Problems

Ted Eisenberg, Section Editor

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This section of the Journal offers readers an opportunity to exchange interesting mathematical problems and solutions. Please send them to Ted Eisenberg, Department of Mathematics, Ben-Gurion University, Beer-Sheva, Israel or fax to: 972-86-477-648. Questions concerning proposals and/or solutions can be sent e-mail to <eisenbt@013.net>. Solutions to previously stated problems can be seen at <http://www.ssma.org/publications>.

Solutions to the problems stated in this issue should be posted before
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5493: *Proposed by Kenneth Korbin, New York, NY*

Convex quadrilateral $ABCD$ is inscribed in a circle with diameter $\overline{AC} = 729$. Sides $\overline{AB}$ and $\overline{CD}$ each have positive integer length. Find the perimeter if $\overline{BD}$=715.

5494: *Proposed by Moshe Stupel, "Shaanan" Academic College of Education and Gordon Academic College of Education, and Avi Sigler, "Shaanan" Academic College of Education, Haifa, Israel*

If $a \geq b \geq c \geq d$ are the lengths of four segments from which an infinite number of convex quadrilaterals can be constructed, calculate the maximal product of the lengths of the diagonals in such quadrilaterals.

5495: *Proposed by D.M. Bătinetu-Giurgiu, "Matei Basarab" National College, Bucharest and Neculai Stanciu, "George Emil Palade" School Buzău, Romania*

Let $\{x_n\}_{n\geq 1},\ x_1 = 1,\ x_n = 1 \cdot \sqrt{3!!} \cdot \sqrt[3]{5!!} \cdot \ldots \sqrt[n]{(2n-1)!!}$.
Find:

$$\lim_{n\to\infty}\left(\frac{(n+1)^2}{\sqrt[n+1]{x_{n+1}}} - \frac{n^2}{\sqrt[n]{x_n}}\right).$$

5496: *Daniel Sitaru, "Theodor Costescu" National Economic College, Drobeta Turnu-Severin, Mehedinti, Romania*

Let a, b, c be real numbers such that $0 < a < b < c$. Prove that:

$$\sum_{cyclic}\left(e^{a-b} + e^{b-a}\right) \geq 2a - 2c + 3 + \sum_{cyclic}\left(\frac{b}{a}\right)^{\sqrt{ab}}.$$

5497: *Proposed by José Luis Díaz-Barrero, Barcelona Tech, Barcelona, Spain*

For all integers $n \geq 2$, show that $\displaystyle\prod_{k=1}^{n-1} 2\sin\left(\frac{k\pi}{n}\right)$ is an integer and determine it.

5498: *Proposed by Ovidiu Furdui and Alina Sîntămărian, both at the Technical University of Cluj-Napoca, Cluj-Napoca, Romania*

Prove that

$$\sum_{n=1}^{\infty} \frac{\{n!e\}}{n} = \int_0^1 \frac{e^x - 1}{x} dx$$

where $\{a\}$ denotes the fractional part of a.

Solutions

5475: *Proposed by Kenneth Korbin, New York, NY*

Given positive integers a, b, c and d such that $\begin{cases} a+b = 14\sqrt{ab-48}, \\ b+c = 14\sqrt{bc-48}, \\ c+d = 14\sqrt{cd-48}, \end{cases}$ with $a < b < c < d$.

Express the values of b, c, and d in terms of a.

Solution 1 by David E. Manes, Oneonta, NY

If $a = 1 < b = 97 < c = 18817 < d = 3650401$, then it is easily verified that these positive integers satisfy the system of equations. For the first equation $a + b = 14\sqrt{ab-48}$, square both sides, simplify and write the equation as a quadratic in b. Then one obtains $b^2 - (194a)b + (a^2 + 9408) = 0$ with roots $b = 97a \pm 56\sqrt{3(a^2-1)}$. Note that if $a = 1$, then $b = 97$. For the second equation, by symmetry, $c = 97b \pm 56\sqrt{3(b^2-1)}$. If $b = 97$, then $c = 97^2 \pm 56\sqrt{3(97^2-1)} = 18817 \,\text{or}\, 1$. Therefore, $c = 18817$ since $b < c$. Writing c in terms of a, we first note that

$$\begin{aligned} b^2 &= \left(97a \pm 56\sqrt{3(a^2-1)}\right)^2 \\ &= 18817a^2 \pm 10864a\sqrt{3(a^2-1)} - 9408. \end{aligned}$$

Therefore,

$$\begin{aligned} c &= 97b \pm 56\sqrt{3(b^2-1)} \\ &= 97\left(97a \pm 56\sqrt{3(a^2-1)}\right) \pm 56\sqrt{3\left(18817a^2 \pm 10864a\sqrt{3(a^2-1)} - 9409\right)}. \end{aligned}$$

If $a = 1$, then $c = 18817$ using the positive radicals. At this point, let $\alpha = 56\sqrt{3(a^2-1)}$ and $\beta = 56\sqrt{3\left(18817a^2 \pm 10864a\sqrt{3(a^2-1)} - 9409\right.}$. With these substitutions, the equation for c reads $c = 97^2 a \pm 97\alpha \pm \beta$ and if $a = 1$, then $\alpha = 0$ and $\beta = 9408$. For the last equation, $d = 97c \pm 56\sqrt{3(c^2-1)}$ again by symmetry. If $c = 18817$, then $d = 3650401$ or 97 and $c < d$ implies $d \neq 97$. Expressing d in terms of a, observe that

$$\begin{aligned} c^2 &= \left(97^2 a \pm 97\alpha \pm \beta\right)^2 \\ &= 97^4 a^2 + 97^2\alpha^2 + \beta^2 \pm 2\cdot 97^3 a\alpha \pm 2\cdot 97^2 a\beta \pm 2\cdot 97\alpha\beta. \end{aligned}$$

Therefore,

$$\begin{aligned} d &= 97c \pm 56\sqrt{3(c^2-1)} \\ &= 97^3 a \pm 97^2\alpha \pm 97\beta \pm 56\sqrt{3\left(97^4a^2 + 97^2\alpha + \beta^2 \pm 2\cdot 97^3 a\alpha \pm 2\cdot 97^2 a\beta \pm 194\alpha\beta - 1\right)}. \end{aligned}$$

Finally, note that if $a = 1$, then $\alpha = 0, \beta = 9408$ and $d = 3650401$ for the positive signs. This completes the solution.

Solution 2 by Ioannis D. Sfikas, National and Kapodistrian University of Athens, Greece

Squaring the equation $x + y = 14\sqrt{xy - 48}$ with x, y positive integers and $x < y$, yields the quadratic equation of $y = y(x)$.

$$y^2 - 194xy + x^2 + 9408 = 0 \tag{1}$$

The discriminant Δ is $\Delta = 37632(x^2 - 1)$. Since x and y are positive integers, hence $x \geq 1$ and $y_1 = 97x + 56\sqrt{3(x^2-1)}$ or $y_2 = 97x - 56\sqrt{3(x^2-1)}$.

