Problems

Ted Eisenberg, Section Editor

****

This section of the Journal offers readers an opportunity to exchange interesting mathematical problems and solutions. Please send them to Ted Eisenberg, Department of Mathematics, Ben-Gurion University, Beer-Sheva, Israel or fax to: 972-86-477-648. Questions concerning proposals and/or solutions can be sent e-mail to <eisenbt@013.net>. Solutions to previously stated problems can be seen at <http://ssmj.tamu.edu>.

Solutions to the problems stated in this issue should be posted before April 15, 2010

- 5098: *Proposed by Kenneth Korbin, New York, NY*

  Given integer-sided triangle ABC with $\angle B = 60^\circ$ and with $a < b < c$. The perimeter of the triangle is $3N^2 + 9N + 6$, where N is a positive integer. Find the sides of a triangle satisfying the above conditions.

- 5099: *Proposed by Kenneth Korbin, New York, NY*

  An equilateral triangle is inscribed in a circle with diameter d. Find the perimeter of the triangle if a chord with length $d - 1$ bisects two of its sides.

- 5100: *Proposed by Mihály Bencze, Brasov, Romania*

  Prove that
  $$\sum_{k=1}^{n} \sqrt{\frac{k}{k+1}} \binom{n}{k} \le \sqrt{\frac{n(2^{n+1}-n)2^{n-1}}{n+1}}$$

- 5101: *Proposed by K. S. Bhanu and M. N. Deshpande, Nagpur, India*

  An unbiased coin is tossed repeatedly until r heads are obtained. The outcomes of the tosses are written sequentially. Let R denote the total number of runs (of heads and tails) in the above experiment. Find the distribution of R.

  Illustration: if we decide to toss a coin until we get 4 heads, then one of the possibilities could be the sequence $T\ T\ H\ H\ T\ H\ T\ H$ resulting in 6 runs.

- 5102: *Proposed by Miquel Grau-Sánchez and José Luis Díaz-Barrero, Barcelona, Spain*

  Let n be a positive integer and let $a_1, a_2, \cdots, a_n$ be any real numbers. Prove that
  $$\frac{1}{1+a_1^2+\ldots+a_n^2} + \frac{1}{F_nF_{n+1}}\left(\sum_{k=1}^{n}\frac{a_kF_k}{1+a_1^2+\ldots+a_k^2}\right)^2 \le 1,$$
  where F_k represents the k^{th} Fibonacci number defined by $F_1 = F_2 = 1$ and for $n \ge 3, F_n = F_{n-1} + F_{n-2}$.

- 5103: *Proposed by Roger Izard, Dallas, TX*

  A number of circles of equal radius surround and are tangent to another circle. Each of the outer circles is tangent to two of the other outer circles. No two outer circles intersect in two points. The radius of the inner circle is a and the radius of each outer circle is b. If

$$a^4 + 4a^3b - 10a^2b^2 - 28ab^3 + b^4 = 0,$$

determine the number of outer circles.

Solutions

- 5080: *Proposed by Kenneth Korbin, New York, NY*

If p is a prime number congruent to 1 (mod 4), then there are positive integers $a, b, c,$ such that

$$\arcsin\left(\frac{a}{p^3}\right) + \arcsin\left(\frac{b}{p^3}\right) + \arcsin\left(\frac{c}{p^3}\right) = 90^\circ.$$

Find $a, b,$ and c if $p = 37$ and if $p = 41$, with $a < b < c$.

Solution 1 by Paul M. Harms, North Newton, KS

The equation in the problem is equivalent to

$$\arcsin\left(\frac{a}{p^3}\right) + \arcsin\left(\frac{b}{p^3}\right) = 90^\circ - \arcsin\left(\frac{c}{p^3}\right).$$

Taking the cosine of both sides yields

$$\begin{aligned} \frac{(p^6 - a^2)^{1/2}(p^6 - b^2)^{1/2}}{p^6} - \frac{ab}{p^6} &= \frac{c}{p^3}. \\ (p^6 - a^2)^{1/2}(p^6 - b^2)^{1/2} - ab &= cp^3. \end{aligned}$$

Since p^3 is a factor on the right side I made some assumptions on a and b so that the left side also had p^3 as a factor.
Assume $a = p^2a_1$ and $b = pb_1$ where all numbers are positive integers. Then we have

$$c = (p^2 - a_1)^{1/2}(p^4 - b_1^2)^{1/2} - a_1b_1.$$

I then looked for perfect squares for $(p^2 - a_1^2)$ and $(p^4 - b_1^2)$.

