Problems

Ted Eisenberg, Section Editor

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This section of the Journal offers readers an opportunity to exchange interesting mathematical problems and solutions. Please send them to Ted Eisenberg, Department of Mathematics, Ben-Gurion University, Beer-Sheva, Israel or fax to: 972-86-477-648. Questions concerning proposals and/or solutions can be sent e-mail to <eisenbt@013.net>. Solutions to previously stated problems can be seen at <http://www.ssma.org/publications>.

Solutions to the problems stated in this issue should be posted before April 15, 2014

- **5289:** *Proposed by Kenneth Korbin, New York, NY*

  Part 1: Thirteen different triangles with integer length sides and with integer area each have a side with length 1131. The angle opposite 1131 is $Arcsin\left(\frac{3}{5}\right)$ in all 13 triangles.

  Find the sides of the triangles.

  Part 2: Fourteen different triangles with integer length sides and with integer area each have a side with length 6409. The size of the angle opposite 6409 is the same in all 14 triangles.

  Find the sides of the triangles.

- **5290:** *Proposed by Tom Moore, Bridgewater State University, Bridgewater, MA*

  Someone wrongly remembered the description of an even perfect number as:
  $N = 2^p\left(2^{p-1} - 1\right)$, where p is a prime number. Classify these numbers correctly. Which are deficient and which are abundant?

  (If n and d are positive integers, $d \neq n$, but $d\Big|n$, then d is called a proper divisor of n. The integer n is called *perfect* if the sum of its proper divisors is equal to n. The number n is called *deficient* if the sum of its proper divisors is less than n; and if the sum of its proper divisors is greater than n, then n is called an *abundant* number. E.g., The proper divisors of 6 are $1, 2,$ and 3. Their sum is 1+2+3=6, and so 6 is a perfect number; all prime numbers are deficient, and the proper divisors of 12 are $1, 2, 4,$ and 6. So 12 is an abundant number.)

- **5291:** *Arkady Alt, San Jose, CA*

  Let $m_a m_b$ be the medians of a triangle with side lengths a, b, c. Prove that:

  $$m_a m_b \leq \frac{2c^2 + ab}{4}.$$

- **5292:** *Proposed by D.M. Bătinetu−Giurgiu, "Matei Basarab" National College, Bucharest, Romania and Neculai Stanciu, "Geroge Emil Palade" General School,* *Buzău, Romania*

Let a and b be real numbers with $a < b$, and let c be a positive real number. If $f : R \longrightarrow R_+$ is a continuous function, calculate:

$$\int_a^b \frac{e^{f(x-a)}\left(f(x-a)\right)^{\frac{1}{c}}}{e^{f(x-a)}\left(f(x-a)\right)^{\frac{1}{c}} + e^{f(b-x)}\left(f(b-x)\right)^{\frac{1}{c}}}dx.$$

- **5293:** *Proposed by José Luis Díaz-Barrero, Barcelona Tech, Barcelona, Spain*

Let ABC be a triangle. Prove that

$$\sqrt[4]{\sin A\cos^2 B} + \sqrt[4]{\sin B\cos^2 C} + \sqrt[4]{\sin C\cos^2 A} \le 3\sqrt[8]{\frac{3}{64}}.$$

- **5294:** *Proposed by Ovidiu Furdui, Technical University of Cluj-Napoca, Cluj-Napoca, Romania*

a) Calculate $\displaystyle\sum_{n=2}^{\infty}\left(n - \zeta(2) - \zeta(3) - \cdots - \zeta(n)\right)$.

b) More generally, for $k \ge 2$ an integer, find the value of the multiple series

$$\sum_{n_1,n_2,\cdots,n_k=1}^{\infty}\left(n_1+n_2+\cdots+n_k - \zeta(2) - \zeta(3) - \cdots - \zeta(n_1+n_2+n_3+\cdots+n_k)\right),$$

where ζ denotes the Riemann Zeta function.

Solutions

- **5271:** *Proposed by Kenneth Korbin, New York, NY*

Given convex cyclic quadrilateral $ABCD$ with $\overline{AB} = x, \overline{BC} = y$, and $\overline{BD} = 2\overline{AD} = 2\overline{CD}$.

Express the radius of the circum-circle in terms of x and y.

