Problems

Ted Eisenberg, Section Editor

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This section of the Journal offers readers an opportunity to exchange interesting mathematical problems and solutions. Please send them to Ted Eisenberg, Department of Mathematics, Ben-Gurion University, Beer-Sheva, Israel or fax to: 972-86-477-648. Questions concerning proposals and/or solutions can be sent e-mail to <eisenbt@013.net>. Solutions to previously stated problems can be seen at <http://www.ssma.org/publications>.

Solutions to the problems stated in this issue should be posted before June 15, 2019

5541: *Proposed by Kenneth Korbin, New York, NY*

A convex cyclic quadrilateral has inradius r and circumradius R. The distance from the incenter to the circumcenter is 169. Find positive integers r and R.

5542: *Proposed by Michel Bataille, Rouen, France*

Evaluate in closed form: $\cos\dfrac{\pi}{13}+\cos\dfrac{3\pi}{13}-\cos\dfrac{4\pi}{13}$.
(Closed form means that the answer should not be expressed as a decimal equivalent.)

5543: *Proposed by Titu Zvonaru, Comănesti, Romania*

Let $ABDC$ be a convex quadrilateral such that
$\angle ABC=\angle BCA=25^\circ, \angle CBD=\angle ADC=45^\circ$. Compute the value of $\angle DAC$. (Note the order of the vertices.)

5544: *Proposed by Seyran Brahimov, Baku State University, Masalli, Azerbaijan*

Solve in $\Re$:

$$\begin{cases}\tan^{-1}x= & \tan y+\tan z\\ \tan^{-1}y= & \tan x+\tan z\\ \tan^{-1}z= & \tan x+\tan y\end{cases}$$

5545: *Proposed by José Luis Díaz-Barrero, Barcelona Tech, Barcelona, Spain*

Let p,q be two twin primes. Show that

$$1+4\left(\sum_{j=1}^{\frac{p-1}{2}}\left\lfloor\frac{jq}{p}\right\rfloor+\sum_{k=1}^{\frac{q-1}{2}}\left\lfloor\frac{kp}{q}\right\rfloor\right)$$

is a perfect square and determine it. (Here $\lfloor x\rfloor$ represents the integer part of x).

5546: *Proposed by Ovidiu Furdui and Alina Sîntămărian, Technical University of Cluj-Napoca, Cluj-Napoca, Romania*

Calculate
$$\sum_{n=1}^{\infty}(-1)^{\lfloor \frac{n}{2} \rfloor}\left(\mathrm{e}^x - 1 - \frac{x}{1!} - \frac{x^2}{2!} - \cdots - \frac{x^n}{n!}\right).$$

Solutions

5523: *Proposed by Kenneth Korbin, New York, NY*

For every prime number P, there is a circle with diameter $4P^4+1$. In each of these circles, it is possible to inscribe a triangle with integer length sides and with area $(2P)(2P+1)(2P-1)(2P^2-1)$. Find the sides of the triangles if $P=2$ and if $P=3$.

Solution 1 by Ed Gray, Highland Beach, FL

Case 1. $P=2$. Then Area $= 4\cdot 5\cdot 3\cdot 7 = 2^2\cdot 3^1\cdot 5^1\cdot 7^1 = 420$.
By Brahmaguptas formula, $A^2 = s(s-a)(s-b)(s-c)$, where a, b, and c are the sides, and s is the semi-perimeter. We note that $(s-a)+(s-b)+(s-c) = 3s-2s = s$. So we seek a factor, s, and three other factors whose sum is s.
$A^2 = (2^4)(3^2)(5^2)(7^2) = 2\cdot 2\cdot 2\cdot 2\cdot 3\cdot 3\cdot 5\cdot 5\cdot 7\cdot 7$. A discerning eye sees that
$49 = 20+20+9$, so
$s = 7\cdot 7$
$s-a = 2\cdot 2\cdot 5$, so $a = 49-20 = 29$.
$s-b = 2\cdot 2\cdot 5$, so $b = 49-20 = 29$.
$s-c = 3\cdot 3$, so $c = 49-9 = 40$.
Each side is less than $4P^4+1 = 65$, and the triangle inequality holds.

Case 2. $P=3$. The area $= 6\cdot 5\cdot 7\cdot 17 = (2^1)(3^1)(5^1)(7^1)(17^1) = 3570$.
$A^2 = s(s-a)(s-b)(s-c) = (2^2)(3^2)(5^2)(7^2)(17^2) = 2\cdot 2\cdot 3\cdot 3\cdot 5\cdot 5\cdot 7\cdot 7\cdot 17\cdot 17$.
Let $s = 2\cdot 3\cdot 5\cdot 7 = 210$. Then
$s-a = 7\cdot 17 = 119,\ a = 210-119 = 91$.
$s-b = 5\cdot 17 = 85,\ b = 210-85 = 125$.
$s-c = 2\cdot 3 = 6, \quad c = 210-6 = 204$.
Each side is less than $4P^4+1 = 325$, and the triangle inequality holds.

Solution 2 by David E. Manes, Oneonta, NY

Given triangle $\triangle ABC$ with side lengths a, b and c opposite the respective vertices A, B and C. Moreover, assume that the triangle has area $[ABC] = (2P)(2P+1)(2P-1)(2P^2-1)$ and is inscribed in a circle with diameter $4P^4+1$, where P is a prime. If $P=2$, then the area $[ABC] = 4\cdot 5\cdot 3\cdot 7 = 420$ and the circle has diameter $4\cdot 2^4+1 = 65$. Therefore, the radius R of the circumscribed circle has value $R = 32.5$. The formula relating the radius R, the area $[ABC]$ and the side lengths a, b and c is $R = abc/(4[ABC])$. With $R = 32.5$, $[ABC] = 420$, one obtains

$abc = 4R[ABC] = 4(32.5)(420) = 54600 = 2^3 \cdot 3 \cdot 5^2 \cdot 7 \cdot 13$. Using the prime factorization of 54600, we then assign values to a, b and c so that $[ABC] = 420$. If $a = 3 \cdot 13 = 39$, $b = 5^2 = 25$ and $c = 2^3 \cdot 7 = 56$, then the semi-perimeter s of $\triangle ABC$ is given by $s = (a+b+c)/2 = (39+25+56)/2 = 60$ and Heron's formula for the area yields

$$[ABC] = \sqrt{s(s-a)(s-b)(s-c)} = \sqrt{60 \cdot 21 \cdot 35 \cdot 4} = 420.$$

Accordingly, if $P = 2$, then the triangle with integer length sides 25, 39 and 56 is inscribed in a circle with diameter $4P^4 + 1 = 65$ and has area $(2P)(2P+1)(2P-1)(2P^2-1) = 420$.