When $p = 37, (37^2 - a_1^2) = (37 - a_1)(37 + a_1)$ and $a_1 = 12$ yields a product of the squares 25 and 49.

When $p = 37, (37^4 - b_1^2) = (37^2 - b_1)(37^2 + b_1)$.

I checked for a number b_1 where both $(37^2 - b_1)$ and$(37^2 + b_1)$ were perfect squares. The numbers b_1 which make $(37^2 - b_1)$ a square are

$$0,\ 37 + 36 = 73,\ 73 + (36 + 35) = 144,\ 144 + (35 + 34) = 213,\ \cdots.$$

When $b_1 = 840$, both factors involving b_1 are perfect squares.

When $p = 37$ a result is $a = (12)37^2 = 16428, b = 840(37) = 31080$ and $c = 27755$.

Since the problem conditions state that $a < b < c$, I will switch notation. One answer is

$$a = 16428, b = 27755, \text{and } = 31080$$

with approximate angles 18.925°, 33.226° and 37.849°.

When $p = 41$, $(41 - a_1)(41 + a_1)$ is a perfect square when $a_1 = 9$ or 40. The product $(41^2 - b_1)(41^2 + b_1)$ is a perfect square when $b_1 = 720$. One answer is

$$a = 9(41^2) = 15129,\ b = 720(41) = 29520 \text{ and } c = 54280$$

with approximate angles 12.757°, 25,361°, and 51.959°.

When $a_1 = 40$ and $b_1 = 720$, c was less than zero so this did not satisfy the problem.

Solution 2 by Tom Leong, Scotrun, PA

Fermat's Two-Square Theorem implies that every prime congruent to 1 mod 4 can be represented as the sum of two distinct squares. We give a solution to the following modest generalization. Suppose the positive integer n is the sum of two distinct squares, say, $n = x^2 + y^2$ with $0 < x < y$. Then a solution to

$$\arcsin\frac{A}{n} + \arcsin\frac{B}{n^2} + \arcsin\frac{C}{n^3} = 90^\circ$$

in positive integers A, B, C is

$$(A, B, C) = \begin{cases} (s, 2st, 2(xs + yt)(xt - ys)) & \text{if } 1 < \dfrac{y}{x} < \sqrt{3} \\ (t, t^2 - s^2, 2(xs + yt)(ys - xt)) & \text{if } \sqrt{3} < \dfrac{y}{x} < 1 + \sqrt{2} \\ (s, s^2 - t^2, (xs + yt)^2 - (ys - xt)^2) & \text{if } 1 + \sqrt{2} < \dfrac{y}{x} < 2 + \sqrt{3} \\ (t, 2st, (ys - xt)^2 - (xs + yt)^2) & \text{if } \dfrac{y}{x} > 2 + \sqrt{3} \end{cases}$$

where $s = y^2 - x^2$ and $t = 2xy$.
We can verify this as follows. Since $\arcsin(A/n) + \arcsin(B/n^2)$ and $\arcsin(C/n^3)$ are complementary,

$$\tan\left(\arcsin\frac{A}{n} + \arcsin\frac{B}{n^2}\right) = \cot\left(\arcsin\frac{C}{n^3}\right).$$

Using the angle sum formula for tangent and $\tan(\arcsin z) = z/\sqrt{1 - z^2}$, this reduces to

$$\frac{A\sqrt{n^4 - B^2} + B\sqrt{n^2 - A^2}}{\sqrt{n^2 - A^2}\sqrt{n^4 - B^2} - AB} = \frac{\sqrt{n^6 - C^2}}{C}.$$

Now verifying the solutions is straightforward using the following identities

$$n = x^2 + y^2, \qquad n^2 = s^2 + t^2, \qquad n^3 = (xs + yt)^2 + (ys - xt)^2$$

and the following inequalities

$$\frac{y}{x} < \sqrt{3} \Leftrightarrow ys < xt, \qquad \frac{y}{x} < 1 + \sqrt{2} \Leftrightarrow s < t, \qquad \frac{y}{x} < 2 + \sqrt{3} \Leftrightarrow ys - xt < xs + yt.$$