Nevertheless, there is $x < y_2$ which holds for $1 \leq x < 7$. Applying (1) to the given system there is

$b_2 = 97a - 56\sqrt{3(a^2-1)},\ c_2 = 97b_2 - 56\sqrt{3(b_2^2-1)}$ and $d_2 = 97c_2 - 56\sqrt{3(c_2^2-1)}$,

with $1 \leq a < 7$.

(1) For $a = 1$, then $b_2 = 97, c_2 = 1$ a contradiction.

(2) For $a = 2$, then $b_2 = 26$, $c_2 = 2$, a contradiction.

(3) For $a = 3, 4, 5,$ or 6, b_2 is not an integer. So, the solution (a, b_2, c_2, d_2) is rejected.

Furthermore, there is $x < y_1$, which holds for $x \geq 1$. We may generalize the problem: given positive integers $\{x_i\}_{i=1}^n$ such that $x_i + x_{i+1} = 14\sqrt{x_i x_{i+1} - 48}$ with $x_i < x_{i+1}$, then we have the recursive relation

$$x_{i+1} = 97x_i + 56\sqrt{3(x_i^2 - 1)}.$$

Again, applying (1) to the given system there are the following recursive relations:

$b_1 = 97a + 56\sqrt{3(a^2-1)}, c_1 = 97b_1 + 56\sqrt{3(b_1^2-1)}$ and $d_1 = 97c_1 + 56\sqrt{3(c_1^2-1)}$, with $a \geq 1$.

So, we may list some solutions:

a	b_1	c_1	d_1
1	97	18817	3650401
2	362	70226	13623482
7	1351	262087	50843527
26	5042	978122	189750626
97	18817	3650401	708158977

We can assume that equation (1) is a Diophantine equation. Then, possible solutions are $(x, y) = (1, 97),\ (2, 26),\ (92, 362),\ (26, 5042),\ (97, 18817)$. Equation (1) reduces to a Diophantine equation of the Pell type. We may write $x^2 - 194xy + y^2$ in the matrix form $\begin{pmatrix} x, & y \end{pmatrix} \begin{pmatrix} 1 & -97 \\ -97 & 1 \end{pmatrix} \begin{pmatrix} x, \\ y \end{pmatrix}$. This matrix has eigen vectors $(1, \pm 1)$, which leads us to consider $u_1 = x + y$ and $v_1 = y - x$. Then, equation (1) becomes $49v_1^2 - 48u_1^2 + 9408 = 0$. Since 9408=$2^6 \cdot 3 \cdot 7^2$, this impies that $u_1 = 7u$ and $v_1 = 12v$ and so, $u^2 - 3v^2 = 4$, a Pell equation. The Pell equation has an infinity of integer solutions in general and the Pell equation implies

$$u = \left(2 - \sqrt{3}\right)^n + \left(2 + \sqrt{3}\right)^n, \qquad v = \frac{\sqrt{3}}{3}\left[\left(2 + \sqrt{3}\right)^n - \left(2 - \sqrt{3}\right)^n\right],$$

with $n \in N$, or,

$$x = \frac{u_1 - v_1}{2} = \frac{7u - 12v}{2} = \frac{1}{2}\left[\left(7 + 4\sqrt{3}\right)\left(2 - \sqrt{3}\right)^n + \left(7 - 4\sqrt{3}\right)\left(2 + \sqrt{3}\right)^n\right].$$

So, we may list some solutions:

n	x	y	c_1	d_1
1	2	362	70226	13623482
2	1	97	18817	3650401
3	2	362	70226	13623482
4	7	1351	262087	50843527
5	26	5042	978122	189750626
6	97	18817	3650401	708158977
7	362	70226	13623482	2642885282
8	1351	262087	50843527	9863382151
9	5042	978122	189750626	36810643322
10	18817	3650401	708158977	137379191137

Solution 3 by Kee-Wai Lau, Hong Kong, China

We show that either $(a, b, c, d) = (2, 26, 5042, 978122)$ or

$$\left(\frac{(2+\sqrt{3})^k + (2-\sqrt{3})^k}{2},\ 97a + 56\sqrt{3(a^2-1)},\ 18817a + 10864\sqrt{3(a^2-1)},\right.$$
$$\left. 3650401a + 2107560\sqrt{3(a^2-1)}\right),$$

for nonnegative integers k, there are k solutions.

Squaring both sides of the given equations, we obtain respectively,

$$\begin{aligned} b^2 - 194ab + a^2 + 9408 &= 0, \qquad (1) \\ c^2 - 194bc + b^2 + 9408 &= 0, \qquad (2) \\ d^2 - 194cd + c^2 + 9408 &= 0. \qquad (3) \end{aligned}$$

By subtracting (1) from (2), we obtain $(c - a)(c + a - 194b) = 0$. Since $c > a$, so $c = 194b - a$. Similarly by subtracting (2) from (3), we obtain $d = 194c - b = 37635b - 194a$. From (1), we obtain $b = 97a \pm 56\sqrt{3(a^2-1)}$. Since b is an integer, so $3(a^2 - 1)$ is a square, which leads to the Pell Equation

$a^2 - 3t^2 = 1$. It is well known that a must be of the form $\dfrac{(2+\sqrt{3})^k + (2-\sqrt{3})^k}{2}$.
We first suppose that: $b = 97a - 56\sqrt{3(a^2-1)}$, with $a > 1$. Since $b > a$, we deduce with some algebra that $a < 7$. This gives $a = 2$ and hence $b = 26$, $c = 5042,\ d = 978122$.
We next suppose that $b = 97a + 56\sqrt{3(a^2-1)}$, where $a \geq 1$. We the obtain the solutions stated at the beginning and this completes the solution.

Solution 4 by Kenneth Korbin (the proposer) NewYork, NY

Sequence $x = (1, 2, 7, 26, 97, 362, 1351, \ldots, x_N, \ldots)$ with $x_{N+1} = 4x_N - x_{N-1}$

$$\begin{aligned} x_N + x_{N+4} &= 14x_{N+2} \\ &= 14\sqrt{(x_N)(x_{N+4}) - 48} \end{aligned}$$

$$\begin{aligned} 1 + 97 &= 14(7) = 14\sqrt{(1)(97) - 48} \\ 2 + 362 &= 14(26) = 14\sqrt{(2)(362) - 48} \\ 7 + 1351 &= 14(97) = 14\sqrt{(7)(1351) - 48} \\ \text{etc.} \end{aligned}$$

If $a = x_N$, then
$b = x_{N+4},\ c = x_{N+8},\ d = x_{N+12},$

Sequence

$$\begin{aligned} y &= (x_N, x_{N+4}, x_{N+8}, x_{N+12}, x_{N+16}, \cdots) \\ y &= (a, b, c, d, x_{N+16}, x_{N+20}, \cdots) \end{aligned}$$

$c = 194b - a$
$ac - b^2 = 9408$
Therefore, $c = \dfrac{b^2 + 9408}{a}$
$194b - a = \dfrac{b^2 + 9408}{a} = c$
Therefore, $b = 97a + 56\sqrt{3a^2 - 3}$
$c = 194b - a$
Therefore, $c = 18817a + 210864\sqrt{3a^2 - 3}$
$d = 194c - b$
Therefore, $= 36590401a + 2107560\sqrt{3a^2 - 3}$

Editor's Comment : As with previous problems, **David Stone and John Hawkins of Georgia Southern University in Statesboro, GA** attached comments about the problem and their solution that I believe are both informative and instructive to the readership. They are listed below.

