Problems

Ted Eisenberg, Section Editor

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This section of the Journal offers readers an opportunity to exchange interesting mathematical problems and solutions. Please send them to Ted Eisenberg, Department of Mathematics, Ben-Gurion University, Beer-Sheva, Israel or fax to: 972-86-477-648. Questions concerning proposals and/or solutions can be sent e-mail to <eisenbt@013.net>. Solutions to previously stated problems can be seen at <http://www.ssma.org/publications>.

Solutions to the problems stated in this issue should be posted before
June 15, 2015

- **5349:** *Proposed by Kenneth Korbin, New York, NY*

  Given angle A with $\sin A = \dfrac{5}{13}$. A circle with radius 1 and a circle with radius x are each tangent to both sides of the angle. The circles are also tangent to each other. Find x.

- **5350:** *Proposed by Kenneth Korbin, New York, NY*

  The four roots of the equation

  $$x^4 - 96x^3 + 206x^2 - 96x + 1 = 0$$

  can be written in the form

  $$x_{1,2} = \left(\frac{\sqrt{a} + \sqrt{b + \sqrt{c}}}{\sqrt{a} - \sqrt{b + \sqrt{c}}}\right)^{\pm 1}$$

  $$x_{3,4} = \left(\frac{\sqrt{a} + \sqrt{b - \sqrt{c}}}{\sqrt{a} - \sqrt{b - \sqrt{c}}}\right)^{\pm 1}$$

  where a, b, and c are positive integers.

  Find a, b, and c if $(a, b, c) = 1$.

- **5351:** *Proposed by D.M. Bătinetu-Giurgiu, "Matei Basarab" National College, Bucharest, Romania and Neculai Stanciu, "George Emil Palade" School, Buzău, Romania*

  Let x, y, z be positive real numbers. Show that

  $$\frac{xy}{x^3 + y^3 + xyz} + \frac{yz}{y^3 + z^3 + xyz} + \frac{zx}{z^3 + x^3 + xyz} \leq \frac{3}{x + y + z}.$$

- **5352:** *Proposed by Arkady Alt, San Jose, CA*

  Evaluate $\sum\limits_{k=0}^{n} x^k - (x-1)\sum\limits_{k=0}^{n-1}(k+1)x^{n-1-k}$.

5353: *Proposed by José Luis Díaz-Barrero, Barcelona Tech, Barcelona, Spain.*

Let $A(z) = \sum_{k=0}^{n} a_k z^k$ be a polynomial of degree n with complex coefficients. Prove that all its zeros lie in the disk $\mathcal{D} = \{z \in \mathbb{C} : |z| < r\}$, where

$$r = \left\{ 1 + \left(\sum_{k=0}^{n-1} \left| \frac{a_k}{a_n} \right|^3 \right)^{1/2} \right\}^{2/3}$$

- **5354:** *Proposed by Ovidiu Furdui, Technical University of Cluj-Napoca, Cluj-Napoca, Romania*

Let $a, b, c > 0$ be real numbers. Prove that the series

$$\sum_{n=1}^{\infty} \left[n \cdot \left(a^{\frac{1}{n}} - \frac{b^{\frac{1}{n}} + c^{\frac{1}{n}}}{2} \right) - \ln \frac{a}{\sqrt{bc}} \right],$$

converges if and only if $2\ln^2 a = \ln^2 b + \ln^2 c$.

Solutions

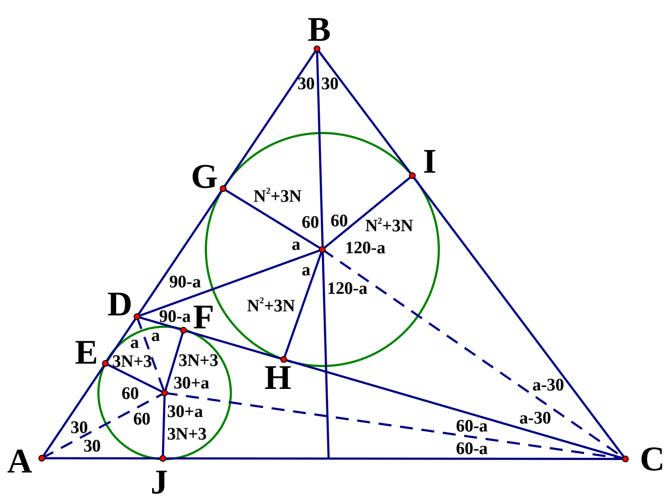
- **5331:** *Proposed by Kenneth Korbin, New York, NY*

Given equilateral $\triangle ABC$ with cevian $\overline{CD}$. Triangle ACD has inradius $3N+3$ and $\triangle BCD$ has inradius N^2+3N where N is a positive integer.

Find lengths $\overline{AD}$ and $\overline{BD}$.

Solution 1 by Ed Gray, Highland Beach, FL

Referring to the diagram, we can derive an equation which relates N and the angle a defined as the bisector of $\angle ADO$. (Points O and P are the centers of the incircles.



It is seen from the diagram that $IC = AG = AE + ED + DG$.

1) $\tan 30 = \dfrac{3N+3}{AE}$, so

2) $AE = \sqrt{3}(3N+3)$

3) $\tan a = \dfrac{3N+3}{ED}$, so

4) $ED = \dfrac{3N+3}{\tan a}$

5) $\tan a = \dfrac{DG}{N^2+3N}$

6) $DG = (N^2+3N)\tan a$

Adding 2), 4), and 6)

7) $AG = (3N+3)\sqrt{3} + \dfrac{3N+3}{\tan a} + \left(N^2+3N\right)\tan a$

To evaluate IC we note:

8) $\tan(a-30) = \dfrac{N^2+3N}{IC}$, or

9) $IC = \dfrac{N^2+3N}{\tan(a-30)}$

Equating 7) and 9) gives the basic equation:

10) $(3N+3)\left(\sqrt{3}+\dfrac{1}{\tan a}\right) + \left(N^2+3N\right)\tan a = \dfrac{N^2+3N}{\tan(a-30)}$.

