Problems

Ted Eisenberg, Section Editor

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This section of the Journal offers readers an opportunity to exchange interesting mathematical problems and solutions. Please send them to Ted Eisenberg, Department of Mathematics, Ben-Gurion University, Beer-Sheva, Israel or fax to: 972-86-477-648. Questions concerning proposals and/or solutions can be sent e-mail to <eisenbt@013.net>. Solutions to previously stated problems can be seen at <http://ssmj.tamu.edu>.

Solutions to the problems stated in this issue should be posted before February 15, 2010

- 5086: *Proposed by Kenneth Korbin, New York, NY*

  Find the value of the sum
  $$\frac{2}{3}+\frac{8}{9}+\cdots+\frac{2N^2}{3^N}.$$

- 5087: *Proposed by Kenneth Korbin, New York, NY*

  Given positive integers $a, b, c,$ and d such that $(a+b+c+d)^2 = 2(a^2+b^2+c^2+d^2)$ with $a < b < c < d$. Rationalize and simplify
  $$\frac{\sqrt{x+y}-\sqrt{x}}{\sqrt{x+y}+\sqrt{x}} \quad \text{if} \quad \begin{cases} x = bc+bd+cd, & \text{and} \\ y = ab+ac+ad. \end{cases}$$

- 5088: *Proposed by Isabel Iriberri Díaz and José Luis Díaz-Barrero, Barcelona, Spain*

  Let a, b be positive integers. Prove that
  $$\frac{\varphi(ab)}{\sqrt{\varphi^2(a^2)+\varphi^2(b^2)}} \le \frac{\sqrt{2}}{2},$$
  where $\varphi(n)$ is Euler's totient function.

- 5089: *Proposed by Panagiote Ligouras, Alberobello, Italy*

  In $\triangle ABC$ let $AB = c, BC = a, CA = b, r =$ the in-radius and $r_a, r_b,$ and $r_c=$ the ex-radii, respectively.

  Prove or disprove that
  $$\frac{(r_a-r)(r_b+r_c)}{r_ar_c+rr_b}+\frac{(r_c-r)(r_a+r_b)}{r_cr_b+rr_a}+\frac{(r_b-r)(r_c+r_a)}{r_br_a+rr_c} \ge 2\left(\frac{ab}{b^2+ca}+\frac{bc}{c^2+ab}+\frac{ca}{a^2+bc}\right).$$

- 5090: *Proposed by Mohsen Soltanifar (student), University of Saskatchewan, Canada*

  Given a prime number p and a natural number n. Calculate the number of elementary matrices $E_{n\times n}$ over the field Z_p.

- 5091: *Proposed by Ovidiu Furdui, Cluj, Romania*

Let $k, p \geq 0$ be nonnegative integers. Evaluate the integral

$$\int_{-\pi/2}^{\pi/2} \frac{\sin^{2p} x}{1 + \sin^{2k+1} x + \sqrt{1 + \sin^{4k+2} x}} dx.$$

Solutions

- 5068: *Proposed by Kenneth Korbin, New York, NY.*

  Find the value of

$$\sqrt{1 + 2009\sqrt{1 + 2010\sqrt{1 + 2011\sqrt{1 + \cdots}}}}.$$

Solution by Dmitri V. Skjorshammer (student, Harvey Mudd College), Claremont, CA

To solve this, we apply Ramanujan's nested radical. Consider the identity $(x+n)^2 = x^2 + 2nx + n^2$, which can be rewritten as

$$x + n = \sqrt{n^2 + x((x+n)+n)}.$$

Now, the $(x+n)+n$ term has the same form as the left-hand side, so we can write it in terms of a radical:

$$x + n = \sqrt{n^2 + x\sqrt{n^2 + (x+n)((x+2n)+n)}}$$

Repeating this process, ad infinitum, yields Ramanujan's nested radical:

$$x + n = \sqrt{n^2 + x\sqrt{n^2 + (x+n)\sqrt{n^2 + \cdots}}}$$

With $n = 1$ and $x = 2009$, the right-hand side becomes the expression in the problem. It follows that the value is 2010.

