Problems

Ted Eisenberg, Section Editor

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This section of the Journal offers readers an opportunity to exchange interesting mathematical problems and solutions. Please send them to Ted Eisenberg, Department of Mathematics, Ben-Gurion University, Beer-Sheva, Israel or fax to: 972-86-477-648. Questions concerning proposals and/or solutions can be sent e-mail to <eisenbt@013.net>. Solutions to previously stated problems can be seen at <http://ssmj.tamu.edu>.

Solutions to the problems stated in this issue should be posted before
April 15, 2009

- 5050: *Proposed by Kenneth Korbin, New York, NY.*

  Given $\triangle ABC$ with integer-length sides, and with $\angle A = 120^o$, and with $(a, b, c) = 1$. Find the lengths of b and c if side $a = 19$, and if $a = 19^2$, and if $a = 19^4$.

- 5051: *Proposed by Kenneth Korbin, New York, NY.*

  Find four pairs of positive integers (x, y) such that $\dfrac{(x-y)^2}{x+y} = 8$ with $x < y$.

  Find a formula for obtaining additional pairs of these integers.

- 5052: *Proposed by Juan-Bosco Romero Márquez, Valladolid, Spain.*

  If $a \geq 0$, evaluate:

  $$\int_0^{+\infty} arctg \frac{2a(1+ax)}{x^2(1+a^2)+2ax+1-a^2}\,\frac{dx}{1+x^2}.$$

- 5053: *Proposed by Panagiote Ligouras, Alberobello, Italy.*

  Let a, b and c be the sides, r the in-radius, and R the circum-radius of $\triangle ABC$. Prove or disprove that

  $$\frac{(a+b-c)(b+c-a)(c+a-b)}{a+b+c} \leq 2rR.$$

- 5054: *Proposed by José Luis Díaz-Barrero, Barcelona, Spain.*

  Let x, y, z be positive numbers such that $xyz = 1$. Prove that

  $$\frac{x^3}{x^2+xy+y^2} + \frac{y^3}{y^2+yz+z^2} + \frac{z^3}{z^2+zx+x^2} \geq 1.$$

- 5055: *Proposed by Ovidiu Furdui, Campia Turzii, Cluj, Romania.*

  Let α be a positive real number. Find the limit

$$\lim_{n\to\infty}\sum_{k=1}^{n}\frac{1}{n+k^{\alpha}}.$$

Solutions

- 5032: *Proposed by Kenneth Korbin, New York, NY.*

Given positive acute angles A, B, C such that

$$\tan A\cdot\tan B+\tan B\cdot\tan C+\tan C\cdot\tan A=1.$$

Find the value of
$$\frac{\sin A}{\cos B\cdot\cos C}+\frac{\sin B}{\cos A\cdot\cos C}+\frac{\sin C}{\cos A\cdot\cos B}.$$

Solution 1 by Brian D. Beasley, Clinton, SC.

Since A, B, and C are positive acute angles with

$$\begin{aligned}1 &= \frac{\sin A\sin B\cos C+\cos A\sin B\sin C+\sin A\cos B\sin C}{\cos A\cos B\cos C}\\ &= \frac{\cos A\cos B\cos C-\cos(A+B+C)}{\cos A\cos B\cos C},\end{aligned}$$

we have $\cos(A+B+C)=0$ and thus $A+B+C=90^\circ$. Then
$$\frac{\sin A}{\cos B\cos C}+\frac{\sin B}{\cos A\cos C}+\frac{\sin C}{\cos A\cos B}=\frac{\sin A\cos A+\sin B\cos B+\sin C\cos C}{\cos A\cos B\cos C}.$$
Letting N be the numerator of this latter fraction, we obtain

$$\begin{aligned}N &= \sin A\cos A+\sin B\cos B+\cos(A+B)\sin(A+B)\\ &= \sin A\cos A+\sin B\cos B+(\cos A\cos B-\sin A\sin B)(\sin A\cos B+\cos A\sin B)\\ &= \sin A\cos A(1+\cos^2 B-\sin^2 B)+\sin B\cos B(1+\cos^2 A-\sin^2 A)\\ &= \sin A\cos A(2\cos^2 B)+\sin B\cos B(2\cos^2 A)\\ &= 2\cos A\cos B(\sin A\cos B+\cos A\sin B)\\ &= 2\cos A\cos B\sin(A+B)\\ &= 2\cos A\cos B\cos C.\end{aligned}$$

Hence the desired value is 2.