Comment 1: The points $(a, b), (b, c)$ and (c, d) all lie on the hyperbola $x^2 - 194xy + y^2 = -9408$.

This is actually the hyperbola $48x^2 - 49y^2 = 4704$, after a 45-degree rotation.
Our hyperbola lies in the first quadrant; it asymptotes are $y = \left(4\sqrt{3}+7\right)^2 x$ and $y = \dfrac{1}{\left(4\sqrt{3}+7\right)^2}x$, and all of the points $(a, b), (b, c)$, and (c, d) lie very close to the steep asymptote whose equation is "almost" $y = 194x$.

Comment 2: Our function f should probably be denoted $f_+(x) = 97x + 56\sqrt{3\,(x^2-1)}$. (See solutions 1, 2 or 3 for the motivation of its derivation.) The companion function $f_-(x) = 97x - 56\sqrt{3\,(x^2-1)}$ actually serves as the inverse of f with suitable restrictions on the domain: $f_+ : [1, \infty) \to [97, \infty)$ is a bijection and $(f_+)^{-1} = f_- : [97, \infty) \to [1, \infty)$. Without the ordering conditions on a, b, c and d, we could use f_+ and f_- randomly to generate solutions based upon an appropriate value for a.

Also solved by Ed Gray, Highland Beach, FL; David Stone and John Hawkins, Georgia Southern University, Statesboro, GA.

5476: *Proposed by Ed Gray, Highland Beach, FL*

Find all triangles with integer area and perimeter that are numerically equal.

Solution 1 by Henry Ricardo, Westchester Area Math Circle, NY

If we assume that any triangle satisfying our condition must have integer sides, then this is an old problem, that of finding all *equable Heronian triangles* [1], [2].

The solution is that *there exist only five triangles with numerically equal area and perimeter—those with sides* $(6, 8, 10), (5, 12, 13), (6, 25, 29), (7, 15, 20), (9, 10, 17)$. (The first two are the only right triangles.)

Prielipp [3] has proved that a triangle has equal area and perimeter if and only if it can be circumscribed about a circle of radius 2. Kilmer [4] uses Prielipp's result to generate triangles of equal area and perimeter. Markowitz [5] shows that there exists an infinite number of right triangles having rational side lengths for which the area equals the perimeter. Bates [6] has shown that a right triangle ΔABC with $\angle C = 90^0$ has numerically equal area and perimeter if and only if $a + b - c = 4$.

References

[1] "Heronian Triangle," *Wikipedia* article.

[2] "Equable Shape," *Wikipedia* article.

[3] "Area = Perimeter," Robert Prielipp, *Math.Teacher* **78** (February 1985), 90; 127.

[4] "Triangles of Equal Area and Perimeter and Inscribed Circles," Jean Kilmer, *Math.Teacher* **71** (January 1988), 65-70

[5] "Area = Perimeter," Lee Markowitz, *Math.Teacher* **74** (March 1981), 222-223.

[6] "Serendipity on the Area of a Triangle," Madelaine Bates, *Math.Teacher* **72** (April 1979), 273-275.

Solution 2 by Ioannis D. Sfikas, National and Kapodistrian University of Athens, Greece

A triangle whose sides and area are rational numbers is called a rational triangle. If the rational triangle is right-angled, it is called a right-angled rational triangle or a numerical right triangle. If the sides of a rational triangle is of integer length, it is called an integer triangle. If further these sides have no common factor greater than unity, the triangle called primitive integer triangle. A Heronian triangle (named after Heron of Alexandria) is an integer triangle with the additional property that its area is also an integer [1].

In 1904, W.A. Whitworth and D. Biddle proved that there are only five Heronian triangles with integer sides whose area equals the perimeter, namely (9,10,17), (7,15,20), (6,25,29), (6,8,10) and (5,12,13) [2]. These Heronian triangles are called perfect triangles. In 1955, R.R. Phelps suggested the problem of finding all Heronian triangles whose integer valued sides add up to twice its area [3]. N. J. Fine solved the last problem and found that there is only one such triangle, namely (3,4,5) [4]. Subbarao [5] proved that the number of integer triangles, whose integer valued-sides add up to λ times their area A is finite, namely $P = \lambda A$, with $\lambda > 0$ and P is perimeter. Furthermore, he showed that the number of integer triangles for $\lambda > \sqrt{8}$ is zero, while he didn't determine the particular values of sides of integer triangles for $\lambda \leq \sqrt{8}$ [specifically, he mentioned other articles about perfect triangles and the right-angled triangle (3,4,5)]. A similar problem concerns the search of the number of Heronian triangles whose integer valued-sides add up to $\frac{1}{n}$ times their area A, namely $A = nP$, with $n \in N$. Markov [6], [7] found an algorithm for the listing of all Heronian triangles with the property $A = nP$. In 1985, Goehl [8] solved that particular problem in the special case of right triangles. In addition, the (3,4,5) right-angled triangle is the integer-sided triangle for which the ratio $\frac{A}{P}$ is a rational number less than 1, it actually has the smallest such ratio [9].

In the usual notation, we have from the hypothesis and the classical area formula of Heron $2s = m$

$$\sqrt{s(s-a)(s-b)(s-c)}. \qquad (1)$$

With $x = s - a, y = s - b, z = s - c$, we have $s = x + y + z$ and (1) becomes

$$4(x + y + z) = xyz. \qquad (2)$$

Suppose (without loss of generality) $x \leq y \leq z$. Then $3 < x$ implies $xyz \geq 16z$ and $x + y + z \leq 3z$ so that

$$4(x + y + z)) \leq 12z < 16z \leq xyz,$$

and (2) would be impossible. Hence, we need try only $x = 1, 2, 3$. (a) For $x = 1$, (2) becomes $(y - 4)(z - 4) = 20$. For $y \geq 9$, then $y - 4 \geq 5$ and $(y - 4)(z - 4) \geq 25$, a contradiction. Furthermore, for $y \leq 4$ and $y = 7$, a contradiction. So, for $y = 5, 6, 8$, we have the following three perfect triangles:
$T_1 = (6, 25, 29), T_2 = (7, 15, 20), T_3 = (9, 10, 17)$.