We expand $\tan(a-30)$

11) $\tan(a-30) = \dfrac{\tan a - \tan 30}{1+\tan a \tan 30} = \dfrac{\tan a - \sqrt{3}/3}{1+\sqrt{3}/3\tan a} = \dfrac{3\tan a - \sqrt{3}}{3+\sqrt{3}\tan a}$. So,

12) $(3N+3)\dfrac{1+\sqrt{3}\tan a}{\tan a} = \left(N^2+3N\right)\left(\dfrac{3+\sqrt{3}\tan a}{3\tan a - \sqrt{3}} - \tan a\right)$.

There is no way to eliminate all of these irrationals except to let:

13) $\tan a = r\sqrt{3}$, where r is, for now, unspecified. Making this substitution, eq-12) becomes:

14) $(3N+3)\dfrac{1+\sqrt{3}r\sqrt{3}}{r\sqrt{3}} = (N^2+3N)\left(\dfrac{3+3r}{3r\sqrt{3}-\sqrt{3}} - r\right)$.

Step 14) simplifies to

15) $\dfrac{(3N+3)(1+3r)}{r} = \dfrac{\left(N^2+3N\right)(9r^2-6r-3)}{(1-3r)}$ and dividing by 3

16) $\dfrac{(N+1)(1+3r)}{r} = (N^2+3N)\dfrac{(3r+1)(r-1)}{1-3r}$, and dividing by $3r+1$

17) $\dfrac{N+1}{r} = (N^2+3N)\dfrac{1-r}{3r-1}$, and simplifying gives

18) $(N+1)(3r-1) = (N^2+3N)(r-r^2)$.

Writing step 18) as a quadratic in r, we obtain,

19) $\left(N^2+3N\right)r^2+\left(3-N^2\right)r-(N+1)=0$, with solution

20) $2(N^2+3N)r=(N^2-3)+\sqrt{(N^2-3)^2+(4N+4)(N^2+3N)}$. The discriminat D^2 is:

21) $D^2=N^4+4N^3+10N^2+12N+9=\left(N^2+2N+3\right)^2$. So

22) $D=N^2+2N+3$, and equation 20) becomes

23) $2(N^2+3N)r=N^2-3+N^2+2N+3$

24) $2(N^2+3N)r=2N^2+2N$

25) $r=\dfrac{N^2+N}{N^2+3N}=\dfrac{N+1}{N+3}$

Then the value of $\tan a$ becomes

26) $\tan a=\dfrac{N+1}{N+3}\sqrt{3}$. So,

Finally, $AD=AE+ED$. So,

27) $AD=(3N+3)\sqrt{3}+(N+3)\sqrt{3}=2\sqrt{3}(2N+3)$, and

Similarly, $DB=DG+GB=2\sqrt{3}\left(N^2+2N\right)$.

Solution 2 by Bruno Salgueiro Fanego, Viveiro, Spain

Let I and J be respectively the inradius of $\triangle ACD$ and $\triangle BCD$, E and F be the tangent points of the incircles of $\triangle ACD$ and $\triangle BCD$ with $\overline{AB}$, respectively, $a=\overline{AB}=\overline{BC}=\overline{CA},\ d=\overline{CD},\ x=\overline{AD},\ e=\overline{AE}$ and $f=\overline{BF}$.

By the cosine theorem in $\triangle ACD$, $d^2=a^2+x^2-2ax\cos(\pi/3)$, d=$\sqrt{a^2-ax+x^2}$.

Since $\overline{AE}$ is a segment of the tangent from A to the incircle of $\triangle ACD$, whose semiperimeter is $\dfrac{a+x+d}{2}$, $e=\dfrac{a+x+d}{2}-d=\dfrac{a+x-d}{2}$ and analogously, $f=\dfrac{2a-x-d}{2}$; on the other hand, in $\triangle IAE$ we have that $\angle IAE=\angle(DAC/2)=\pi/6$, and $IE\perp AD$, so $e=(3N+3)\cot(\pi/6)=3\sqrt{3}(N+1)$ and analogously $f=\sqrt{3}N(N+3)$.

Thus, $a+x-\sqrt{x^2+a^2-xa}=6\sqrt{3}\,(N+1)$ and
$2a-x-\sqrt{x^2+a^2-xa}=2\sqrt{3}N\,(N+3)$.

Subtracting the first equation from the second one, we obtain that $a=2x+2\sqrt{3}\left(N^2-3\right)$, and isolating the square root and squaring the first equation we obtain that

$$(a+x)^2-121\sqrt{3}(N+1)(a+x)+108(N+1)^2 \quad = \quad x^2+a-xa, \text{ or equivalently}$$

$$ax-4\sqrt{3}(N+1)(a+x)+36(N+1)^2 \quad = \quad 0$$

Substituting here the obtained value of a as a function of x we deduce that $x^2+\sqrt{3}(N^2-6N-9)x-12N^2+6N^2+72N+54=0$, which is a quadratic equation with solutions

$$
\begin{aligned}
x &= \frac{1}{2}\left(-\sqrt{3}\left(N^2-6N-9\right) \pm \sqrt{3\left(N^2-6N-9\right)^2 - 4\left(-12N^2+6N^2+72N+54\right)}\right) \\
&= \frac{1}{2}\left(-\sqrt{3}\left(N^2-6N-9\right) \pm \sqrt{3\left(N^2+2N+3\right)^2}\right) \\
&= \frac{\sqrt{3}}{2}\left(-N^2+6N+9 \pm \left(N^2+2N+3\right)\right) \in \left\{-\sqrt{3}(N+1)(N-3),\ (2\sqrt{3}(2N+3)\right\},
\end{aligned}
$$

from where, being $a = 2x + 2\sqrt{3}(N^2-3)$, we deduce that $a \in \left\{4\sqrt{3}N,\ 2\sqrt{3}\left(N+1\right)\left(N+3\right)\right\}$, respectively, that is, $\overline{AD} = 2\sqrt{3}\left(2N+3\right)$ and $\overline{BD} = 2\sqrt{3}N\left(N+2\right)$.

Note that N is a positive integer, so the first case would be possible if $(N+1)(3-N)$ and $(N-1)(N+3)$ are positive, which is impossible, hence, $\overline{AD} = 2\sqrt{3}(2N+3)$ and $BD = 2\sqrt{3}N\left(N+2\right)$.