Solution 2 by Kee-Wai Lau, Hong Kong, China.

The condition $\tan A\tan B+\tan B\tan C+\tan C\tan A=1$ is equivalent to $\cot A+\cot B+\cot C=\cot A\cot B\cot C$. Since it is well known that

$$\cos(A+B+C)=-\sin A\sin B\sin C\Big(\cot A+\cot B+\cot C-\cot A\cot B\cot C\Big),$$

so $\cos(A+B+C)=0$ and $A+B+C=\dfrac{\pi}{2}$. Hence,

$$\sin 2A+\sin 2B+\sin 2C \quad = \quad 2\sin(A+B)\cos(A-B)+2\sin C\cos C$$

$$
\begin{aligned}
&= 2\cos C\big(\cos(A-B)+\cos(A+B)\big)\\
&= 4\cos A\cos B\cos C.
\end{aligned}
$$

If follows that

$$\frac{\sin A}{\cos B\cos C}+\frac{\sin B}{\cos A\cos C}+\frac{\sin C}{\cos A\cos B}=\frac{\sin 2A+\sin 2B+\sin 2C}{2\cos A\cos B\cos C}=2.$$

Solution 3 by Boris Rays, Chesapeake, VA.

$\tan A\tan B+\tan B\tan C+\tan C\tan A=1$ implies,

$$
\begin{aligned}
\tan B(\tan A+\tan C) &= 1-\tan A\tan C\\
\frac{\tan A+\tan C}{1-\tan A\tan C} &= \frac{1}{\tan B}\\
\tan(A+C) &= \cot B=\tan(90^o-B).
\end{aligned}
$$

Similarly, we obtain:

$$
\begin{aligned}
\tan(B+C) &= \frac{1}{\tan A}=\cot A=\tan(90^o-A)\\
\tan(A+B) &= \frac{1}{\tan C}=\cot C=\tan(90^o-C), \text{which implies}\\
A &= 90^o-(B+C)\\
B &= 90^o-(A+C)\\
C &= 90^o-(A+B).
\end{aligned}
$$

Therefore,

$$\frac{\sin A}{\cos B\cos C}+\frac{\sin B}{\cos A\cos C}+\frac{\sin C}{\cos A\cos B}$$

$$
\begin{aligned}
&= \frac{\sin(90^o-(B+C))}{\cos B\cos C}+\frac{\sin(90^o-(A+C))}{\cos A\cos C}+\frac{\sin(90^o-(A+B))}{\cos A\cos B}\\
&= \frac{\cos(B+C)}{\cos B\cos C}+\frac{\cos(A+C)}{\cos A\cos C}+\frac{\cos(A+B)}{\cos A\cos B}\\
&= \frac{\cos B\cos C-\sin B\sin C}{\cos B\cos C}+\frac{\cos A\cos C-\sin A\sin C}{\cos A\cos C}+\frac{\cos A\cos B-\sin A\sin B}{\cos A\cos B}\\
&= \Big(1-\tan B\tan C\Big)\quad+\quad\Big(1-\tan A\tan C\Big)\quad+\quad\Big(1-\tan A\tan B\Big)\\
&= 1+1+1-(\tan A\tan B+\tan B\tan C+\tan A\tan C)\\
&= 3-1=2.
\end{aligned}
$$

Also solved by Elsie M. Campbell, Dionne T. Bailey and Charles Diminnie (jointly), San Angelo, TX; Bruno Salgueiro Fanego, Viveiro, Spain; Paul M.

Harms, North Newton, KS; John Hawkins, and David Stone (jointly), Statesboro, GA; Valmir Krasniqi, Prishtin, Kosovo; David E. Manes, Oneonta, NY; Charles McCracken, Dayton, OH; David C.Wilson, Winston-Salem, NC, and the proposer.