If $P = 3$, then $\triangle ABC$ has area $[ABC] = (2P)(2P+1)(2P-1)(2P^2-1) = 6 \cdot 7 \cdot 5 \cdot 17 = 3570$ and is inscribed in a circle with diameter $4P^4 + 1 = 4 \cdot 3^4 + 1 = 325$, whence the radius R of the circumscribed circle is $R = 162.5$. Therefore, the product of the side lengths a, b and c satisfies the equation $abc = 4R[ABC] = 4(162.5)(3570) = 2320500 = 2^2 \cdot 3 \cdot 5^3 \cdot 7 \cdot 13 \cdot 17$. For this case, let $a = 2^2 \cdot 3 \cdot 17 = 204$, $b = 5^3 = 125$ and $c = 7 \cdot 13 = 91$. Then the semi-perimeter $s = (a+b+c)/2 = (204+125+91)/2 = 210$ so that the area of $\triangle ABC$ is given by

$$[ABC] = \sqrt{s(s-a)(s-b)(s-c)} = \sqrt{210 \cdot 6 \cdot 85 \cdot 119} = 3570.$$

Therefore, if $P = 3$, then the triangle with integer side lengths 91, 125 and 204 is inscribed in a circle with diameter $4P^4 + 1 = 325$ and the triangle $\triangle ABC$ has area $[ABC]$ given by $[ABC] = (2P)(2P+1)(2P-1)(2P^2-1) = 3570$.

Solution 3 by David Stone and John Hawkins, Georgia Southern University, Statesboro, GA

The condition that P be prime is not necessary.

Note: This problem is similar to SSM Problem 5356 published May 2015. Our solution is based on Brian Beasley's solution (December 2015) to that problem.

We show that for a positive integer P, the triangle with sides given by

$$\begin{aligned} a &= (2P+1)\left(2P^2 - 2P + 1\right) = 4P^3 - 2P^2 + 1 \\ b &= (2P-1)\left(2P^2 + 2P + 1\right) = 4P^3 + 2P^2 - 1 \\ c &= 4P\left(2P^2 + 1\right) = 8P^3 - 4P \end{aligned}$$

has area $2P(2P+1)(2P-1)\left(2P^2-1\right)$ and can be inscribed in a circle with diameter $4P^4 + 1$.

In particular:

For $P = 2$, the sides of the triangle are $25, 39$ and 56; the diameter of the circle is 65 and the area of the triangle is 420.

For $P = 3$, the sides of the triangle are $91, 125$ and 204; the diameter of the circle is 325 and the area of the triangle is 3570.

We do not know whether our formula produces all such triangles. We used a computer program to determine that it does produce the unique triangle for each positive integer P from 1 through 12.

SOLUTION:

We let a, b, and c be the sides of the triangle, A its area, and R its circumradius. It is known that R is given by $R = \dfrac{abc}{4A}$.

Thus since we are given $A = 2P(2P+1)(2P-1)\left(2P^2-1\right)$ and diameter $4P^4+1$, we have

$$\begin{aligned} abc &= 4AR = 4 \cdot 2P(2P+1)(2P-1)\left(2P^2-1\right)\frac{4P^4+1}{2} \\ &= 4P(2P+1)(2P-1)\left(2P^2-1\right)\left(4P^4+1\right) \\ &= 4P(2P+1)(2P-1)\left(2P^2-1\right)\left(2P^2+2P+1\right)\left(2P^2-2P+1\right). \end{aligned}$$

We found a, b, c (as given above) by judiciously selecting the above factors of abc so that $1 \le a, b, c \le 2R = 4P^4+1$ and the sum of any two of them exceeds the third.

It is easy to verify that our a, b, and c are ≥ 1.

We must show that a, b, c satisfy the requirements of the problem. Note that

$$\begin{aligned} a+b+c &= 16P^3 - 4P = 4P(2P-1)(2P+1); \\ a+b-c &= 4P > 0, so\ a+b > c; \\ a+c-b &= 8P^3 - 4P^2 - 4P + 2 = 2(2P-1)\left(2P^2-1\right) > 0, \text{so } a+c > b; \\ b+c-a &= 8P^3 + 4P^2 - 4P - 2 = 2(2P+1)\left(2P^2-1\right) > 0, \text{so } b+c > a. \end{aligned}$$

This shows that a, b, c do form a triangle. It also puts us in position to calculate the area by Herons Formula;

$$\begin{aligned} A^2 &= \frac{1}{16}(a+b+c)(a+b-c)(a+c-b)(b+c-a) \\ &= \frac{1}{16}4P(2P-1)(2P+1)(4P)\left[2(2P-1)\right]\left(2P^2-1\right)\Big]\left[2(2P+1)\left(2P^2-1\right)\right] \\ &= 4P^2(2P-1)^2(2P+1)^2\left(2P^2-1\right)^2. \end{aligned}$$

Therefore, $A - 2P(2P-1)(2P_1)(2P^2-1)$, as desired.

Finally, we calculate the diameter of the circumscribed circle:

$$\begin{aligned} D &= 2R = \frac{abc}{2A} = \frac{4P(2P+1)(2P-1)\left(2P^2-1\right)\left(2P^2+2P+1\right)\left(2P^2-2P+1\right)}{4P(2P-1)(2P+1)\left(2P^2-1\right)} \\ &= \left(2P^2+2P+1\right)\left(2P^2-2P+1\right) = 4P^4+1, \text{as desired.} \end{aligned}$$

Because the sides a, b, c produce the appropriate circumradius, we know that the sides actually fit into the circle: each is $\le D$.

Here are the results for $P = 1, 2, \ldots, 12$. Each of these is the unique triangle satisfying the given conditions.

P	a	b	c	$Area$	$Diameter$
1	3	5	4	6	5
2	25	39	56	420	65
3	91	125	204	3570	325
4	225	287	496	15624	1025
5	451	549	980	48510	2501
6	793	935	1704	121836	5185
7	1275	1469	2716	264810	9605
8	1921	2175	4064	518160	16385
9	2755	3077	5796	936054	26245
10	3801	4199	7960	1588020	40001
11	5083	5565	10604	2560866	58565
12	6625	7199	13776	3960600	82945

Comment: There are other ways to factor $abc = 4P(2P+1)(2P-1)\left(2P^2-1\right)\left(2P^2+2P+1\right)\left(2P^2-2P+1\right)$ so the sides form a triangle, but which do not give the desired area. For example, with $P=2$, the sides $50, 39, 28$ form a triangle whose area is not the desired area (420).

Ditto for $35, 39, 40$.

Also these triangles do not have the desired circumradius of $4P^4+1$.

Also solved by Brian D. Beasley, Presbyterian College, Clinton, SC; Kee-Wai Lau, Hong Kong, China, and the proposer.

5524: *Proposed by Michael Brozinsky, Central Islip, NY*

A billiard table whose sides obey the law of reflection is in the shape of a right triangle ABC with legs of length a and b where $a > b$ and hypotenuse c. A ball is shot from the right angle and rebounds off the hypotenuse at point P on a path parallel to leg CB that hits CAat point Q. Find the ratio $\dfrac{\overline{AQ}}{\overline{QC}}$.

Solution 1 by Ed Gray, Highland Beach,FL

Usually in a triangle, especially right triangles, sides are labeled with small letters, and the vertices are labeled with capital letters, the same letter being used to designate a side being opposite a vertex. To make this problem work, the drawing must be as follows (not withstanding rotations). The right angle C is at lower right, the hypotenuse is c. Vertex A is North of C, and B is at the left of C. However, $\overline{AC} > \overline{BC}$ to accommodate the law of reflection. So, if $a > b, \overline{AC} = a$, and $\overline{BC} = b$.

Let D be a point on the hypotenuse such that $\overline{DC}$ is perpendicular to the hypotenuse. Point P is on the hypotenuse where the ball strikes and $\overline{BD} < \overline{BP}$, (i.e., P is between D and A). Let $\overline{PF}$ be the normal to the hypotenuse where F is a point on $\overline{AC}$.

