Problems

Ted Eisenberg, Section Editor

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This section of the Journal offers readers an opportunity to exchange interesting mathematical problems and solutions. Please send them to Ted Eisenberg, Department of Mathematics, Ben-Gurion University, Beer-Sheva, Israel or fax to: 972-86-477-648. Questions concerning proposals and/or solutions can be sent e-mail to <eisenbt@013.net>. Solutions to previously stated problems can be seen at <http://www.ssma.org/publications>.

Solutions to the problems stated in this issue should be posted before
February 15, 2012

- **5182:** *Proposed by Kenneth Korbin, New York, NY*

  Part I: An isosceles right triangle has perimeter P and its Morley triangle has perimeter x. Find these perimeters if $P = x + 1$.

  Part II: An isosceles right triangle has area K and its Morley triangle has area y. Find these areas if $K = y + 1$

- **5183:** *Proposed by Kenneth Korbin, New York, NY*

  A convex pentagon ABCDE, with integer length sides, is inscribed in a circle with diameter $\overline{AE}$.

  Find the minimum possible perimeter of this pentagon.

- **5184:** *Proposed by Neculai Stanciu, Buzău, Romania*

  If x, y and z are positive real numbers, then prove that

$$\frac{(x+y)(y+z)(z+x)}{(x+y+z)(xy+yz+zx)} \geq \frac{8}{9}.$$

- **5185:** *Proposed by José Luis Díaz-Barrero, Barcelona, Spain*

  Calculate, without using a computer, the value of

$$\sin\left[\arctan\left(\frac{1}{3}\right)+\arctan\left(\frac{1}{5}\right)+\arctan\left(\frac{1}{7}\right)+\arctan\left(\frac{1}{11}\right)+\arctan\left(\frac{1}{13}\right)+\arctan\left(\frac{111}{121}\right)\right].$$

- **5186:** *Proposed by John Nord, Spokane, WA*

  Find k so that $\displaystyle\int_0^k\left(-\frac{b}{a}x+b\right)^n dx = \frac{1}{2}\int_0^a\left(-\frac{b}{a}x+b\right)^n dx.$

- **5187:** *Proposed by Ovidiu Furdui, Cluj, Romania*

Let $f:[0,1]\to(0,\infty)$ be a continuous function. Find the value of

$$\lim_{n\to\infty}\left(\frac{\sqrt[n]{f(\frac{1}{n})}+\sqrt[n]{f(\frac{2}{n})}+\cdots+\sqrt[n]{f(\frac{n}{n})}}{n}\right)^n.$$

Solutions

- **5164:** *Proposed by Kenneth Korbin, New York, NY*

A triangle has integer length sides (a,b,c) such that $a-b=b-c$. Find the dimensions of the triangle if the inradius $r=\sqrt{13}$.

Solution 1 by Albert Stadler, Herrliberg, Switzerland

If a,b and c are the side lengths of the triangle then the inradius r is given by the formula

$r=\dfrac{1}{2}\sqrt{\dfrac{(b+c-a)(c+a-b)(a+b-c)}{a+b+c}}$. (see, e.g., http//mathworld.wolfram.com/Inradius.html).

By assumption, $c=2b-a$. So

$$\sqrt{13}=\frac{1}{2}\sqrt{\frac{(3b-2a)(2a-b)}{3}},\ \text{or equivalently}$$

$$(3b-2a)(2a-b)=156.$$

Obviously b is even. (If b were odd, then both $3b-2a$ and $2a-b$ are odd, and therefore their product would be odd, which is not true.) So $b=2b'$ and this gives the equation

$$(3b'-a)(a-b')=39.$$

Note that $39=xy$ is the product of two integers. So,

$$(x,y)\in\{(1,39),(3,13),(13,3),(39,1),(-1,-39),(-3,-13),(-13,-3),(-39,-1)\}.$$

If $3b'-a=x$ and $a-b'=y$, then

$$b'=\frac{x+y}{2},\ \text{and}$$
$$a=\frac{x+3y}{2}.$$

We find $(a,b,c)\in\{(59,40,21),(21,16,11),(11,16,21),(21,40,59)\}$, and we easily verify that each triplet satisfies the triangle inequality.