Also solved by Bruno Salgueiro Fanego, Viveiro, Spain; Pat Costello, Richmond, KY; Michael N. Fried, Kibbutz Revivim, Israel; David E. Manes, Oneonta, NY; Paolo Perfetti, Department of Mathematics, University Tor Vergata, Rome, Italy; David Stone and John Hawkins (jointly), Statesboro, GA; Nguyen Van Vinh (student, Belarusian State University), Minsk, Belarus, and the proposer.

- 5069: *Proposed by Kenneth Korbin, New York, NY.*

  Four circles having radii $\frac{1}{14}$, $\frac{1}{15}$, $\frac{1}{x}$ and $\frac{1}{y}$ respectively, are placed so that each of the circles is tangent to the other three circles. Find positive integers x and y with $15 < x < y < 300$.

Solution by Bruno Salgueiro Fanego, Viveiro, Spain

If all the circles are tangent in a point, the problem is not interesting because x and y can take on any value for which $15 < x < y < 300$. So we assume that the circles are not mutually tangent at a point.
By Descarte's circle theorem with ϵ_1, ϵ_2 and ϵ_3 being the curvature of the first three circles, the curvature ϵ_4 of the fourth circle can be obtained with Soddy's formula:

$$\begin{aligned} \epsilon_4 &= \epsilon_1 + \epsilon_2 + \epsilon_3 \pm 2\sqrt{\epsilon_1\epsilon_2 + \epsilon_2\epsilon_3 + \epsilon_3\epsilon_1}, \text{ that is,} \\ y &= 14 + 15 + x \pm 2\sqrt{14 \cdot 15 + 15 \cdot x + x \cdot 14} \\ y &= 29 + x + \pm 2\sqrt{210 + 29x} \end{aligned}$$

Then, $210 + 29x$ must be a perfect square, say a^2. Since, $15 < x < 300$,

$$25^2 < 210 + 29x < 95^2, \text{ so}$$
$$26 \le a \le 94.$$

Thus,

$$29 \Big| (a^2 - 210).$$

The only integers a, $26 \le a \le 94$, which satisfy this condition are $35, 52, 64, 81$, and 93. Taking into account that $15 < x < y < 300$, we have:

$$\begin{aligned} \text{For } a &= 35,\ x = \mathit{35} \text{ and so } y = \mathit{29} + x \pm \mathit{2a} = \mathit{134} \\ \text{For } a &= 52,\ x = 86 \text{ and } y = \mathit{219}; \\ \text{For } a &= 64,\ x = 134 \text{ and } y = \mathit{291}; \end{aligned}$$

and for $a \in \{81, 93\}$, none of the obtained values of y is valid.

Thus the only pairs of integers x and y with $15 < x < y < 300$ are

$$(x, y) \in \Big\{(35, 134), (86, 219), (134, 291)\Big\}.$$

Also solved by Michael N. Fried, Kibbutz Revivim, Israel; Paul M. Harms, North Newton, KS; John Hawkins and David Stone (jointly), Statesboro, GA; Antonio Ledesma Vila, Requena-Valencia, Spain, and the proposer.

- **5070:** *Proposed by Isabel Iriberri Díaz and José Luis Díaz- Barrero, Barcelona, Spain.*

Find all real solutions to the system

$$\left.\begin{aligned} 9(x_1^2 + x_2^2 - x_3^2) &= 6x_3 - 1, \\ 9(x_2^2 + x_3^2 - x_4^2) &= 6x_4 - 1, \\ &\cdots\cdots\cdots \\ 9(x_n^2 + x_1^2 - x_2^2) &= 6x_2 - 1. \end{aligned}\right\}$$