As for the original problem, for $n = 37$, since $37 = 1^2 + 6^2$, we have
$x = 1, y = 6, s = 35, t = 12$ which gives

$$\arcsin\frac{12}{37} + \arcsin\frac{840}{37^2} + \arcsin\frac{27755}{37^3} = \arcsin\frac{16428}{37^3} + \arcsin\frac{31080}{37^3} + \arcsin\frac{27755}{37^3} = 90^\circ.$$

For $n = 41$, since $41 = 4^2 + 5^2$, we have $x = 4, y = 5, s = 9, t = 40$ which gives

$$\arcsin\frac{9}{41} + \arcsin\frac{720}{41^2} + \arcsin\frac{54280}{41^3} = \arcsin\frac{15129}{41^3} + \arcsin\frac{29520}{41^3} + \arcsin\frac{54280}{41^3} = 90^\circ.$$

Comment by editor: David Stone and John Hawkins of Statesboro, GA developed equations:

$$\begin{aligned} b &= \sqrt{\frac{p^3(p^3-c)}{2}} \\ a &= \frac{-bc + \sqrt{b^2c^2 + p^6\left(p^6 - b^2 - c^2\right)}}{p^3}. \end{aligned}$$

Using Matlab they found four solutions for $p = 37$,

$$\begin{array}{lll} a = 16428 & b = 27755 & c = 31080 \\ a = 3293 & b = 32157 & c = 36963 \\ a = 7363 & b = 27188 & c = 38332 \\ a = 352 & b = 25123 & c = 43808 \end{array}$$

and two solutions for $p = 41$,

$$\begin{array}{lll} a = 15129 & b = 29520 & c = 54280 \\ a = 5005 & b = 31529 & c = 58835. \end{array}$$

Also solved by Brian D. Beasley, Clinton, SC, and the proposer.

- **5081:** *Proposed by Kenneth Korbin, New York, NY*

Find the dimensions of equilateral triangle ABC if it has an interior point P such that $\overline{PA} = 5, \overline{PB} = 12$, and $\overline{PC} = 13$.

Solution 1 by Kee-Wai Lau, Hong Kong, China

Let the length of the sides of the equilateral triangle be x. We show that $x = \sqrt{169 + 60\sqrt{3}}$.

Applying the cosine formula to triangles $APB,\ BPC$, and CPA respectively, we obtain

$$\cos\angle APB = \frac{169 - x^2}{120},\ \cos\angle BPC = \frac{313 - x^2}{312},\ \cos\angle CPA = \frac{194 - x^2}{130}.$$

Since

$$\begin{aligned} \angle APB + \angle BPC + \angle CPA &= 360^\circ \text{ so} \\ \cos\angle CPA &= \cos(\angle APB + \angle BPC) \text{ and} \\ \sin\angle APB \sin\angle BPC &= \cos\angle APB \cos\angle BPC - \cos\angle CPA. \end{aligned}$$

Hence,

$$\left(\frac{\sqrt{338x^2 - x^4 - 14161}}{120}\right)\left(\frac{\sqrt{626x^2 - x^4 - 625}}{312}\right) = \left(\frac{169 - x^2}{120}\right)\left(\frac{313 - x^2}{312}\right) - \frac{194 - x^2}{130} \text{ or}$$

$$
\begin{aligned}
\sqrt{338x^2 - x^4 - 14161}\sqrt{626x^2 - x^4 - 625} &= (169 - x^2)(313 - x^2) - 288(194 - x^2).
\end{aligned}
$$

Squaring both sides and simplifying, we obtain

$$
\begin{aligned}
576x^6 - 194668x^4 + 10230336x^2 &= 0 \text{ or} \\
576x^2(x^4 - 338x^2 + 17761) &= 0.
\end{aligned}
$$

It follows that $x = \sqrt{169 - 60\sqrt{3}},\ \sqrt{160 + 60\sqrt{3}}$. Since $\angle APB,\ \angle BPC, \angle CPA$ are not all acute, the value of $\sqrt{169 - 60\sqrt{3}}$ must be rejected.

This completes the solution.