Solution 2 by Arkady Alt, San Jose, CA

Let F and s be the area and semiperimeter. Since $a+c=2b$ then $s=\dfrac{a+b+c}{2}=\dfrac{3b}{2}$, and using $F=\sqrt{s(s-a)(s-b)(s-c)}=sr$ we obtain

$$(s-a)(s-b)(s-c)=sr^2 \iff \left(\frac{3b}{2}-a\right)\left(\frac{3b}{2}-b\right)\left(\frac{3b}{2}-c\right)=13\cdot\frac{3b}{2}$$

$$\iff \left(\frac{3b}{2}-a\right)\left(\frac{3b}{2}-c\right)=39$$

$$\iff \left(\frac{9b^2}{4}-(a+c)\frac{3b}{2}+ac\right)=39 \iff \left(\frac{9b^2}{4}-2b\cdot\frac{3b}{2}+ac\right)=39$$

$$\iff 4ac-3b^2=12\cdot 13.$$

Thus we have

$$\begin{cases} a+c=2b \\ 4ac-3b^2=156 \end{cases} \implies \begin{cases} 4a(2b-a)-3b^2=156 \\ c=2b-a \end{cases} \quad \text{if, and only if,}$$

$$\begin{cases} 4a(2b-a)-3b^2=156 \\ c=2b-a \end{cases} \iff \begin{cases} 8ab-a^2-3b^2=156 \\ c=2b-a. \end{cases}$$

Since $8ab-a^2-3b^2=(3b-2a)(2a-b)$ and

$$\begin{cases} a<s \\ b<s \\ c<s \end{cases} \iff \begin{cases} 2a<3b \\ c<s \end{cases} \iff \begin{cases} 2a<3b \\ 2(2b-a)<3b \end{cases} \iff b<2a<3b$$

then the problem is equivalent to the system

$$(1) \qquad \begin{cases} (3b-2a)(2a-b)=156 \\ b<2a<3b. \end{cases}$$

Since $3b-2a\equiv 2a-b \pmod 2$ and $156=2^2\cdot 3\cdot 13=2\cdot 78=6\cdot 26$ then (1) in positive integers is equivalent to

$$\begin{cases} 3b-2a=k \\ 2a-b=m \end{cases} \iff \begin{cases} 2b=k+m \\ 4a=k+3m \end{cases} \iff \begin{cases} a=\dfrac{k+3m}{4} \\ b=\dfrac{k+m}{2} \end{cases},$$

where $(k,m)\in\{(2,78),(78,2),(6,26),(26,6)\}$.

Noting that the inequality $b<2a<3b \iff \dfrac{k+m}{2}<\dfrac{k+3m}{2}<\dfrac{3(k+m)}{2}$ holds for any positive k,m we finally obtain

$$(a,b)\in\{(59,40),(21,40),(21,16),(11,16)\}.$$

Thus, $(a,b,c) \in \{(59,40,21),(21,40,59),(21,16,11),(11,16,21)\}$ are all solutions of the problem.

Comment by David Stone and John Hawkins, Statesboro, GA. In their featured solutions to SSM 5146 (May 2011 issue) both Kee-Wai Lau and Brian Beasley found all integral triangles with in-radius $\sqrt{13}$. Note that the condition $a-b=b-c$ is equivalent to $b=(a+c)/2$. That is, irrespective of how one might label or order the sides, the side b must be the "middle-length" side , the average of the other two sides.

Also solved by Brain D. Beasley, Clinton, SC; Valmir Bucaj (student, Texas Lutheran University), Seguin, TX; Elsie M. Campbell, Dionne T. Bailey and Charles Diminnie (jointly), San Angelo TX; Bruno Salgueiro Fanego, Viveiro, Spain; Tania Moreno García, University of Holguín (UHO), Holguín, Cuba jointly with José Pablo Suárez Rivero, University of Las Palmas de Gran Canaria (ULPGC), Spain; Paul M. Harms, North Newton, KS; Enkel Hysnelaj, University of Technology, Sydney, Australia jointly with Elton Bojaxhiu, Kriftel, Germany; Sugie Lee, John Patton, and Matthew Fox (jointly; students at Taylor University), Upland, IN; Kee-Wai Lau, Hong Kong, China; David E. Manes, Oneonta, NY; Charles McCracken, Dayton, OH; Boris Rays, Brooklyn, NY; David Stone and John Hawkins (jointly), Statesboro, GA; Jim Wilson, Athens, GA, and the proposer.

- **5165:** *Proposed by Thomas Moore, Bridgewater, MA*

"Dedicated to Dr. Thomas Koshy, friend, colleague and fellow Fibonacci enthusiast."

Let $\sigma(n)$ denote the sum of all the different divisors of the positive integer n. Then n is perfect, deficient, or abundant according as $\sigma(n)=2n, \sigma(n)<2n$, or $\sigma(n)>2n$. For example, 1 and all primes are deficient; 6 is perfect, and 12 is abundant. Find infinitely many integers that are not the product of two deficient numbers.

Solution 1 by Kee-Wai Lau, Hong Kong, China

Let $p_1=2,\ p_2=3,\ p_3=5,\ \ldots$ be the sequence of primes. We show that for any positive integer n, the integer $\prod_{k=1}^{n+10} p_k$ is not the product of two deficient numbers.