Solution 3 by Kee-Wai Lau, Hong Kong, China

We show that $\left(\overline{AD}, \overline{BD}\right) = \left(2\sqrt{3}(2N+3), 2\sqrt{3}N(N+2)\right)$.

Let $\overline{AD} = x$ and $\overline{BD} = y$ so that $\overline{AC} = \overline{BC} = x+y$. The area of $\triangle ACD = \dfrac{x(x+y)\sin 60^\circ}{2} = \dfrac{\sqrt{3}x(x+y)}{4}$ and the area of $\triangle BCD = \dfrac{\sqrt{3}y(x+y)}{4}$.

Applying the cosine formula to $\triangle ACD$, we obtain $\overline{CD} = \sqrt{x^2+xy+y^2}$.

Since the area of a triangle equals the product of its semiperimeter with its inradius, so

$$
\begin{aligned}
\frac{\sqrt{3}x(x+y)}{2\left(2x+y+\sqrt{x^2+xy+y^2}\right)} &= N+3, \qquad (1) \text{ and} \\
\frac{\sqrt{3}y(x+y)}{2\left(x+2y+\sqrt{x^2+xy+y^2}\right)} &= N^2+3N. \qquad (2)
\end{aligned}
$$

Since the left side of (1) equals $\dfrac{\sqrt{3}\left(2x+y-\sqrt{x^2+xy+y^2}\right)}{6}$ and the left side of (2) equals $\dfrac{\sqrt{3}\left(x+2y-\sqrt{x^2+xy+y^2}\right)}{6}$, so we obtain respectively from (1) and (2) that

$$
\begin{aligned}
\sqrt{x^2+xy+y^2} &= 2x+y-6\sqrt{3}(N+1) \qquad (3) \text{ and} \\
\sqrt{x^2+xy+y^2} &= x+2y-2\sqrt{3}N(N+3) \qquad (4)
\end{aligned}
$$

From (3) and (4), we obtain $y = x + 2\sqrt{3}\left(N^2 - 3\right)$. Substituting y back into (3), squaring and simplifying, we obtain,

$x^2 + \sqrt{3}\left(N^2 - 6N - 9\right)X - 12N^3 + 6N^2 + 72N + 54 = 0$. Hence either

$x = 2\sqrt{3}(2N+3),\ y = 2\sqrt{3}N(N+2)$ or $x = \sqrt{3}(3-N)(1+N),\ y = \sqrt{3}(N-1)(N+3)$.

Since only the former solution satisfies (3) and (4), so we obtain the claimed solution.

Also solved by Albert Stadler, Herrliberg, Switzerland, and the proposer.

- **5332:** *Proposed by Tom Moore, Bridgewater State University, Bridgewater, MA*

Inspired by the prime number 1000000000000066600000000000001, known as *Belphegor's prime* where there are thirteen consecutive zeros to the left and right of 666, we consider the numbers $100\ldots0201500\ldots01$ where there are $k-$zeros left and right of 2015. For $k < 28$ only $k = 9$ and $k = 27$ yield prime numbers.

(a) Prove that the sequence $120151,\ 10201501,\ 1002015001, \ldots$ has an infinite subsequence of all composite numbers.

(b) Find the next prime in both the sequences $100\ldots066600\ldots01$ and $100\ldots0201500\ldots01$, after the ones noted above.

Solution 1 by Albert Stadler, Herrliberg, Switzerland

(a) The sequence can be expressed as $a_k = 10^{2k+5} + 2015\cdot 10^{k+1} + 1,\ k = 0, 1, \ldots,$ where k denotes the number of consecutive zeros to the right and to the left of 2015.

We note that $1000 \equiv -1 \pmod{13},\ 2015 \equiv 0 \pmod{13},\ 10^{13} \equiv 1 \pmod{53}$. So

$$a_{3n+2} = 10^{3(2n+3)} + 2015\cdot 10^{3n+3} + 1 \equiv 1000^{2n+3} + 2015\cdot 10^{3n+3} + 1 \equiv -1 + 0 + 1 = 0 \pmod{13},$$

$$a_{13n} = 10^{26n+5} + 2015\cdot 10^{13n+1} + 1 \equiv 10^5 + 20150 + 1 \equiv 0 \pmod{53},$$

$$a_{13n+8} = 10^{26n+21} + 2015\cdot 10^{13n+9} + 1 \equiv 10^8 + 2015\cdot 10^9 + 1 \equiv 0 \pmod{53}.$$

So there are infinitely many indices k for which a_k is composite.

(b) Tom Moore is wrong in saying that

$$1\underbrace{0\ldots0}_{9\ zeros}2015\underbrace{0\ldots0}_{9\ zeros}1 \text{ and}$$

$$1\underbrace{0\ldots0}_{27\ zeros}2015\underbrace{0\ldots0}_{27\ zeros}1$$

are primes. The correct statement is that

$$1\underbrace{0\ldots0}_{7\ zeros}2015\underbrace{0\ldots0}_{7\ zeros}1 \text{ and}$$

$$1\underbrace{0\ldots0}_{25\ zeros}2015\underbrace{0\ldots0}_{25\ zeros}1$$

are primes.

Let $b_k = 10^{2k+4} + 666 \cdot 10^{k+1} + 1$. Then b_{13} is Belphegore's prime. Using the PrimeQ function of Mathematica we find that

- b_{42} is prime,
- b_k is composite for $14 \le k \le 41$,
- a_k is composite for$0 \le k \le 7000,$ except for $k = 7$ and $k = 25$.

I was not able to find a $k > 25$ for which a_k is prime.

Solution 2 by Pat Costello, Eastern Kentucky University, Richmond, KY

(a) The number 2015 is divisible by 13 and so starting with 1002015001, every third number in the sequence is divisible by 13 (the leading 1 is a $10^{3(2x+3)} \equiv -1 \ (\text{mod}13)$ which cancels with the final 1).

(b) The next primes in the sequence $100\ldots066600\ldots01$ are when then the number of zeroes is $k = 42$ and $k = 506$ (probably prime according to *Mathematica*).

In the sequence $100\ldots0201500\ldots-1$, I believe the k values that give primes should be $k = 7$ and $k = 25$ (not 9 and 27) and *Mathematica* did not find any more primes (or probably primes) in the sequence with $k < 2000$.

