{t + ts + s}\right]^2 \left(1 + t^4 + s^4\right) \text{ or equivalently,}$$

$$(t + ts + s)^2 \ = \ t^2 + 2st^2 + s^2t^2 + 2st + 2s^2t + s^2 \leq 3\left(1 + t^4 + s^4\right) \text{ for } 1 \leq t \leq s, \text{ i.e.,}$$

$$0 \ \leq \ 3t^4 + 3s^4 - t^2 - s^2 - 2t^2s - 2s^2t - s^{t2} - 2st + 3 \text{ for } 1 \leq \ t \leq s.$$

Let $f(t, s)$ be the right side of the last inequality. We use partial derivatives to help find the minimum of the function in the appropriate domain.

Subtracting the equations $f_t(t, s) = 0$ and $f_s(t, s) = 0$ we obtain:

$$12(t^3 - s^3) - 2(t - s) - 2(s^2 - t^2) - 2st(s - t) - 2(s - t) = 0 = 2(t - s)\left[6t^2 + 7st + 6s^2 + s + t\right].$$

The part in the brackets is clearly positive for $1 \leq t \leq s$ so we must check $t = s$ and other boundary points of the domain for a minimum of the function.

When $t = s$,

$$f(t,s) = f(s,s) = 5s^4 - 4s^3 - 4s^2 + 3 = (s-1)^2\left[5s^2 + 6s + 3\right].$$

The function has a minimum in this case for $t = s = 1$. For the boundary $t = 1$ with t=1$\leq s$,

$$f(1,s) = (s-1)^2\left[3s^2 + 6s + 5\right]$$

which again has a minimum for $t = s = 1$. Since $f(s,t) \geq f(1,1) = 0$ for $1 \leq t \leq s$, the inequality of the problem has been proved.

Solution 2 by Dionne Bailey, Elsie Campbell, and Charles Diminnie, San Angelo, TX

Using the well-known result $x^2 + y^2 \geq 2xy$, with equality if and only if $x = y$, we obtain

$$\begin{aligned} xy + yz + zx &\leq \frac{1}{2}\left[\left(x^2+y^2\right) + \left(y^2+z^2\right) + \left(z^2+x^2\right)\right] \\ &= x^2 + y^2 + z^2, \qquad (1) \end{aligned}$$

and consequently,

$$\begin{aligned} (x+y+z)^2 &= x^2 + y^2 + z^2 + 2\left(xy + yz + zx\right) \\ &= \leq 3\left(x^2+y^2+z^2\right). \qquad (2) \end{aligned}$$

Further, equality is attained in (1) or (2) if and only if $x = y = z$.

By (1),

$$\begin{aligned} a\sqrt{bc} + b\sqrt{ca} + c\sqrt{ab} &= \sqrt{ab}\sqrt{ca} + \sqrt{bc}\sqrt{ab} + \sqrt{ca}\sqrt{bc} \\ &\leq ab + bc + ca \\ &= 3, \qquad (3) \end{aligned}$$

with equality if and only if $\sqrt{ab} = \sqrt{bc} = \sqrt{ca}$, i.e., if and only if $a = b = c = 1$.

Also, since $a, b, c > 0$, (1) and (2) imply that

$$\begin{aligned} 9 &= (ab + bc + ca)^2 \\ &\leq \left(a^2 + b^2 + c^2\right)^2 \\ &\leq 3\left(a^4 + b^4 + c^4\right) \end{aligned}$$

and hence,

$$a^4 + b^4 + c^4 \geq 3. \qquad (4)$$

Once again, equality is achieved in (4) if and only if $a = b = c = 1$.

Therefore, by (3) and (4),

$$\frac{a\sqrt{bc}+b\sqrt{ca}+c\sqrt{ab}}{a^4+b^4+c^4} \le \frac{3}{3} = 1,$$

with equality if and only if $a=b=c=1$.