Solution by Antonio Ledesma Vila, Requena -Valencia, Spain

Add all

$$\begin{aligned}9(x_1^2+x_2^2-x_3^2) &= 6x_3-1\\ 9(x_2^2+x_3^2-x_4^2) &= 6x_4-1\\ &\cdots\\ 9(x_n^2+x_1^2-x_2^2) &= 6x_2-1\end{aligned}$$

$$9\left(\sum_{i=1}^n x_i^2+\sum_{i=1}^n x_i^2-\sum_{i=1}^n x_i^2\right)=6\sum_{i=1}^n x_i-n$$

$$9\sum_{i=1}^n x_i^2=6\sum_{i=1}^n x_i-n$$

$$\sum_{i=1}^n (3x_i)^2=2\sum_{i=1}^n (3x_i)-n$$

$$\sum_{i=1}^n (3x_i)^2-2\sum_{i=1}^n (3x_i)+n=0$$

$$\sum_{i=1}^n (3x_i-1)^2=0,$$

$$x_i=\frac{1}{3} \text{ for all } i$$

Also solved by Elsie M. Campbell, Dionne T. Bailey, and Charles Diminnie (jointly), San Angelo, TX; Bruno Salgueiro Fanego, Viveiro, Spain; Kee-Wai Lau, Hong Kong; China; David E. Manes, Oneonta, NY; John Nord, Spokane, WA; Paolo Perfetti, Department of Mathematics, University Tor Vergata, Rome, Italy; Boris Rays, Brooklyn, NY; Dmitri V. Skjorshammer (student, Harvey Mudd College), Claremont, CA, and the proposer.

- 5071: *Proposed by José Luis Díaz-Barrero, Barcelona, Spain.*

Let h_a, h_b, h_c be the altitudes of $\triangle ABC$ with semi-perimeter s, in-radius r and circum-radius R, respectively. Prove that

$$\frac{1}{4}\left(\frac{s(2s-a)}{h_a}+\frac{s(2s-b)}{h_b}+\frac{s(2s-c)}{h_c}\right)\le\frac{R^2}{r}\left(\sin^2 A+\sin^2 B+\sin^2 C\right).$$

Solution by Charles McCracken, Dayton, OH

Multiply both sides of the inequality by 4 to obtain

$$\frac{s(2s-a)}{h_a}+\frac{s(2s-b)}{h_b}+\frac{s(2s-c)}{h_c} \le \frac{(2R)^2}{r}\left[\sin^2 A+\sin^2 B+\sin^2 C\right]$$

$$\frac{s(2s-a)}{h_a}+\frac{s(2s-b)}{h_b}+\frac{s(2s-c)}{h_c} \le \frac{1}{r}\left[(2R)^2\sin^2 A+(2R)^2\sin^2 B+(2R)^2\sin^2 C\right].$$

Now $2R = \dfrac{a}{\sin A} = \dfrac{b}{\sin B} = \dfrac{c}{\sin C}$ so the inequality becomes

$$\frac{s(2s-a)}{h_a} + \frac{s(2s-b)}{h_b} + \frac{s(2s-c)}{h_c} \le \frac{1}{r}\Big(a^2+b^2+c^2\Big).$$

From Johnson (Roger A. Johnson, Advanced Euclidean Geometry, Dover, 2007, p. 11) we have

$$h_a = \frac{2\Delta}{a},\ h_b = \frac{2\Delta}{b},\ h_c = \frac{2\Delta}{c},\ \text{where } \Delta \text{ represents the area of the triangle.}$$

The inequality now takes the form

$$\frac{as(2s-a)}{2\Delta} + \frac{bs(2s-b)}{2\Delta} + \frac{cs(2s-c)}{2\Delta} \le \frac{1}{r}\Big(a^2+b^2+c^2\Big).$$

Since $\Delta = rs$, we now have our inequality in the form

$$\begin{aligned}
\frac{as(2s-a)}{2rs} + \frac{bs(2s-b)}{2rs} + \frac{cs(2s-c)}{2rs} &\le \frac{1}{r}\Big(a^2+b^2+c^2\Big)\\
\frac{a(2s-a)}{2} + \frac{b(2s-b)}{2} + \frac{c(2s-c)}{2} &\le \Big(a^2+b^2+c^2\Big)
\end{aligned}$$