Solution 3 by Ashland University Undergraduate Problem Solving Group, Ashland, OH

a) We begin by noting $a_k = 10^{5+2k} + 2015(10^{k+1}) + 1$ is an explicit formula for the number with k-zeros to the left and right of 2015.

Suppose $k \equiv 2 \pmod 3$ so $k = 3n+2$ for some integer n. Then
$a_{3n+2} = 10^{6n+9} + 2015(10^{3n+3}) + 1$. Since $2015 \equiv 0 \pmod{13}$, we have
$a_{3n+2} \equiv 10^{6n+9} + 1 \pmod{13}$. Thus $a_{3n} \equiv (10^3)^{2n+3} + 1 \pmod{13}$. Note
$10^3 = 1000 \equiv -1 \pmod{13}$ and clearly $2n+3$ is odd, so
$a_{3n+2} \equiv (-1)^{2n+3} + 1 \equiv -1 + 1 \equiv 0 \pmod{13}$ and hence $13 \big| a_{3n+2}$ and a_{3n+2} is composite. Thus the subsequence $\{a_{_n}\}$ where $k_n = 3n+2$ for $n = 0, 1, 2, 3, \ldots$ is an infinite subsequence of all composite numbers.

b) For the sequence $10\ldots0666001,\ a_k = 10^{2k+4} + 666(10^{k+1}) + 1$ and we used MAPLE to find that the next prime occurs when $k = 42$, .i.e., there are 42 zeros to the left and right of 666. (The only additional primes in this sequence with $k \le 1000$ occur when $k = 506$ and $k = 608$).

For the sequence $10\ldots020150\ldots01,\ a_k = 10^{5+2k} + 2015(10^{k+1}) + 1$ and were unable to find the next prime in the sequence, using MAPLE to check all terms with $k \le 7000$ were composite.

Also solved by Brain D. Beasley, Presbyterian College, Clinton, SC; Ed

Gray, Highland Beach, FL; Haroun Meghaichi (student, University of Science and Technology, Houari Boumediene) Algeria, and the proposer.

- **5333:** *Proposed by Paolo Perfetti, Department of Mathematics, Tor Vergata Roma University, Rome, Italy.*

Evaluate

$$\int_{-\pi/2}^{\pi/2} \frac{\left(\ln\left(1+\tan x+\tan^2 x\right)\right)^2}{1+\sin x\cos x} dx.$$

Solution 1 by Kee-Wai Lau, Hong Kong, China

Denote the integral of the problem by I. We show that

$$I = \frac{2\sqrt{3}\pi\left(\pi^2+3\ln^2 3\right)}{9}. \qquad (1)$$

Let $J = \int_o^{\pi/2} \ln(\cos x)dx$ and $K = \int_0^{\pi/2} \ln^2(\cos x)dx$. It is known [1], p. 531, section 4.224, entries 6 and 8) that

$$J = \frac{-\pi\ln 2}{2} \qquad (2), \text{ and}$$

$$K = \frac{-\pi(\pi^2+12\ln^2 2)}{24}. \qquad (3)$$

By means of the substitution $\tan x = \dfrac{\sqrt{3}\tan y - 1}{2}$, we see that

$$I = \frac{2}{\sqrt{3}} \int_{-\pi/2}^{\pi/2} \ln^2\left(\frac{3\sec^2 y}{4}\right) dy = \frac{4}{\sqrt{3}} \int_0^{\pi/2} \ln^2\left(\frac{3\sec^2 y}{4}\right) dy.$$

Since $\ln^2\left(\dfrac{3\sec^2 y}{4}\right) = \ln^2\left(\dfrac{3}{4}\right) - 4\ln\left(\dfrac{3}{4}\right)\ln(\cos y) + 4\ln^2(\cos y)$, so

$$I = \frac{4}{\sqrt{3}}\left(\ln^2\left(\frac{3}{4}\right)\frac{\pi}{2} - 4\ln\left(\frac{3}{4}\right)J + 4K\right).$$

Using (2) and (3), we obtain (1).

Reference

1. I.S. Gradshteyn and I.M. Ryzhik: *Table of Integrals, Series, and Products,* seventh edition, Elsevier, Inc. 2007.

Solution 2 by Albert Stadler, Herrliberg, Switzerland

We claim that the integral equals $\dfrac{2\pi(\pi^2+3\ln^2 3)}{3\sqrt{3}}$.

We perform a change of variables and put $y = \tan x$. The $dy = \dfrac{1}{\cos^2 x} dx = (1+y^2)dx$ and

$$I = \int_{-\pi/2}^{\pi/2} \frac{\left(\ln\left(1+\tan x+\tan^2 x\right)\right)^2}{1+\sin x\cos x} dx = \int_{-\infty}^{\infty} \frac{\left(\ln\left(1+y+y^2\right)\right)^2}{1+\dfrac{y}{1+y^2}} \frac{dy}{1+y^2} = \int_{-\infty}^{\infty} \frac{\left(\ln\left(1+y+y^2\right)\right)^2}{1+y+y^2} dy =$$

$$= \int_{-\infty}^{\infty} \frac{\left(\ln\left(1+\left(y-\frac{1}{2}\right)+\left(y-\frac{1}{2}\right)^2\right)\right)^2}{1+\left(y-\frac{1}{2}\right)+\left(y-\frac{1}{2}\right)^2} dy = \int_{-\infty}^{\infty} \frac{\left(\ln\left(\frac{3}{4}+y^2\right)\right)^2}{\frac{3}{4}+y^2} dy = 2\int_0^{\infty} \frac{\left(\ln\left(\frac{3}{4}+y^2\right)\right)^2}{\frac{3}{4}+y^2} dy.$$

Put $f(s) = 2\displaystyle\int_0^{\infty} \frac{1}{\left(\frac{3}{4}+y^2\right)^s} dy$ for $\Re(s) > \dfrac{1}{2}$.