Suppose , on the contrary, that $\prod_{k=1}^{n+10} p_k = ab$, where both a and b are deficient numbers. Clearly a and b are relatively prime and so

$$4\left(\prod_{k=1}^{n+10} p_i\right) = 4ab > \sigma(a)\sigma(b) = \sigma(ab) = \sigma\left(\prod_{k=1}^{n+10} p_k\right) = \prod_{k=1}^{n+10}\left(1+p_k\right).$$

Hence,

$$4 > \prod_{k=1}^{n+10}\left(1+\frac{1}{p_k}\right) \geq \prod_{k=1}^{11}\left(1+\frac{1}{p_k}\right) = \frac{3822059520}{955049953} = 4.0019\ldots,$$

which is a contradiction. This completes the solution.

Solution 2 by Stephen Chou, Talbot Knighton, and Tom Peller (students at Taylor University), Upland, IN

All negative numbers have the same numerical divisors as their positive counterparts;

however, the negatives also include all the negative forms of those divisors. For instance, -6 has divisor of $1, 2, 3, 6, -1 - 2, -3, -6$. Therefore $\sigma(n) = 0$ because the divisors will all negate themselves. Knowing that $2n$ of any negative will result in a lower negative, we see that all the negatives are abundant. Since the negatives are all abundant numbers and the only way to have a negative product is to multiply a negative by a positive, then at most a negative number can have only one deficient factor. Therefore, there are infinitely many integers, namely the negatives, that are not the product of two deficient numbers.

Editor's comment: Once again the students have out smarted the professors; the intent of the problem was to find infinitely many *positive* integers that are not the product of two deficient numbers. But the problem wasn't explicitly stated that way, and so the students win; mea culpa.

Solution 3 by Pat Costello, Richmond, KY

Let $n = 2^k \cdot 3780 = 2^{k+2} \cdot 3^3 \cdot 5 \cdot 7 = 2^{k+2} \cdot 945$ for any non-negative integer k. We want to show that for any divisor d of n and pair $(d, n/d)$, one of the two values is either perfect or abundant. Since the σ function is multiplicative, we have

$$\begin{aligned}
\sigma(945) &= \sigma(3^3 \cdot 5 \cdot 7) \\
&= \sigma(3^3) \cdot \sigma(5) \cdot \sigma(7) \\
&= 40 \cdot 6 \cdot 8 \\
&= 1920 \\
&> 2 \cdot 945.
\end{aligned}$$

So 945 is abundant. Then in the pair $(945, n/945)$, the 945 is abundant.

By multiplicativity,

$$\begin{aligned}
\sigma\left(2^{k+2} \cdot 945\right) &= \sigma\left(2^{k+2}\right) \cdot \sigma(945) \\
&> \sigma\left(2^{k+2}\right) \cdot 2 \cdot 945, \text{by the above} \\
&> 2^{k+2} \cdot 2 \cdot 945, \text{since } \sigma(m) > m \text{ for } m > 1 \\
&= 2\left(2^{k+2} \cdot 945\right).
\end{aligned}$$

This means all the n values are themselves abundant so in the pair $(1, n)$, the value n is the abundant value. This argument also shows that in the pair $(2^j, n/2^j)$, the second value is the abundant value.

In the following table, we list the divisors $d > 1$ of 945 and the values of the fractions $\sigma(d)/d$.

d	3	5	7	9	15	21	27	35	45	63	105	135	189	315	945
$\sigma(d)/d$	1.3	1.2	1.14	1.4	1.6	1.5	1.48	1.37	1.73	1.65	1.82	1.77	1.69	1.98	2.03

The key thing we want to see from the table is that the minimum value in the second row corresponds to $d = 7$.

Suppose that d is a divisor of $2^{k+2} \cdot 945$ that is of the form $2^j \cdot m$ where $j \geq 2$ and m is a divisor of 945 greater than 1. The fractions $\sigma(2^j)/2^j$ are easily seen to be strictly increasing with a limit of 2. Then

$$\begin{aligned}
\sigma\left(2^j \cdot m\right)/2^j \cdot m &= \sigma\left(2^j\right) \cdot \sigma(m)/2^j \cdot m \\
&= \sigma(2^j)/2^j \cdot \sigma(m)/m \\
&\geq \frac{7}{4} \cdot \frac{8}{7} \text{ from the table and that } j \geq 2 \\
&= 2.
\end{aligned}$$

Hence the divisor $2^j \cdot m$ is perfect or abundant.

Suppose that d is a divisor of 945 and less than 945, say $d = 945/m$ for an $m \geq 1$. Then the pair is $(945/m, 2^{k+2} \cdot m)$ and the second value is perfect or abundant.