Let $r =$ the angle of incidence $= CPF$. The angle of reflection $= r = \angle FPQ$. Since $\angle PQC$ is a right angle, then $\angle QCP = 90^\circ - 2r$. Note that $\angle PCD = r$ since $\overline{PF} \parallel \overline{CD}$ and alternate interior angles are equal.

Therefore,

$$\angle DCB + r + (90^\circ - 2r) = \angle ACB = 90^\circ, \text{so } \angle DCB = r,$$

and $\angle DBC = 90^\circ - r,\ \angle APQ = 90^\circ - r$ by corresponding angles, so $\angle BAC = r$.

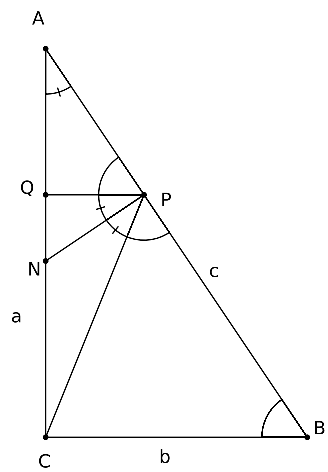
Then $\tan(\angle APQ) = \tan(90^\circ - r) = \dfrac{1}{\tan(r)} = \dfrac{\overline{AQ}}{\overline{PQ}}$, and $\tan(\angle QPC) = \tan(2r) = \dfrac{\overline{CQ}}{\overline{PQ}}$.

So, $\dfrac{\overline{AQ}}{\overline{CQ}} = \dfrac{\dfrac{1}{\tan(r)}}{\tan(2r)} = \dfrac{1 - \tan^2(r)}{2\tan^2(r)}$. From $\triangle ABC, \tan(r) = \dfrac{b}{a}$. So, $\dfrac{\overline{AQ}}{\overline{CQ}} = \dfrac{a^2 - b^2}{2b^2}$.

Editor′s comment : Ed's comment that nonstandard labeling was being used in this problem is absolutely correct. I wrote to the proposer and he acknowledged the mix up, but stated that everything will still work out with standard notation but then we must state that $a < b$.

Solution 2 by Michel Bataille, Rouen, France

Since PQ is parallel to BC, we have $\angle PQA = 90^\circ$, hence $\angle QPA = B$. Let the perpendicular to AB at P intersect the line AC at N. Then $\angle NPC = \angle QPN = 90^\circ - B = A$ and $\angle ACP = \angle QCP = 90^\circ - 2A$. Thus, we must have $A \leq 45^\circ$ and so $B \geq 45^\circ \geq A$. Therefore the longest leg is $a = CA$ while $b = CB$ (see figure).
Now, $\angle PCB = 2A$ and $\angle CPB = B = \angle PBC$, from which we deduce $PB = 2CB\cos B = 2b\cos B = 2b \cdot \frac{b}{c} = \frac{2b^2}{c}$. It follows that $AP = c - \frac{2b^2}{c} = \frac{c^2 - 2b^2}{c} = \frac{a^2 - b^2}{c}$ and so $\frac{AP}{PB} = \frac{a^2 - b^2}{2b^2}$. Since PQ is parallel to BC, we have $\frac{AQ}{QC} = \frac{AP}{PB}$ and we can conclude that $\frac{AQ}{QC} = \frac{a^2 - b^2}{2b^2}$.



Also solved by Kenneth Korbin, NY, NY; David Stone and John Hawkins, Georgia Southern University Statesboro GA, and the proposer.

5525: *Proposed by Daniel Sitaru, National Economic College "Theodor Costescu", Drobeta Turnu-Severin, Mehedinti, Romania*

Find real values for x and y such that:

$$4\sin^2(x+y) = 1 + 4\cos^2 x + 4\cos^2 y.$$

Solution 1 by Albert Stadler, Herrliberg, Switzerland

Put $u = e^{2ix}, v = e^{2iy}$. Then the given equation reads as

$$\begin{aligned} 0 &= \left(e^{2ix+2iy} + e^{-2ix-2iy} - 2\right) + 1 + \left(e^{2ix} + e^{-2ix} + 2\right) + \left(e^{2iy} + e^{-2iy} + 2\right) = \\ &= u\frac{1}{uv} + u + \frac{1}{u} + v + \frac{1}{v} + 3 = \frac{(uv+u+1)(uv+v+1)}{uv}. \end{aligned}$$

So either $v = -\frac{1}{u} - 1$ or $\frac{1}{v} = -u - 1$. If x and y run through the real numbers v and $\frac{1}{v}$ represent circles in the complex plane with radius 1 and center 0, while $-u-1$ and $\frac{-1}{u} - 1$ represent circles with radius 1 and center -1. Therefore $(u,v) \in \left\{\left(e^{2\pi i/3}, e^{2\pi i/3}\right), \left(e^{-2\pi i/3}, e^{-2\pi i/3}\right)\right\}$ which translates to $x \equiv y \equiv \pm\frac{\pi}{3} (\text{mod } \pi)$.

Solution 2 by Michael C. Faleski, University Center, MI

Let's rewrite the statement of the problem using several trigonometric indentities. This leads to

$$4(\sin x \cos y + \sin x \cos y)^2 = 1 + 4\cos^2 x + 4\cos^2 y$$

$$4(\sin^2 x \cos^2 y + \sin^2 y \cos^2 x + 2\sin x \sin y \cos x \cos y) = 1 + 4\cos^2 x + 4\cos^2 y$$

$$4\left((1-\cos^2 x)\cos^2 y+\cos^2 x(1-\cos^2 y)+2\sin x\sin y\cos x\cos y\right)=1+4\cos^2 x+4\cos^2 y$$

$$-8\cos^2 x\cos^2 y+8\sin x\sin y\cos x\cos y=1$$

$$-8\left(\frac{1}{2}+\frac{1}{2}\cos(2x)\right)\left(\frac{1}{2}+\frac{1}{2}\cos(2y)\right)+2\sin 2x\sin 2y=1$$

$$-2(1+\cos 2x+\cos 2y+\cos 2x\cos 2y)+2\sin 2x\sin 2y=1$$

$$-2-2\cos 2x-2\cos 2y-2\cos 2x\cos 2y+2\sin 2x\sin 2y=1$$

$$-2\cos 2x-2\cos 2y-2(\cos 2x\cos 2y-\sin 2x\sin 2y)=3$$

$$\cos 2x+\cos 2y+\cos(2x+2y)=-\frac{3}{2}.$$

And now we use $\cos a=\cos b=2\cos\left(\frac{1}{2}(a+b)\right)\cos\left(\frac{1}{2}(a-b)\right)$
to produce $2\cos(x+y)\cos(x-y)+(2\cos^2(x+y)-1)=-\frac{3}{2}$,
and so we have $2\cos^2(x+y)+2\cos(x-y)\cos(x+y)+\frac{1}{2}=0$, or
$\cos^2(x+y)+\cos(x-y)\cos(x+y)+\frac{1}{4}=0$.
We will now use the quadratic formula to solve for $\cos(x+y)$.

$$\cos(x+y)=\frac{-\cos(x-y)\pm\sqrt{\cos^2(x-y)-1}}{2}.$$

As we are required to have real solutions, this means that
$\cos^2(x-y)-1\geq 0\longrightarrow\cos^2(x-y)\geq 1$. This condition is only true for
$\cos^2(x-y)=1\longrightarrow\cos(x-y)=1$.
Letting $y=x-a$, we find $\cos a=1\longrightarrow a=2n\pi,\forall n\in Z$.

$$\cos(x+y)=-\frac{\cos(x-y)}{2}=-\frac{1}{2}.$$

Since $y=\pm 2n\pi$, then for $0\leq x\leq 2\pi, x=y$. Hence, $\cos 2x=-\frac{1}{2}$, which leads to
$2x=\frac{2}{3}\pi,\frac{4}{3}\pi\longrightarrow x=\left(\frac{1}{3}\pi,\frac{2}{3}\pi\right)$. So, for $0\leq x,y\leq 2\pi$, $(x,y)=\left(\frac{1}{3}\pi,\frac{1}{3}\pi\right),\left(\frac{2}{3}\pi,\frac{2}{3}\pi\right)$.