- 5033: *Proposed by Kenneth Korbin, New York, NY.*

Given quadrilateral $ABCD$ with coordinates $A(-3,0), B(12,0), C(4,15)$, and $D(0,4)$. Point P is on side $\overline{AB}$ and point Q is on side $\overline{CD}$. Find the coordinates of P and Q if area $\triangle PCD$ = area $\triangle QAB = \frac{1}{2}$area quadrilateral $ABCD$. (1)

Solution by Bruno Salgueiro Fanego, Viveiro, Spain.

P is on side $\overline{AB} : y = 0 \Rightarrow P(p, 0)$.

Q is on side $\overline{CD} : y = \dfrac{11}{4}x + 4 \Rightarrow Q(4q,\ 11q + 4)$.

Area quadrilateral $ABCD$=area $\triangle ABD$+area $\triangle BCD$, so

$$(1) \quad \Leftrightarrow \quad \frac{1}{2}\left|\det\begin{pmatrix} p & 0 & 1 \\ 4 & 15 & 1 \\ 0 & 4 & 1 \end{pmatrix}\right| = \frac{1}{2}\left|\det\begin{pmatrix} 4q & 11q+4 & 1 \\ -3 & 0 & 1 \\ 12 & 0 & 1 \end{pmatrix}\right|$$

$$= \frac{1}{2}\left|\det\begin{pmatrix} -3 & 0 & 1 \\ 12 & 0 & 1 \\ 0 & 4 & 1 \end{pmatrix}\right| + \frac{1}{2}\left|\det\begin{pmatrix} 12 & 0 & 1 \\ 4 & 15 & 1 \\ 0 & 4 & 1 \end{pmatrix}\right|$$

$$\Leftrightarrow \quad \left|11p + 16\right| = 30 + 74 = 15\left|11q + 4\right| \Leftrightarrow 11p + 16 = \pm 104 = 15(11q + 4)$$

$$\Leftrightarrow \quad P_1(8, 0) \text{ or } \mathrm{P}_2(-120/11, 0) \text{ and } \mathrm{Q}_1(16/15, 104/15) \text{ or } \mathrm{Q}_2(-656/165, -104/15).$$

Observations by Ken Korbin. The following four points are on a straight line: midpoint of $\overline{AC}$, midpoint of $\overline{BD}$, P_1, and Q_1. Moreover, the midpoint of $\overline{P_1 P_2}$ = the midpoint of $\overline{Q_1, Q_2}$ = the intersection point of lines AB and CD.

Also solved by Brian D. Beasley, Clinton, SC; Michael N. Fried, Kibbutz Revivim, Israel; John Hawkins and David Stone (jointly), Statesboro, GA; Paul M. Harms, North Newton, KS; Peter E. Liley, Lafayette, IN; David E. Manes, Oneonta, NY; Charles McCracken, Dayton, OH; Boris Rays, Chesapeake, VA; David C.Wilson, Winston-Salem, NC, and the proposer.

- 5034: *Proposed by Roger Izard, Dallas, TX.*

In rectangle $MDCB$, $MB \perp MD$. F is the midpoint of BC, and points N, E and G lie on line segments DC, DM and MB respectively, such that $NC = GB$. Let the area of quadrilateral $MGFC$ be A_1 and let the area of quadrilateral $MGFE$ be A_2. Determine the area of quadrilateral $EDNF$ in terms of A_1 and A_2.

Solution by Paul M. Harms, North Newton, KS.

Put the rectangle $MDCB$ on a coordinate system. Assume all nonzero coordinates are positive with coordinates

$$M(0,0), B(0,b), C(c,b), D(c,0) \text{ and } E(e,0), F(c/2,b), G(0,g), N(c,g).$$

The coordinates satisfy $e < c$ and $g < b$. The area A_1 of the quadrilateral $MGFC =$ the area of $\triangle MGF+$ area of $\triangle MFC$. Then

$$A_1 = \frac{1}{2}g(c/2) + \frac{1}{2}(c/2)b = \frac{1}{2}(c/2)(b+g).$$

The area A_2 of the quadrilateral $MGFE =$ area of $\triangle MGF+$ area of $\triangle MEF$. Then

$$A_2 = \frac{1}{2}g(c/2) + \frac{1}{2}eb.$$

The area of the quadrilateral $EDNF =$ area of $\triangle EFD+$ area of $\triangle FDN$. The area of the quadrilateral $EDNF$ is then

$$\begin{aligned} &= \frac{1}{2}(c-e)b + \frac{1}{2}g(c/2) \\ &= 2(\frac{1}{2})(c/2)b - \frac{1}{2}eb + \frac{1}{2}g(c/2) \\ &= 2A_1 - A_2. \end{aligned}$$