Substituting $a+b+c$ for $2s$ we have

$$\begin{aligned}
a(b+c)+b(c+a)+c(a+b) &\le 2a^2+2b^2+2c^2\\
ab+ac+bc+ba+ca+cb &\le 2a^2+2b^2+2c^2\\
ab+bc+ca &\le a^2+b^2+c^2
\end{aligned}$$

This last inequality, $ab+bc+ca \le a^2+b^2+c^2$, can be readily proved true for any triple of positive numbers a, b, c by letting $b = a+\delta$ and $c = a+\epsilon$ with $0 < \delta < \epsilon$. Hence the original inequality holds.

Also solved by Bruno Salgueiro Fanego, Viveiro, Spain; Paul M. Harms, North Newton, KS; Kee-Wai Lau, Hong Kong, China; Boris Rays, Brooklyn, NY; David Stone and John Hawkins (jointly), Statesboro, GA, and the proposer.

- 5072: *Proposed by Panagiote Ligouras, Alberobello, Italy.*

  Let a, b and c be the sides, l_a, l_b, l_c the bisectors, m_a, m_b, m_c the medians, and h_a, h_b, h_c the heights of $\triangle ABC$. Prove or disprove that

  a) $\dfrac{(-a+b+c)^3}{a} + \dfrac{(a-b+c)^3}{b} + \dfrac{(a+b-c)^3}{c} \ge \dfrac{4}{3}\Big(m_a \cdot l_a + l_b \cdot h_b + h_c \cdot m_c\Big)$

  b) $3\sum_{cyc}\dfrac{(-a+b+c)^3}{a} \ge 2\sum_{cyc}[m_a(l_a+h_a)].$

Solution by proposer

We have

$$\frac{(-a+b+c)^3}{a}+\frac{(a-b+c)^3}{b}+\frac{(a+b-c)^3}{c}\geq a^2+b^2+c^2. \qquad (1)$$

In fact, the equality is homogeneous and putting $a+b=c=1$ gives

$$\frac{(-a+b+c)^3}{a}+\frac{(a-b+c)^3}{b}+\frac{(a+b-c)^3}{c}\geq a^2+b^2+c^2 \Leftrightarrow \sum_{cyc}\frac{(1-2a)^3}{a}\geq\sum_{cyc}a^2. \qquad (2)$$

Applying Chebyshev's Inequality gives

$$\sum_{cyc}\frac{(1-2a)^3}{a}=\sum_{cyc}\frac{1}{a}(1-2a)^3\geq\frac{1}{3}\Big(\sum_{cyc}\frac{1}{a}\Big)\cdot\Big[\sum_{cyc}(1-2a)^3\Big]. \qquad (3)$$

Using the well known equalities

$$\sum x^3=\Big(\sum x\Big)^3-3(x+y)(y+z)(z+x). \quad (4)$$

$$(a+b+c)\Big(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\Big)\geq 3^2=9 \qquad (5)$$

and applying (4), (3), and (5) we have

$$\begin{aligned}
&\sum_{cyc}\frac{(1-2a)^3}{a}\geq\frac{1}{3}\Big(\sum_{cyc}\frac{1}{a}\Big)\cdot\Big[\sum_{cyc}(1-2a)^3\Big]\\
=\ &\frac{1}{3}\Big(\sum_{cyc}\frac{1}{a}\Big)\cdot\Big[(1-2a+1-2b+1-2c)^3-3(1-2a+1-2b)(1-2b+1-2c)(1-2c+1-2a)\Big]\\
=\ &\frac{1}{3}\Big(\sum_{cyc}\frac{1}{a}\Big)\cdot[1-24abc]\\
=\ &\frac{1}{3}\Big(\sum_{cyc}\frac{1}{a}\Big)\cdot(\sum a)-\frac{24}{3}\Big(\sum_{cyc}ab\Big)\\
\geq\ &\frac{1}{3}\cdot 9-8\Big(\sum_{cyc}ab\Big)\\
\Leftrightarrow\ &\sum_{cyc}\frac{(1-2a)^3}{a}\geq 3-8\Big(\sum_{cyc}ab\Big). \qquad (6)
\end{aligned}$$