Solution 3 by Bruno Salgueiro Fanego, Viveiro, Spain

$$4\sin^2((x+y)=1+4\cos^2 x+4\cos^2 y\iff 4\left(1-\cos^2(x+y)\right)=1+2\cos(2x)+2+2\cos(2y)$$

$$\iff\quad 4-4\cos^2(x+y)=5+4\cos\left(\frac{2x+2y}{2}\right)\cos\left(\frac{2x-2y}{2}\right)$$

$$\iff\quad 0=4-4\cos^2(x+y)+4\cos(x+y)\cos(x-y)+1$$

$$\iff \quad 0 = (2\cos(x+y) + \cos(x-y))^2 - \cos^2(x-y) + 1$$

$$\iff \quad 0 = (2\cos(x+y) + \cos(x-y))^2 + \sin^2(x-y)$$

$$\iff \quad 2\cos(x+y) + \cos(x-y) = 0 = \sin(x-y) \iff x - y = k\pi,\ k \ \in Z$$

$$\cos(x+y) + \cos(k\pi) = 0 \iff x - y = k\pi; \quad \cos(x+y) = \frac{(-1)^{k+1}}{2},\ k \in Z$$

$$\iff \quad x - y = k\pi;\ x + y = \arccos\frac{(-1)^{k+1}}{2},\ \in Z$$

$$\iff \quad x = \frac{1}{2}\left(\arccos\frac{(-1)^{k+1}}{2} + k\pi\right),\ y = \frac{1}{2}\left(\arccos\frac{(-1)^{k+1}}{2} - k\pi\right),\ k \in Z.$$

Solution 4 by Kee-Wai Lau, Hong Kong, China

Since $\sin(x+y) = \sin x \cos y + \cos x \sin y$, so the given equation is equivalent to $1 - 8\sin x \cos x \sin y \cos y + 8\cos^2 x \cos^2 y = 0$. Clearly $\cos x \neq 0$ and $\cos y \neq 0$. So dividing both sides of the last equation by $\cos^2 x \cos^2 y$, we obtain $\sec^2 x \sec^2 y - 8\tan x \tan y + 8 = 0$ or $(1 + \tan^2 x)(1 + \tan^2 y) - 8\tan x \tan y + 8 = 0$, or

$$(\tan x - \tan y)^2 + (\tan x \tan y - 3)^2 = 0.$$

Thus $\tan x = \tan y$ and $\tan x \tan y = 3$, so that $\tan x = \tan y = \sqrt{3}$ or $\tan x = \tan y = -\sqrt{3}$. It follows that

$$(x, y) = \left(\frac{\pi}{3} + m\pi, \frac{\pi}{3} + n\pi\right),\ \left(\frac{2\pi}{3} + m\pi, \frac{2\pi}{3} + n\pi\right),$$

where m and n are arbitrary integers.

Solution 5 by Ulrich Abel, Technische Hochschule Mittelhessen, Germany

Using $\cos(2x) = 2\cos^2(x) - 1 = 1 - 2\sin^2(x)$ we see that the equation

$$4\sin^2(x+y) = 1 + 4\cos^2(x) + 4\cos^2(y)$$

is equivalent to

$$0 = 3 + 2\cos(2x+2y) + 2\cos(2x) + 2\cos(2y) =: f(x, y).$$

Using $\sin(2a) + \sin(2b) = 2\sin(a+b)\cos(a-b)$ we obtain

$$\begin{aligned} \text{grad} f(x,y) &= -4 \cdot (\sin(2x+2y) + \sin(2x), \sin(2x+2y) + \sin(2y)) \\ &= -8 \cdot (\sin(2x+y)\cos y, \sin(x+2y)\cos x). \end{aligned}$$

Therefore, $\text{grad} f(x,y) = (0,0)$ happens if

- $2x = \pi \pmod{2\pi}$ and $2y = \pi \pmod{2\pi}$. The critical points $\left(\frac{2n+1}{2}\pi, \frac{2m+1}{2}\pi\right)$ with integers n, m satisfy

$$f\left(\frac{2n+1}{2}\pi, \frac{2m+1}{2}\pi\right) = 3 + 2 \cdot 1 + 2(-1)^{n+1} + 2(-1)^{m+1} > 0.$$

- $2x=\pi\,(\text{mod }2\pi)$ and $2x+y=0\,(\text{mod }\pi)$. The critical points $\left(\dfrac{2n+1}{2}\pi, m\pi-(2n+1)\pi\right)$ with integers n, m satisfy

$$f\left(\frac{2n+1}{2}\pi, m\pi-(2n+1)\pi\right)=3+2\cdot(-1)+2(-1)^{n+1}+2\cdot 1>0.$$

- 2y=π (mod 2π) and $x+2y=0$ (mod π) is symmetrical to the preceding case.
- $2x+y=0\,(\text{mod}\pi)$ and $x+2y=0$ (modπ). This implies $3x+3y=(n+m)\pi$ and $x-y=(n-m)\pi$ with integers n,m. We infer that $(x,y)=\dfrac{\pi}{3}(2n-m, 2m-n)$ are the remaining critical points of f.

$$\begin{aligned}
& f\left(\frac{2n-m}{3}\pi, \frac{2m-n}{3}\pi\right) \\
= \;& 3+2\cos\frac{2(n+m)\pi}{3}+2\cos\frac{(4n-2m)\pi}{3}+2\cos\frac{(4m-2n)\pi}{3} \\
= \;& 3+2\left(2\cos^2\frac{(n+m)\pi}{3}-1\right)+4\cos\frac{(n+m)\pi}{3}\cos(n-m)\pi \\
= \;& 1+4\cos^2\frac{N\pi}{3}+4(-1)^N\cos\frac{N\pi}{3}=\left(1+2(-1)^N\cos\frac{N\pi}{3}\right)^2\geq 0
\end{aligned}$$

with $N:=n+m$. Consequently, the function value is equal to zero iff N is not a multiple of 3.

In total, we have $f(x,y)\geq 0$ on R^2 and $f(x,y)=0$ if and only if $(x,y)=(2n-m, 2m-n)\dfrac{\pi}{3}$, for all integers n,m satisfying $n+m\neq 0$ (mod 3). The solutions of the above trigonometric identity are exactly the zeros of f.