Solution 1 by Andrea Fanchini, Cantú, Italy

Method I

In a cyclic quadrilateral with successive vertices A, B, C, D and sides $a = \overline{AB}, b = \overline{BC}, c = \overline{CD}, d = \overline{DA}$, the length of the diagonal $q = \overline{BD}$ can be expressed in terms of the sides as:

$$q = \sqrt{\frac{(ac+bd)(ab+cd)}{ad+bc}}$$

Let $t = \overline{AD} = \overline{CD}$. Then in our case we have

$$2t = \sqrt{\frac{(xt + yt)(xy + t^2)}{xt + yt}} \Rightarrow t = \sqrt{\frac{xy}{3}}$$

Let $p = \overline{AC}$ and according Ptolemy's theorem

$$p = \frac{ac + bd}{q} = \frac{x + y}{2}$$

Then we denote $\angle ABD = \angle DBC = \beta$, so $\angle ABC = 2\angle ABD = 2\angle DBC = 2\beta$.
Furthermore, from the angle at the center theorem $\angle AOD = \angle ABC = 2\beta$.
Now with the Carnot's theorem at the side $\overline{AC}$ of the $\triangle ABC$, we have

$$p^2 = x^2 + y^2 - 2xy \cos 2\beta \Rightarrow \cos 2\beta = \frac{3x^2 + 3y^2 - 2xy}{8xy}$$

Using another time Carnot's theorem at the side $\overline{AD}$ of the $\triangle AOD$, we obtain

$$t^2 = R^2 + R^2 - 2R^2 \cos 2\beta$$

from which, we finally obtain, the radius R of the circum-circle in terms of x and y

$$R = \frac{2xy}{\sqrt{3\,(10xy - 3x^2 - 3y^2)}}$$

Method II

Applying Parameshvara's formula, a cyclic quadrilateral with successive sides a, b, c, d and semiperimeter s has the circumradius R given by

$$R = \frac{1}{4}\sqrt{\frac{(ab + cd)(ac + bd)(ad + bc)}{(s - a)(s - b)(s - c)(s - d)}}$$

In our case we have $a = x, b = y$ and $c = d = \sqrt{\frac{xy}{3}}$. Substituting, we obtain the formula requested.

Solution 2 by Kee-Wai Lau, Hong Kong, China

Let $\overline{BD} = 2z$ and $\angle BAD = \theta = \pi - \angle BCD$. Applying the cosine formula to triangles BAD and BCD respectively, we obtain,

$$\cos\theta = \frac{x^2 - 3z^2}{2xz} \text{ and } -\cos\theta = \cos(\pi - \theta) = \frac{y^2 - 3z^2}{2yz}.$$

Hence,

$$z = \sqrt{\frac{xy}{3}}, \quad \cos\theta = \frac{\sqrt{3}(x - y)}{2\sqrt{xy}}, \text{and } \sin\theta = \frac{1}{2}\sqrt{\frac{(3x - y)(3y - x)}{xy}}.$$

It is easy to check that $\sin\theta$ is a positive real number not exceeding 1 if and only if $\frac{1}{3} < \frac{x}{y} < 3$. Subject to this condition, we obtain

that the radius of the circum-cirlce $= \dfrac{\overline{BD}}{2\sin\theta} = \dfrac{2xy}{\sqrt{3(3x - y)(3y - x)}}$.

Also solved by Bruno Salgueiro Fanego, Viveiro, Spain; Ed Gray, Highland Beach, FL; Neculai Stanciu, Buzău, Romania and Titu Zvonaru, Comănesti, Romania; David Stone and John Hawkins, Georgia Southern University, Statesboro, GA; Ercole Suppa, Teramo, Italy, and the proposer.

- **5272:** *Proposed by Tom Moore, Bridgewater State University, Bridgewater, MA*

The Jacobsthal numbers begin $0, 1, 1, 3, 5, 11, 21, \cdots$ with general term $J_n = \dfrac{2^n - (-1)^n}{3}$, $\forall n \geq 0$. Prove that there are infinitely many Pythagorean triples like $(3, 4, 5)$ and $(13, 84, 85)$ that have "hypotenuse" a Jacobsthal number.

Solution 1 by Dionne Bailey, Elsie Campbell, and Charles Diminnie, Angelo State University, San Angelo, TX

For $n \geq 1$, $\left(2^{2n} - 1, 2^{n+1}, 2^{2n} + 1\right)$ is a primitive Pythagorean triple since $\gcd\left(2^{2n} - 1, 2^{n+1}\right) = 1$ and

$$\begin{aligned} \left(2^{2n} - 1\right)^2 + \left(2^{n+1}\right)^2 &= 2^{4n} - 2^{2n+1} + 1 + 2^{2n+2} \\ &= 2^{4n} + 2^{2n+1} + 1 \\ &= \left(2^{2n} + 1\right)^2. \end{aligned}$$

It follows that for any positive integer m, $\left(\left(2^{2n} - 1\right) m, 2^{n+1} m, \left(2^{2n} + 1\right) m\right)$ is also a Pythagorean triple. In particular, when $n \geq 1$, $\left(\left(2^{2n} - 1\right) J_{2n}, 2^{n+1} J_{2n}, \left(2^{2n} + 1\right) J_{2n}\right)$ is a Pythagorean triple with

$$\begin{aligned} \left(2^{2n} + 1\right) J_{2n} &= \left(2^{2n} + 1\right) \cdot \frac{2^{2n} - 1}{3} \\ &= \frac{2^{4n} - 1}{3} \\ &= J_{4n}. \end{aligned}$$

Hence, for $n \geq 1$, $\left(\left(2^{2n} - 1\right) J_{2n}, 2^{n+1} J_{2n}, J_{4n}\right)$ is a Pythagorean triple whose "hypotenuse" is a Jacobsthal number.