All pairs $(d, n/d)$ have at least one value which is perfect or abundant. Since k is an arbitrary nonnegative integer, we have the desired infinite set.

Solution 4 by Brian D. Beasley, Clinton, SC

We make use of the following three facts:

(1) 945 is abundant (in fact, it is the smallest odd abundant number);

(2) any nontrivial multiple of a perfect number is abundant;

(3) any multiple of an abundant number is abundant.

Given any integer $k \geq 2$, we show that $n_k = 2^k \cdot 945$ is not the product of two deficient numbers. For contradiction, if $n_k = 2^k \cdot 3^3 \cdot 5 \cdot 7 = xy$ for deficient numbers x and y, then the perfect number 6 divides neither x nor y, so without loss of generality, we assume that 2^k divides x and 3^3 divides y. Next, we consider cases:

(a) If 5 divides x, then x is abundant, since it is a multiple of the abundant number 20.

(b) If 7 divides x, then x is either perfect or abundant, since it is a multiple of the perfect number 28.

(c) If neither 5 nor 7 divides x, then $y = 3^3 \cdot 5 \cdot 7 = 945$ is abundant.

Since each case leads to a contradiction, we are done. In fact, it follows that for $k \geq 3$, if $n_k = xy$, then at least one of x or y is abundant.

Addendum. Facts (1) and (2) above may be found in Burton's *Elementary Number Theory* (6th edition) on page 235, while fact (3) follows by applying an argument similar to that used to prove fact (2).

Solution 5 by proposer

A computer program shows that there are 55 such numbers below 10^5, the smallest being 3780. The canonical factorization of these numbers is revealing. One notices that the list includes all numbers of the from $3780p$ were $p \in \{11, 13, 17, 19, 23\}$. This suggest that $N = 3780p$ is such a number, for all primes $p \geq 11$.

To prove this, lent $N = 3780p = ab$ with $1 \leq a \leq b \leq N$. Now $3780p = 2^2 \cdot 3^3 \cdot 5 \cdot 7 \cdot p$ has 96 divisors, many of which are multiples of 12. But 12 is an abundant number and so is any multiple of 12. (More generally, any multiple of an abundant number is also abundant.) So we need only consider factorizations $N = ab$ where neither a nor b is a multiple of 12. We list all these factorizations in the tables below showing the companion factors a and b, along with their type (P: perfect; D: deficient; A: abundant).

a	*type*	b	*type*
1	D	$3780p$	A
2	D	$1890p$	A
4	D	$945p$	A
6	P	$630p$	A
10	D	$378p$	A
14	D	$270p$	A
18	A	$210p$	D
20	A	$189p$	D
27	D	$140p$	A
28	P	$135p$	D
30	A	$126p$	D
42	A	$90p$	A
54	A	$70p$	A

a	*type*	b	*type*
p	D	3780	A
$2p$	D	1890	A
$4p$	D	945	A
$6p$	A	630	A
$10p$	D	378	A
$14p$	D	270	A
$18p$	A	210	D
$20p$	A	189	D
$27p$	D	140	A
$28p$	A	135	D
$30p$	A	126	D
$42p$	A	90	A
$54p$	A	70	A

Also solved by David E. Manes, Oneonta, NY; Albert Stadler, Herrliberg, Switzerland, and David Stone and John Hawkins (jointly), Statesboro, GA.

- **5166:** *Proposed by José Luis Díaz-Barrero, Barcelona, Spain*

Let a, b, c be lengths of the sides of a triangle ABC. Prove that

$$\left(3^{a+b} + \frac{c}{b}3^{-b}\right)\left(3^{b+c} + \frac{a}{c}3^{-c}\right)\left(3^{c+a} + \frac{b}{a}3^{-a}\right) \geq 8.$$

Solution by Boris Rays, Brooklyn, NY

By the Arithmetic-Geometric-Mean Inequality for each expression in the parentheses above we have:

$$\begin{aligned} 3^{a+b} + \frac{c}{b}3^{-b} &\geq 2\sqrt{3^{a+b} \cdot \frac{c}{b}3^{-b}} = 2\sqrt{\frac{c}{b}3^{a}} \\ 3^{b+c} + \frac{a}{c}3^{-c} &\geq 2\sqrt{3^{b+c} \cdot \frac{a}{c}3^{-c}} = 2\sqrt{\frac{a}{c}3^{b}} \\ 3^{c+a} + \frac{b}{a}3^{-a} &\geq 2\sqrt{3^{c+a} \cdot \frac{b}{a}3^{-a}} = 2\sqrt{\frac{b}{a}3^{c}}. \end{aligned}$$