Also solved by Hatef I. Arshagi, Guilford Technical Community College, Jamestown, NC; Michel Bataille, Rouen, France; Brian D. Beasley, Presbyterian College, Clinton, SC; Ed Gray, Highland Beach, FL; David E. Manes, Oneonta, NY; Adrian Naco, Polytechnic University, Tirana, Albania; Ioannis D. Sfikas, National and Kapodistrian University of Athens, Greece; David Stone and John Hawkins, Georgia Southern University, Statesboro, GA; Marian Ursărescu, "Roman Vodă College," Roman, Romania, and the proposer.

5526: *Proposed by Ioannis D. Sfikas, National and Kapodistrian University of Athens, Greece*

The lengths of the sides of a triangle are 12, 16 and 20. Determine the number of straight lines which simultaneously halve the area and the perimeter of the triangle.

Solution 1 by Albert Stadler, Herrliberg, Switzerland

We claim that there is exactly one straight line which simultaneously halves the area and the perimeter of the triangle.

If the line passes through the sides of length 12 and 16, and its intersection with side 12 is x units from the acute angle on that side, then the line cuts off a right triangle of base $12-x$

and height $12 + x$. The area of this triangle is $\dfrac{144 - x^2}{2}$. Setting this equal to 48, we would have $x = \pm 4\sqrt{3}$, but the construction of this line requires $0 \le x \le 4$, so there is no such line that cuts the triangle's area in half.

If the line passes through the sides of length 16 and 20, and its intersection with side 16 is x units from the acute angle on that side, then it cuts off a triangle with base x and height $\frac{3}{5}(24 - x)$. The area of this triangle is $\frac{1}{2}x\frac{3}{5}(24 - x) = \frac{3}{10}x(24 - x)$, which takes a maximum value $\frac{432}{10} < 48$ at $x = 12$, so no such line can cut the triangle's area in half.

The remaining case is a line through sides 12 and 20. Let the line intersect side 12 at a point x units from the right angle. Then it cuts off a triangle of base $12 - x$ and height $\frac{4}{5}(12 + x)$, which has area $\frac{2}{5}(144 - x^2)$. Setting this equal to 48, we find that $x = \pm 2\sqrt{6}$, but $0 \le x \le 8$ by the construction of the line, so we have one solution, $x = 2\sqrt{6}$.

Comment: this problem is not new. It was discussed (for instance) in an internet site called "Problem of the Month", run by the University of Regina in Regina, Saskatchewan, Canada.

The problem of the month of April 2012 stated
(see http://mathcentral.uregina.ca/mp/previous2011/apr12sol.php):

Recall that the incenter I of a triangle is the point where the three internal angle bisectors meet. Prove that any line through I that divides the area of the triangle in half also divides its perimeter in half; conversely, any line through I that divides the perimeter of the triangle in half also divides its area in half.

In the solution the problem editor referred to a theorem of Verena Haider which states that for any triangle ABC and any line l, l divides the area and the perimeter of $\triangle ABC$ in the same ratio if and only if it passes through the triangle's incenter. Furthermore the problem editor made the statement that it is not hard to prove that every triangle has exactly one, two, or three bisecting lines, and no other values are possible, and provided a few references.

Solution 2 by Adrian Naco, Polytechnic University, Tirana, Albania

Let be a right angle triangle ABC where $AB = 20, AC = 12, BC = 16$.

Case 1. The straight line intersect the sides AC and AB in the points M and N respectively. Let us sign $AM = x, AN = y$.
The area of the traiangle AMN (we sign the area of the triangle by $[AMN]$) is half the area of the triangle ABC ($[ABC]$), that is'

$$\begin{aligned} [AMN] = \frac{1}{2}[ABC] \quad &\Rightarrow \quad \frac{AM \cdot AN \cdot \sin\angle MAN}{2} = \frac{1}{2} \cdot \frac{AC \cdot AB \cdot \sin\angle CAB}{2} \\ &\Rightarrow \quad \frac{xy \sin\angle MAN}{2} = \frac{1}{2} \cdot \frac{12 \cdot 20 \cdot \sin\angle CAB}{2} \\ &\Rightarrow \quad xy = 120 \end{aligned}$$

Furthermore, the straight line MN halve the perimeter of the triangle ABC, that is, $x+y=24$. So, the lengths x, y of the respective sides AM and AN of the triangle AMN are roots of the following quadratic equation,

$$\begin{aligned} t^2-24t+120=0 \quad &\Rightarrow \quad \{x,y\}=\{10,14\} \\ &\Rightarrow \quad x=AM=10,\ \ y=AN=14 \end{aligned}$$

Case 2. The straight line intersect the sides BC and AB in the points M and N respectively. Let us sign $BM=x, BN=y$.
The area of the traiangle BMN is half the area of the triangle ABC, that is,

$$\begin{aligned} [BMN]=\frac{1}{2}[ABC] \quad &\Rightarrow \quad \frac{BM\cdot BN\cdot \sin\angle MBN}{2}=\frac{1}{2}\cdot\frac{BC\cdot AB\cdot \sin\angle CBA}{2} \\ &\Rightarrow \quad \frac{xy\sin\angle MBN}{2}=\frac{1}{2}\cdot\frac{16\cdot 20\cdot \sin\angle CAB}{2} \\ &\Rightarrow \quad xy=160 \end{aligned}$$

Furthermore, the straight line MN halve the perimeter of the triangle ABC that is, $x+y=24$. So, the lengths x, y of the respective sides BM and BN of the triangle BMN are roots of the following quadratic equation,

$$t^2-24t+160=0 \qquad \Leftrightarrow \qquad (t-12)^2+16=0$$

which have no solution. So this case is not possible. Case 3. The straight line intersect the sides AC and BC in the points M and N respectively. Let us sign $CM=x, CN=y$.
The area of the traiangle CMN is half the area of the triangle ABC, that is,

$$\begin{aligned} [CMN]=\frac{1}{2}[ABC] \quad &\Rightarrow \quad \frac{CM\cdot CN\cdot \sin\angle MCN}{2}=\frac{1}{2}\cdot\frac{AC\cdot BC\cdot \sin\angle ACB}{2} \\ &\Rightarrow \quad \frac{xy\sin\angle MCN}{2}=\frac{1}{2}\cdot\frac{12\cdot 16\cdot \sin\angle ACB}{2} \\ &\Rightarrow \quad xy=96 \end{aligned}$$

Furthermore, the straight line MN halve the perimeter of the triangle ABC, that is, $x+y=24$. So, the lengths x, y of the respective sides CM and CN of the triangle CMN are roots of the following quadratic equation,

$$\begin{aligned} t^2-24t+96=0 \quad &\Rightarrow \quad \{x,y\}=\{12-2\sqrt{7}, 12+2\sqrt{7}\} \\ &\Rightarrow \quad x=CM=12-2\sqrt{7},\ \ y=CN=12+2\sqrt{7} \end{aligned}$$

This case is not possible since $12+2\sqrt{7}>16=BC$.

Finally, the only possible case is when the straight line intersect the sides $AC=12$ and $AB=20$ in the respective points M and N such that $AM=10$ and $AN=14$.