Solution 2 by Ed Gray, Highland Beach, FL

1) $2^2 \equiv (-1) \pmod 5$
2) $2^{2k} \equiv (-1)^k \pmod 5$
3) If k is even, $2^{2k} - 1 \equiv 0 \pmod 5$
4) If k is odd, $2^{2k} + 1 \equiv 0 \pmod 5$, in either case
5) $\left(2^{2k} - 1\right)\left(2^{2k} + 1\right) \equiv 0 \pmod 5$, or
6) $2^{4k} - 1 \equiv 0 \pmod 5$.

Suppose
7) $n = 4k$.
Then
8) $J_n = J_{4k} = \dfrac{2^{4k} - 1}{3} \equiv 0 \pmod 5$ by (6).

Therefore,
9) If $n = 4k$, let $J_n = J_{4k} = r(2^2 + 1^2)$.
Let this be the "hypotenuse." The formulae for a Pythagorean triple are:
10) $x = r(2ab), y = r(a^2 - b^2), z = r(a^2 + b^2)$.
From (9), let $a = 2, b = 1$.
Then (10) becomes:
11) $x = r(2ab),\ y = r(a^2 - b^2),\ z = r(a^2 + b^2)$, or
12) $x = 4r,\ y = 3r,\ z = 5r$, where r is defined by (9).
13) Hence $x^2 + y^2 = z^2$.

Solution 3 by Kenneth Korbin, New York, NY

If a positive integer is a multiple of 5, then it is the length of the hypotenuse of at least one Pythagorean triangle.

In the J series, every fourth term is a multiple of 5.
For example, $J_4 = 5,\ J_8 = 85,\ J_{12} = 1365$, and in general $J_{4n} = 16J_{4(n-1)} + 5$.

We have

$$\begin{aligned} J_n &= \frac{2^n - (-1)^n}{3}. \text{ Then,} \\ J_{4n} &= \frac{2^{4n} - (-1)^{4n}}{3} = \frac{16^n - 1^n}{3}. \\ 16^n - 1 &\equiv 15 \pmod{15} \\ \frac{16^n - 1}{3} &\equiv 5 \pmod{5}. \end{aligned}$$

The J sequence (mod 10) is

$$(1, 1, 3, 5, 1, 1, 3, 5, \ldots, 1, 1, 3, 5, \ldots)$$

If a and b are positive integers and if $a|b$, then $J_{4a}|J_4b$.

Also solved by Jahangeer Kholdi and Farideh Firoozbakht, University of Isfahan, Khansar, Iran; Carl Libis, Lane College, Jackson, TN; Bob Sealy, Sackville, NB, Canada; David Stone and John Hawkins, Georgia Southern University, Statesboro, GA, and the proposer.

- **5273:** *Proposed by Titu Zvonaru, Comănesti, Romania and Neculai Stanciu, "Geroge Emil Palade" General School, Buzău, Romania*

Solve in the positive integers the equation $abcd + abc = (a + 1)(b + 1)(c + 1)$.

Solution 1 by Adrian Naco, Polytechnic University,Tirana, Albania.

We have that,

$$2 \le d + 1 = \left(1 + \frac{1}{a}\right)\left(1 + \frac{1}{b}\right)\left(1 + \frac{1}{c}\right) \le \left(1 + \frac{1}{1}\right)\left(1 + \frac{1}{1}\right)\left(1 + \frac{1}{1}\right) = 8, \text{ or } 1 \le \text{d} \le 7.$$

Let us suppose that $1 \le c \le b \le a$, then,

$$2 \le (d+1) = \left(1+\frac{1}{a}\right)\left(1+\frac{1}{b}\right)\left(1+\frac{1}{c}\right) \le \left(1+\frac{1}{c}\right)^3$$
$$\Rightarrow \quad \sqrt[3]{2} \le 1+\frac{1}{c} \quad \Rightarrow \quad c \le \frac{1}{\sqrt[3]{2}-1}$$
$$\Rightarrow \quad c \in \{1,2,3\}$$

Case 1. $c = 1$. Thus,

$$ab(d+1) = 2(a+1)(b+1) \quad \Rightarrow \quad d+1 = 2\left(\frac{a+1}{a}\right)\left(\frac{b+1}{b}\right) > 2$$