Also solved by Bruno Salgueiro Fanego, Viveiro, Spain; John Hawkins and David Stone (jointly), Statesboro,GA; Kenneth Korbin, New York, NY; Peter E. Liley, Lafayette, IN; David E. Manes, Oneonta, NY; Boris Rays, Chesapeake, VA, and the proposer.

- 5035: *Proposed by José Luis Díaz-Barrero, Barcelona, Spain.*

  Let a, b, c be positive numbers. Prove that

$$(a^a b^b c^c)^2 (a^{-(b+c)} + b^{-(c+a)} + c^{-(a+b)})^3 \geq 27.$$

Solution 1 by David E. Manes, Oneonta, NY.

Note that the inequality is equivalent to

$$\frac{3}{a^{\frac{1}{b+c}} + b^{\frac{1}{c+a}} + c^{\frac{1}{b+c}}} \leq \sqrt[3]{a^{2a}b^{2b}c^{2c}}.$$

Since the problem is symmetrical in the variables a, b, and c, we can assume $a \geq b \geq c$. Therefore, $\ln a \geq \ln b \geq \ln c$. By the Rearrangement Inequality

$$\begin{aligned} a\ln a + b\ln b + c\ln c &\geq b\ln a + c\ln b + a\ln c \text{ and} \\ a\ln a + b\ln b + c\ln c &\geq c\ln a + a\ln b + b\ln c. \end{aligned}$$

Adding the two inequalities yields

$$2a\ln a + 2b\ln b + 2c\ln c \geq (b+c)\ln a + (c+a)\ln b + (a+b)\ln c.$$

Therefore,

$$\begin{aligned}
\ln\left(a^{2a}b^{2b}c^{2c}\right) &\geq \ln\left(a^{b+c}b^{c+a}c^{a+b}\right) \text{ or} \\
a^{2a}b^{2b}c^{2c} &\geq a^{b+c}b^{c+a}c^{a+b} \text{ and so} \\
\sqrt[3]{a^{2a}b^{2b}c^{2c}} &\geq \sqrt[3]{a^{b+c}b^{c+a}c^{a+b}}.
\end{aligned}$$

By the Harmonic-Geometric Mean Inequality

$$\frac{3}{a^{\frac{1}{b+c}} + b^{\frac{1}{c+a}} + c^{\frac{1}{b+c}}} \leq \sqrt[3]{a^{b+c}b^{c+a}c^{a+b}} \leq \sqrt[3]{a^{2a}b^{2b}c^{2c}}.$$

Solution 2 by Paolo Perfetti, Mathematics Department, University "Tor Vergata," Rome, Italy.

Taking the logarithm we obtain,

$$2\sum_{\text{cyc}} \ln a + 3\ln\left(\sum_{\text{cyc}} a^{-(b+c)}\right) \geq 3\ln 3.$$

The concavity of the logarithm yields,

$$2\sum_{\text{cyc}} \ln a + 3\left(\ln 3 - \sum_{\text{cyc}}(b+c)\ln a\right) \geq 3\ln 3.$$

Defining $s = a + b + c$ gives,

$$\sum_{\text{cyc}}(3a - s)\ln a \geq 0.$$

Since the second derivative of the function $f(x) = (3x - s)\ln x$ is positive for any x and s, ($f''(x) = 3/x + s/x^2$) it follows that,

$$\sum_{\text{cyc}}(3a - s)\ln a \geq \sum_{\text{cyc}}(3a - s)\ln a\Big|_{a=s/3} = 0.$$

Also solved by Bruno Salgueiro Fanego, Viveiro, Spain; Kee-Wai Lau, Hong Kong, China; Boris Rays, Chesapeake, VA, and the proposer.