Comments and Solutions 2 & 3 by Tom Leong, Scotrun, PA

Comments: This problem is not new and has appeared in, e.g., the 1998 Irish Mathematical Olympiad and T. Andreescu & R. Gelca, *Mathematical Olympiad Challenges*, Birkhäuser, 2000, p5. A nice elementary solution to this problem uses a rotation argument (Solution 2 below). A quick solution to a more general problem can be found using a somewhat obscure result of Euler on tripolar coordinates (Solution 3 below).

Solution 2

Rotate the figure about the point C by $60°$ so that B maps onto A. Let P' denote the image of P under this rotation. Note that triangle PCP' is equilateral since $PC = P'C$ and $\angle PCP' = 60°$. So $\angle P'PC = 60°$. Furthermore, since $PP' = 13$, triangle APP' is a 5-12-13 right triangle. Consequently,

$$
\cos\angle APC = \cos(\angle APP' + 60°) = \frac{5}{13}\cdot\frac{1}{2} - \frac{12}{13}\cdot\frac{\sqrt{3}}{2} = \frac{5 - 12\sqrt{3}}{26}.
$$

So by the Law of Cosines,

$$
AC = \sqrt{5^2 + 13^2 - 2\cdot 5\cdot 13\cdot \frac{5 - 12\sqrt{3}}{26}} = \sqrt{169 + 60\sqrt{3}}
$$

Solution 3

A generalization follows from a result of Euler on *tripolar coordinates* (see, e.g., van Lamoen, Floor and Weisstein, Eric W. "Tripolar Coordinates" From MathWorld–A Wolfram Web Resource.
`http://mathworld.wolfram.com/TripolarCoordinates.html`.) Suppose triangle ABC is equilateral with side length s, and P is a point in the plane of ABC. The triple $(x, y, z) = (PA, PB, PC)$ is the tripolar coordinates of P in reference to triangle ABC. A result of Euler implies these tripolar coordinates satisfy

$$
s^4 - (x^2 + y^2 + z^2)s^2 + x^4 + y^4 + z^4 - x^2y^2 - y^2z^2 - z^2x^2 = 0
$$

which gives the positive solutions

$$
s = \sqrt{\frac{x^2 + y^2 + z^2 \pm \sqrt{(x^2 + y^2 + z^2)^2 - 2(x-y)^2 - 2(y-z)^2 - 2(z-x)^2}}{2}}.
$$

The larger solution refers to the case where P is interior to the triangle, while the smaller solution refers to the case where P is exterior to the triangle. In the case where (x, y, z) is a Pythatgorean triple with $x^2 + y^2 = z^2$, this simplifies to the surprisingly terse

$$s = \sqrt{z^2 \pm xy\sqrt{3}}.$$

In the original problem, with $(x, y, z) = (5, 12, 13)$, we find

$$s = \sqrt{169 \pm 60\sqrt{3}}$$

with the larger solution $s = \sqrt{169 + 60\sqrt{3}}$ being the desired answer.

A conjecture by David Stone and John Hawkins, Statesboro, GA

If a, b, c form a *right* triangle with $a^2 + b^2 = c^2$, then

1. the side length of the unique equilateral triangle ABC having an *interior* point P such that $\overline{PA} = a, \overline{PB} = b$, and $\overline{PC} = c$ is $s\sqrt{c^2 + ab\sqrt{3}}$, and
2. the side length of the unique equilateral triangle with an *exterior* point P satisfying $\overline{PA} = a, \overline{PB} = b$, and $\overline{PC} = c$ is $s\sqrt{c^2 - ab\sqrt{3}}$.

Also solved by Scott H. Brown, Montgomery, AL; Valmir Bucaj (student, Texas Lutheran University), Seguin, TX; Pat Costello, Richmon, KY; Paul M. Harms, North Newton, KS; Antonio Ledesma López, Requena-Valencia, Spain; David E. Manes, Oneonta, NY; Charles McCracken, Dayton, OH; John Nord, Spokane, WA; Boris Rays, Brooklyn, NY; Armend Sh. Shabani, Republic of Kosova; David Stone and John Hawkins, Statesboro, GA, and the proposer.