(b) For $x = 2$, (2) becomes $(y - 2)(z - 2) = 8$. For $y \geq 5$, then $y - 2 \geq 3$ and $(y - 2)(z - 2) \geq 9$, a contradiction. Furthermore, for $y = 1, 2$, a contradiction. So, for $y = 3, 4$ we have the following two perfect triangles: $T_4 = (5, 12, 13)$ and $T_5 = (6, 8, 10)$. Notice that each of the pairs (T_1, T_5) and (T_3, T_5) have a common side. These pairs can be placed along their common sides to form a large triangle in each case [10]. In particular, the perfect triangle T_4 and T_5 are right-angled triangles [11].

(c) For $x = 3$, (2) becomes $(3y - 4)(3z - 4) = 52$. Then, $y \neq 3$, since $3y - 4 = 5$ is not a factor of 52. Furthermore, $y \leq 3$, since $4 \leq y \leq z$ implies $8 \leq 3y - 4$, whence

$(3y-4)(3-4) \geq 64$, a contradiction.

References

[1] Carmichael, Robert D. (1915). Diophantine analysis. New York: John Wiley and Sons.

[2] Dickson, Leonard Eugene (2005). History of the theory of numbers, Volume II: Diophantine Analysis. Dover Publications.

[3] Bankoff, Leon; Olds, C. D.; Phelps, R. R.; Lehner, Joseph and Linis, Viktors (1955). Problems for solution: E1166-E1170. The American Mathematical Monthly, 62 (5): 364-365.

[4] Phelps, R. R. and Fine, N. J. (1956). E1168. The American Mathematical Monthly, 63 (1): 43-44.

[5] Subbarao, M. V. (1971). Perfect triangles. The American Mathematical Monthly, 78 (4): 384-385.

[6] Markov, Lubomir P. (2006). Pythagorean triples and the problem A = mP for triangles. Mathematics Magazine, 79 (2): 114-121.

[7] Markov, Lubomir (2007). Heronian triangles whose areas are integer multiples of their perimeters. Forum Geometricorum, 7: 129-135.

[8] Goehl Jr., John F. (1985). Area = k(perimeter). The Mathematics Teacher, 78 (5): 330-332.

[9] Dolan, Stan (2016). Less than equable Heronian triangles. The Mathematical Gazette, 100 (549): 482-489.

[10] Rabinowitz, Stanley (1992). Index to Mathematical Problems 1980-1984 (Indexes to mathematical problems). Mathpro Press.

[11] Markowitz, Lee (1981). Area = Perimeter. The Mathematics Teacher, 74 (3): 222-223.

Comment submitted by Dionne Bailey, Elsie Campbell, Charles Diminnie, and Trey Smith, Angelo State University, San Angelo, TX

There are only five examples of triangles with integer sides for which the area and perimeter are equal and integer-valued. These are the triangles for which $(a, b, c) = (5, 12, 13)$, $(6, 8, 10)$, $(6, 25, 29)$, $(7, 15, 20)$, and $(9, 10, 17)$. These can be found by using an algorithm described in Reference [1].

However, it's possible to find an infinite number of examples of triangles where at least one side is irrational and yet the area and perimeter are equal and integer-valued. For example, if n is an integer and $n \geq 4$, let $a = 2n - \sqrt{n^2 - 12}$, $b = 2n$, and $c = 2n + \sqrt{n^2 - 12}$. When $n = 4$, $(a, b, c) = (6, 8, 10)$. It is easily shown that when $n > 4$, $n^2 - 12$ cannot be a perfect square and hence, a and c are irrational. We note also that for all $n \geq 4$, $a < b < c$ and $a + b - c = 2\left(n - \sqrt{n^2 - 12}\right) > 0$. Consequently, we have $a < b < c$ and $a + b > c$, which guarantees that there is a non-degenerate triangle with sides a, b, and c. For this triangle, the perimeter $P = 6n$ and the semi-perimeter $s = 3n$. Then, Heron's Formula for the area A yields

$$\begin{aligned}
A &= \sqrt{s\,(s-a)\,(s-b)\,(s-c)} \\
&= \sqrt{(3n)\left(n+\sqrt{n^2-12}\right)(n)\left(n-\sqrt{n^2-12}\right)} \\
&= \sqrt{3n^2\left[n^2-\left(n^2-12\right)\right]} \\
&= \sqrt{36n^2} \\
&= 6n \\
&= P.
\end{aligned}$$

References:

[1] T. Leong, D. Bailey, E. Campbell, C. Diminnie, and P. Swets, *Another Approach to Solving $A = mP$ for Triangles*, Mathematics Magazine **80,** pp. 363 - 368, 2007**.**

Editor′s comment : Some readers asked if the side lengths had to be integers, and from the history of the problem we can see that that was the intent originally. But as mentioned in the comment by Bailey, Campbell, Diminnie, and Smith, the side lengths need not be integers and this is the territory where the solution of Stone and Hawkins took them.

Daivd Stone and John Hawkins of Georgia Southern University, seem to have rediscovered a version of the result cited by Bailey, et.al., that the side lengths need not be rational. In their solution they stated and proved the following algorithm:

For any integer $P \geq 21$, there are infinitely many triangles with $A = P$. All such triangles are given by the following prescription;

Let $P \geq 21$ be an integer. Choose b such that $b > 4$ and $2b^3 - Pb^2 + 16P \leq 0$.

Compute $z = b^2 - \dfrac{16P}{P-2b}$.

Let $a = \dfrac{P-b}{2} - \dfrac{1}{2}\sqrt{z}$ and c= $\dfrac{P-b}{2} + \dfrac{1}{2}\sqrt{z}$.

Then a, b, c form a triangle with area and perimeter P.

(After verifying the above algorithm they presented the following examples.)

Example 1: The first example has two irrational sides, but still has $A = P =$ integer.

Let $P = 26$. Then we must choose b such that $2b^3 - 26b^2 + 432 \leq 0$.

That is, $5.146 < b < 11.399$.

Let $b = 8$, then $z = \dfrac{112}{5}$, so $\sqrt{z} = \dfrac{4\sqrt{35}}{5}$.

Thus the other two sides of our triangle are

$a = \dfrac{26-8}{2} - \dfrac{1}{2}\dfrac{4\sqrt{35}}{5} = 9 - \dfrac{2\sqrt{35}}{5} = \dfrac{45-2\sqrt{35}}{5} \approx 6.6336$

and

$c = \dfrac{45+2\sqrt{35}}{5} \approx 11.3664$

Example 2: This example demonstrates that the minimum value for $P, 21$ is actually achieved.