We evaluate $f(s)$ in terms of the beta function

$B(x,y) = \displaystyle\int_0^1 t^{x-1}(1-t)^{y-1}\,dt = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}$, by performing a change of variables in the defining integral of $f(s)$. Letting $z = \dfrac{1}{1+y^2}$, $y = \sqrt{\dfrac{1}{z}-1}$, $dy = \dfrac{-1}{2z^2\sqrt{\dfrac{1}{z}-1}} dz$ we obtain

$$f(s) = 2\int_0^{\infty} \frac{1}{\left(\frac{3}{4}+y^2\right)^s} dy = 2\sqrt{\frac{3}{4}}\int_0^{\infty} \frac{1}{\left(\frac{3}{4}+\frac{3}{4}y^2\right)^s} dy = 2\left(\frac{3}{4}\right)^{\frac{1}{2}-s}\int_0^{\infty} \frac{1}{(1+y^2)^s} dy =$$

$$= \left(\frac{3}{4}\right)^{\frac{1}{2}-s}\int_0^1 z^{s-\frac{3}{2}}\frac{1}{\sqrt{1-z}} dz =$$

$$= \left(\frac{3}{4}\right)^{\frac{1}{2}-s}\frac{\Gamma\left(s-\frac{1}{2}\right)\Gamma\left(\frac{1}{2}\right)}{\Gamma(s)} = \frac{\sqrt{3\pi}}{2}\left(\frac{4}{3}\right)^s\frac{\Gamma\left(s-\frac{1}{2}\right)}{\Gamma(s)},$$

where we have used that $\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$.

We have $\dfrac{d^2}{ds^2}\dfrac{1}{\Gamma(s)} = \dfrac{d}{ds}\dfrac{-\Gamma'(s)}{\Gamma^2(s)} = -\dfrac{\Gamma''(s)}{\Gamma^2(s)} + 2\dfrac{\left(\Gamma'(s)\right)^2}{\Gamma^3(s)}$,

$$\frac{\frac{d^2}{ds^2}u(s)v(s)w(s)}{u(s)v(s)w(s)} = \frac{u''(s)}{u(s)} + \frac{v''(s)}{v(s)} + \frac{w''(s)}{w(s)} + 2\frac{u'(s)v'(s)}{u(s)v(s)} + 2\frac{v'(s)w'(s)}{v(s)w(s)} + 2\frac{w'(s)u'(s)}{w(s)u(s)}.$$

So

$$I = f''(1) = \frac{\sqrt{3\pi}}{2}\frac{4}{3}\Gamma\left(\frac{1}{2}\right)\ln^2\left(\frac{4}{3}\right) + \frac{\sqrt{3\pi}}{2}\frac{4}{3}\Gamma''\left(\frac{1}{2}\right) +$$

$$+\frac{\sqrt{3}\pi}{2}\frac{4}{3}\Gamma''\left(\frac{1}{2}\right)\left(-\Gamma''(1)+2\left(\Gamma'(1)\right)^2\right)+2\frac{\sqrt{3}\pi}{2}\frac{4}{3}\Gamma'\left(\frac{1}{2}\right)\ln\left(\frac{4}{3}\right)$$

$$+2\frac{\sqrt{3}\pi}{2}\frac{4}{3}\Gamma'\left(\frac{1}{2}\right)\left(-\Gamma'(1)\right)+2\frac{\sqrt{3}\pi}{2}\frac{4}{3}\ln\left(\frac{4}{3}\right)\Gamma\left(\frac{1}{2}\right)\left(-\Gamma'(1)\right) \qquad (1)$$

To evaluate $\Gamma'(1)$, $\Gamma''(1)$, $\Gamma'\left(\frac{1}{2}\right)$, $\Gamma''\left(\frac{1}{2}\right)$ we use the well known equations,

$$\frac{\Gamma'(z)}{\Gamma(z)}=\frac{1}{z}+\gamma+\sum_{n\geq 1}\left(\frac{1}{n+z}-\frac{1}{n}\right).$$

$$\frac{\Gamma''(z)}{\Gamma(z)}-\left(\frac{\Gamma'(z)}{\Gamma(z)}\right)^2=\sum_{n\geq 0}\frac{1}{(n+z)^2},$$

from which we deduce

$$(i)\quad \Gamma(1) \;=\; -\gamma,$$

$$(ii)\quad \Gamma''(1) \;=\; \gamma^2+\sum_{n\geq 0}\frac{1}{(n+1)^2}=\gamma^2+\frac{\pi^2}{6},$$

$$\begin{aligned}(iii)\quad \Gamma'\left(\frac{1}{2}\right) &= -\Gamma\left(\frac{1}{2}\right)\left(2+\gamma+\sum_{n\geq 1}\left(\frac{1}{n+\frac{1}{2}}-\frac{1}{n}\right)\right)=-\sqrt{\pi}\left(2+\gamma+2\sum_{n\geq 1}\left(\frac{1}{2n+1}-\frac{1}{2n}\right)\right)=\\ &= -\sqrt{\pi}\left(\gamma+2\ln 2\right),\end{aligned}$$

$$\begin{aligned}(iv)\quad \Gamma'\left(\frac{1}{2}\right) &= -\Gamma\left(\frac{1}{2}\right)\left(\left(\frac{\Gamma'\left(\frac{1}{2}\right)}{\Gamma\left(\frac{1}{2}\right)}\right)^2+\sum_{n\geq 0}\frac{4}{(2n+1)^2}\right)=\\ &= \sqrt{\pi}\left((\gamma+2\ln 2)^2+4\left(\sum_{n\geq 0}\frac{1}{n^2}-\sum_{n\geq 0}\frac{1}{4n^2}\right)\right)=\sqrt{\pi}\left((\gamma+2\ln 2)^2+\frac{\pi^2}{2}\right).\end{aligned}$$

We plug $(i)-(iv)$ into (1) and get

$$I \;=\; \frac{2}{3}\sqrt{3}\pi\ln^2\left(\frac{4}{3}\right)+\frac{2}{3}\sqrt{3}\pi\left((\gamma+2\ln 2)^2+\frac{\pi^2}{2}\right)+\frac{2}{3}\sqrt{3}\pi\left(\gamma^2-\frac{\pi^2}{6}\right)+$$

$$- \frac{4}{3}\sqrt{3}\pi\left(\gamma + 2\ln 2\right)\ln\left(\frac{4}{3}\right) - \frac{4}{3}\sqrt{3}\pi\gamma\left(\gamma + 2\ln 2\right) + \frac{4}{3}\sqrt{3}\pi\gamma\ln\left(\frac{4}{3}\right) =$$

$$= \frac{2\pi\left(\pi^2 + 3\ln^2 3\right)}{3\sqrt{3}}.$$

Comment by editor. The numerical answer to this problem can be approximated to whatever degree of accuracy one wishes by piecing together various integrating techniques for power series expansions over specific domains and for estimating the area under the graph of a positively valued curve. This method of computing the value of the integral was employed by **Ed Gray of Highland Beach, FL** *in his 10 page solution that gave him a numerical answer that was correct to several decimal places. But as one can see from the above solutions, the problem was not as straight-forward as I had initially thought.*

This problem was also solved by its proposer.