- 5082: *Proposed by David C. Wilson, Winston-Salem, NC*

Generalize and prove:

$$\begin{aligned}
\frac{1}{1\cdot 2} + \frac{1}{2\cdot 3} + \cdots + \frac{1}{n(n+1)} &= 1 - \frac{1}{n+1}\\
\frac{1}{1\cdot 2\cdot 3} + \frac{1}{2\cdot 3\cdot 4} + \cdots + \frac{1}{n(n+1)(n+2)} &= \frac{1}{4} - \frac{1}{2(n+1)(n+2)}\\
\frac{1}{1\cdot 2\cdot 3\cdot 4} + \frac{1}{2\cdot 3\cdot 4\cdot 5} + \cdots + \frac{1}{n(n+1)(n+2)(n+3)} &= \frac{1}{18} - \frac{1}{3(n+1)(n+2)(n+3)}\\
\frac{1}{1\cdot 2\cdot 3\cdot 4\cdot 5} + \cdots + \frac{1}{n(n+1)(n+2)(n+3)(n+4)} &= \frac{1}{96} - \frac{1}{4(n+1)(n+2)(n+3)(n+4)}
\end{aligned}$$

Solution 1 by Paolo Perfetti, Department of Mathematics, Tor Vergata University, Rome, Italy

We will give two different proofs, each relies on the telescoping property.

First proof:

Our quantity may be written as $\displaystyle\sum_{k=1}^{n} \frac{1}{k(k+1)\cdots(k+m)}$ where m is a positive integer.

Next we observe

$$\frac{1}{k(k+1)\cdots(k+m-1)} - \frac{1}{(k+1)\cdots(k+m)} = \frac{m}{k(k+1)\cdots(k+m)}$$

yielding, also by telescoping,

$$\begin{aligned}
\sum_{k=1}^{n} \frac{1}{k(k+1)\cdots(k+m)} &= \frac{1}{m}\sum_{k=1}^{n}\left(\frac{1}{k(k+1)\cdots(k+m-1)} - \frac{1}{(k+1)\cdots(k+m)}\right) \\
&= \frac{1}{m}\left(\frac{1}{m!} - \frac{1}{(n+1)\cdots(n+m)}\right)
\end{aligned}$$

Second proof:

If $a_k = \dfrac{1}{k(k+1)\cdots(k+m)}$, then $\dfrac{a_{k+1}}{a_k} = \dfrac{k\cdot(k+1)\cdots(k+m)}{(k+1)(k+2)\cdots(k+m)} = \dfrac{k}{k+1+m}$
and then $ma_k = ka_k - (k+1)a_{k+1}$ and therefore

$$\begin{aligned}
m\sum_{k=1}^{n} a_k = m\sum_{k=0}^{n-1} a_{k+1} &= m\sum_{k=0}^{n-1}(ka_k - (k+1)a_{k+1}) \\
&= \frac{1}{m!} - \frac{1}{(n+1)(n+2)\cdots(n+m)}
\end{aligned}$$

and the result is immediate.

Solution 2 by G. C. Greubel, Newport News, VA

It can be seen that all the series in question are of the form

$$S_n^m = \sum_{k=1}^{n} \frac{(k-1)!}{(k+m)!}.$$

Making a slight change we have

$$S_n^m = \frac{1}{m!}\sum_{k=1}^{n}\frac{(k-1)!m!}{(k+m)!} = \frac{1}{m!}\sum_{k=1}^{n} B(k, m+1),$$

where $B(x,y)$ is the Beta function. By using an integral form of the Beta function, namely,

$$B(x,y) = \int_0^1 t^{x-1}(1-t)^{y-1}dt,$$

the series becomes

$$\begin{aligned}
S_n^m &= \frac{1}{m!}\sum_{k=1}^{n}\int_0^1 t^m(1-t)^{k-1}dt \\
&= \frac{1}{m!}\int_0^1 t^m(1-t)^{-1}\cdot\frac{(1-t)(1-(1-t)^n)}{t}dt \\
&= \frac{1}{m!}\int_0^1 t^{m-1}(1-(1-t)^n)dt
\end{aligned}$$

$$= \frac{1}{m!}\left(\int_0^1 t^{m-1}dt - B(n+1,m)\right)$$

$$= \frac{1}{m!}\left(\frac{1}{m} - B(n+1,m)\right)$$

$$= \frac{1}{m}\left[\frac{1}{m!} - \frac{n!}{(n+m)!}\right].$$

The general result is given by

$$\sum_{k=1}^{n}\frac{(k-1)!}{(k+m)!} = \frac{1}{m}\left[\frac{1}{m!} - \frac{n!}{(n+m)!}\right].$$

As examples let $m = 1$ to obtain

$$\sum_{k=1}^{n}\frac{1}{k(k+1)} = 1 - \frac{1}{n+1}$$

and when $m = 2$ the series becomes

$$\sum_{k=1}^{n}\frac{1}{n(n+1)(n+2)} = \frac{1}{4} - \frac{1}{2(n+1)(n+2)}.$$

The other series follow with higher values of m.