Editor′s comment: The proposer, **Ioannis D. Sfikas of National and Kapodistrian University in Athens, Greece** accompanied his solution with an interesting discussion of the problem's history. He wrote the following:

Comments. An interesting issue arising from classical Euclidean geometry concerns the existence of lines called "equalizers" that bisect both the area and the perimeter of a triangle. The search for such lines can be seen as a trivial process, but this abstains from the real picture. The complete study concerning the special case of a triangle was conducted by Kontokostas (2010). The possibility of the existence of an equalizer that can be applied to an arbitrary planar shape is an important parameter. However, a general method may not exist in order to solve this problem.

In general, an equalizer can be applied to any body and that is a fact that came up from a useful topology theorem: the Ham-Sandwich Theorem, also called the Stone-Tukey Theorem (after Arthur H. Stone and John W. Tukey). The theorem states that, given $d \geq 2$ measurable solids in $\Re^d$, it is possible to bisect all of them in half with a single $(d-1)$-dimensional hyperplane. In other words, the Ham-Sandwich Theorem provides the following paraphrased statement: *Take a sandwich made of a slice of ham and two slices of bread. No matter where one places the pieces of the sandwich in the kitchen, or house, or universe, so long as one's knife is long enough one can cut all three pieces in half in only one pass.* Proving the theorem for $d = 2$ (known as the *Pancake Theorem)* is simple and can be found in Courant and Robbins (1996, p. 267).

In 1994, Alexander Shen, professor at the Independent University of Moscow, published in *The Mathematical Intelligencer* a selection of problems, known as "coffin problems," which were offered to "undesirable" applicants at the entrance examinations at the Department of Mechanics and Mathematics (Mekh-mat) of Moscow University at 1970s and 1980s. Four examinations were held at the Mekh-Mat: written math, oral math, literature essay composition, and oral physics (Frenkel, 2013, p. 28). These problems appear to resemble greatly with the Olympiad problems. It should be noted that these problems also differ from the Olympiad problems by being, in many cases, either false or poorly stated. Their solution does not require knowledge of a higher level of mathematics, but require, however, ingenuity, creativity and unorthodox attitudes. Solutions to these problems were thoroughly analyzed by Ilan Vardi (2005a, 2005b, 2005c).

The Mathematics Department of Moscow State University, the most prestigious mathematics school in Russia, had at that time been actively trying to keep Jewish students (and other "undesirables") from enrolling in the department (Vershik, 1994, p. 5). One of the methods they used for doing this was to give the unwanted students a different set of problems on their oral exam. These problems were carefully designed to have elementary solutions (so that the Department could avoid scandals) that were nearly impossible to find. Any student who failed to answer could be easily rejected, so this system was an effective method of controlling admissions. These kinds of math problems were informally referred to as "Jewish" problems or "coffins." Coffins is the literal translation from Russian (Khovanova and Radul, 2012, p. 815). These problems along with their solutions were, of course, kept as a secret, but Valera Senderov and his friends had managed to collect a list. In 1975, they approached us to solve these problems, so that they could train the Jewish students following these mathematical ideas. *Problem 5* of Shen's catalogue, which had been proposed by Podkolzin in 1978, states: *Draw a straight line that halves the area and perimeter of a triangle.* A solution was included in the first chapter of Mikhail Shifman's book (2005, pp. 50-51).

The Canadian Mathematical Olympiad is an annual premier national advanced mathematics competition sponsored by the Canadian Mathematical Society. In 1985, 17th Canadian Mathematical Olympiad was held, and the first problem was:

17th Canadian Mathematical Olympiad 1985, Problem 1
A triangle has sides 6, 8, 10. Show that there is a unique line, which bisects the area and the perimeter.

The solution to the above problem is given in detail by Doob (1993, p. 169). The same subject seems to appear as *Problem 9* at the Canadian mathematical magazine *Crux Mathematicorum* destined for students. Readers are invited to search for the number of equalizers included on a right triangle whose sides differ from those presented in Problem 1 (Woodrow, 1991, p. 72):

Problem 9, Crux Mathematicorum 1991

The lengths of the sides of a triangle are 3, 4 and 5. Determine the number of straight lines which simultaneously halve the area and the perimeter of the triangle.

A solution to the magazine's *Problem 9* was given by Michael Selby from the University of Windsor. A solution was also already given to *Problem 1* of the Canadian Mathematical Olympiad stating that the questioned right triangle contains only one equalizer. The solution of the particular problem doesn't abstain from *Problem 1.* A relative problem was also proposed by the Flemish Mathematical Olympiad in 2004 in Belgium. It states:

Flanders Mathematics Olympiad 2004, Problem 1

Consider a triangle with side lengths 501 m, 668 m, 835 m. How many lines can be drawn with the property that such a line halves both area and perimeter?

[1] Courant, Richard and Robbins, Herbert (1996). *What is Mathematics? An elementary approach to ideas and methods,* second edition. Oxford, England: Oxford University Press.

[2] Doob, Michael (1993).*The Canadian Mathematical Olympiad (1969-1993): celebrating the first twenty-five years.* Canadian Mathematical Society.

[3] Frenkel, Edward (2013). *Love and Math: the heart of hidden reality* . BasicBooks.

[4] Khovanova, Tanya and Radul, Alexey (2012). Killer problems. *The American Mathematical Monthly,* 119 (10): 815-823.

[5] Kodokostas, Dimitrios (2010). Triangle equalizers. *Mathematics Magazine,* 83 (2): 141-146.

[6] Shen, Alexander (1994). Entrance examinations to the Mekh-mat. *The Mathematical Intelligencer,* 16 (4): 6-10.

[7] Shifman, Mikhail (2005). *You failed your math test,* Comrade Einstein: adventures and misadventures of young mathematicians or test your skills in almost recreational mathematics. World Scientific.

[8] Vardi, Ilan (2005a). Mekh-mat entrance examination problems. In Shifman, Mikhail A. (2005). *You failed your math test,* Comrade Einstein: adventures and misadventures of young mathematicians or test your skills in almost recreational mathematics, pp. 22-95. World Scientific.

[9] Vardi, Ilan (2005b). Solutions to the year 2000 International Mathematical Olympiad. In Shifman, Mikhail A. (2005). *You failed your math test,* Comrade Einstein: adventures and misadventures of young mathematicians or test your skills in almost recreational mathematics, pp. 96-121. World Scientific.

[10] Vardi, Ilan (2005c). My role as an outsider. In Shifman, Mikhail A. (2005). *You failed your math test,* Comrade Einstein: adventures and misadventures of young mathematicians or test your skills in almost recreational mathematics, pp. 122-125. World Scientific.

[11] Vershik, Anatoly (1994). Admission to the mathematics faculty in Russia in the 1970s and 1980s. *Mathematical Intelligencer,* 16 (4): 4-5.

[12] Woodrow, Robert E. (March 1991). The Olympiad Corner: No 123. *Crux Mathematicorum,* 17(3), 65-74.

Also solved by Ed Gray, Highland Beach, FL; Kee-Wai Lau, Hong Kong, China; David Stone and John Hawkins, Georgia Southern University, Statesboro, GA, and the proposer.