- **5334:** *Proposed by José Luis Díaz-Barrero, Barcelona Tech, Barcelona, Spain*

Let $x_{ij},\ (1 \le i \le m, 1 \le j \le n)$ be nonnegative real numbers. Prove that

$$\prod_{j=1}^{n}\left(1 - \prod_{i=1}^{m}\frac{\sqrt{x_{ij}}}{1+\sqrt{x_{ij}}}\right) + \prod_{i=1}^{m}\left(1 - \prod_{j=1}^{n}\frac{1}{1+\sqrt{x_{ij}}}\right) \ge 1.$$

Solution by Kee-Wai Lau, Hong Kong, China

For $1 \le i \le m,\ 1 \le j \le n$, let y_{ij} be real numbers satisfying $0 \le y_{ij} \le 1$. We prove by induction on $m+n$ that

$$\prod_{j=1}^{n}\left(1 - \prod_{i=1}^{m} y_{ij}\right) + \prod_{i=1}^{m}\left(1 - \prod_{j=1}^{n}\left(1 - y_{ij}\right)\right) \ge 1. \qquad (1)$$

For $m+n=2$, we have $m=n=1$, and (1) becomes an equality. So suppose that (1) holds for $m+n=k\ge 2$. We now consider $m+n=k+1$.

Denote the left side of (1) by $f\left(y_{mn}\right)$. Then

$$f\left(y_{mn}\right) \ge \prod_{j=1}^{n}\left(1 - \prod_{i=1}^{m} y_{ij}\right) + \prod_{i=1}^{m}\left(1 - \prod_{j=1}^{n-1}\left(1 - y_{ij}\right)\right), \qquad (2)$$

and

$$f\left(y_{mn}\right) \ge \prod_{j=1}^{n}\left(1 - \prod_{i=1}^{m-1} y_{ij}\right) + \prod_{i=1}^{m}\left(1 - \prod_{j=1}^{n}\left(1 - y_{ij}\right)\right). \qquad (3)$$

Here we assign the value 1 to any empty products. From (2), we obtain by the induction assumption that

$$f(0) \geq \prod_{j=1}^{n-1}\left(1-\prod_{i=1}^{m} y_{ij}\right)+\prod_{i=1}^{m}\left(1-\prod_{j=1}^{n-1}(1-y_{ij})\right) \geq 1, \qquad (4)$$

and from (3), we obtain by the induction assumption that

$$f(1) \geq \prod_{j=1}^{n}\left(1-\prod_{i=1}^{m-1} y_{ij}\right)+\prod_{i=1}^{m-1}\left(1-\prod_{j=1}^{n}(1-y_{ij})\right) \geq 1. \qquad (5)$$

Since $f(y_{mn})$ is a polynomial in y_{mn} with degree 0 or 1, so from (4) and (5), we see that $f(y_{mn}) \geq 1$, and (1) holds also for $m+n=k+1$. Hence (1) holds in general and the inequality of the problem follows by the substitution $y = \dfrac{\sqrt{x_{ij}}}{1+\sqrt{x_{ij}}}$.

Also solved by Bruno Salgueiro Fanego, Viveiro, Spain; Ethan Gegner (student), Taylor University, Upland, IN; Albert Stadler, Herrliberg, Switzerland, and the proposer.

- **5335:** *Proposed by Arkady Alt, San Jose, CA*

Prove that for any real $p > 1$ and $x > 1$ that

$$\frac{\ln x}{\ln(x+p)} \leq \left(\frac{\ln(x+p-1)}{\ln(x+p)}\right)^p.$$

Solution 1 by Ethan Gegner (student), Taylor University, Upland, IN

The weighted AM-GM inequality, followed by Jensen's inequality applied to the concave function $\ln x$ yields

$$\begin{aligned}
(\ln x)^{1/p}(\ln(x+p))^{\frac{p-1}{p}} &\leq \frac{1}{p}\ln x + \frac{p-1}{p}\ln(x+p) \\
&\leq \ln\left(\frac{1}{p}x + \frac{p-1}{p}(x+p)\right) \\
&= \ln(x+p-1).
\end{aligned}$$

Exponentiation by p and then rearranging yields the desired result.

Solution 2 by Bruno Salgueiro Fanego, Viveiro, Spain

The inequality is true for any real $p \geq 1$ and $x > 1$, because

$$\left(\frac{\ln(x+p-1)}{\ln(x+p)}\right)^p - \frac{\ln x}{\ln(x+p)} \geq 1 + p\left(\frac{\ln(x+p-1)}{\ln(x+p)} - 1\right) - \frac{\ln x}{\ln(x+p)}$$

$$= \frac{p\ln\left(\dfrac{x+p-1}{x+p}\right) - \ln\left(\dfrac{x}{x+p}\right)}{\ln(x+p)}$$

$$= \frac{qy\ln(1-y^{-1}) - \ln(1-q)}{\ln y}$$

$$= \frac{q}{\ln y}\left(-\sum_{k=1}^{\infty} k^{-1}y^{1-k} + \sum_{k=1}^{\infty} k^{-1}q^{k-1}\right)$$

$$= \frac{q}{\ln y}\sum_{k=1}^{\infty} k^{-1}\left(q^{k-1} - y^{1-k}\right) \geq 0,$$

where we have used Bernoulli's inequality

$$(1+t)^p \geq 1 + pt \text{ for } t = \frac{\ln(x+p-1)}{\ln(x+p)} - 1 \geq -1.$$

Note that $p \geq 1, x > 1 \Rightarrow x+p-1 > 1, x+p > 1 \Rightarrow \ln(x+p-1), \ln(x+p) > 0$, the notation $y = x+p$ and $q = \dfrac{p}{y}$, the series expansion $\ln(1-u) = -\sum_{k=1}^{\infty} k^{-1}u^k$ for $u = y^{-1}$ and $u = q$ (observe that $0 < y^{-1}, q < 1$) and the fact that $q \geq y^{-1}$ with equality iff $p = 1 \Rightarrow q^{k-1} \geq \left(y^{-1}\right)^{k-1}$ for any integer $k \geq 1$.