Comments by Tom Leong, Scotrun, PA

This series is well-known and has appeared in the literature in several places. Some references include

1. Problem 241, *College Mathematics Journal* (Nov 1984, p448–450)
2. Problem 819, *College Mathematics Journal* (Jan 2007, p65–66)
3. K. Knopp, *Theory and Application of Infinite Series*, 2nd ed., Blackie & Son, 1951, p233
4. D.O. Shklarsky, N.N. Chentzov, and I.M. Yaglom, *The USSR Olympiad Problem Book*, W.H. Freeman and Company, 1962, p30

In the first reference above, four different perspectives on this series are given.

Also solved by Dionne Bailey, Elsie Campbell, and Charles Diminnie (jointly), San Angelo, TX; Paul M. Harms, North Newton, KS; N. J. Kuenzi, Oshkosh, WI; Kee-Wai Lau, Hong Kong, China; Antonio Ledesma López, Requena-Valencia, Spain; Tom Leong, Scotrun, PA; David E. Manes, Oneonta, NY; Boris Rays, Brooklyn, NY; Raúl A. Simón, Santiago, Chile; Armend Sh. Shabani, Republic of Kosova; David Stone and John Hawkins (jointly), Statesboro, GA, and the proposer.

- **5083:** *Proposed by José Luis Díaz-Barrero, Barcelona, Spain*

Let $\alpha > 0$ be a real number and let $f : [-\alpha, \alpha] \to \Re$ be a continuous function two times derivable in $(-\alpha, \alpha)$ such that $f(0) = 0$ and f'' is bounded in $(-\alpha, \alpha)$. Prove that the sequence $\{x_n\}_{n\geq 1}$ defined by

$$x_n = \begin{cases} \displaystyle\sum_{k=1}^{n} f\left(\frac{k}{n^2}\right), & n > \dfrac{1}{\alpha}; \\ 0, & n \leq \dfrac{1}{\alpha} \end{cases}$$

is convergent and determine its limit.

Solution 1 by Michael N. Fried, Kibbutz Revivim, Israel

Clearly, for n large enough, we will have $n > \dfrac{1}{\alpha}$. Therefore, we only need to show that $\displaystyle\sum_{k=1}^{n} f\left(\frac{k}{n^2}\right)$ converges and to find *its* limit as $n \to \infty$.

Since $f(0) = 0$ and $f'(x)$ exist in $[0, k/n^2] \subset [0, 1/n] \subset [-\alpha, \alpha]$, there is some $\xi_k \in [0, k/n^2]$ such that $f\left(\dfrac{k}{n^2}\right) = f'\left(\xi_k\right)\dfrac{k}{n^2}$ by the mean value theorem.

Let $f'(M_n) = \max_k f'(\xi_k)$ and $f'(m_n) = \min_k f'(\xi_k)$.

Then, since $\displaystyle\sum_{k=1}^{n} f\left(\frac{k}{n^2}\right) = \sum_{k=0}^{n} f'(\xi_k)\frac{k}{n^2}$, we have:

$$f'(m_n)\sum_{k=1}^{n}\frac{k}{n^2} \leq \sum_{k=1}^{n} f\left(\frac{k}{n^2}\right) \leq f'(M_n)\sum_{k=1}^{n}\frac{k}{n^2}, \text{ or}$$

$$f'(m_n)\left(\frac{1}{2}+\frac{1}{2n}\right) \leq \sum_{k=1}^{n} f\left(\frac{k}{n^2}\right) \leq f'(M_n)\left(\frac{1}{2}+\frac{1}{2n}\right).$$