Therefore,

$$\begin{aligned} \left(3^{a+b} + \frac{c}{b}3^{-b}\right)\left(3^{b+c} + \frac{a}{c}3^{-c}\right)\left(3^{c+a} + \frac{b}{a}3^{-a}\right) &\geq 8\sqrt{\frac{c}{b} \cdot \frac{a}{c} \cdot \frac{b}{a} \cdot 3^{a} \cdot 3^{b} \cdot 3^{c}} \\ &= 8\sqrt{3^{a+b+c}} \end{aligned}$$

$$= 8 \cdot 3^{\dfrac{a+b+c}{2}}.$$

The factor $3^{(a+b+c)/2}$ is an exponential expression with base 3 ($3 > 1$) and exponent $(a+b+c)/2 > 0$. Hence, $3^{(a+b+c)/2} > 1$. Therefore,

$$\left(3^{a+b} + \frac{c}{b}3^{-b}\right)\left(3^{b+c} + \frac{a}{c}3^{-c}\right)\left(3^{c+a} + \frac{b}{a}3^{-a}\right) \geq 8.$$

Also solved by Arkady Alt, San Jose, CA; Bruno Salgueiro Fanego Viveiro Spain; Enkel Hysnelaj, University of Technology, Sydney, Australia jointly with Elton Bojaxhiu, Kriftel, Germany; Valmir Bucaj (student, Texas Lutheran University), Seguin, TX; Elsie M. Campbell, Dionne T. Bailey and Charles Diminnie (jointly), San Angelo TX; Kee-Wai Lau, Hong Kong, China; David E. Manes, Oneonta, NY; Paolo Perfetti, Department of Mathematics, University "Tor Vergata," Rome, Italy; Albert Stadler, Herrliberg, Switzerland: Neculai Stanciu, Buzău, Romania; David Stone and John Hawkins (jointly), Statesboro, GA, and the proposer.

- **5167:** *Paolo Perfetti, Department of Mathematics, University "Tor Vergata," Rome, Italy*

Find the maximum of the real valued function

$$f(x,y) = x^4 - 2x^3 - 6x^2y^2 + 6xy^2 + y^4$$

defined on the set $D = \{(x,y) \in \Re^2 : \; x^2 + 3y^2 \leq 1\}$.

Solution 1 by Michael Brozinsky, Central Islip, NY

We note that the given constraint $x^2 + 3y^2 \leq 1$ implies that $-1 \leq x \leq 1$ and $y^2 \leq \dfrac{1}{3}$. Now, $f(-1,0) = 3$, and to show that 3 is the maximum it suffices to show that $f(x,y) \leq 3 \cdot (x^2 + 3y^2)$. That is

$$x^4 - 2x^3 - 6x^2y^2 + 6xy^2 + y^4 \quad \leq \quad 3 \cdot (x^2 + 3y^2) \text{ or equivalently,}$$

$$x^2 \cdot \left(x^2 - 2x - 3\right) + y^2 \cdot \left(y^2 + 6x\right) \quad \leq \quad y^2 \left(6x^2 + 9\right) \text{ when } (x,y) \text{ is in } D. \qquad (1)$$

Now $x^2 - 2x - 3 \leq 0$ if $-1 \leq x \leq 3$ and $y^2 + 6x \leq \dfrac{1}{3} + 6x \leq 6x^2 + 9$ for all x, (as the minimum of $6x^2 - 6x + 9$ is 7.5), and so (1) is obvious as $x^2\left(x^2 - 2x - 3\right) \leq 0$ and $y^2 \cdot (y^2 + 6x) \leq y^2 \cdot (x^2 + 9)$ when (x,y) is in D.

Solution 2 by Enkel Hysnelaj, University of Technology, Sydney, Australia and Elton Bojaxhiu, Kriftel, Germany

First we will look for extreme points inside the region, which is the set $\{(x,y) \in \Re^2 : x^2 + 3y^2 < 1\}$, and such points will be the critical points of the function $f(x,y)$. The partial derivatives of the function $f(x,y)$ will be

$$\begin{cases} f_x = 4x^3 - 6x^2 - 12xy^2 + 6y^2 \\ f_y = -12x^2y + 12xy + 4y^3 \end{cases}$$

Solving the system of the equations

$$\begin{cases} 4x^3 - 6x^2 - 12xy^2 + 6y^2 = 0 \\ -12x^2y + 12xy + 4y^3 = 0, \end{cases}$$

we have that the only critical point inside the region will be $(x, y) = (0, 0)$, which will be considered a point where the function might get the maximum value.