Moreover, equality is attained iff it occurs in Bernouilli's inequality and in the inequality $q \geq y^{-1}$. Since there is equality in this last inequality iff $p = 1$ and in this case also in Bernoulli's inequality, we conclude that equality occurs iff $p = 1$.

Solution 3 by Paul M. Harms, North Newton, KS

All logarithms involved with the inequality are positive. Then the inequality is correct if the logarithm of the left side is less than the logarithm of the right side. Taking the natural logarithm of both sides and dividing by p the problem inequality is equivalent t o

$$\frac{\ln\ln x - \ln\ln(x+p)}{p} \leq \frac{\ln\ln(x+p-1) - \ln\ln(x+p)}{1},$$

Let $f(x) = \ln\ln x$ where $x > 1$. Multiplying both sides of the inequality by (-1) we can write the resulting inequality as

$$\frac{f(x+p) - f(x)}{(x+p) - x} \geq \frac{f(x+p) - f(x+p-1)}{(x+p) - (x+p-1)},$$

forms often associated with the Mean Value Theorem for derivatives.

Let the following letters and points be associated with each other:

$$A\left(x, f(x)\right),\ B\left((x+p), f(x+p)\right),\ C\left((x+p), f(x)\right),$$

$$E\left((x+p-1), f(x+p-1)\right),\ F\left((x+p), f(x+p-1)\right),$$

and let D be intersection of the line segment between A and B with the line segment between E and F.

Consider the right triangle $\triangle BEF$ and the similar right triangles $\triangle ABC$ and $\triangle DBF$. The left side of the last inequality is the ratio of the distances $\dfrac{BC}{AC} = \dfrac{BF}{DF}$ and the right side equals the ratio $\dfrac{BF}{EF}$.

Since $f'(x) = \dfrac{1}{x\ln x} > 0$, and $f''(x) = \dfrac{-1\,(1+\ln x)}{(x\ln x)^2} < 0$ for $x > 1$, the line segment from A to B is below the graph of $y = f(x)$. Point D then satisfies the distance inequality $DF < EF$ so we have $\dfrac{BF}{DF} \geq \dfrac{BF}{EF}$. The problem inequality is correct.

Solution 4 by Nicusor Zlota, "Traian Vuia" Technical College, Focsani, Romania

The inequality in the statement of the problem is equivalent to

$$\frac{\ln x}{\ln(x+p)} \leq \left(\frac{\ln(x+p-1)}{\ln(x+p)}\right)^p \iff \ln\left(\ln\left(x+p\right)\right)^{p-1} \leq \left(\ln\left(x+p-1\right)\right)^p. \qquad (*)$$

Knowing that $\ln x > 0$ and using the AM-GM inequality, we have:

$$\ln x\left(\ln\left(x+p\right)\right)^{p-1} \leq \left(\frac{\ln x + (p-1)\ln(x+p)}{p}\right)^p = \left(\ln\sqrt[p]{x(x+p)^{p-1}}\right)^p \leq \left(\ln(x+p-1)\right)^p$$

for every $p > 1$ and $x > 1$. Using the fact that $\ln x$ is an increasing function, we deduce that $(*)$ is true and also the equivalent inequality in the statement of the problem.

Also solved by Ed Gray, Highland Beach, FL; Kee-Wai Lau, Hong Kong, China; Haroun Meghaichi (student, University of Science and Technology, Houari Boumediene), Algiers, Algeria; Paolo Perfetti, Department of Mathematics, Tor Vergata University, Rome, Italy; Albert Stadler, Herrliberg, Switzerland, and the proposer.

- **5336:** *Proposed by Ovidiu Furdui, Technical University of Cluj-Napoca, Cluj-Napoca, Romania*

Caculate:

$$\sum_{k=1}^{\infty}\left(1+\frac{1}{2}+\cdots+\frac{1}{k}-\ln\left(k+\frac{1}{2}\right)-\gamma\right).$$

Solution 1 by Perfetti Paolo, Department of Mathematics, Tor Vergata University, Rome, Italy

The first item we employ is

$$\sum_{k=1}^{n}\frac{1}{k} = \ln n + \gamma_n,\ \gamma_n = \gamma + o(1), \qquad n(\gamma_n - \gamma) \to 1/2.$$

The second item we use is the content of problem 1781 of Mathematics Magazine, vol.80–5, 2007.

Rearranging the sum up to n we get

$$\sum_{k=1}^{n}\frac{n}{k}-\sum_{k=1}^{n-1}\frac{k}{k+1}-\sum_{k=1}^{n}\ln\left(k+\frac{1}{2}\right)-n\gamma=$$
$$=n(\lg n+\gamma_n)-\sum_{k=1}^{n-1}\left(1-\frac{1}{k+1}\right)-\ln\prod_{k=1}^{n}\frac{2k+1}{2}-n\gamma=$$
$$=n\ln n+n(\gamma_n-\gamma)-(n-1)+\ln n+\gamma_n-1-\ln\frac{(2n+1)!}{2^{2n}n!}$$

Stirling's formula $n!=(n/e)^n\sqrt{2\pi n(1+o(1))}$ and $\ln(1+x)\sim x$ for $x\to 0$ yields

$$n\ln n+n(\gamma_n-\gamma)-n+\ln n+\gamma_n-(2n+1)\ln(2n+1)+(2n+1)+$$
$$-\frac{1}{2}\ln(2\pi(2n+1))+o(1)+2n\ln 2+n\ln n-n+\frac{1}{2}\ln(2\pi n)+o(1)$$