But f' is bounded in $[-\alpha, \alpha]$ and, thus, in every subinterval of $[-\alpha, \alpha]$. Therefore, f' is continuous in every subinterval of $[-\alpha, \alpha]$. Hence,

$$\lim_{n\to\infty} f'(m_n) = \lim_{n\to\infty} f'(M_n) = f'(0), \text{ so that}$$

$$\lim_{n\to\infty}\sum_{k=1}^{n} f\left(\frac{k}{n^2}\right) = \frac{f'(0)}{2}$$

Heuristically, we can approach the problem in a slightly different way. Keeping in mind that $f(0) = 0$, write:

$$\sum_{k=1}^{n} f\left(\frac{k}{n^2}\right) = n^2\sum_{k=0}^{n}\left(\frac{k}{n}\times\frac{1}{n}\right)\frac{1}{n^2} \approx n^2\int_0^{\frac{1}{n}} f\left(\xi\right)d\xi.$$

The approximation become exact as $n \to \infty$ (this is the heuristic part!)
Since f' is bounded in $(0, \alpha)$ (being bounded in $(-\alpha, \alpha)$), and since $f(0) = 0$ we can write, for some $s \in (0, 1/n)$:

$$n^2\int_0^{\frac{1}{n}} f(\xi)d\xi = n^2\int_0^{\frac{1}{n}}\left(f'(0)\xi + \frac{f''(s)}{2}\xi^2\right)d\xi$$

$$= n^2 \left(\frac{f'(0)}{2} \frac{1}{n^2} + \frac{f''(s)}{6} \frac{1}{n^3} \right)$$

$$= \frac{f'(0)}{2} + \frac{f''(s)}{6} \frac{1}{n}$$

$$= \frac{f'(0)}{2} \text{ as } n \to \infty.$$

Solution 2 by Ovidiu Furdui, Cluj, Romania

The limit equals

$$\frac{f'(0)}{2}.$$

We have, since $f(0) = 0$, that for all $n > \dfrac{1}{\alpha}$ one has

$$\begin{aligned} x_n = \sum_{k=1}^{n} f\left(\frac{k}{n^2}\right) &= \sum_{k=1}^{n} \left(f\left(\frac{k}{n^2}\right) - f(0) \right) \\ &= \sum_{k=1}^{n} \frac{k}{n^2} f'\left(\theta_{k,n}\right) \\ &= \sum_{k=1}^{n} \frac{k}{n^2} \left(f'(\theta_{k,n}) - f'(0) \right) + \sum_{k=1}^{n} \frac{k}{n^2} f'(0) \\ &= \sum_{k=1}^{n} \frac{k}{n^2} \theta_{k,n} f''(\beta_{k,n}) + \frac{f'(0)(n+1)}{2n}. \end{aligned} \tag{1}$$

We used, in the preceding calculations, the **Mean Value Theorem** twice where $0 < \beta_{k,n} < \theta_{k,n} < \dfrac{k}{n^2}$. Now,

$$\left| \sum_{k=1}^{n} \frac{k}{n^2} \theta_{k,n} f''(\beta_{k,n}) \right| \le M \sum_{k=1}^{n} \frac{k}{n^2} \theta_{k,n} \le M \sum_{k=1}^{n} \frac{k^2}{n^4} = M \frac{(n+1)(2n+1)}{6n^3},$$

where $M = \sup_{x \in (-\alpha, \alpha)} |f''(x)|$. Thus,

$$\lim_{n \to \infty} \sum_{k=1}^{n} \frac{k}{n^2} \theta_{k,n} f''(\beta_{k,n}) = 0. \tag{2}$$

Combining (1) and (2) we get that the desired limit holds and the problem is solved.

Also solved by Dionne Bailey, Elsie Campbell and Charles Diminnie (jointly), San Angelo, TX; Tom Leong, Scotrun, PA; Paolo Perfetti, Department of Mathematics, Tor Vergata Universtiy, Rome, Italy, and the proposer.

- 5084: *Charles McCracken, Dayton, OH*

A natural number is called a "repdigit" if all of its digits are alike.
Prove that regardless of positive integral base b, no natural number with two or more digits when raised to a positive integral power will produce a repdigit.

Comments by David E. Manes, Oneonta, NY; Michael N. Fried, Kibbutz Revivim, Israel, the proposer, and the editor.