5527: *Proposed by José Luis Díaz-Barrero, Barcelona Tech, Barcelona, Spain*

Let a, b and c be positive real numbers such that $a+b+c=3$. Prove that for all real $\alpha > 0$, holds:

$$\frac{1}{2}\left(\frac{1-a^{\alpha+1}b^{\alpha}}{a^{\alpha}b^{\alpha}}+\frac{1-b^{\alpha+1}c^{\alpha}}{b^{\alpha}c^{\alpha}}+\frac{1-c^{\alpha+1}a^{\alpha}}{c^{\alpha}a^{\alpha}}\right)$$
$$\leq \sqrt{\left(\frac{1-a^{\alpha+1}}{a^{\alpha}}+\frac{1-b^{\alpha+1}}{b^{\alpha}}+\frac{1-c^{\alpha+1}}{c^{\alpha}}\right)\left(\frac{1-a^{\alpha}b^{\alpha}c^{\alpha}}{a^{\alpha}b^{\alpha}c^{\alpha}}\right)}.$$

Editor's comment : A mistake was detected in the statement of the problem by **Michel Bataille of Rouen, France.** He noticed the following:

The inequality easily rewrites as

$$A := a^{\alpha}+b^{\alpha}+c^{\alpha}-3a^{\alpha}b^{\alpha}c^{\alpha} \leq B := 2\sqrt{(1-a^{\alpha}b^{\alpha}c^{\alpha})(a^{\alpha}b^{\alpha}+b^{\alpha}c^{\alpha}+a^{\alpha}c^{\alpha}-3a^{\alpha}b^{\alpha}c^{\alpha})}. \quad (1)$$

We take $a=\frac{1}{2},\ b=1,\ c=\frac{3}{2}$ and first consider the case $\alpha=2$. We obtain $A=1.8125$ and $B=\frac{\sqrt{154}}{8}=1.55....$, hence (1) does not hold.
In the case $\alpha=1$, we find $A=0.75$ and $B=\frac{\sqrt{2}}{2}=0.707...$, hence (1) does not hold.
In the case $\alpha=1/2$, $A=0.333..$ and $B=0.327..$, hence (1) does not hold.
However, we prove the reverse inequality in the case $\alpha=1$, that is,

$$3-3abc \geq 2\sqrt{(1-abc)(ab+bc+ca-3abc)}. \quad (2)$$

Since $3=a+b+c\geq 3\sqrt[3]{abc}$, we have $1-abc\geq 0$ and (2) will certainly holds if $3\sqrt{1-abc}\geq 2\sqrt{ab+bc+ca-3abc}$ or, squaring and arranging,

$$9+3abc-4(ab+bc+ca)\geq 0. \quad (3)$$

Now, From Schur's inequality $a(a-b)(a-c)+b(b-c)(b-a)+c(c-a)(c-b)\geq 0$, we obtain

$$(a+b+c)(ab+bc+ca)-3abc\leq(a+b+c)\left((a+b+c)^2-3(ab+bc+ca)\right)+6abc$$

or since $a+b+c=3$, $3(ab+bc+ca)-3abc\leq 3(9-3(ab+bc+ca))+6abc$, that is, $4(ab+bc+ca)\leq 9+3abc$ and (3) holds.
Perhaps the reverse inequality does hold when $\alpha>0,\alpha\neq 1$ but I have not been able to find a proof.

Editor again : With respect to the above, the solution to this problem remains open.

5528: *Proposed by Ovidiu Furdui, Technical University of Cluj-Napoca, Cluj-Napoca, Romania*

Let $a>0$. Calculate $\displaystyle\int_a^\infty\int_a^\infty\frac{\mathrm{d}x\mathrm{d}y}{x^6(x^2+y^2)}$.

Solution 1 by Albert Stadler, Herrliberg, Switzerland

We easily verify that

$$\int\int\frac{dxdy}{x^6(x^2+y^2)}=\frac{1}{6xy^5}-\frac{1}{18x^3y^3}+\frac{1}{30x^5y}+\frac{\arctan\frac{x}{y}}{6y^6}-\frac{\arctan\frac{y}{x}}{6x^6}+C.$$

Therefore,

$$\int_a^\infty\int_a^\infty\frac{dxdy}{x^6(x^2+y^2)}=\frac{13}{90a^6}.$$

Solution 2 by Michael C. Faleski, University Center, MI

We start by evaluating the y−integral using trigonometric substitution with $y=x\tan\theta, dy=x\sec^2\theta d\theta$ to give

$$\int_a^\infty\frac{dy}{x^6(x^2+y^2)}\to\int\frac{1}{x^6}\left(\frac{x}{x^2}\right)d\theta\to\frac{1}{x^7}\tan^{-1}\left(\frac{y}{x}\right)\Big|_a^\infty=\frac{1}{x^7}\left(\frac{\pi}{2}-\tan^{-1}\left(\frac{a}{x}\right)\right).$$

This quantity is now integrated with respect to x by braking it into two terms written as

$$\int_a^\infty\frac{\pi}{2}\frac{dx}{x^7}-\int_a^\infty\frac{\tan^{-1}\left(\frac{a}{x}\right)}{x^7}dx.$$

The first term evaluates as

$$\int_a^\infty\frac{\pi dx}{2x^7}=-\frac{\pi}{12}\frac{1}{x^6}\Big|_a^\infty=\frac{\pi}{12a^6}.$$

For the second term, we start with integration by parts using $u=-\tan^{-1}\left(\frac{a}{x}\right)\to du=\dfrac{a}{x^2+a^2}$ and $dv=\dfrac{1}{x^7}dx\to v=-\dfrac{1}{6x^6}$ which yields

$$\frac{\tan^{-1}\left(\frac{a}{x}\right)}{6x^6}\Big|_a^\infty-\left(-\frac{a}{6}\right)\int_a^\infty\frac{dx}{x^6(x^2+a^2)}=\left(0-\frac{\pi}{24a^6}\right)+\frac{a}{6}\int_a^\infty\frac{dx}{x^6(x^2+a^2)}.$$

For the last term, one approach would be to make a $u-$substitution 0f $x = a\tan\theta \to dx = a\sec^2\theta d\theta$ leading to

$$\frac{a}{6}a\int_a^\infty \frac{dx}{x^6(x^2+a^2)} \to \frac{a}{6}\int_{\pi/4}^{\pi/2} \frac{1}{a^8}\frac{a\sec^2\theta\ d\theta}{\tan^6\theta\sec^2\theta} =$$

We can use (which is easily shown using $\cot^2 x = (\csc 2x - 1)$ repeatedly) that

$$\int \cot^6 x dx = \frac{\cot^5 x}{5} + \frac{\cot^3 x}{3} - \frac{\cot x}{1} - x + C.$$

For our scenario, we have

$$\frac{1}{6a^6}\int_{\pi/4}^{\pi/2} \cot^6\theta d\theta = \frac{1}{6a^6}\left(-\frac{\cot^5\theta}{5} + \frac{\cot^3\theta}{3} - \frac{\cot\theta}{1} - \theta\right)\Bigg|_{\pi/4}^{\pi/2} = \frac{1}{6a^6}\left(\frac{1}{5} - \frac{1}{3} + 1 - \frac{\pi}{4}\right).$$

So finally, putting putting all of the numerical terms together yields:

$$\int_a^\infty\int_a^\infty \frac{dxdy}{x^6(x^2+y^2)} = \frac{\pi}{12a^6} - \frac{\pi}{24a^6} + \frac{1}{6a^6}\left(\frac{13}{15} - \frac{\pi}{4}\right) = \frac{13}{90a^6}.$$