$(2n+1)\ln(2n+1)=(2n+1)(\ln 2+\ln n+o(\frac{1}{n})=2n\ln n+2n\ln 2+\ln n+\ln 2+o(1).$

The sum becomes

$$n(\gamma_n-\gamma)+n\ln n(1-2+1)+n(-1-2\ln 2+2\ln 2+2-1)+$$
$$+\ln n(1-1-\frac{1}{2}+\frac{1}{2})+(\gamma_n-\ln 2-\frac{1}{2}\ln(4\pi)+\frac{1}{2}\ln(2\pi)$$

and in the limit we obtain $\frac{1}{2}+\gamma-\frac{3}{2}\ln 2.$

Solution 2 by Anastasios Kotronis, Athens, Greece

Let

$$S_n:=\sum_{k=1}^{n}\left(1+\frac{1}{2}+\cdots+\frac{1}{k}-\ln\left(k+\frac{1}{2}\right)-\gamma\right).$$

Summing by parts we have

$$S_n=\sum_{k=1}^{n}(k+1-k)H_k-\ln\left(\prod_{k=1}^{n}\frac{2k+1}{2}\right)-n\gamma$$
$$=kH_k\Big|_1^{n+1}-\sum_{k=1}^{n}(k+1)\left(H_{k+1}-H_k\right)-\ln\left(\prod_{k=1}^{n}\frac{2k(2k+1)}{2^2k}\right)-n\gamma$$
$$=(n+1)\left(H_{n+1}-1\right)-\ln\left(\frac{(2n+1)!}{2^{2n}n!}\right)-n\gamma$$
$$\to\frac{1}{2}+\gamma-\frac{3\ln 2}{2}$$

by Stirling's approximation.

Solution 3 by Haroun Meghaichi, (student, University of Science and Technology Houari Boumediene), Algiers, Algeria.

Let H_n be the n-th harmonic number then for any integer $n > 1$ we have

$$\begin{aligned} a_n = \sum_{k=1}^{n} H_k &= \sum_{k=1}^{n} (k+1)H_{k+1} - kH_k - 1 \\ &= (n+1)\left(H_{n+1} - 1\right) \\ &= n\left(\ln n + \gamma - 1\right) + \ln n + \gamma + \frac{1}{2} + o(1). \end{aligned}$$

And

$$\begin{aligned} b_n &= \sum_{k=1}^{n} \ln\left(\frac{2k+1}{2}\right) = \ln\left(\frac{(2n+1)!!}{2^n}\right) \\ &= \ln\left(\frac{(2n+1)!}{4^n (n!)}\right) = \ln(2n+1)! - \ln n! - 2n\ln 2 \\ &= n\left(\ln n - 1\right) + \ln n + \frac{3}{2}\ln 2 + o(1). \end{aligned}$$

The last line comes directly from Stirling approximation, then we have

$$\sum_{k=1}^{n}\left(H_k - \ln\left(k + \frac{1}{2}\right) - \gamma\right) = a_n - b_n - n\gamma = \gamma + \frac{1}{2} - \frac{3}{2}\ln 2 + o(1)$$

Hence, the answer is $\boxed{\gamma + \frac{1}{2} - \frac{3}{2}\ln 2 = \frac{1}{2}\ln\frac{e^{2\gamma+1}}{8}}$.

Also solved by Bruno Salgueiro Fanego, Viveiro, Spain; Kee -Wai Lau, Hong Kong, China; Albert Stadler, Herrliberg, Switzerland, and the proposer.

Comments

Editor's note: The following comment was sent to me by **Henry Ricardo of the NY Math Circle.** In the March, 2015 solutions, Solution 1 of problem 5330 is incorrect. By throwing in the factor B(1), the solvers have replaced the original problem by one whose solution is almost trivial. The proposer (Ovidiu Furdui) no doubt specified that the matrix product start with B(2) to make it more challenging. The extra factor does not provide a generalization or extension but, rather, a simplification that is contrary to the spirit of the problem as proposed.

Solution 1 was solved by looking at a few examples, guessing a general form for the product and then proving the product held by induction. I thought it was a nice simple way to solve the problem. Henry disagreed. So I sent his comment on to Ovidiu Furdui, the proposer of the problem and asked him if the published solution were on a test, would he give full credit. Here is his response.

The reader is right, solution 2 is the correct one. On one hand, the problem asks for the calculation of the product starting from $B(2)$ up to $B(n)$, for $n \geq 2$ and in solution 1 basically the solvers have computed a product which simplifies very much the problem, so from a mathematical point of view the problem asks for one thing and the solvers give another. The product $A(n) = B(1)B(2)\cdots B(n)$ as they give it is correct but this is not what the problem asks for. (Me, Ted, speaking again; I don't see it this way– as I see it, they did answer the question. Now back to Ovidiu.)

On the other hand, to answer your question, if this problem would have been an exam problem and the student(s) would have solved the problem as in solution 1, then certainly I would give partial credit for this solution, but not full credit due to the fact that, strictly speaking the solution is not what the problem asks for. However, I would offer partial credit to the student for calculating the product A(n) (for observing its form and for proving that by induction) but not full credit.

Solution 2 is the correct solution of this problem.

(Editor again:) But still I wasn't satisfied that the solution was incorrect, and so I explained the solution to Michael Fried, and he agreed with Henry and with Ovidui. His reasoning was that the authors of the Solution 1 had changed the initial conditions of the sequence by saying that the sequence started with B(1) and not B(2). But I argued that the authors of Solution 1 stated in their argument, "we have shown, by mathematical induction that (1) holds for all integers $n \geq 2$," and again I felt that that they had shown that. To my way of thinking, we had the product of matrices $B(1)B(2)\cdots B(n)$. The authors of Solution 1 could obtain the correct answer by a simple translation. I also thought that they could obtain the answer by multiplying the product by the inverse of B(1), and therein I made a mistake. Matrix B(1) is not invertible. Anyway, at this point the score was two against me, nobody for me. I then sent the question (Was the published solution 1 incorrect?) to Albert Stadler, and he agreed with the others, and he pointed out my mistake that matrix B(1) was not invertible. The score was now 3-0, and I am now siding with the majority.

Solution 1 to 5332 misses the spirit of the intended problem; once again, mea culpa.