Let $P = 21$. Then we must choose b such that $2b^3 - 21b^2 + 336 \leq 0$.

That is, $6.405 < b < 7.562$.

Let $b = 7$. Then $z = 1$ so $a = 7 - \frac{1}{2} = \frac{13}{2}$ and $= 7 + \frac{1}{2} = \frac{15}{2}$.

Here we have the triangle $\left(\frac{13}{2}, \frac{14}{2}, \frac{15}{2}\right)$ with rational, non-integer sides with $A = P = 21$.

Example 3: Note that the previous example used $b = P/3$. This is always a valid value for b. In this case, we have the triangle

$\left(\frac{P}{3} - \frac{1}{6}\sqrt{P^2 - 432}, \frac{P}{3}, \frac{P}{3} + \frac{1}{6}\sqrt{P^2 - 432}\right)$

which has $A = P$. This triangle has rational sides only for $P = 21, 24, 31, 39, 56$ and 109.

Also solved by Kee-Wai Lau, Hong Kong, China David E. Manes, Oneonta, NY; Albert Stadler, Herrliberg, Switzerland, and the proposer

5477: *Proposed by Daniel Sitaru, "Theodor Costescu" National Economic College, Drobeta Turnu-Sevrin, Meredinti, Romania*

Compute:

$$L = \lim_{n\to\infty}\left(\ln n + \lim_{x\to 0}\frac{1 - \sqrt{1+x^2}\sqrt[3]{1+x^2}\cdot\ldots\cdot\sqrt[n]{1+x^2}}{x^2}\right).$$

Solution 1 by Ed Gray, Highland Beach, FL

We rewrite the expression as:

1. $\lim_{x\to 0}\frac{\left[1 - (1+x^2)^{\frac{1}{2}+\frac{1}{3}+\frac{1}{n}+\ldots+\frac{1}{n}}\right]}{x^2}$.

2. Let $N = \sum_{k=2}^{k=n}\frac{1}{k}$, i.e., the harmonic series -1

3. Now consider $\lim_{x\to 0}\frac{\left[1 - \left(1+x^2\right)^N\right]}{x^2}$.

We expand $(1+x^2)^N$ by the Binomial Theorem:

4. $(1+x^2)^N = 1 + Nx^2 + \frac{N(N-1)}{2!}x^4 + \ldots$

Then

5. $\lim_{x\to 0}\frac{\left[1 - \left(1 + Nx^2 + \frac{N(N-1)}{2}x^4 + \ldots\right)\right]}{x^2}$, or

6. $\lim_{x\to 0}\frac{\left[-Nx^2 + \frac{-N(N-1)}{2}x^4 + \ldots\right]}{x^2} = \frac{-Nx^2}{x^2} = -N$.

The original limit becomes

7. $\lim_{n\to\infty}(\ln(n) - N) = \lim_{n\to\infty}\left(\ln(n) - \sum_{k=2}^{k=n}\frac{1}{k}\right) = \lim_{n\to\infty}(\ln(n) + 1 - \text{Harmonic series})$.

The Euler-Mascheroni Constant is defined as $\gamma = \lim_{n\to\infty} \textit{The Harmonic series} - \ln(n)$. Therefore our expression in step 7 equals $1 - \gamma$.

Solution 2 by Bruno Salgueiro Fanego, Viveiro, Spain

Since

$$\lim_{x\to 0}\frac{1-\left(1+x^2\right)^{\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{n}}}{x^2}=\lim_{x\to 0}\frac{1-\left(1+x^2\right)^{H_n-1}}{x^2}\left[\frac{0}{0}=Indet.\right]\underset{L'Hospital}{=}$$

$$\lim_{x\to 0}\frac{0-\left(H_n-1\right)\left(1+x\right)^{H_n-2}2x}{2x}=\left(1-H_n\right)\lim_{x\to 0}\left(1+x^2\right)^{H_n-2}=1-H_n,$$

$$L=\lim_{n\to\infty}\left(\ln n+1-H_n\right)=1-\lim_{n\to\infty}\left(H_n-\ln n\right)=1-\gamma,$$

where H_n is the n-th harmonic number and γ is the Euler-Mascheroni constant.

Solution 3 by Paolo Perfetti, Department of Mathematics, Tor Vergata University, Rome, Italy

$$\sqrt{1+x^2}\,\sqrt[3]{1+x^2}\cdots\sqrt[n]{1+x^2}=(1+x^2)^{\frac{1}{2}+\frac{1}{3}+\frac{1}{n}}=(1+x^2)^{H_n-1}$$

We have

$$\lim_{n\to\infty}\left(\ln n+\lim_{x\to 0}\frac{1-(1+x^2)^{H_n-1}}{x^2}\right)$$

Now

$$\lim_{x\to 0}\frac{1-(1+x^2)^{H_n-1}}{x^2}=-H_n+1$$

thus

$$L=\lim_{n\to\infty}(\ln-\ln n-\gamma+o(1)+1)=-\gamma+1$$

Solution 4 by Julio Cesar Mohnsam and Mateus Gomes Lucas, both from IFSUL, Campus Pelots-RS, Brazil, and Ricardo Capiberibe Nunes of E.E. Amlio de Caravalho Bas, Campo Grande-MS, Brazil

$$L=\lim_{n\to\infty}\left(\ln n+\lim_{x\to 0}\frac{1-(1+x^2)^{H_n-1}}{x^2}\right)=\lim_{n\to\infty}\left(\ln n+\lim_{x\to 0}(1-H_n)(1+x^2)^{H_n-2}\right).$$

because,

$$\lim_{x\to 0}\frac{[1-(1+x^2)^{H_n-1}]}{(x^2)}\overset{\frac{0}{0}}{=}\lim_{x\to 0}\frac{[1-(1+x^2)^{H_n-1}]'}{(x^2)'}=\lim_{x\to 0}(-H_n+1)(1+x^2)^{H_n-2}=-H_n+1$$

Therefore:

$$L=\lim_{n\to\infty}(\ln n-H_n+1)=\lim_{n\to\infty}(\ln n-H_n)+1=1+\lim_{n\to\infty}(\ln n-H_n)=1-\gamma$$

Note: γ is Euler-Mascheroni constant and $H_n=1+\frac{1}{2}+\cdots+\frac{1}{n}$.

Also solved by Yen Tung Chung, Taichung, Taiwan; Serban George Florin, Romania; Kee-Wai Lau, Hong Kong, China; Moti Levy, Rehovot, Israel;

Angel Plaza, University of Las Palmas de Granada Canaria Spain; Ravi Prakash, New Delhi, India; Henry Ricardo, Westchester Area Math Circle, NY; Ioannis D. Sfikas, National and Kapodistrian University of Athens, Greece; Shivam Sharma, New Delhi, India; Albert Stadler, Herrliberg, Switzerland, and the proposer.