Solution 3 by Ulrich Abel, Technische Hochschule Mittelhessen, Germany

We show that, for $a > 0$,

$$I := \int_a^\infty\int_a^\infty \frac{dxdy}{x^6\left(x^2+y^2\right)} = \frac{13}{90a^6}.$$

Integrating both sides of the identity

$$\frac{1}{x^6\left(x^2+y^2\right)} + \frac{1}{y^6\left(x^2+y^2\right)} = \frac{x^6+y^6}{x^6y^6\left(x^2+y^2\right)} = \frac{x^4-x^2y^2+y^4}{x^6y^6} = \frac{1}{x^2y^6} - \frac{1}{x^4y^4} + \frac{1}{x^6y^2}$$

we conclude that

$$\begin{aligned} 2I &= \int_a^\infty\int_a^\infty \left(\frac{1}{x^2y^6} - \frac{1}{x^4y^4} + \frac{1}{x^6y^2}\right)dxdy \\ &= \left(\frac{1}{x\cdot 5y^5} - \frac{1}{3x^3\cdot 3y^3} + \frac{1}{5x^5\cdot y}\right)\Bigg|_{x=a}^\infty\Bigg|_{y=a}^\infty \\ &= -\left(\frac{1}{5} - \frac{1}{9} + \frac{1}{5}\right)\frac{1}{a^6} = \frac{13}{45a^6}. \end{aligned}$$

Solution 4 by Brian Bradie, Christopher Newport University, Newport, News, VA

Let $a > 0$, n be a positive integer, and consider

$$\int_a^\infty\int_a^\infty \frac{dx\,dy}{x^n(x^2+y^2)}.$$

With the substitutions $u = x/a$ and $v = y/a$,

$$\begin{aligned}
\int_a^\infty \int_a^\infty \frac{dx\,dy}{x^n(x^2+y^2)} &= \frac{1}{a^n}\int_1^\infty \int_1^\infty \frac{du\,dv}{u^n(u^2+v^2)} \\
&= \frac{1}{a^n}\int_1^\infty \frac{1}{u^n}\left.\frac{\tan^{-1}(v/u)}{u}\right|_1^\infty du \\
&= \frac{1}{a^n}\int_1^\infty \frac{1}{u^{n+1}}\left(\frac{\pi}{2} - \tan^{-1}\frac{1}{u}\right) du \\
&= \frac{1}{a^n}\int_1^\infty \frac{1}{u^{n+1}}\tan^{-1} u\,du.
\end{aligned}$$

By integration by parts, we next find

$$\int_a^\infty \int_a^\infty \frac{dx\,dy}{x^n(x^2+y^2)} = \frac{1}{na^n}\left(\frac{\pi}{4} + \int_1^\infty \frac{u^{-n}}{1+u^2}\,du\right);$$

the substitution $u = 1/w$ then yields

$$\int_a^\infty \int_a^\infty \frac{dx\,dy}{x^n(x^2+y^2)} = \frac{1}{na^n}\left(\frac{\pi}{4} + \int_0^1 \frac{u^{n}}{1+u^2}\,du\right).$$

Let

$$I_n = \int_0^1 \frac{u^n}{1+u^2}\,du$$

Then

$$\begin{aligned}
I_1 &= \int_0^1 \frac{u}{1+u^2}\,du = \frac{1}{2}\ln 2; \\
I_2 &= \int_0^1 \frac{u^2}{1+u^2}\,du = \int_0^1 \left(1 - \frac{1}{1+u^2}\right) du = 1 - \frac{\pi}{4};
\end{aligned}$$

and, for $n > 2$,

$$I_n = \frac{1}{n-1} - \int_0^1 \frac{u^{n-2}}{1+u^2}\,du = \frac{1}{n-1} - I_{n-2}.$$

Thus,

$$\begin{aligned}
I_3 &= \frac{1}{2} - I_1 = \frac{1}{2} - \frac{1}{2}\ln 2; \\
I_4 &= \frac{1}{3} - I_2 = \frac{\pi}{4} - \frac{2}{3}; \\
I_5 &= \frac{1}{4} - I_3 = \frac{1}{2}\ln 2 - \frac{1}{4};\ \text{and} \\
I_6 &= \frac{1}{5} - I_4 = \frac{13}{15} - \frac{\pi}{4}.
\end{aligned}$$

Finally,

$$\begin{aligned}
\int_a^\infty \int_a^\infty \frac{dx\,dy}{x(x^2+y^2)} &= \frac{1}{a}\left(\frac{\pi}{4}+\frac{1}{2}\ln 2\right) \\
\int_a^\infty \int_a^\infty \frac{dx\,dy}{x^2(x^2+y^2)} &= \frac{1}{2a^2}\left(\frac{\pi}{4}+1-\frac{\pi}{4}\right) = \frac{1}{2a^2} \\
\int_a^\infty \int_a^\infty \frac{dx\,dy}{x^3(x^2+y^2)} &= \frac{1}{3a^3}\left(\frac{\pi}{4}+\frac{1}{2}-\frac{1}{2}\ln 2\right) \\
\int_a^\infty \int_a^\infty \frac{dx\,dy}{x^4(x^2+y^2)} &= \frac{1}{4a^4}\left(\frac{\pi}{4}+\frac{\pi}{4}-\frac{2}{3}\right) = \frac{1}{4a^4}\left(\frac{\pi}{2}-\frac{2}{3}\right) \\
\int_a^\infty \int_a^\infty \frac{dx\,dy}{x^5(x^2+y^2)} &= \frac{1}{5a^5}\left(\frac{\pi}{4}+\frac{1}{2}\ln 2-\frac{1}{4}\right) \\
\int_a^\infty \int_a^\infty \frac{dx\,dy}{x^6(x^2+y^2)} &= \frac{1}{6a^6}\left(\frac{\pi}{4}+\frac{13}{15}-\frac{\pi}{4}\right) = \frac{13}{90a^6}
\end{aligned}$$

Solution 5 by Kee-Wai Lau, Hong Kong, China

We show that the integral of the problem, denoted by I, equals $\dfrac{13}{90a^6}$.

Since

$$\int_a^\infty \frac{dy}{x^2+y^2} = \frac{1}{x}\left[\arctan\left(\frac{y}{x}\right)\right]_a^\infty = \frac{\arctan\left(\frac{x}{a}\right)}{x} \text{ for } x>0, \text{ so } I = \int_a^\infty \frac{\arctan\left(\frac{x}{a}\right)}{x^7}.$$

By the substitution $t = \dfrac{x}{a}$, we obtain $I = \dfrac{1}{a^6}\displaystyle\int_1^\infty \frac{\arctan t}{t^7}dt$. Integrating by parts, we obtain $I = \dfrac{\pi}{24a^6} + \dfrac{J}{6a^6}$, where $J = \displaystyle\int_1^\infty \frac{dt}{(1+t^2)t^6}$. We now substitute $t = \cot\theta$ to reduce J to the standard integral $\displaystyle\int_0^{\pi/4} \tan^6\theta d\theta$. which equals $\dfrac{13}{15} - \dfrac{\pi}{4}$. Hence our result for I.

Also solved by Michel Bataille, Rouen, France; Pat Costello, Eastern Kentucky University, Richmond, KY; Ed Gray, Highland Beach, FL; Ioannis D. Sfikas, National and Kapodistrian University, Athens, Greece, and the proposer.