Now we will find the extremes on the contour of the region, which is $\{(x, y) \in \Re^2 : x^2 + 3y^2 = 1\}$. For any point on the contour we have $y^2 = \dfrac{1 - x^2}{3}$ and substituting this into the formula of $f(x, y)$ we obtain the function $g(x)$ such that

$$\begin{aligned} g(x) &= x^4 - 2x^3 - 6x^2\frac{1 - x^2}{3} + 6x\frac{1 - x^2}{3} + \left(\frac{1 - x^2}{3}\right)^2 \\ &= \frac{1}{9} + 2x - \frac{20x}{9} - 4x^3 + \frac{28x^4}{9} \end{aligned}$$

so, we have to find the extremes of the function $g(x)$ on the segment $[-1, 1]$.

If $x = \pm 1$ we have $f(-1, 0) = 3$ and $f(1, 0) = 1$, and so far, we shown that a local maximum point is when $(x, y) = (-1, 0)$. Now we much check to see if there is a maximum point inside the segment $[-1, 1]$. Taking the derivative of the function $g(x)$ we obtain

$$g'(x) = 2 - \frac{40x}{9} - 12x^2 + \frac{112x^3}{9}.$$

The equation $g'(x) = 0$ has no solution inside the segment $[-1, 1]$, which implies that there is no extreme point inside this segment. And so we may conclude that 3 is the absolute maximum of the real valued function $f(x, y)$ on the given domain and that it is achieved at the point $(x, y) = (-1, 0)$.

Solution 3 byÁngel Plaza, University of Las Palmas de Gran Canaria, Spain

Function $f(x, y)$ is harmonic. Then, by the maximum principle its maximum (and minimum) is attained at the boundary of compact subset D. Since the boundary of D is an ellipse, by using its parametrization the problem is reduced to a one variable optimization problem.

The parametric equations of the given ellipse are

$$x = \cos t; \;\; y = \frac{1}{\sqrt{3}} \sin t$$

and the problem yields to maximizing the function

$$g(t) = \cos^4 t - 2\cos^3 t - 2\cos^2 t \sin^2 t + 2\cos^2 t \sin^2 t + \frac{\sin^4 t}{9} = -2\cos^3 t + \cos^4 t + \frac{\sin^4 t}{9}.$$

Since $g'(t) = -\frac{2}{9} \cos t \sin t \left(-27\cos t + 18\cos^2 t - 2\sin^2 t\right)$ it is deduced that the maximum is attained at $t = \pi$ with the value $f(\pi) = 3$.

Also solved by Pat Costello, Richmond, KY; Bruno Salgueiro Fanego, Viveiro, Spain; Paul M. Harms, North Newton, KS; Kee-Wai Lau, Hong Kong, China; David E. Manes, Oneonta, NY; Boris Rays, Brooklyn, NY; Albert Stadler, Herrliberg, Switzerland; David Stone and John Hawkins (jointly), Statesboro, GA, and the proposer.

- **5168:** *Proposed by G. C. Greubel, Newport News, VA*

  Find the value of a_n in the series

$$\frac{7t+2t^2}{1-36t+4t^2} = a_0 + \frac{a_1}{t} + \frac{a_2}{t^2} + \cdots + \frac{a_n}{t^n} + \cdots.$$

Solution 1 by Bruno Salgueiro Fanego, Viveiro, Spain

By direct division, $4t^2-36t+1|\overline{2t^2+7t}$ we see that $a_0=\frac{1}{2}, a_1=\frac{25}{4}$. Moreover, the characteristic equation of the denominator is $1-36r+4r^2=0$, whose roots are $r_1=\frac{9-4\sqrt{5}}{2}$, $r_2=\frac{9+4\sqrt{5}}{2}$, so $a_n = Ar_1^n + Br_2^n$ for some real numbers A and B . Taking $n=0$, we obtain

$$A+B = A\cdot 1 + B\cdot 1 = Ar_1^0 + Br_2^0 = a_0 = \frac{1}{2},$$

and taking $n=1$ we obtain

$$A\frac{9+4\sqrt{5}}{2} + B\frac{9-4\sqrt{5}}{2} = Ar_1^1 + Br_2^1 = a_1 = \frac{25}{4}.$$

So, by solving the system of equations $\begin{cases} A+B=\frac{1}{2} \\ 18(A+B)+8\sqrt{5}(A-B)=25 \end{cases}$ we obtain

$$A=\frac{5-4\sqrt{5}}{20},\ B=\frac{5+4\sqrt{5}}{20}.$$

Hence,

$$a_n = Ar_1^n + Br_2^n = \frac{5-4\sqrt{5}}{20}\left(\frac{9-4\sqrt{5}}{2}\right)^n + \frac{5+4\sqrt{5}}{20}\left(\frac{9+4\sqrt{5}}{2}\right)^n.$$