We have

$$3-8\Big(\sum_{cyc}ab\Big)\geq\sum_{cyc}a^2. \qquad (7)$$

In fact,

$$3-8\Big(\sum_{cyc}ab\Big)\geq\sum_{cyc}a^2\Leftrightarrow 3-6\Big(\sum_{cyc}ab\Big)\geq\sum_{cyc}a^2+2\Big(\sum_{cyc}ab\Big)$$

$$\Leftrightarrow \quad 3-6\Big(\sum_{cyc} ab\Big) \geq \Big(\sum_{cyc} a\Big)^2 = 1 \Leftrightarrow 3-6\Big(\sum_{cyc} ab\Big) \geq 1-3$$

$$\Leftrightarrow \quad \sum_{cyc} ab \leq \frac{1}{3} = \frac{\Big(\sum a\Big)^2}{3}$$

$$\Leftrightarrow \quad \sum_{cyc}(a-b)^2 \geq 0, \text{ and this last statement is true.}$$

Using (6) and (7) we have

$$\sum_{cyc} \frac{(1-2a)^3}{a} \geq 3-8\Big(\sum_{cyc} ab\Big) \geq \sum_{cyc} a^2$$
$$\Leftrightarrow \sum_{cyc} \frac{(1-2a)^3}{a} \geq \sum_{cyc} a^2, \text{ and (1) is true.}$$

Is well known that

$$\begin{aligned} a^2+b^2 &= 2m_c^2+\frac{1}{2}c^2 \qquad \text{(A)} \\ c^2+b^2 &= 2m_a^2+\frac{1}{2}a^2 \qquad \text{(B)} \\ c^2+a^2 &= 2m_b^2+\frac{1}{2}b^2 \qquad \text{(C)} \end{aligned}$$

For (A),(B), and (C)

$$\begin{aligned} m_a^2+m_b^2+m_c^2 &= \frac{3}{4}(a^2+b^2+c^2) \text{ and} \\ a^2+b^2+c^2 &= \frac{4}{3}(m_a^2+m_b^2+m_c^2) \quad (8) \end{aligned}$$

It is also well known that

$$m_a \geq l_a \geq h_a, \quad m_b \geq l_b \geq h_b, \quad m_c \geq l_c \geq h_c. \quad (9)$$

Using (9) we have

$$m_a^2 \geq m_a \cdot l_a \geq m_a \cdot h_a, \quad m_b^2 \geq m_b \cdot l_b \geq m_b \cdot h_b, \quad m_c^2 \geq m_c \cdot l_c \geq m_c \cdot h_c \quad \text{(D)}$$

$$m_a^2 \geq l_a \cdot h_a, \quad m_b^2 \geq l_b \cdot h_b, \quad m_c^2 \geq l_c \cdot h_c, \quad \text{(E)}$$

Using (8) and (D) we have

$$a^2+b^2+c^2 \geq \frac{4}{3}(m_a l_a + m_b l_b + m_c l_c). \quad (10)$$

$$a^2+b^2+c^2 \geq \frac{4}{3}(m_a h_a + m_b h_b + m_c h_c). \quad (11)$$

And using (8), (D), and (E) we have

$$a^2+b^2+c^2 \geq \frac{4}{3}(m_a l_a + l_b h_b + h_c m_c). \quad (12)$$

For part a of the problem, using (1) and (12) we have

$$\frac{(-a+b+c)^3}{a}+\frac{(a-b+c)^3}{b}+\frac{(a+b-c)^3}{c} \geq \frac{4}{3}\left(m_a \cdot l_a + l_b \cdot h_b + h_c \cdot m_c\right)$$