- 5036: *Proposed by José Luis Díaz-Barrero, Barcelona, Spain.*

Find all triples (x, y, z) of nonnegative numbers such that

$$\begin{cases} x^2 + y^2 + z^2 = 1 \\ 3^x + 3^y + 3^z = 5 \end{cases}$$

Solution 1 by John Hawkins and David Stone, Statesboro, GA.

We are looking for all first octant points of intersection of the unit sphere with the surface $3^x + 3^y + 3^z = 5$. Clearly, the intercept points $(1, 0, 0), (0, 1, 0)$ and $(0, 0, 1)$ are solutions. We claim there no other solutions.

Consider the traces of our two surfaces in the xy-plane: the unit circle and the curve give by $3^x + 3^y = 4$. Our only concern is in the first quadrant, where we have a unit quarter circle and the curve $y = \dfrac{\ln(4-3^x)}{\ln 3}$. The two curves meet on the coordinate axes; otherwise graphing software shows that the logarithmic curve lies *inside* the quarter circle.

By the symmetry of the variables, we have the same behavior when we look at the traces in the xz- and yz-planes. That is, at our boundaries of concern, the exponential surface starts inside the sphere. By implicit differentiation of $3^x + 3^y + 3^z = 5$, we have the partial derivatives $\dfrac{\partial z}{\partial x} = -\dfrac{3^x}{3^z}$ and $\dfrac{\partial z}{\partial y} = -\dfrac{3^y}{3^z}$, which are both negative for nonnegative x, y and z. Therefore, the exponential surface *descends* from a trace inside the sphere to a trace which lies within the sphere. So the two surfaces have no points of intersection within the interior of the first octant.

Solution 2 by Ovidiu Furdui, Campia Turzii, Cluj, Romania.

Such triples are $(x, y, z) = (1,0,0), (0,1,0), (0,0,1)$. We note that the first equation implies that $x, y, z \in [0,1]$. On the other hand, using Bernoulli's inequality we obtain that

$$\begin{cases} 3^x = (1+2)^x \le 1+2x \\ 3^y = (1+2)^y \le 1+2y \\ 3^z = (1+2)^z \le 1+2, \end{cases}$$

and hence, $5 = 3^x + 3^y + 3^z \le 3 + 2(x+y+z)$. It follows that $1 \le x+y+z$. This implies that $x^2+y^2+z^2 \le x+y+z$, and hence, $x(1-x)+y(1-y)+z(1-z) \le 0$. Since the left hand side of the preceding inequality is nonnegative we obtain that $x(1-x) = y(1-y) = z(1-z) = 0$ from which it follows that x, y, z are either 0 or 1. This combined with the first equation of the system shows that exactly one of x, y, and z is 1 and the other two are 0, and the problem is solved.

Solution 3 by the proposer.

By inspection we see that $(1,0,0), (0,1,0)$ and $(0,0,1)$ are solutions of the given system. We claim that they are the only solutions of the system. In fact, for all $t \in [0,1]$ the function $f(t) = 3^t$ is greater than or equal to the function $g(t) = 2t^2+1$, as can be easily proven, for instance, by drawing their graphs when $0 \le t \le 1$.

Since $x^2+y^2+z^2 = 1$, then $x \in [0,1], y \in [0,1]$ and $z \in [0,1]$. Therefore

$$\begin{aligned} 3^x &\ge 2x^2+1, \\ 3^y &\ge 2y^2+1, \\ 3^z &\ge 2z^2+1. \end{aligned}$$

Adding up the preceding expressions yields

$$3^x + 3^y + 3^z \ge 2(x^2+y^2+z^2) + 3 \ge 5$$

and we are done

Also solved by Charles McCracken, Dayton, OH; Paolo Perfetti, Mathematics Department, University "Tor Vergata," Rome, Italy, and Boris Rays, Chesapeake,VA.

- **5037:** *Ovidiu Furdui, Campia Turzii, Cluj, Romania*

Let k, p be natural numbers. Prove that

$$1^k + 3^k + 5^k + \cdots + (2n+1)^k = (1 + 3 + \cdots + (2n+1))^p$$

for all $n \geq 1$ if and only if $k = p = 1$.