5478: *Proposed by D. M. Btinetu-Giurgiu, "Matei Basarab" National Collge, Bucharest, Romania and Neculai Stanciu, "George Emil Palade" Secondary School, Buzu, Romania*

Compute:

$$\int_0^{\pi/2} \cos^2 x \left(\sin x \sin^2\left(\frac{\pi}{2}\cos x\right) + \cos x \sin^2\left(\frac{\pi}{2}\sin x\right)\right) dx.$$

Solution 1 by Karl Havlak, Angelo State University, San Angelo, TX

Let $u = \cos x$. Then $du = -\sin x dx$ and $\sin(x) = \sqrt{1-u^2}$. We may rewrite the given integral as

$$\int_0^1 \left(u^2 \sin^2\left(\frac{\pi}{2}u\right) + \frac{u^3}{\sqrt{1-u^2}} \sin^2\left(\frac{\pi}{2}\sqrt{1-u^2}\right)\right) du.$$

Considering the second term in the integrand, we let $v = \sqrt{1-u^2}$ so that $dv = -\dfrac{u}{\sqrt{1-u^2}} du$ and $u^2 = 1 - v^2$. We may write the integral above as

$$\int_0^1 u^2 \sin^2\left(\frac{\pi}{2}u\right) du + \int_0^1 (1-v^2) \sin^2\left(\frac{\pi}{2}v\right) dv.$$

This reduces to $\displaystyle\int_0^1 \sin^2\left(\frac{\pi}{2}v\right) dv$, which can be easily shown to be $\frac{1}{2}$.

Solution 2 by Moti Levy, Rehovot, Israel)

$$\begin{aligned} I &:= \int_0^{\frac{\pi}{2}} \left(\cos^2 x\right) \left(\sin x \sin^2\left(\frac{\pi}{2}\cos x\right) + \cos x \sin^2\left(\frac{\pi}{2}\sin x\right)\right) dx \\ &= \int_0^{\frac{\pi}{2}} \cos^2 x \sin x \sin^2\left(\frac{\pi}{2}\cos x\right) dx + \int_0^{\frac{\pi}{2}} \cos^2 x \cos x \sin^2\left(\frac{\pi}{2}\sin x\right) dx \end{aligned}$$

Change of variables, $u = \cos x$ in the first integral and $v = \sin x$ in the second integral gives

$$\begin{aligned} I &= \int_0^1 u^2 \sin^2\left(\frac{\pi}{2}u\right) du + \int_0^1 \left(1-u^2\right) \sin^2\left(\frac{\pi}{2}u\right) du \\ &= \int_0^1 \sin^2\left(\frac{\pi}{2}u\right) du = \frac{1}{2}. \end{aligned}$$

Solution 3 by Bruno Salgueiro Fanego, Viveiro, Spain

$$I = \int_0^{\pi/2} \cos^2 x \left(\sin x \sin^2\left(\frac{\pi}{2}\cos x\right) + \cos x \sin^2\left(\frac{\pi}{2}\sin x\right)\right) dx$$

$$
\begin{aligned}
&= \int_0^{\pi/2} \left(1-\sin^2 x\right)\sin x \sin^2\left(\frac{\pi}{2}\cos x\right) + \int_0^{\pi/2} \cos^3 x \sin^2\left(\frac{\pi}{2}\sin x\right) dx \\
&= I_1 - I_2 + I_3 \text{ where}
\end{aligned}
$$

$$
\begin{aligned}
I_1 &= \int_0^{\pi/2} \sin x \sin^2\left(\frac{\pi}{2}\cos x\right), \quad I_2 = \int_0^{\pi/2} \sin^3 x \sin^2\left(\frac{\pi}{2}\cos x\right) dx \text{ and} \\
I_3 &= \int_0^{\pi/2} \cos^3 x \sin^2\left(\frac{\pi}{2}\sin x\right) dx.
\end{aligned}
$$

Since

$$
\begin{aligned}
I_1 &= \int_0^{\pi/2} \sin x \frac{1-\cos(\pi\cos x)}{2} dx = \int_0^{\pi/2} \frac{\sin x}{2} - \frac{\sin x(\cos\pi\cos x)}{2} dx \\
&= \left[-\frac{\cos x}{2} - \frac{\sin(\pi\cos x)}{2\pi}\right]_{x=0}^{x=\pi/2} \\
&= \frac{\cos(\pi/2)}{2} - \frac{\sin(\pi\cos(\pi/2))}{2\pi} - \left(-\frac{\cos 0}{2} - \frac{\sin(\pi\cos 0)}{2\pi}\right) \\
&= 0 - 0 + \frac{1}{2} + 0 = \frac{1}{2}.
\end{aligned}
$$

With the substitution $t = \frac{\pi}{2} - x$, one obtains that

$$
\begin{aligned}
I_2 &= \int_0^{\pi/2} \sin^3 x \sin^2\left(\frac{\pi}{2}\cos x\right) dx = \int_{\pi/2}^{0} \sin^3\left(\frac{\pi}{2}-t\right)\sin^2\left(\frac{\pi}{2}\cos\left(\frac{\pi}{2}-t\right)\right)(-dt) \\
&= \int_0^{\pi/2} \cos^3 t \sin^2\left(\frac{\pi}{2}\sin t\right) dt = I_3.
\end{aligned}
$$

The value of the given integral is therefore $I = I_1 = \frac{1}{2}$.

Also solved by Yen Tung Chung, Taichung, Taiwan; Ed Gray, Highland Beach, FL; Kee-Wai Lau, Hong Kong, China; Paolo Perfetti, Department of Mathematics, Tor Vergata, Rome, Italy; Ángel Plaza, University of Las Palmas de Gran Canaria, Spain; Ravi Prakash, New Delhi, India; Ioannis D. Sfikas, National and Kapodistrian University of Athens, Greece; Shivam Sharma, New Delhi, India; Albert Stadler, Herrliberg, Switzerland, and the proposers.