Solution 2 by Kee-Wai Lau, Hong Kong, China

It can be checked readily that

$$\frac{7t+2t^2}{1-36t+4t^2} = \frac{5+4\sqrt{5}}{20}\left(\frac{1}{1-\frac{9+4\sqrt{5}}{2t}}\right) + \frac{5-4\sqrt{5}}{20}\left(\frac{1}{1-\frac{9-4\sqrt{5}}{2t}}\right).$$

For $t > \frac{9+4\sqrt{5}}{2}$, we have $\frac{1}{1-\frac{9+4\sqrt{5}}{2}} = \sum_{n=0}^{\infty}\left(\frac{9+4\sqrt{5}}{2t}\right)^n$ and $\frac{1}{1-\frac{9-4\sqrt{5}}{2t}} = \sum_{n=0}^{\infty}\left(\frac{9-4\sqrt{5}}{2t}\right)^n$. Hence for positive integer n

$$a_n = \frac{5+4\sqrt{5}}{20}\left(\frac{9+4\sqrt{5}}{2}\right)^n + \frac{5-4\sqrt{5}}{20}\left(\frac{9-4\sqrt{5}}{2}\right)^n.$$

Solution 3 by Dionne Bailey, Elsie Campbell, and Charles Diminnie, San Angelo, TX

Let $\{f_n\}$ be the Fibonacci sequence defined by $f_0 = 0$, $f_1 = 1$, and $f_{n+2} = f_{n+1} + f_n$ for $n \geq 0$. Also, let $\phi = \frac{1+\sqrt{5}}{2}$ and $\overline{\phi} = \frac{1-\sqrt{5}}{2}$. Then, we will use Binet's Formula

$$f_n = \frac{\phi^n - \overline{\phi}^n}{\sqrt{5}}$$

for $n \geq 0$ and the known results

$$\phi^n = f_n\phi + f_{n-1}, \overline{\phi}^n = f_n\overline{\phi} + f_{n-1}, \text{and } \phi\overline{\phi} = -1 \tag{1}$$

for $n \geq 1$.

To begin, make the change of variable $s = \frac{1}{t}$ and simplify to get

$$\frac{7t + 2t^2}{1 - 36t + 4t^2} = \frac{7s + 2}{s^2 - 36s + 4}.$$

Note that (1) implies that $\phi^6 = f_6\phi + f_5 = 8\phi + 5 = 9 + 4\sqrt{5}$ and similarly, $\overline{\phi}^6 = 9 - 4\sqrt{5}$. Then, the roots of $s^2 - 36s + 4$ are $s = 18 \pm 8\sqrt{5} = 2\phi^6, 2\overline{\phi}^6$ and we have

$$\frac{7s + 2}{s^2 - 36s + 4} = \frac{7s + 2}{\left(s - 2\phi^6\right)\left(s - 2\overline{\phi}^6\right)}.$$

If we perform a partial fraction expansion and use Binet's Formula, (1), and the formula for a geometric series, we obtain

$$\begin{aligned}
\frac{7s + 2}{s^2 - 36s + 4} &= \frac{7\phi^6 + 1}{8\sqrt{5}}\frac{1}{s - 2\phi^6} - \frac{7\overline{\phi}^6 + 1}{8\sqrt{5}}\frac{1}{s - 2\overline{\phi}^6} \\
&= \frac{1}{8\sqrt{5}}\left[-\frac{7\phi^6 + 1}{2\phi^6}\frac{1}{1 - \left(\frac{s}{2\phi^6}\right)} + \frac{7\overline{\phi}^6 + 1}{2\overline{\phi}^6}\frac{1}{1 - \left(\frac{s}{2\overline{\phi}^6}\right)}\right] \\
&= \frac{1}{16\sqrt{5}}\left[\left(7 + \phi^6\right)\sum_{n=0}^{\infty}\frac{1}{\left(2\overline{\phi}^6\right)^n}s^n - \left(7 + \overline{\phi}^6\right)\sum_{n=0}^{\infty}\frac{1}{\left(2\phi^6\right)^n}s^n\right] \\
&= \frac{1}{16\sqrt{5}}\sum_{n=0}^{\infty}\frac{1}{2^n}\left[\frac{7 + \phi^6}{\overline{\phi}^{6n}} - \frac{7 + \overline{\phi}^6}{\phi^{6n}}\right]s^n \\
&= \frac{1}{16\sqrt{5}}\sum_{n=0}^{\infty}\frac{1}{2^n}\frac{\left(7 + \phi^6\right)\phi^{6n} - \left(7 + \overline{\phi}^6\right)\overline{\phi}^{6n}}{(-1)^{6n}}s^n \\
&= \sum_{n=0}^{\infty}\frac{1}{2^{n+4}}\left[7\left(\frac{\phi^{6n} - \overline{\phi}^{6n}}{\sqrt{5}}\right) + \left(\frac{\phi^{6n+6} - \overline{\phi}^{6n+6}}{\sqrt{5}}\right)\right]s^n
\end{aligned}$$