Solution 3 by Albert Stadler, Herrliberg, Switzerland

The homogeneous form of this inequality reads as

$$\left(a\sqrt{bc}+b\sqrt{ca}+c\sqrt{ab}\right)(ab+bc+ca) \le 3\left(a^4+b^4+c^4\right) \text{ or equivalently as}$$

$$a^2b^{\frac{3}{2}}c^{\frac{1}{2}}+a^{\frac{3}{2}}b^2c^{\frac{1}{2}}+a^{\frac{3}{2}}b^{\frac{3}{2}}c+ab^{\frac{3}{2}}c^{\frac{3}{2}}+a^{\frac{1}{2}}b^2c^{\frac{3}{2}}+a^{\frac{1}{2}}b^{\frac{3}{2}}c^2+a^2b^{\frac{1}{2}}c^{\frac{3}{2}}+a^{\frac{3}{2}}bc^{\frac{3}{2}}+a^{\frac{3}{2}}b^{\frac{1}{2}}c^2 \le 3\left(a^4+b^4+c^4\right).$$

By the weighted AM-GM inequality

$$a^{4r}b^{4s}c^{4t} \le ra^4+sb^4+tc^4 \qquad (1)$$

for all tuples (r,s,t) of positive real numbers r,s, and t such that $r+s+t=1$. We write down the nine inequalities that result from (1) by choosing:

$$\begin{aligned}(r,s,t) \quad = \quad & \left(\frac{1}{2},\frac{3}{8},\frac{1}{8}\right), \left(\frac{3}{8},\frac{1}{2},\frac{1}{8}\right), \left(\frac{3}{8},\frac{3}{8},\frac{1}{4}\right), \\ & \left(\frac{1}{4},\frac{3}{8},\frac{3}{8}\right), \left(\frac{1}{8},\frac{1}{2},\frac{3}{8}\right), \left(\frac{1}{8},\frac{3}{8},\frac{1}{2}\right), \\ & \left(\frac{1}{2},\frac{1}{8},\frac{3}{8}\right), \left(\frac{3}{8},\frac{1}{4},\frac{3}{8}\right), \left(\frac{3}{8},\frac{1}{8},\frac{1}{2}\right).\end{aligned}$$

and add them up. The result follows.

Also solved by Arkady Alt, San Jose, CA; Kee-Wai Lau, Hong Kong, China; David E. Manes, Oneonta, NY; Ercole Suppa, Teramo, Italy; Paolo Perfetti, Department of Mathematics, University "Tor Vergata," Rome, Italy; Boris Rays, Brooklyn, NY; Titu Zvonaru, Comănesti, Romania jointly with Neculai Stanciu, Buzău, Romania, and the proposer.

- **5192:** *Proposed by G. C. Greubel, Newport News, VA*

Let $[n]=[n]_q=\dfrac{1-q^n}{1-q}$ be a q number and $\ln_q(x)=\displaystyle\sum_{n=1}^{\infty}\frac{x^n}{[n]}$ be a q-logarithm. Evaluate the following series:

$$i) \qquad \sum_{k=0}^{\infty}\frac{q^{mk}}{[mk+1][mk+m+1]} \text{ and}$$

$$ii) \qquad \sum_{k=1}^{\infty}\frac{x^k}{[k][k+m]}.$$

Solution 1 by Kee-Wai Lau, Hong Kong, China

For $0<|q|<1$ and for $0<|x|<1$, we have

i)

$$\begin{aligned}
&\sum_{k=0}^{\infty} \frac{q^{mk}}{[mk+1][mk+m+1]} \\
= \quad & \frac{1}{q[m]} \sum_{k=0}^{\infty} \left(\frac{1}{[mk+1]} - \frac{1}{[mk+m+1]} \right) \\
= \quad & \frac{1}{q[m]} \left(1 - (1-q)\right) \\
= \quad & \frac{1}{[m]}.
\end{aligned}$$

and ii)

$$\begin{aligned}
&\sum_{k=1}^{\infty} \frac{x^k}{[k][k+m]} \\
= \quad & \frac{1}{[m]} \sum_{k=1}^{\infty} \left(\frac{x^k}{[k]} - \frac{q^m x^k}{[k+m]} \right) \\
= \quad & \frac{1}{[m]} \left(\sum_{k=1}^{\infty} \frac{x^k}{[k]} - \left(\frac{q}{x}\right)^m \sum_{k=1}^{\infty} \frac{x^{k+m}}{[k+m]} \right) \\
= \quad & \frac{1}{[m]} \left(\ln_q(x) - \left(\frac{q}{x}\right)^m \ln_q(x) + \left(\frac{q}{x}\right)^m \sum_{k=1}^{m} \frac{x^k}{[k]} \right).
\end{aligned}$$