5479: *Proposed by José Luis Díaz-Barrero, Barcelona Tech, Barcelona, Spain*

Let $f : [0,1] \to \Re$ be a continuous convex function. Prove that

$$\frac{2}{5}\int_0^{1/3} f(t)dt + \frac{3}{10}\int_0^{2/3} f(t)dt \geq \frac{5}{8}\int_0^{8/15} f(t)dt.$$

Solution 1 by Dionne Bailey, Elsie Campbell, Charles Diminnie, and Trey Smith, Angelo State University, San Angelo, TX

If $c \in (0,1)$, then for all $t \in [0,1]$, $ct \in [0,c] \subset [0,1]$ and hence, $f(ct)$ is continuous on $[0,1]$. Further, by making the substitution $u = ct$, we get

$$\int_0^1 f(ct)\,dt = \frac{1}{c}\int_0^c f(u)\,du = \frac{1}{c}\int_0^c f(t)\,dt$$

and thus,

$$\int_0^c f(t)\,dt = c\int_0^1 f(ct)\,dt. \tag{1}$$

By (1),

$$\begin{aligned}
&\frac{2}{5}\int_0^{\frac{1}{3}} f(t)\,dt + \frac{3}{10}\int_0^{\frac{2}{3}} f(t)\,dt \\
&= \left(\frac{2}{5}\right)\left(\frac{1}{3}\right)\int_0^1 f\left(\frac{1}{3}t\right)dt + \left(\frac{3}{10}\right)\left(\frac{2}{3}\right)\int_0^1 f\left(\frac{2}{3}t\right)dt \\
&= \frac{1}{3}\int_0^1 \left[\frac{2}{5}f\left(\frac{1}{3}t\right) + \frac{3}{5}f\left(\frac{2}{3}t\right)\right]dt. \qquad (2)
\end{aligned}$$

Since $f(t)$ is convex on $[0,1]$ and $\frac{1}{3}t,\ \frac{2}{3}t \in [0,1]$ for all $t \in [0,1]$, Jensen's Theorem implies that

$$\begin{aligned}
\frac{2}{5}f\left(\frac{1}{3}t\right) + \frac{3}{5}f\left(\frac{2}{3}t\right) &\geq f\left[\left(\frac{2}{5}\right)\left(\frac{1}{3}t\right) + \left(\frac{3}{5}\right)\left(\frac{2}{3}t\right)\right] \\
&= f\left(\frac{8}{15}t\right) \qquad (3)
\end{aligned}$$

for all $t \in [0,1]$. By combining (1), (2), and (3), we obtain

$$\begin{aligned}
\frac{2}{5}\int_0^{\frac{1}{3}} f(t)\,dt + \frac{3}{10}\int_0^{\frac{2}{3}} f(t)\,dt &= \frac{1}{3}\int_0^1 \left[\frac{2}{5}f\left(\frac{1}{3}t\right) + \frac{3}{5}f\left(\frac{2}{3}t\right)\right]dt \\
&\geq \frac{1}{3}\int_0^1 f\left(\frac{8}{15}t\right)dt \\
&= \left(\frac{1}{3}\right)\left(\frac{15}{8}\right)\int_0^{\frac{8}{15}} f(t)\,dt \\
&= \frac{5}{8}\int_0^{\frac{8}{15}} f(t)\,dt.
\end{aligned}$$

Solution 2 by Michael Brozinsky, Central Islip, NY

We have by the Mean Value Theorem for Integrals that there exists constants C on $\left(0,\frac{1}{3}\right)$, D on $\left(0,\frac{2}{3}\right)$, E on $\left(0,\frac{8}{15}\right)$, S on $\left(\frac{1}{3},\frac{2}{3}\right)$ such that

$$\int_0^{1/3} f(t)dt = \frac{1}{3}f(C),\ \int_0^{2/3} f(t)dt = \frac{2}{3}f(D),\ \int_0^{8/15} f(t)dt = \frac{8}{15}f(E) \text{ and so}$$

$$\frac{1}{5}f(S) = \int_{1/3}^{8/15} f(t)dt = \int_0^{8/15} f(t)dt - \int_0^{1/3} f(t)dt = \frac{8}{15}f(E) - \frac{1}{3}f(C) \text{ and}$$

$$\frac{2}{15}f(T) = \int_{8/15}^{2/3} f(t)dt = \int_0^{2/3} f(t)dt - \int_0^{8/15} f(t)dt = \frac{2}{3}f(D) - \frac{8}{15}f(E)dt.$$

The first of these last two equations gives $f(E) = \frac{3}{8}f(S) + \frac{5}{8}f(C)$ and so the second then gives $f(D) = \frac{1}{5}f(T) + \frac{3}{10}f(S) + \frac{1}{2}f(C)$.

The desired inequality $\frac{2}{5}\int_0^{1/3} f(t)dt + \frac{3}{10}\int_0^{12/3} f(t)dt \geq \frac{5}{8}\int_0^{8/15} f(t)dt$ can be written as
$\frac{2}{15}f(C) + \frac{3}{10}\cdot\frac{2}{3}\cdot f(D) \geq \frac{5}{8}\cdot\frac{8}{15}\cdot f(E)$, or equivalently as
$\frac{2}{15}f(C) + \frac{1}{5}\left(\frac{1}{5}f(T) + \frac{3}{10}f(S) + \frac{1}{2}f(C)\right) \geq \frac{1}{3}\left(\frac{3}{8}f(S) + \frac{5}{8}f(C)\right)$ which is equivalent to
$\left(\frac{2}{15} + \frac{1}{10} - \frac{5}{24}\right)f(C) + \frac{1}{25}f(T) \geq \left(\frac{1}{8} - \frac{3}{50}f(S)\right)$ and then to
$\frac{1}{40}f(C) + \frac{1}{25}f(T) \geq \frac{13}{200}f(S)$ and finally to
$\frac{5}{13}f(C) + \frac{8}{13}f(T) \geq f(S)$ which is true by the convexity since $C < S < T$.

Note: A function is convex on $[a,b]$ means that for all $0 \leq \lambda \leq 1$ whenever $a \leq X < Z \leq b$ we have

$f(X) + \lambda\left(f(Z) - f(X)\right) \leq f(X + \lambda(Z - X))$ which can be cast as
$(1-\lambda)f(X) + \lambda f(Z) \geq f\left(X + \lambda(Z - X)\right)$ and so in the above taking $X = C,\ Z = T$, **and** $\lambda = \frac{S - C}{T - C}$ we have $X + \lambda(Z - X) = S$.

Solution 3 by Henry Ricardo, Westchester Area Math Circle, NY

If f is convex on $[0,1]$, then for all $x, y \in [0,1]$ and for all $\lambda \in [0,1]$, we have

$$\lambda f(x) + (1-\lambda)f(y) \ \geq \ f(\lambda x + (1-\lambda)y). \qquad (1)$$

Setting $\lambda = 2/5, x = t/3$, and $y = 2t/3$ in (1), $0 \leq t \leq 3/2$, we get

$$\frac{2}{5}f(t/3) \ + \ \frac{3}{5}f(2t/3) \ \geq \ f(8t/15).$$

Integrating from $t = 0$ to $t = 1$ yields

$$\frac{2}{5}\int_0^1 f(t/3)dt \ + \ \frac{3}{5}\int_0^1 f(2t/3)dt \geq \int_0^1 f(8t/15)dt. \quad (2)$$