$$= \sum_{n=0}^{\infty} \frac{7f_{6n} + f_{6n+6}}{2^{n+4}} s^n$$

$$= \sum_{n=0}^{\infty} \frac{2f_{6n+1} + 3f_{6n}}{2^{n+2}} s^n$$

$$= \sum_{n=0}^{\infty} \frac{2f_{6n+1} + 3f_{6n}}{2^{n+2}} \frac{1}{t^n}.$$

Also, since $\left|\overline{\phi}\right| < \phi$, the series converges when

$$|s| < \min\left\{2\left|\overline{\phi}\right|^6, 2\phi^6\right\} = 2\left|\overline{\phi}\right|^6,$$

i.e., when

$$|t| > \frac{1}{2\left|\overline{\phi}\right|^6} = \frac{\phi^6}{2}.$$

Therefore,

$$a_n = \frac{2f_{6n+1} + 3f_{6n}}{2^{n+2}}$$

for $n \geq 0$.

Also solved by Arkady Alt, San Jose, CA; Brian D. Beasley, Clinton, SC; Paul M. Harms, North Newton, KS; Enkel Hysnelaj, University of Technology, Sydney, Australia jointly with Elton Bojaxhiu, Kriftel, Germany; David E. Manes, Oneonta, NY; Ángel Plaza (University of Las Palmas de Gran Canaria), Spain; Boris Rays, Brooklyn, NY; Albert Stadler, Herrliberg, Switzerland; David Stone and John Hawkins (jointly), Statesboro, GA; Boris Rays, Brooklyn, NY, and the proposer.

- **5169:** *Proposed by Ovidiu Furdui, Cluj, Romania*

  Let $n \geq 1$ be an integer and let i be such that $1 \leq i \leq n$. Calculate:

$$\int_0^1 \cdots \int_0^1 \frac{x_i}{x_1 + x_2 + \cdots + x_n} dx_1 \cdots dx_n.$$

Solutions 1 and 2 by Albert Stadler, Herrliberg, Switzerland

1) Let $I_i = \int_0^1 \cdots \int_0^1 \frac{x_i}{x_1 + x_2 + \cdots + x_n} dx_1 \cdots dx_n$. Then by symmetry, $I_1 = I_2 = \cdots I_n$. So,

$$I_1 + I_2 + \cdots + I_n = \int_0^1 \cdots \int_0^1 \frac{x_1 + x_2 + \cdots x_n}{x_1 + x_2 + \cdots + x_n} dx_1 \cdots dx_n = 1,$$

and $I_i = \dfrac{1}{n}$ for $1 \leq i \leq n$.

2) Another albeit less elegant proof runs as follows:

$$\begin{aligned}
\int_0^1 \cdots \int_0^1 \frac{x_i}{x_1+x_2+\cdots+x_n} dx_1 \cdots dx_n &= \int_0^\infty \int_0^1 \cdots \int_0^1 x_i e^{-t(x_1+x_2+\cdots x_n)} dx_1 \cdots dx_n dt \\
&= \int_0^\infty \frac{\left(1-e^{-t}\right)^{n-1}\left(1-(1+t)e^{-t}\right)}{t^{n+1}} dt \\
&= -\frac{1}{n}\int_0^\infty \frac{d}{dt}\frac{(1-e^{-t})^n}{t^n} dt \\
&= \frac{1}{n}\lim_{t\to 0}\frac{(1-e^{-t})^n}{t^n} = \frac{1}{n}.
\end{aligned}$$

The above is so because:

$$\int_0^1 e^{-tx_j} dx_j = \frac{1-e^{-t}}{t}, \quad \int_0^1 x_i e^{-tx_i} dx_i = \frac{1-(1+t)e^{-t}}{t^2},$$

$$\frac{d}{dt}\frac{(1-e^{-t})^n}{t^n} = -\frac{n(1-e^{-t})^n}{t^{n+1}} + \frac{n(1-e^{-t})^{n-1}e^{-t}}{t^n} = -n\frac{(1-e^{-t})^{n-1}(1-(1+t)e^{-t})}{t^{n+1}}.$$

Also solved by Michael N. Fried, Kibbutz Revivim, Israel; Enkel Hysnelaj, University of Technology, Sydney, Australia jointly with Elton Bojaxhiu, Kriftel, Germany; Kee-Wai Lau, Hong Kong, China; Paolo Perfetti, Department of Mathematics, University "Tor Vergata," Rome, Italy; David Stone and John Hawkins (jointly), Statesboro, GA, and the proposer.