Solution 2 by Albert Stadler, Herrliberg, Switzerland

i)

$$\begin{aligned}
\sum_{k=0}^{\infty} \frac{q^{mk}}{[mk+1]\,[mk+m+1]} &= (1-q)^2 \sum_{k=0}^{\infty} \frac{q^{mk}}{(1-q^{mk+1})\,(1-q^{mk+m+1})} \\
&= (1-q)^2 \sum_{k=0}^{\infty} q^{mk} \left(\frac{1}{1-q^{mk+1}} - \frac{1}{1-q^{mk+m+1}} \right) \frac{1}{q^{mk+1} - q^{mk+m+1}} \\
&= \frac{(1-q)^2}{q - q^{m+1}} \sum_{k=0}^{\infty} \left(\frac{1}{1-q^{mk+1}} - \frac{1}{1-q^{mk+m+1}} \right) \\
&= \frac{(1-q)^2}{q - q^{m+1}} \sum_{k=0}^{\infty} \left(\frac{1}{1-q^{mk+1}} - \frac{1}{1-q^{m(k+1)+1}} \right)
\end{aligned}$$

$$= \frac{(1-q)^2}{q-q^{m+1}} \cdot \left(\frac{(1)}{1-q} - 1\right)$$

$$= \frac{(1-q)}{1-q^m}$$

ii)

$$\sum_{k=1}^{\infty} \frac{x^k}{[k][k+m]} = (1-q)^2 \sum_{k=1}^{\infty} \frac{x^k}{(1-q^k)(1-q^{k+m})}$$

$$= (1-q)^2 \sum_{k=1}^{\infty} x^k \left(\frac{1}{1-q^k} - \frac{1}{1-q^{k+m}}\right) \frac{1}{q^k - q^{k+m}}$$

$$= \frac{(1-q)^2}{1-q^m} \sum_{k=1}^{\infty} \left(\frac{x}{q}\right)^k \left(\frac{1}{1-q^k} - \frac{1}{1-q^{k+m}}\right)$$

$$= \frac{(1-q)^2}{1-q^m} \sum_{k=1}^{\infty} \left[x^k \left(\frac{1}{q^k} + \frac{1}{1-q^k}\right) - x^k q^m \left(\frac{1}{q^{k+m}} + \frac{1}{1-q^{k+m}}\right)\right]$$

$$= \frac{(1-q)^2}{1-q^m} \sum_{k=1}^{\infty} \left[x^k \left(\frac{1}{1-q^k}\right) - x^k q^m \left(\frac{1}{1-q^{k+m}}\right)\right]$$

$$= \frac{(1-q)}{1-q^m} \sum_{k=1}^{\infty} \frac{x^k}{[k]} - \frac{(1-q)^2 q^m}{(1-q^m)x^m} \sum_{k=1}^{\infty} x^{k+m} \left(\frac{1}{1-q^{k+m}}\right)$$

$$= \frac{(1-q)}{1-q^m} \sum_{k=1}^{\infty} \frac{x^k}{[k]} - \frac{(1-q)^2 q^m}{(1-q^m)x^m} \left[-\frac{x}{1-q} - \frac{x^2}{1-q^2} - \cdots - \frac{x^m}{1-q^m} + \sum_{k=1}^{\infty} x^k \left(\frac{1}{1-q^k}\right)\right]$$