Thus, we have that $2 \le d \le 7$.
a) If $a = b$, then it implies that,

$$d = 1 + 2 \cdot \frac{2a+1}{a^2} \quad \Rightarrow \quad a = 1 = b,\ d = 7$$

b) If $a \ge b+1$, then,

$$\begin{aligned} 3ab \le ab(d+1) = 2(a+1)(b+1) \quad &\Rightarrow \quad 3ab \le 2ab + 2a + 2b + 2 \\ &\Rightarrow \quad ab \le 2a + 2b + 2 \\ &\Rightarrow \quad b \le 2 + \frac{2(b+1)}{a} \le 2 + 2 = 4 \\ &\Rightarrow \quad b \in \{1,2,3,4\} \end{aligned}$$

Thus, we have the following solutions

$$\begin{gathered} b = 1,\ a = 2,\ d = 5 \\ b = 1,\ a = 4,\ d = 4 \\ b = 3,\ a = 8,\ d = 2 \\ b = 4,\ a = 5,\ d = 2 \end{gathered}$$

Case 2. If $c = 2$, then,

$$2ab(d+1) = 3(a+1)(b+1).$$

a) If $a = b$, then it implies that,

$$2a^2(d+1) = 2(a+1)^2 \quad \Rightarrow \quad a^2/3 \quad \Rightarrow \quad a = 1 < 2 = c \le a \quad \Rightarrow \quad a < a!$$

b) If $a \geq b+1$, then,

$$\begin{aligned} 4ab \leq 2ab(d+1) = 3(a+1)(b+1) \quad &\Rightarrow \quad 4ab \leq 3ab + 3a + 3b + 3 \\ &\Rightarrow \quad ab \leq 3a + 3b + 3 \\ &\Rightarrow \quad b \leq 3 + 3\frac{(b+1)}{a} \leq 3 + 3 = 6 \\ &\Rightarrow \quad b \in \{2,3,4,5,6\} \end{aligned}$$

Thus, we have the following solutions

$$\begin{aligned} &b = 2,\ a = 3,\ \ d = 2 \\ &b = 4,\ a = 15,\ d = 1 \\ &b = 6,\ a = 7,\ \ d = 1. \end{aligned}$$

Case 3. If $c = 3$, then,

$$\begin{aligned} 6ab \leq 3ab(d+1) = 4(a+1)(b+1) \quad &\Rightarrow \quad 6ab \leq 4ab + 4a + 4b + 4 \\ &\Rightarrow \quad ab \leq 2a + 2b + 2 \\ &\Rightarrow \quad b \leq 2 + 2\frac{b+1}{a} \leq 2 + 2 = 4 \\ &\Rightarrow \quad b \in \{3,4\} \end{aligned}$$

Thus, we have the following solutions

$$\begin{aligned} &b = 3,\ a = 8,\ d = 1 \\ &b = 4,\ a = 5,\ d = 1. \end{aligned}$$

Finally, the solutions (a,b,c,d), of the given equality are,

$$\begin{aligned} &Case\ 1:\ \ (1,1,1,8) \\ &(1,1,2,5),(1,2,1,5),(2,1,1,5) \\ &(1,1,4,4),(1,4,1,4),(4,1,1,4) \\ &(1,3,8,2),(1,8,3,2),(3,1,8,2),(3,8,1,2),(8,1,3,2),(8,3,1,2) \end{aligned}$$

$(1,4,5,2),(1,5,4,2),(4,1,5,2),(4,5,1,2),(5,1,4,2),(5,4,1,2).$

$Case\ 2:$ $(2,2,3,2),(2,3,2,2),(3,2,2,2)$
$(2,4,15,1),(2,15,4,1),(4,2,15,1),(4,15,2,1),(15,2,4,1),(15,4,2,1)$
$(2,6,7,1),(2,7,6,1),(6,2,7,1),(6,7,2,1),(7,2,6,1),(7,6,2,1).$

$Case\ 3:$ $(3,3,8,1),(3,8,3,1),(8,3,3,1)$
$(3,4,5,1),(3,5,4,1),(4,3,5,1),(4,5,3,1),(5,3,4,1),(5,4,3,1).$

Solution 2 by Kee-Wai Lau, Hong Kong, China

We show that the solutions are given by

$$\begin{aligned}(a,b,c,d) &= (1,1,1,7),(1,1,2,5),\ (1,1,4,4),\ (1,2,3,3),\ (1,3,8,2),\ (1,4,5,2),\\ &\quad (2,2,3,2),\ (2,4,15,1),\ (2,5,9,1),\ (2,6,7,1),\ (3,3,8,1),\ (3,4,5,1).\end{aligned}$$

together with solutions obtained by permutations of entries a,b,c.

