Problems

Ted Eisenberg, Section Editor

****

This section of the Journal offers readers an opportunity to exchange interesting mathematical problems and solutions. Please send them to Ted Eisenberg, Department of Mathematics, Ben-Gurion University, Beer-Sheva, Israel or fax to: 972-86-477-648. Questions concerning proposals and/or solutions can be sent e-mail to <eisenbt@013.net>. Solutions to previously stated problems can be seen at <http://www.ssma.org/publications>.

Solutions to the problems stated in this issue should be posted before April 15, 2018

5481: *Proposed by Kenneth Korbin, New York, NY*

A triangle with integer area has integer length sides $(3, x, x+1)$. Find five possible values of x with $x > 4$.

5482: *Proposed by Daniel Sitaru, "Theodor Costescu" National Economic College, Drobeta Turnu-Severin, Mehedinti, Romania*

Prove that if n is a natural number then

$$\frac{(tan5^\circ)^n}{(tan4^\circ)^n+(tan3^\circ)^n}+\frac{(tan4^\circ)^n}{(tan3^\circ)^n+(tan2^\circ)^n}+\frac{(tan3^\circ)^n}{(tan2^\circ)^n+(tan1^\circ)^n}\geq\frac{3}{2}.$$

5483: *Proposed by D.M. Bătinetu-Giurgiu, "Matei Basarab" National College, Bucharest and Neculai Stanciu, "George Emil Palade" School Buzău, Romania*

If $a, b > 0$, and $x \in \left(0, \frac{\pi}{2}\right)$ then show that

$$(i) \qquad (a+b)\cdot\frac{\sin x}{x}+\frac{2ab}{a+b}\cdot\frac{\tan x}{x}\geq\frac{6ab}{a+b}.$$

$$(ii) \qquad a\cdot\tan x+\ b\cdot\sin x>2x\sqrt{ab}.$$

5484: *Proposed by Mohsen Soltanifar, Dalla Lana School of Public Health, University of Toronto, Canada*

Let X_1, X_2 be two continuous positive valued random variables on the real line with corresponding mean, median, and mode $\overline{x_1},\ \tilde{x}_1,\ \hat{x}_1$ and $\overline{x_2},\ \tilde{x}_2,\ \hat{x}_2$ respectively. Assume for their associated CDFs, (Cummulative Distribution Functions) we have

$$F_{X_1}(t)\leq F_{X_2}(t) \qquad (t>0).$$

Prove or give a counter example:

$$\text{(i) } \overline{x_2} \le \overline{x_1}, \quad \text{(ii) } \tilde{x_2} \le \tilde{x_1}, \quad \text{(iii) } \hat{x}_2 \le \hat{x}_1.$$

5485: *Proposed by José Luis Díaz-Barrero, Barcelona Tech, Barcelona, Spain*

Let x, y, z be three positive real numbers. Show that

$$\prod_{cyclic} (2x + 3y + z + 1) \sum_{cyclic} (4x + 2y + 1)^{-3} \ge 3.$$

5486: *Proposed by Ovidiu Furdui, Technical University of Cluj-Napoca, Cluj-Napoca, Romania*

Let $(x_n)_{n\ge 0}$ be the sequence defined by $x_0 = 0, x_1 = 1, x_2 = 1$ and $x_{n+3} = x_{n+2} + x_{n+1} + x_n + n,\ \forall n \ge 0$. Prove that the series $\sum_{n=1}^{\infty} \frac{x_n}{2^n}$ converges and find its sum.

Solutions

5463: *Proposed by Kenneth Korbin, New York, NY*

Let N be a positive integer. Find triangular numbers x and y such that $x^2 + 14xy + y^2 = \left(72N^2 - 12N - 1\right)^2$.

Solution 1 by Dionne Bailey, Elsie Campbell, and Charles Diminnie, Angelo State University, San Angelo, TX

The n^{th} triangular number T_n is given by $T_n = \dfrac{n(n+1)}{2}$. To simplify matters, we will assume that $x \le y$. Then, by trial and error, we found the following solutions for the first four values of N.

N	$72N^2 - 12N - 1$	x	y
1	59	$T_4 = 10$	$T_6 = 21$
2	263	$T_{10} = 55$	$T_{12} = 78$
3	611	$T_{16} = 136$	$T_{18} = 171$
4	1103	$T_{22} = 253$	$T_{24} = 300$

This leads to the conjecture that one solution consists of

$$x = T_{6N-2} = \frac{(6N-2)(6N-1)}{2} = (3N-1)(6N-1) = 18N^2 - 9N + 1 \tag{1}$$

and

$$y = T_{6N} = \frac{6N(6N+1)}{2} = 3N(6N+1) = 18N^2 + 3N. \tag{2}$$

After some algebraic simplification, we obtain

$$\begin{aligned} x^2+14xy+y^2 &= \left(18N^2-9N+1\right)^2+14\left(18N^2-9N+1\right)\left(18N^2+3N\right) \\ &\quad +\left(18N^2+3N\right)^2 \\ &= 5184N^4-1728N^3+24N+1 \\ &= \left(72N^2-12N-1\right)^2 \end{aligned}$$

and hence, (1) and (2) provide a solution for each $N \geq 1$.