Setting $u = t/3$ in the first integral of (2), we have

$$\frac{2}{5}\int_0^1 f(t/3)dt \ = \ \frac{6}{5}\int_0^{1/3} f(u)du.$$

Similarly, setting $u = 2t/3$ in the second integral, we get

$$\frac{3}{5}\int_0^1 f(2t/3)dt \ = \ \frac{9}{10}\int_0^{2/3} f(u)du.$$

Finally, setting $u = 8t/15$, we find that

$$\int_0^1 f(8t/15)dt \ = \ \frac{15}{8}\int_0^{8/15} f(u)du.$$

Substituting these into (2) and dividing by 3, we obtain

$$\frac{2}{5}\int_0^{1/3} f(u)du \ + \ \frac{3}{10}\int_0^{2/3} f(u)du \ \geq \ \frac{5}{8}\int_0^{8/15} f(u)du.$$

Solution 4 by Ángel Plaza, University of Las Palmas de Gran Canaria, Spain

By changing variables it follows that $\alpha\displaystyle\int_0^a f(t)\ dt = \int_0^{\alpha a} f(s/\alpha)\ ds$. Therefore, the left-hand side of the proposed inequality, say LHS, is

$$\begin{aligned} LHS &= \frac{2}{5}\int_0^{1/3} f(t)\ dt + \frac{3}{5}\int_0^{1/3} f(2t)\ dt \\ &\geq \int_0^{1/3} f\left(\frac{2}{5}t + \frac{6}{5}t\right)\ dt \\ &= \int_0^{1/3} f\left(\frac{8}{5}t\right)\ dt \end{aligned}$$

where we have used that f is convex, so $\dfrac{2}{5}f(t) + \dfrac{3}{5}f(2t) \geq f\left(\dfrac{2}{5}t + \dfrac{6}{5}t\right)$. Since the right-hand side is $\dfrac{5}{8}\displaystyle\int_0^{8/15} f(t)\ dt = \int_0^{1/3} f\left(\frac{8}{5}t\right)\ dt$, the conclusion follows.

Also solved by Bruno Salgueiro Fanego, Viveiro, Spain; Ed Gray, Highland Beach, FL; Kee-Wai Lau, Hong Kong, China; Moti Levy, Rehovot, Israel; Paolo Perfetti, Department of Mathematics, Tor Vergata University, Rome, Italy; Albert Sadler, Herrliberg, Switzerland; Ioannis D. Sfikas, National and Kapodistrian University of Athens, Greece, and the proposer.

5480: *Proposed by Ovidiu Furdui, Technical University of Cluj-Napoca, Cluj-Napoca, Romania*

Let $n \geq 1$ be a nonnegative integer. Prove that in $C[0, 2\pi]$

$$span\{1, \sin x, \sin(2x), \ldots, \sin(nx)\} = span\{1, \sin x, \sin^2 x, \ldots, \sin^n x\}$$

if and only if $n = 1$.

We mention that $span\{v_1, v_2, \ldots, v_k\} = \sum_{j=1}^{k} a_j v_j,\ a_j \in \Re, j = 1, \ldots, k$, denotes the set of all linear combinations with $v_1, v_2, \ldots, v_k$.

Solution 1 by Moti Levy, Rehovot, Israel

If $n = 1$ then the spans are trivially equal.

Let $n > 1$. Suppose that $\sin^2 x$ can be expressed as a linear combination of the functions $\{1, sinx, sin(2x), ..., sin(nx)\}$,

$$\sin^2 x = a_0 + \sum_{k=1}^{n} a_k \sin(kx). \tag{1}$$

By setting $x = 0$, we have $a_0 = 0$.

The following definite integral vanish for integer k.

$$\begin{aligned} \int_0^{2\pi} sin^2 x \cdot sin(kx)dx &= \frac{1}{2}\int_0^{2\pi} (1 - \cos(2x)) \cdot sin(kx)dx \\ &= \frac{1}{2}\int_0^{2\pi} sin(kx)dx - \frac{1}{2}\int_0^{2\pi} \cos(2x) \cdot sin(kx)dx = 0. \end{aligned} \tag{2}$$

Now we multiply both sides of (1) by $\sin^2 x$ and integrate from 0 to 2π,

$$\int_0^{2\pi} \sin^4(x)\, dx = \sum_{k=1}^{n} a_k \int_0^{2\pi} sin^2 x \sin(kx)\, dx. \tag{3}$$

The right hand side of (3) is equal to $\frac{3}{4}\pi$ but the left hand side is equal to zero, by (2). This contradiction leads to the conclusion that $\sin^2 x$ cannot be expressed as a linear combination of the functions $\{1, sinx, sin(2x), ..., sin(nx)\}$, hence the spans are not equal for $n > 1$.

Solution 2 by David Stone and John Hawkins, Georgia Southern University, Statesboro, GA

For $n = 1$ it is certainly true that $span\{1, \sin x\} = span\{1, \sin x\}$.

For $n > 1$,

$$span\{1, \sin x, \sin(2x), \sin(3x), \ldots, \sin(nx)\} \neq span\{1, \sin x, \sin^2 x, \sin^3 x, \ldots, \sin^n x\}$$

because $\sin^2 x$ cannot be written as a linear combination of $1, \sin x, \sin(2x), \sin(3x), \ldots, \sin(nx)$.

To see this, suppose that $\sin^2 x = c_0 \cdot (1) + c_1 \sin(x) + c_2 \sin(2x) + c_3 \sin(3x) + \ldots, +c_n \sin(nx)$.

Then this equation must hold for all values of x in $C[0, 2\pi]$.

Letting $x = 0$, shows that $c_0 = 0$.

Letting $x = \dfrac{\pi}{2}$ gives us $1 = 0 + c_1 \cdot 1 + c_2 \cdot 0 + c_3 \cdot (-1) + c_4 \cdot 0 + c_5 \cdot 1 + \ldots$
or (1) $1 = c_1 - c_3 + c_5 - c_7 + c_9 + \ldots$.

Letting $x = \frac{3\pi}{2}$ gives us $1 = 0 + c_1 \cdot (-1) + c_2 \cdot 0 + c_3 \cdot 1 + c_4 \cdot 0 + c_5 \cdot (-1) + \ldots$
or (2) $1 = -c_1 + c_3 - c_5 + c_7 - c_9 + \ldots$.

The final term in each summation depends upon the of parity of n, but the terms on the right hand sides match up in any case. So, adding (1) + (2) gives us $2 = 0$, which is certainly a contradiction.

Thus, $\sin^2 x$ cannot be written in terms of $1, \sin x, \sin(2x), \sin(3x), \ldots, \sin(nx)$.

Also solved by Kee-Wai Lau, Hong Kong, China; Ioannis D. Sfikas, National and Kapodistrian University of Athens, Greece, and the proposer

Mea Culpa

The names Dionne Bailey, Elsie Campbell, and Charles Diminnie all at Angelo State University in San Angelo, TX were inadvertently omitted from the list of those who had solved problem 5470.

Paolo Perfetti of the Mathematics Department at Tor Vergata University in Rome, Italy should be credited for having solved 5472.