Clearly it suffices to consider the case $a \le b \le c$. We have

$$1 \le d = \left(1+\frac{1}{a}\right)\left(1+\frac{1}{b}\right)\left(1+\frac{1}{c}\right) - 1 \le \left(1+\frac{1}{a}\right)^3 - 1 \text{ so that } a \le \frac{1}{2^{\frac{1}{3}}-1} < 4.$$

Hence, for $a = 1,2,3$, we have respectively $1 \le d \le 7,\ 1 \le d \le 2,\ d = 1$. We then obtain the following table readily:

a	d	c in terms of b	Solutions (b,c) in positive integers with $a \leq b \leq c$
1	1	$-b-1$	No solutions
	2	$2+\dfrac{6}{b-2}$	(3,8),(4,5)
	3	$1+\dfrac{2}{b-1}$	(2,3)
	4	$1+\dfrac{4-b}{3b-2}$	(1,4)
	5	$1+\dfrac{2-b}{2b-1}$	(1,2)
	6	$1+\dfrac{4-3b}{5b-2}$	No solutions
	7	$1+\dfrac{2(b-1)}{3b-1}$	(1,1)
2	1	$3+\dfrac{12}{b-3}$	(4,15), (5,9),(6,7)
	2	$1+\dfrac{2}{b-1}$	(2,3)
3	1	$2+\dfrac{6}{b-2}$	(3,8), (4,5)

Also solved by Ed Gray, Highland Beach, FL; Jahangeer Kholdi and Farideh Firoozbakht, University of Isfahan, Khansar, Iran; Kenneth Korbin, NY, NY, and by David Stone and John Hawkins, Georgia Southern University, Statesboro, GA, and the proposers.

- **5274:** *Proposed by Enkel Hysnelaj, University of Technology, Sydney, Australia*

Let x, y, z, α be real positive numbers. Show that if

$$\sum_{cycl} \frac{(n+1)\,x^3 + nx}{x^2+1} = \alpha$$

then

$$\sum_{cycl} \frac{1}{x} > \frac{9n}{\alpha} - \frac{\alpha}{n} + \frac{9n\alpha}{9n^2+\alpha^2}$$

where n is a natural number.

Solution by proposer

Doing easy manipulations we have

$$\alpha = \sum_{cycl} \frac{(n+1)x^3 + nx}{x^2+1} = \sum_{cycl} \frac{1}{x} + \sum_{cycl} \frac{-1+(n-1)x^2+(n+1)x^4}{x(x^2+1)}$$

Let $f(x) = \dfrac{-1+(n-1)x^2+(n+1)x^4}{x(x^2+1)}$. One can easy observe that

$$\begin{aligned} f'(x) &= \frac{1+(n+2)x^2+(2n+4)x^4+(n+1)x^6}{x^2(1+x^2)^2} \\ f''(x) &= -\frac{2(1+3x^2+2x^6)}{x^3(1+x^2)^3} \end{aligned}$$

It is obvious that $f'(x) > 0$ and $f''(x) < 0$ for any real positive number x, which implies that the function $f(x)$ is an increasing and concave function in the real positive domain. Applying Jensen's inequality we have

$$\sum_{cycl} \frac{-1+(n-1)x^2+(n+1)x^4}{x(x^2+1)} = \sum_{cycl} f(x) \le 3f\left(\frac{1}{3}\sum_{cycl} x\right)$$

Doing easy manipulations, one can easy observe that

$$\alpha = \sum_{cycl} \frac{(n+1)x^3+nx}{x^2+1} = \sum_{cycl} nx + \sum_{cycl} \frac{x^3}{x^2+1} > n\sum_{cycl} x$$

Finally, using the above results we have

$$\begin{aligned} \sum_{cycl} \frac{1}{x} &= \alpha - \sum_{cycl} \frac{-1+(n-1)x^2+(n+1)x^4}{x(x^2+1)} \\ &\ge \alpha - 3f\left(\frac{1}{3}\sum_{cycl} x\right) \\ &> \alpha - 3f\left(\frac{\frac{\alpha}{n}}{3}\right) \\ &= \alpha - 3f\left(\frac{\alpha}{3n}\right) \\ &= \frac{9n}{\alpha} - \frac{\alpha}{n} + \frac{9n\alpha}{9n^2+\alpha^2} \end{aligned}$$

and this is the end of the proof.