$$= \frac{1-q}{1-q^m} \sum_{k=1}^{\infty} \frac{x^k}{[k]} + \frac{(1-q)^2 q^m}{(1-q^m)x^m} \left[\frac{x}{1-q} + \frac{x^2}{1-q^2} + \cdots + \frac{x^m}{1-q^m}\right] - \frac{(1-q)q^m}{(1-q^m)x^m} \sum_{k=1}^{\infty} \frac{x^k}{[k]}$$

$$= \frac{1-q}{1-q^m} \left(1 - \left(\frac{q}{x}\right)^m\right) \ln_q(x) + \frac{(1-q)q^m}{(1-q^m)} \left[\frac{x^{1-m}}{1} + \frac{x^{2-m}}{1+q} + \cdots + \frac{1}{1+q+q^2+\cdots q^{m-1}}\right].$$

Also solved by Arkady Alt, San Jose, CA, and the proposer.

- **5193:** *Proposed by Ovidiu Furdui, Cluj-Napoca, Romania*

  Let f be a function which has a power series expansion at 0 with radius of convergence R.

  a) Prove that $\displaystyle\sum_{n=1}^{\infty} n f^{(n)}(0) \left(e^x - 1 - \frac{x}{1!} - \frac{x^2}{2!} \cdots - \frac{x^n}{n!}\right) = \int_0^x e^{x-t} t f'(t)dt, \quad |x| < R.$

b) Let α be a non-zero real number. Calculate $\displaystyle\sum_{n=1}^{\infty} n\alpha^n \left(e^x - 1 - \frac{x}{1!} - \frac{x^2}{2!} \cdots - \frac{x^n}{n!}\right)$.

Solution 1 by Ángel Plaza, University of Las Palmas de Gran Canaria, Spain.

a) Let $S(x)$ be the sum of the series. Then, by differentiation, and for $|x| < R$,

$$S'(x) = \sum_{n=1}^{\infty} nf^{(n)}(0)\left(e^x - 1 - \frac{x}{1!} - \frac{x^2}{2!} \cdots - \frac{x^{n-1}}{(n-1)!}\right) = S(x) + \sum_{n=1}^{\infty} nf^{(n)}(0) \cdot \frac{x^n}{n!}.$$

It follows that $S'(x) = S(x) + xf'(x)$, and hence

$$S(x) = \int_0^x e^{x-t} tf'(t)dt + Ce^x,$$

where C is a constant of integration. Because $S(0) = 0$, we have $C = 0$ and

$$S(x) = \int_0^x e^{x-t} tf'(t)dt.$$

b) Note that if $f(x) = e^{\alpha x}$ then $f^{(n)}(0) = \alpha^n$, for $n \geq 1$. Hence, by part a), the sum of the given series is $\displaystyle\int_0^x e^{x-t} te^t dt = \frac{x^2 e^2}{2}$ if $\alpha = 1$. If $\alpha \neq 1$, the sum of the series is $\displaystyle\int_0^x e^{x-t} t\alpha e^{\alpha t} dt = \frac{\alpha x e^{\alpha x}}{\alpha - 1} + \frac{\alpha(e^x - e^{\alpha x})}{(\alpha - 1)^2}$.

Solution 2 by Anastasios Kotronis, Athens, Greece

a) From the problem's assumptions we have that

$$f(x) = \sum_{n=0}^{+\infty} \frac{f^{(n)}(0)}{n!} x^n \quad \text{and} \quad f'(x) = \sum_{n=1}^{+\infty} \frac{f^{(n)}(0)}{(n-1)!} x^{n-1} \quad \text{for} \quad |x| < R,$$

so, for $|x| < R$ we obtain

$$\begin{aligned} \int_0^x e^{x-t} tf'(t)\, dt &= \int_0^x e^{x-t} \sum_{n=1}^{+\infty} \frac{f^{(n)}(0)}{(n-1)!} t^n \, dt \\ &= e^x \sum_{n=1}^{+\infty} \frac{f^{(n)}(0)}{(n-1)!} \int_0^x t^n e^{-t}\, dt \\ &= e^x \sum_{n=1}^{+\infty} \frac{f^{(n)}(0)}{(n-1)!} I_n. \qquad (1) \end{aligned}$$