Solution 1 by Carl Libis, Kingston, RI.

Since $\Big(1 + 3 + \cdots + (2n+1)\Big)^p = \Big[(n+1)^2\Big]^p = (n+1)^{2p}$, it is clear that $\Big(1 + 3 + \cdots + (2n+1)\Big)^p$ is a monic polynomial of degree $2p$.

Let $S_k^{2n+1} = \sum\limits_{i=1}^{2n+1} i^k$. Then

$$S_k^{2n+1} = \sum_{i=1}^{n+1}(2i-1)^k + \sum_{i=1}^{n}(2i)^k = \sum_{i=0}^{n}(2i+1)^k + 2^k\sum_{i=1}^{n} i^k = \sum_{i=0}^{n}(2i+1)^k + 2^k S_k^n.$$

Then $\sum\limits_{i=0}^{n}(2i+1)^k = S_k^{2n+1} - 2^k S_k^n$. It is well known for sums of powers of integers S_k^n, that the leading term of S_k^n is $\dfrac{n^{k+1}}{k+1}$. Thus the leading term of $1^k + 3^k + 5^k + \cdots + (2n+1)^k$ is

$$\frac{(2n+1)^{k+1}}{k+1} - \frac{2^k n^{k+1}}{k+1} = \frac{2^{k+1}n^{k+1} - 2^k n^{k+1}}{k+1} = \frac{2^k n^{k+1}}{k+1}.$$

This is monic if, and only if, $k = 1$. When $k = 1$ we have that

$$\sum_{i=0}^{n}(2i+1) = S_1^{2n+1} - 2S_1^n = \frac{(2n+1)(2n+2)}{2} - 2\frac{n(n+1)}{2} = (n+1)^2.$$

For k, p natural numbers we have that $1^k + 3^k + 5^k + \cdots + (2n+1)^k = \Big(1 + 3 + \cdots + (2n+1)\Big)^p$ for all $n \geq 1$ if, and only if, $k = p = 1$.

Solution 2 by Kee-Wai Lau, Hong Kong, China.

If $k = p = 1$, the equality $1^k + 3^k + 5^k + \cdots + (2n+1)^k = (1 + 3 + \cdots + (2n+1))^p$ is trivial. Now suppose that the equality holds for all $n \geq 1$. By putting $n = 1, 2$, we obtain $1 + 3^k = 4^p$ and $1 + 3^k + 5^k = 9^p$. Hence

$$\begin{aligned} 3^k &= 4^p - 1 \text{ and} \\ 5^k &= 9^p - 4^p. \end{aligned}$$

Eliminating k from the last two equations, we obtain $9^p = 4^p + (4^p - 1)^{(\ln 5/\ln 3)}$. Hence,

$$\begin{aligned} 9^p &< 2\Big(4^{p(\ln 5/\ln 3)}\Big) \\ p\ln 9 &< \ln 2 + \frac{p(\ln 4)(\ln 5)}{\ln 3}, \text{and} \end{aligned}$$

$$p \quad < \quad \frac{(\ln 2)(\ln 3)}{(\ln 3)(\ln 9) - (\ln 4)(\ln 5)} = 4.16\cdots.$$

Thus $p = 1, 2, 3, 4$. But it is easy to check that only the case $p = 1$ and $k = 1$ admits solutions in the natural numbers for the equation $1 + 3^k = 4^p$, and this completes the solution.

Solution 3 by Paul M. Harms, North Newton, KS.

Clearly if $k = p = 1$, the equation holds for all appropriate integers n. For the *only if* part of the statement consider the contrapostive statement:

If $p \neq 1$ or $k \neq 1$, then for some $n \geq 1$ the equation does not hold.

Consider $n = 1$. Then the equation in the problem is $1^k + 3^k = (1 + 3)^p = 4^p$. If $k = 1$ with $p > 1$, then $4 < 4^p$ so the equation does not hold.

If $k > 1$ with $p = 1$, then $1^k + 3^k > 4$ so the equation does not hold.

Now consider both $p > 1$ and $k > 1$ using the equation in the form
$3^k = 4^p - 1^k = (2^p - 1)(2^p + 1)$.