Manes: The website <http://www.research.att.com/ njas/sequences/A158235> appears to have many counterexamples to problem 5084.
Editor: Following are some examples and comments from the above site.

$$11, 20, 39, 40, 49, 78, 133, 247, 494, 543, 1086, 1218,$$

$$1651, 1729, 2172, 2289, 2715, 3097, 3258, 3458, 3801,$$

$$171, 4344, 4503, 4578, 4887, 5187, 5430, 6194, 6231.$$

(And indeed, each number listed above can be written as repdigit *in some base.* For example:)

$$\begin{aligned}
11^2 &= 11111 \text{ in base 3}\\
20^2 &= 1111 \text{ in base 7}\\
39^2 &= 333 \text{ in base 22}\\
40^2 &= 4444 \text{ in base 7}\\
49^2 &= 777 \text{ in base 18}\\
78^2 &= (12)(12)(12) \text{ in base 22}\\
1218^2 &= (21)(21)(21)(21) \text{ in base 41}
\end{aligned}$$

McCracken: When I wrote the problem I intended that the number and it's power be written *in the same base.*

Editor: Charles McCracken sent in a proof that was convincing to me that the statement, as he had intended it to be, was indeed correct. No natural number with two or more digits (written in base b), when raised to a positive integral power, will produce a repdigit (in base b). I showed the problem, its solution, and Manes' comment, to my colleague Michael Fried, and he finally convinced me that although the intended statement might be true, the proof was in error.

Fried: The Sloan Integer Sequence site (mentioned above) also cites a paper which among other things, refers to Catalan's conjecture, now proven, stating that the only solution to $x^k - y^n = 1$ is $3^2 - 2^3 - 9 - 8 = 1$. This is the fact one needs to show that Charles' claim is true for base 2 repdigits. For in base 2 only numbers of the form $11111\ldots 1$ are repdigits. These numbers are equal to $2^n - 1$. So if one of these numbers were equal to x^k, we would have $2^n - 1 = x^k$ or $2^n - x^k = 1$. But by the proven Catalan conjecture, the latter can never be satisfied.

Editor: So, dear readers, let's rephrase the problem: Prove or disprove that regardless of positive integral base b, no natural number with two or more digits when raised to a positive integral power will produce a repdigit in base b.

- 5085: *Proposed by Valmir Krasniqi, (student, Mathematics Department,) University of Prishtinë, Kosova*

  Suppose that $a_k,\ (1 \le k \le n)$ are positive real numbers. Let $e_{j,k} = (n-1)$ if $j = k$ and $e_{j,k} = (n-2)$ otherwise. Let $d_{j,k} = 0$ if $j = k$ and $d_{j,k} = 1$ otherwise.

  Prove that
$$\prod_{j=1}^{n}\sum_{k=1}^{n} e_{j,k}a_k^2 \ge \prod_{j=1}^{n}\left(\sum_{k=1}^{n} d_{j,k}a_k\right)^2.$$

Solution by proposer

On expanding each side and reducing, the inequality becomes

$$\prod_{k=1}^{n}\left[(n-2)S + a_k^2\right] \quad \ge \quad \prod_{k=1}^{n}(T - a_k)\,,\ \text{where}$$

$$S = \sum_{k=1}^{n} a_k^2 \quad \text{and} \quad T = \sum_{k=1}^{n} a_k.$$

Since $(T-a_1)^2 \le (n-1)(S-a_1^2)$, etc., it suffices to prove that

$$\prod_{k=1}^{n}\left[(n-2)S + a_k^2\right] \ge (n-1)^n \prod_{k=1}^{n}(S - a_k)\,. \qquad (1)$$

If we now let $x_k = S - a_k^2$ where $k = 1, 2, 3, \ldots, n$ so that $S = \dfrac{x_1 + x_2 + \ldots + x_n}{n-1}$ and $a_k^2 = S - x_k$, then (1) becomes

$$\prod_{k=1}^{n}\left(S' - x_k\right) \ge (n-1)^n \cdot x_1 \cdot x_2 \cdot \ldots \cdot x_n,\ \text{where } S' = \sum_{k=1}^{n} x_k.$$

The result now follows by applying the AM-GM inequality to each of the factors $(S' - x_k)$ on the left hand side. There is equality if, and only if, all the a_k's are equal.

Also solved by Tom Leong, Scotrun, PA