For part b of the problem, using (1), (10) and (11) we have

$$2\left[\frac{(-a+b+c)^3}{a}+\frac{(a-b+c)^3}{b}+\frac{(a+b-c)^3}{c}\right] \geq$$
$$\frac{4}{3}\left(m_a \cdot l_a + m_b \cdot l_b + m_c \cdot l_c\right)+\frac{4}{3}\left(m_a \cdot h_a + m_b \cdot h_b + m_c \cdot h_c\right)$$

$$\Leftrightarrow \quad \frac{(-a+b+c)^3}{a}+\frac{(a-b+c)^3}{b}+\frac{(a+b-c)^3}{c}$$

$$\geq \quad \frac{2}{3}\left(m_a \cdot l_a + m_b \cdot l_b + m_c \cdot l_c + m_a \cdot h_a + m_b \cdot h_b + m_c \cdot h_c\right)$$

$$\Leftrightarrow \quad \frac{(-a+b+c)^3}{a}+\frac{(a-b+c)^3}{b}+\frac{(a+b-c)^3}{c}$$

$$\geq \quad \frac{2}{3}\left[m_a \cdot (l_a+h_a) + m_b \cdot (l_b+h_b) + m_c \cdot (l_c+h_c)\right]$$

- 5073: *Proposed by Ovidiu Furdui, Campia-Turzii, Cluj, Romania.*

Let $m > -1$ be a real number. Evaluate

$$\int_0^1 \{\ln x\} x^m dx,$$

where $\{a\} = a - [a]$ denotes the fractional part of a.

Solution 1 by Bruno Salgueiro Fanego, Viveiro, Spain

$$I_m = \int_0^1 \{\ln x\} x^m dx = \int_0^1 (\ln x - [\ln x]) x^m dx = \int_0^1 (\ln x) x^m dx - \int_0^1 [\ln x] x^m dx = A - B$$

where $A = \displaystyle\int_0^1 (\ln x) x^m dx$ and $B = \displaystyle\int_0^1 [\ln x] x^m dx$. Integrating by parts $\left(\int udv = uv - \int vdu \text{ with } u = \ln x \text{ and } dv = x^m dx\right)$, and by using Barrow's and L'Hospital's rule we obtain,

$$\int (\ln x) x^m dx \quad = \quad \frac{(\ln x) x^{m+1}}{m+1} - \int \frac{x^m}{m+1} dx = \frac{(\ln x) x^{m+1}}{m+1} - \frac{x^{m+1}}{(m+1)^2}$$

$$\Longrightarrow A = \frac{(\ln x)x^{m+1}}{m+1} - \frac{x^{m+1}}{(m+1)^2}\Big|_0^1$$

$$= \frac{(\ln 1)1^{m+1}}{m+1} - \frac{1^{m+1}}{(m+1)^2} - \left(\lim_{x\to 0^+}\frac{(\ln x)x^{m+1}}{m+1} - \frac{0^{m+1}}{(m+1)^2}\right)$$

$$= \frac{-1}{(m+1)^2} - \lim_{x\to 0^+}\frac{(\ln x)}{(m+1)x^{-(m+1)}}$$

$$= \frac{-1}{(m+1)^2} - \lim_{x\to 0^+}\frac{x^{-1}}{-(m+1)^2x^{-(m+2)}}$$

$$= \frac{-1}{(m+1)^2} + \lim_{x\to 0^+}\frac{x^{m+1}}{(m+1)^2}$$

$$= \frac{-1}{(m+1)^2}$$

With the partition $\left\{\ldots, e^{-n},\ e^{-n+1},\ e^{-n+2}, \ldots,\ e^{-2},\ e^{-1},\ e^0 = 1\right\}$ of $(0,1]$, being $[\ln x] = -n$ for $e^{-n} \le x < e^{-n+1}$, and $\left|e^{-m-1}\right| < 1$,

$$B = \int_0^1 [\ln x]x^m dx = \sum_{n=1}^{\infty}\int_{e^{-n}}^{e^{-n+1}} [\ln x]x^m dx$$