Now $I_n = -\displaystyle\int_0^x t^n (e^{-t})'\, dt = -x^n e^{-x} + nI_{n-1}$, so it is easily verified by induction that

$$I_n = -e^{-x}\left(x^n + nx^{n-1} + \cdots + n!x^0\right) + n!$$

With the above, (1) will give

$$\int_0^x e^{x-t} t f'(t)\,dt \;=\; e^x \sum_{n=1}^{+\infty} \frac{f^{(n)}(0)}{(n-1)!}\left(-e^{-x}\left(x^n + nx^{n-1} + \cdots + n!x^0\right) + n!\right)$$

$$= \sum_{n=1}^{+\infty} \frac{f^{(n)}(0)}{(n-1)!}\left(n!e^x - x^n - nx^{n-1} - \cdots - n!x^0\right)$$

$$= \sum_{n=1}^{+\infty} n f^{(n)}(0)\left(e^x - 1 - \frac{x}{1!} - \cdots - \frac{x^n}{n!}\right).$$

2) From (1) with $f(x) = e^{\alpha x}$ we obtained that

$$\sum_{n=1}^{+\infty} n\alpha^n \left(e^x - 1 - \frac{x}{1!} - \cdots - \frac{x^n}{n!}\right) \;=\; \int_0^x e^{x-t}\alpha t e^{\alpha t}\,dt$$

$$= I_\alpha.$$

So,

$$\begin{cases} \displaystyle\int_0^x e^{x-t} t e^t \; dt = \frac{x^2 e^x}{2}, & \text{for } \alpha = 1 \\ I_\alpha = \alpha e^x \left(\int_0^x t\left(\frac{e^{(\alpha-1)t}}{\alpha-1}\right)\; dt\right), & \text{for} \alpha \neq 1 \\ \quad = \frac{\alpha e^{\alpha x}}{\alpha-1}\left(x - \frac{1}{\alpha-1}\right) + \frac{\alpha e^x}{(\alpha-1)^2}. & \end{cases}$$

Solution 3 by Arkady Alt, San Jose, CA

a) Let

$a_n(x) = e^x - 1 - \frac{x}{1!} - \frac{x^2}{2!} - \ldots - \frac{x^n}{n!}, n \in N \cup \{0\}$ and $F(x) = \sum_{n=1}^{\infty} n f^{(n)}(0)\, a_n(x)$.

Noting that

$$a_n'(x) \;=\; e^x - 1 - \frac{x}{1!} - \frac{x^2}{2!} - \ldots - \frac{x^{n-1}}{(n-1)!}$$

$$= a_{n-1}(x), n \in N$$

we obtain

$$F'(x) \;=\; \left(\sum_{n=1}^{\infty} n f^{(n)}(0)\, a_n(x)\right)'$$

$$= \sum_{n=1}^{\infty} n f^{(n)}(0)\, a_n'(x)$$

$$= \sum_{n=1}^{\infty} n f^{(n)}(0)\, a_{n-1}(x).$$

Therefore,

$$\begin{aligned}
\mathrm{F}(x) - F'(x) &= \sum_{n=1}^{\infty} n f^{(n)}(0)\left(a_n(x) - a_{n-1}(x)\right) \\
&= \sum_{n=1}^{\infty} n f^{(n)}(0)\left(-\frac{x^n}{n!}\right) \\
&= -\sum_{n=1}^{\infty} f^{(n)}(0)\frac{x^n}{(n-1)!} \\
&= -x\sum_{n=0}^{\infty} f^{(n+1)}(0)\frac{x^n}{n!} \\
&= -xf'(x).
\end{aligned}$$

Multiplying equation $F'(x) - F(x) = xf'(x)$ by e^{-x} we obtain

$$\begin{aligned}
F'(x)e^{-x} - F(x)e^{-x} &= e^{-x}xf'(x) \iff \left(F(x)e^{-x}\right)' \\
&= e^{-x}xf'(x).
\end{aligned}$$