- **5275:** *Proposed by José Luis Díaz-Barrero, Barcelona Tech, Barcelona, Spain*

Find all real solutions to the following system of equations

$$\left.\begin{array}{c} \underbrace{\sqrt{2+\sqrt{2+\ldots+\sqrt{2+x_1}}}}_{n} + \underbrace{\sqrt{2-\sqrt{2+\ldots+\sqrt{2+x_1}}}}_{n} = x_2\sqrt{2}, \\ \underbrace{\sqrt{2+\sqrt{2+\ldots+\sqrt{2+x_2}}}}_{n} + \underbrace{\sqrt{2-\sqrt{2+\ldots+\sqrt{2+x_2}}}}_{n} = x_3\sqrt{2}, \\ \ldots\ldots\ldots\ldots\ldots\ldots \\ \underbrace{\sqrt{2+\sqrt{2+\ldots+\sqrt{2+x_{n-1}}}}}_{n} + \underbrace{\sqrt{2-\sqrt{2+\ldots+\sqrt{2+x_{n-1}}}}}_{n} = x_n\sqrt{2}, \\ \underbrace{\sqrt{2+\sqrt{2+\ldots+\sqrt{2+x_n}}}}_{n} + \underbrace{\sqrt{2-\sqrt{2+\ldots+\sqrt{2+x_n}}}}_{n} = x_1\sqrt{2}, \end{array}\right\}$$

where $n \geq 2$.

Solution by Arkady Alt, San Jose, CA

Let $h(x) := \sqrt{2+x}$. Then $h(x)$ is a function defined on $[-2,\infty)$ with range $[0,\infty)$.

Since $h : [-2,\infty) \longrightarrow [0,\infty)$ then for any $n \in N$ we can define recursively $n-$iterated function $h_n : [-2,\infty) \longrightarrow [0,\infty)$, namely $h_1 = h$ and $h_{n+1} = h \circ h_n, n \geq 1$.

Let $f(x) := \dfrac{h_n(x) + \sqrt{2 - h_{n-1}(x)}}{\sqrt{2}}$ for $x \in [-2,\infty)$ such that $h_{n-1}(x) \leq 2$.

Since $h_{n-1}(x) \leq 2 \iff h_{n-1}^2(x) \leq 4 \iff h_{n-2}(x) \leq 2 \iff .... \iff h_1(x) \leq 2 \iff x \leq 2$

then $Dom(f) = [-2, 2]$. Moreover, applying inequality $\dfrac{a+b}{\sqrt{2}} \leq \sqrt{a^2+b^2}$ to $a = h_n(x)$ and $b = \sqrt{2 - h_{n-1}(x)}$ we obtain $f(x) \leq 2$ and since by definition $f(x) \geq 0$ for $x \in Dom(f)$
then $range(f) \subset [0, 2]$.

Using f we can rewrite original system as follow:

(1) $\begin{cases} x_{k+1} = f(x_k), k = 1, 2, ..., n-1 \\ x_1 = f(x_n) \end{cases}$.

Since $x_k \in [0, 2]$, $k = 1, 2, ..., n$ then by setting $t_k := \cos^{-1}\left(\dfrac{x_k}{2}\right), k = 1, 2, ..., n$
we obtain $t_k \in \left[0, \dfrac{\pi}{2}\right]$, $x_k = 2\cos t_k$, $k = 1, 2, ..., n$.

Noting that $h(2\cos t) = 2\cos t2$ for $t \in \left[0, \dfrac{\pi}{2}\right]$ by Math. Induction we obtain
$h_k(2\cos t) = 2\cos\dfrac{t}{2^k}, k = 1, 2, ..., ...$and, therefore, $f(2\cos t) =$
$\dfrac{1}{\sqrt{2}}\left(2\cos\dfrac{t}{2^n} + \sqrt{2 - 2\cos\dfrac{t}{2^{n-1}}}\right) = 2\left(\dfrac{1}{\sqrt{2}}\cos\dfrac{t}{2^n} + \dfrac{1}{\sqrt{2}}\sin\dfrac{t}{2^n}\right) = 2\cos\left(\dfrac{\pi}{4} - t2^n\right)$.

Since $\dfrac{\pi}{4} - \dfrac{t}{2^n} \in \left[0, \dfrac{\pi}{2}\right]$ for $t \in \left[0, \dfrac{\pi}{2}\right]$ then $\dfrac{\pi}{4} - \dfrac{t_k}{2^n} \in \left[0, \dfrac{\pi}{2}\right]$ as well as $t_k \in \left[0, \dfrac{\pi}{2}\right]$ for any $k = 1, 2, ..., n$ and, therefore, **(1)**

$\iff \begin{cases} 2\cos t_{k+1} = 2\cos\left(\dfrac{\pi}{4} - \dfrac{t_k}{2^n}\right), k = 1, 2, ..., n-1 \\ 2\cos t_1 = 2\cos\left(\dfrac{\pi}{4} - \dfrac{t_n}{2^n}\right) \end{cases} \iff$

(2) $\begin{cases} t_{k+1} = \dfrac{\pi}{4} - \dfrac{t_k}{2^n}, k = 1, 2, ..., n-1 \\ t_1 = \dfrac{\pi}{4} - t_n 2^n \end{cases}$.