$$= \sum_{n=1}^{\infty}\int_{e^{-n}}^{e^{-n+1}} (-n)x^m dx = \sum_{n=1}^{\infty}\frac{-nx^{m+1}}{m+1}\Big|_{e^{-n}}^{e^{-n+1}}$$

$$= \sum_{n=1}^{\infty}\frac{-n\left(e^{(-n+1)(m+1)} - e^{-n(m+1)}\right)}{m+1}$$

$$= \sum_{n=1}^{\infty}\frac{-n\left(e^{m+1}e^{(-n)(m+1)} - e^{-n(m+1)}\right)}{m+1}$$

$$= \sum_{n=1}^{\infty}\frac{-n\left(e^{(m+1)} - 1\right)e^{-n(m+1)}}{m+1}$$

$$= \sum_{n=1}^{\infty}\frac{\left(1 - e^{(m+1)}\right)n\left(e^{-m-1}\right)^n}{m+1}$$

$$= \frac{\left(1-e^{m+1}\right)e^{-m-1}}{m+1}\sum_{n=1}^{\infty}\left(e^{-m-1}\right)^{n-1}$$

$$= \frac{e^{-m-1}-1}{m+1}\sum_{n=1}^{\infty}\frac{d}{dx}x^n\Big|_{x=e^{-m-1}}$$

$$= \frac{e^{-m-1}-1}{m+1}\frac{d}{dx}\sum_{n=1}^{\infty}x^n\Big|_{x=e^{-m-1}}$$

$$= \frac{e^{-m-1}-1}{m+1}\frac{d}{dx}\frac{x}{1-x}\Big|_{x=e^{-m-1}}$$

$$= \frac{e^{-m-1}-1}{(m+1)(1-x)^2}\Big|_{x=e^{-m-1}}$$

$$= \frac{e^{-m-1}-1}{(m+1)(e^{-m-1}-1)^2} = \frac{1}{(m+1)(e^{-m-1}-1)}, \text{ so}$$

$$I_m = A-B = -\frac{1}{(m+1)^2} - \frac{1}{(m+1)(e^{-m-1}-1)}$$

$$= \frac{me^{m+1}+1}{(m+1)^2(e^{m+1}-1)}.$$

Solution 2 by the proposer

The integral equals

$$\frac{e^{m+1}}{(m+1)(e^{m+1}-1)} - \frac{1}{(1+m)^2}.$$

We have, by making the substitution $\ln x = y$, that

$$\int_0^1 \{\ln x\}\, x^m dx = \int_{-\infty}^{0} \{y\}\, e^{(m+1)y} dy$$

$$= \sum_{k=0}^{\infty}\int_{-k-1}^{-k} \{y\}\, e^{(m+1)y} dy$$

$$= \sum_{k=0}^{\infty}\int_{-k-1}^{-k} (y-(-k-1))\, e^{(m+1)y} dy$$

$$= \sum_{k=0}^{\infty}\int_{-k-1}^{-k} (y+k+1))\, e^{(m+1)y} dy$$

$$= \sum_{k=0}^{\infty}\left(\frac{y+k+1}{m+1}e^{(m+1)y}\Big|_{-k-1}^{-k} - \frac{e^{(m+1)y}}{(m+1)^2}\Big|_{-k-1}^{-k}\right)$$

$$= \sum_{k=0}^{\infty}\frac{e^{-(m+1)k}}{m+1} - \frac{1}{(m+1)^2}\sum_{k=0}^{\infty}\left(e^{-(m+1)k} - e^{-(m+1)(k+1)}\right)$$

$$= \frac{e^{m+1}}{(m+1)(e^{m+1}-1)} - \frac{1}{(1+m)^2},$$

and the problem is solved.

Also solved by Valmir Bucaj (student, Texas Lutheran University), Seguin, TX; Paul M. Harms, North Newton, KS; Kee-Wai Lau, Hong Kong, China; Paolo Perfetti, Department of Mathematics, University Tor Vergata, Rome, Italy; and David Stone and John Hawkins (jointly), Statesboro, GA.