If $p > 1$, then $2^p - 1 > 1$ and $2^p + 1 > 1$. Also, the expressions $2^p - 1$ and $2^p + 1$ are 2 units apart so that if 3 is a factor of one of these expressions then 3 is not a factor of the other expression. Since both expressions are greater than one, if 3 is a factor of one of the expressions, then the other expression has a prime number other than 3 as a factor. Thus $(2^p - 1)(2^p + 1)$ has a prime number other than 3 as a factor and cannot be equal to 3^k, a product of just the prime number 3. Thus the equation does not hold when both $p > 1$ and $k > 1$.

Solution 4 by John Hawkins and David Stone, Statesboro, GA.

Denote $1^k + 3^k + 5^k + \cdots + (2n + 1)^k = (1 + 3 + \cdots + (2n + 1))^p$ by (#). The condition requesting *all* $n \geq 1$ is overkill. Actually, we can prove the following are equivalent:

(a) condition (#) holds for all $n \geq 1$,

(b) condition (#) holds for all $n = 1$,

(c) $k = p = 1$.

Clearly, (a)$\Rightarrow$ (b).

Also (c) $\Rightarrow$ (a), for if $k = p = 1$, then (#) becomes the identity

$$1 + 3 + 5 + \cdots + (2n + 1) = (1 + 3 + \cdots + (2n + 1)).$$

Finally, we prove that (b) $\Rightarrow$ (c). Assuming the truth of (#) for $n = 1$ tells us that $3^k = 4^p - 1$.

If $k = 1$, we immediately conclude that $p = 1$ and we are finished.

Arguing by contradiction, suppose $k \geq 2$, so 3^k is actually a multiple of 9. Thus $4^p \equiv 1 (\text{mod } 9)$. Now consider the powers of 4 modulo 9:

$$4^0 \equiv 1 (\text{mod } 9)$$
$$4^1 \equiv 4 (\text{mod } 9)$$

$$4^2 \equiv 7 (\text{mod } 9)$$
$$4^3 \equiv 1 (\text{mod } 9)$$

That is, 4 has order 3(mod 9), so $4^p \equiv 1(\text{mod } 9)$ if and only if p is a multiple of 3. Based upon some numerical testing, we consider 4^p modulo 7: $4^p = 4^{3t} \equiv 64^t \equiv 1^t \equiv 1(\text{mod } 7)$. That is, 7 divides $4^p - 1$, so $4^p - 1$ cannot be a power of 3. We have reached a contradiction.

Solution 5 by the proposer.

One implication is easy to prove. To prove the other implication we note that

$$1+3+\cdots+(2n+1) = \sum_{k=1}^{n+1}(2k-1) = 2\sum_{k=1}^{n+1} k-(n+1) = (n+1)(n+2)-(n+1) = (n+1)^2.$$

It follows that

$$1^k + 3^k + 5^k + \cdots + (2n+1)^k = (n+1)^{2p}.$$

We multiply the preceding relation by $2/(2n+1)^{k+1}$ and we get that

$$\frac{2}{2n+1}\left(\left(\frac{1}{2n+1}\right)^k + \left(\frac{3}{2n+1}\right)^k + \cdots + \left(\frac{2n+1}{2n+1}\right)^k\right) = 2\frac{(n+1)^{2p}}{(2n+1)^{k+1}}. \qquad (1)$$

Letting $n \to \infty$ in (1) we get that

$$\int_0^1 x^k dx = \frac{1}{k+1} = \lim_{n\to\infty} 2\frac{(n+1)^{2p}}{(2n+1)^{k+1}}.$$

It follows that $2p = k+1$ and that $\frac{1}{k+1} = \frac{1}{2^k}$. However, the equation $k+1 = 2^k$ has a unique positive solution namely $k = 1$. This can be proved by applying Bernouli's inequality as follows

$$2^k = (1+1)^k \geq 1 + k \cdot 1 = k+1,$$

with equality if and only if $k = 1$. Thus, $k = p = 1$ and the problem is solved.

Also solved by Boris Rays, Chesapeake, VA.

Late Solutions

Late solutions were received from **David C. Wilson of Winston-Salem, NC** to problems 5026, 5027, and 5028.