Lemma:

Let a, b be real numbers such that $|a| \neq 1$. Then system of equations
$\begin{cases} t_{k+1} = b + at_k, k = 1, 2, ..., n-1 \\ t_1 = b + at_n \end{cases}$
have only solution $t_1 = t_2 = ... = t_n = \dfrac{b}{1-a}$.

Proof: Noting that $\dfrac{b}{1-a} = b + a \cdot \dfrac{b}{1-a}$ and denoting $c := \dfrac{b}{1-a}$ we obtain
$t_{k+1} = b + at_k \iff t_{k+1} - c = a\left(t_k - c\right), k = 1, 2, ...n-1$
and $t_1 = b + at_n \iff t_1 - c = a\left(t_n - c\right)$. Since $t_k - c, k = 1, 2, ...$
is geometric sequence we have $t_k - c = a^{k-1}\left(t_1 - c\right), k = 1, 2, ...n-1$ and therefore,

$t_1 - c = a \cdot a^{n-1}\left(t_1 - c\right) \iff t_1 - c = a^n\left(t_1 - c\right) \iff \left(t_1 - c\right)\left(1 - a^n\right) = 0 \iff t_1 = c.$

That yield $t_k - c = a^{k-1}\left(t_1 - c\right) = 0 \iff t_k = c, k = 2, ..., n.$

Thus, $t_1 = t_2 = ... = t_n = c = \dfrac{b}{1-a}$.

Applying the Lemma with $a = -\dfrac{1}{2^n}$ and $b = \dfrac{\pi}{4}$ we obtain the only solution of **(2),**

$t_1 = t_2 = ... = t_n = \dfrac{2^{n-2}\pi}{2^n + 1}$ and then $x_1 = x_2 = ... = x_n = 2\cos\left(\dfrac{2^{n-2}\pi}{2^n + 1}\right)$ is the only solution of original system.

Also solved by Adrian Naco, Polytechnic University, Tirana, Albania; Paolo Perfetti, Department of Mathematics, "Tor Vergata" University, Rome, Italy; and the proposer.

- **5276:** *Proposed by Ovidiu Furdui, Technical University of Cluj-Napoca, Cluj-Napoca, Romania*

(a) Let $a \in (0, 1]$ be a real number. Calculate

$$\int_0^1 a^{\left\lfloor \frac{1}{x} \right\rfloor} dx,$$

where $\lfloor x \rfloor$ denotes the floor of x.

(b) Calculate

$$\int_0^1 2^{-\left\lfloor \frac{1}{x} \right\rfloor} dx.$$

Solution 1 by Angel Plaza, Universidad de Las Palmas de Gran Canaria, Spain

(a) Using the substitution $1/x = y$, the integral becomes $I = \displaystyle\int_1^\infty a^{\lfloor y \rfloor}/y^2 dy$. For any positive integer k and $y \in [k, k+1)$ we have $\lfloor y \rfloor = k$. Then

$$\begin{aligned} I &= \sum_{k=1}^{\infty} \int_k^{k+1} a^k/y^2 dy = \sum_{k=1}^{\infty} a^k \left(\frac{1}{k} - \frac{1}{k+1}\right) \\ &= \sum_{k=1}^{\infty} \frac{a^k}{k} - \sum_{k=1}^{\infty} \frac{a^k}{k+1} (\text{ since both series are absolutely convergent}) \\ &= -\ln(1-a) + \frac{\ln(1-a) + a}{a}. \end{aligned}$$

Since $\displaystyle\sum_{k=1}^{\infty} a^k = \frac{1}{1-a}$, and $\dfrac{a^k}{k} = \displaystyle\int_0^a x^{k-1} dx$ for $k \geq 1$.

(b) Since $2^{-\lfloor \frac{1}{x} \rfloor} = \left(\frac{1}{2}\right)^{\lfloor \frac{1}{x} \rfloor}$, then by part (a) we have

$$\int_0^1 2^{-\lfloor \frac{1}{x} \rfloor} dx = -\ln(1/2) + 2\ln(1/2) + 1 = 1 - \ln 2.$$

Solution 2 by Paolo Perfetti, Department of Mathematics, "Tor Vergata" University, Rome, Italy

Proof (a). We change $y = 1/x$.

$$\int_0^1 a^{\lfloor \frac{1}{x} \rfloor} dx = \int_1^\infty \frac{a^{\lfloor y \rfloor}}{y^2} dy = \sum_{k=1}^\infty \int_k^{k+1} \frac{a^k}{y^2} dy = \sum_{k=1}^\infty a^k \left(\frac{1}{k} - \frac{1}{k+1}\right)$$

If $a = 1$ we have telescoping

$$\sum_{k=1}^\infty \left(\frac{1}{k} - \frac{1}{k+1}\right) = 1.$$

If $a < 1$ we

$$\sum_{k=1}^\infty a^k \left(\frac{1}{k} - \frac{1}{k+1}\right) = \sum_{k=1}^\infty \int_0^a y^{k-1} dy - \frac{1}{a} \sum_{k=1}^\infty \int_0^a y^k dy$$

$$= e = \int_0^a \frac{dy}{1-y} - \frac{1}{a} \int_0^a \frac{y}{1-y} dy = \int_0^a \frac{dy}{1-y} + \frac{1}{a} \int_0^a dy - \frac{1}{a} \int_0^a \frac{1}{1-y} dy$$

$$= -\ln(1-a) + a + \frac{1}{a} \ln(1-a) = 1 + \frac{1-a}{a} \ln(1-a).$$

(b). If $a = 1/2$ we have $1 - \ln 2$.