Hence,

$$\begin{aligned}
F(x)e^{-x} &= \int_o^x e^{-t}tf'(t)\,dt \\
&\iff F(x) = \int_o^x e^{x-t}tf'(t)\,dt.
\end{aligned}$$

b) Let $f(x) = e^{\alpha x}$ then $f^{(n)}(0) = \alpha^n$ and, using the result we obtained in part (a) we get,

$$\begin{aligned}
\sum_{n=1}^{\infty} n\alpha^n\left(e^x - 1 - \frac{x}{1!} - \frac{x^2}{2!} - \dots - \frac{x^n}{n!}\right) &= \int_0^x e^{x-t}t\alpha e^{\alpha t}dt \\
&= \alpha e^x\int_0^x te^{t(\alpha-1)}dt.
\end{aligned}$$

If $\alpha = 1$ then $\int_0^x te^{t(\alpha-1)}dt = \frac{x^2}{2}$ and, therefore,

$$\begin{aligned}
\sum_{n=1}^{\infty} n\alpha^n\left(e^x - 1 - \frac{x}{1!} - \frac{x^2}{2!} - \dots - \frac{x^n}{n!}\right) &= \sum_{n=1}^{\infty} n\left(e^x - 1 - \frac{x}{1!} - \frac{x^2}{2!} - \dots - \frac{x^n}{n!}\right) \\
&= \frac{\alpha e^x x^2}{2}.
\end{aligned}$$

If $\alpha \neq 1$ then

$$\int_0^x te^{t(\alpha-1)}dt = \frac{xe^{(\alpha-1)x}}{\alpha-1} - \frac{e^{(\alpha-1)x}}{(\alpha-1)^2}$$

$$= \frac{e^{(\alpha-1)x}\left(x\left(\alpha-1\right)-1\right)}{\left(\alpha-1\right)^2}.$$

Hence,

$$\sum_{n=1}^{\infty} n\alpha^n \left(e^x - 1 - \frac{x}{1!} - \frac{x^2}{2!} - \dots - \frac{x^n}{n!}\right) = \frac{\alpha e^{\alpha x}\left(x\left(\alpha-1\right)-1\right)}{\left(\alpha-1\right)^2}.$$

Solution 4 by Paolo Perfetti, Department of Mathematics, University "Tor Vergata," Rome, Italy

a) We need the two lemmas:

Lemma 1 $m!n! \le (n+m)!$

Proof by Induction. Let m be fixed. If $n=0$ evidently holds true. Let's suppose that the statement is true for any $1 \le n \le r$. For $n = r+1$ we have

$$m!(r+1)! = m!r!(r+1) \le (m+r)!(r+1) \le (m+r)!(m+r+1) = (m+r+1)!$$

which clearly holds for any $m \ge 0$. Since the inequality is symmetric, the induction on m proceeds along the same lines. q.e.d.

Lemma 2 The power series

$$\sum_{n=1}^{\infty} nf^{(n)}(0)\left(e^x - 1 - \frac{x}{1!} - \frac{x^2}{2!} - \ldots - \frac{x^n}{n!}\right) = \sum_{n=1}^{\infty} nf^{(n)}(0) \sum_{k=n+1}^{\infty} \frac{x^k}{k!}$$

converges for $|x| < R$ and is differentiable.

Proof:

$$\sum_{k=n+1}^{\infty} \frac{x^k}{k!} = \frac{x^{n+1}}{(n+1)!} \sum_{k=n+1}^{\infty} x^{k-n-1}\frac{(n+1)!}{k!}.$$

By using the Lemma 1 we can bound

$$\sum_{k=n+1}^{\infty} |x|^{k-n-1}\frac{(n+1)!}{k!} \le \sum_{k=n+1}^{\infty} \frac{|x|^{k-n-1}}{(k-n-1)!} = \sum_{k=0}^{\infty} \frac{|x|^k}{k!} = e^{|x|} \le e^R.$$

Thus we can write

$$\sum_{n=0}^{\infty} n|f^{(n)}(0)| \sum_{k=n+1}^{\infty} \frac{|x|^k}{k!} \le e^R |x| \sum_{n=0}^{\infty} n|f^{(n)}(0)| \frac{|x|^n}{n!}\frac{n!}{(n+1)!}$$