Solution 3 by David Stone and John Hawkins, Georgia Southern University, Statesboro,GA

The solutions:

$$(a) \quad \int_0^1 a^{\lfloor \frac{1}{x} \rfloor} dx = \begin{cases} 1, & \text{if } a = 1 \\ 1 + \dfrac{1-a}{a} \ln(1-a), & \text{if } \ 0 < a < 1 \end{cases}$$

$$(b) \quad \int_0^1 2^{\lfloor \frac{1}{x} \rfloor} dx = 1 - \ln 2.$$

For part (a), note first that if $a = 1$, then $\int_0^1 a^{\lfloor \frac{1}{x} \rfloor} dx = 1$.

Henceforth, we assume $0 < a < 1$.

We shall use the following sums, for $x \in (0, 1]$.

By integrating $\sum_{k=0}^{\infty} x^k = \frac{1}{1-x}$ and re-indexing, we have the well-known sum:

(1) $\sum_{k=1}^{\infty} \frac{1}{k} x^k = -\ln(1-x).$

Then, by some algebraic manipulations, we have

(2) $\sum_{k=1}^{\infty} \frac{1}{k+1} x^k = -1 - \frac{1}{x} \ln(1-x).$

If we partition the interval $(0, 1]$ into subintervals $\left(\frac{1}{k+1}, \frac{1}{k}\right]$, our integral can be written as a sum: $\int_0^1 a^{\lfloor \frac{1}{x} \rfloor} dx = \sum_{k=1}^{\infty} \int_{1/k+1}^{1/k} a^{\lfloor \frac{1}{x} \rfloor} dx.$

We see that

$$
\begin{aligned}
& \frac{1}{k+1} < x \leq \frac{1}{k} \\
\iff \quad & \frac{1}{k+1} < x \text{ and } x \leq \frac{1}{k} \\
\iff \quad & \frac{1}{x} < k+1 \text{ and } k \leq \frac{1}{x} \\
\iff \quad & k \leq \frac{1}{x} < k+1 \\
\iff \quad & \left\lfloor \frac{1}{x} \right\rfloor = k.
\end{aligned}
$$

Thus

$$\int_{1/k+1}^{1/k} a^{\lfloor \frac{1}{x} \rfloor} dx = \int_{1/k+1}^{1/k} a^k dx = a^k \left(\frac{1}{k} - \frac{1}{k+1} \right).$$

Therefore, summing and applying (1) and (2),

$$
\begin{aligned}
\int_{1/k+1}^{1/k} a^{\lfloor \frac{1}{x} \rfloor} dx &= \sum_{k=1}^{\infty} \frac{a^k}{k} - \sum_{k=1}^{\infty} \frac{a^k}{k+1} \\
&= -\ln(1-a) - \left\{ -1 - \frac{1}{a} \ln(1-a) \right\} \\
&= -\ln(1-a) + 1 + \frac{1}{a} \ln(1-a)
\end{aligned}
$$

$$= \quad 1 + \frac{1-a}{a}\ln(1-a).$$

For part (b), note that $\displaystyle\int_0^1 2^{-\left\lfloor \frac{1}{x}\right\rfloor} dx = \int_0^1 \left(\frac{1}{2}\right)^{\left\lfloor \frac{1}{x}\right\rfloor} dx.$

Applying the result for (a), this equals

$$1 + \frac{1-\frac{1}{2}}{\frac{1}{2}}\ln\left(1-\frac{1}{2}\right) = 1 + \ln\left(\frac{1}{2}\right) = 1 - \ln 2.$$

Also solved by Ed Gray, Highland Beach, FL; G.C. Greubel, Newport News, VA; Adrian Naco, Polytechnic University,Tirana, Albania, and the proposer.

Mea Culpa (once again)

When Enkel Hysnelaj of the University of Technology in Sydney, Australia submitted problem 5264, it came to me in several versions, with the successor version correcting an error he noticed in the previous version. Foolishly I kept all versions of the problem, and when I posted 5264, I posted an incorrect version of it. Problem 5274 is the corrected statement of the problem. Thanks to Ed Gray for coming up with a counter-example to 5264, and to Enkel for setting things straight in 5274.