Since

$$\limsup_{n\to\infty} \left|\frac{f^{(n)}(0)}{n!}\right|^{1/n} = R^{-1} \implies \limsup_{n\to\infty} \left|\frac{f^{(n)}(0)}{n!}\frac{n}{n+1}\right|^{1/n} = R^{-1}$$

the series

$$\sum_{n=1}^{\infty} nf^{(n)}(0) \sum_{k=n+1}^{\infty} \frac{x^k}{k!}$$

converges for any $|x| < R$. Its differentiability is a consequence of the standard theory on power–series so we don't write it here. q.e.d.

The function $\int_0^x e^{x-t} t f'(t) dt$ is also differentiable by the fundamental theorem of calculus and the derivative yields

$$\left(\int_0^x e^{x-t} t f'(t) dt \right)' = x f'(x) + \int_0^x e^{x-t} t f'(t) dt$$

namely it satisfies the ordinary differential equation $Q'(x) = Q(x) + x f'(x)$, $Q(0) = 0$.

The derivative of the series in question a) is

$$\sum_{n=1}^{\infty} n f^{(n)}(0) \left(e^x - 1 - \frac{x}{1!} - \frac{x^2}{2!} - \ldots - \frac{x^{n-1}}{(n-1)!} \right)$$

that is

$$\sum_{n=1}^{\infty} n f^{(n)}(0) \sum_{k=n+1}^{\infty} \frac{x^k}{k!} + \sum_{n=1}^{\infty} n f^{(n)}(0) \frac{x^n}{n!}$$

which is in turn equals

$$= \sum_{n=1}^{\infty} n f^{(n)}(0) \sum_{k=n+1}^{\infty} \frac{x^k}{k!} + x f'(x)$$

Moreover $\left(\sum_{n=1}^{\infty} n f^{(n)}(0) \sum_{k=n+1}^{\infty} \frac{x^k}{k!} \right) \Big|_{x=0} = 0$ thus the functions $\int_0^x e^{x-t} t f'(t) dt$ and $\sum_{n=1}^{\infty} n f^{(n)}(0) \left(e^x - 1 - \frac{x}{1!} - \frac{x^2}{2!} \ldots - \frac{x^n}{n!} \right)$ satisfy the same differential equation with the same initial condition. By the uniqueness theorem for ODE, they are the same function. This concludes the proof.

b) $\alpha^n = (e^{\alpha x})^{(n)} \Big|_{x=0}$ thus

$$\sum_{n=1}^{\infty} n \alpha^n \left(e^x - 1 - \frac{x}{1!} - \frac{x^2}{2!} \ldots - \frac{x^n}{n!} \right) = \int_0^x e^{x-t} t \alpha e^{\alpha t} dt$$

If $\alpha = 1$ we obtain $\int_0^x t e^x \alpha dt = \alpha \frac{x^2}{2} e^x$.
If $\alpha \neq 1$ we obtain integrating by parts

$$\begin{aligned} \alpha e^x \int_0^x t e^{t(\alpha-1)} dt &= \alpha e^x \left(\frac{1}{\alpha - 1} t e^{t(\alpha-1)} \Big|_0^x - \frac{1}{\alpha - 1} \int_0^x e^{t(\alpha-1)} dt \right) \\ &= \frac{\alpha x e^{\alpha x}}{\alpha - 1} - \frac{\alpha e^{\alpha x}}{(\alpha - 1)^2} + \frac{\alpha e^x}{(\alpha - 1)^2}. \end{aligned}$$

Also solved by Dionne T. Bailey, Elsie M. Campbell, Charles Diminnie, and Andrew Siefker, San Angelo, TX; Kee-Wai Lau, Hong Kong, China; Albert Stadler, Herrliberg, Switzerland, and the proposer.

Mea Culpa

The name of **Achilleas Sinefakopoulos of Larissa, Greece** was inadvertently omitted in the March issue of the column as having solved problem 5184. I am terrible sorry for this oversight–Ted.