Solution 2 by Albert Stadler, Herrliberg, Switzerland

We put $x=\dfrac{(aN+b)(aN+b-1)}{2},\ y=\dfrac{(cN+d)(cN+d-1)}{2}$.

$$\frac{(aN+b)^2(aN+b-1)^2}{4}+14\frac{(aN+b)(aN+b-1)(cN+d)^2(cN+d-1)^2}{4}+\frac{(cN+d)^2(cN+d-1)^2}{4}$$
$$=(72N^2-12N-1)^2.$$

By comparing the coefficients of N^4, N^3, N^2, N and the statement of the problem we find the solutions

$$\begin{aligned} (x,y) &= \left(\frac{(6N-1)(6N-2)}{2}, \frac{(6N+1)6N}{2}\right) \text{ and} \\ (x,y) &= \left(\frac{(6N+1)6N}{2}, \frac{(6N-1)(6N-2)}{2}\right). \end{aligned}$$

Also solved by Brian D. Beasley, Presbyterian College, Clinton, SC; Anthony Bevelacqua, University of North Dakota, Grand Forks, ND; Jeremiah Bartz, University of North Dakota, Grand Forks, ND; Bruno Salgueiro Fanego, Viveiro, Spain; Ed Gray, Highland Beach, FL; Kee-Wai Lau, Hong Kong, China; David E. Manes, Oneonta, NY; Ioannis D. Sfikas, National and Kapodistrian University of Athens, Greece; Titu Zvonaru, Comănesti and Neculai Stanciu,"George Emil Palade" School Buzău, Romania; David Stone and John Hawkins, Georgia Southern University, Statesboro, GA, and the proposer.

5464: *Proposed by Ed Gray, Highland Beach, FL*

Let ABC be an equilateral triangle with side length s that is colored white on the front side and black on the back side. Its orientation is such that vertex A is at lower left, B is its apex, and C is at lower right. We take the paper at B and fold it straight down along the bisector of angle B, thus exposing part of the back side which is black. We continue to fold until the black part becomes $1/2$ of the existing figure, the other half being white. The problem is to determine the position of the fold, the distance defined by x (as a function of s) which is the distance from B to the fold.

Solution 1 by David E. Manes, Oneonta, NY

If $x=\dfrac{\sqrt{3}}{2}(2-\sqrt{2})s$, then the area of the resulting black triangle equals the sum of the areas of the two resulting white triangles.

Introduce the coordinates $A(-s/2,0), B(0,\sqrt{3}s/2)$ and $C(s/2,0)$. Then triangle ABC is an equilateral triangle with side length s and altitude $\sqrt{3}s/2)$. In view of the coordinates x and y, let t denote the distance from vertex B to the fold. The equation of the line L containing the points B and C is $y=-\sqrt{3}\left(x-\frac{s}{2}\right)$. Note that for a given value of t, the value of y is given by $y=\frac{\sqrt{3}}{2}s-t$. For example, let $t=\frac{\sqrt{3}s}{4}$. Then $y=\frac{\sqrt{3}s}{4}$ and vertex B has been moved to the origin, thus creating three equilateral triangles two of which are white. Substituting the above value of y in the equation for L yields $x=\frac{s}{4}$ so that the point $P\left(\frac{s}{4},\frac{\sqrt{3}}{4}s\right)$ is a base vertex for the black triangle and an apex for one of the white triangles with side PC. By symmetry, the side length for the black triangle is $2\left(\frac{s}{4}\right)=\frac{s}{2}$ so that its area A_B is given by $A_B=\left(\frac{\sqrt{3}}{4}\right)\left(\frac{s}{2}\right)^2=\frac{\sqrt{3}}{16}s^2$. However, the side length $PC=\sqrt{\left(\frac{s}{4}\right)^2+\left(\frac{\sqrt{3}s}{4}\right)^2}=\frac{s}{2}$, hence the sum of the areas A_W of the two white triangles is $A_W=2\left(\frac{\sqrt{3}}{4}\right)\left(\frac{s}{2}\right)^2=\frac{\sqrt{3}}{8}s^2$. Since $A_W>A_B$, it follows that the value of t is greater than $\frac{\sqrt{3}s}{4}$. For this reason, let $t=\frac{\sqrt{3}}{4}s+k$, where k is a real number such that $0<k<\frac{\sqrt{3}}{4}s$. Then

$$y=\frac{\sqrt{3}}{2}s-t=\frac{\sqrt{3}}{2}-\left(\frac{\sqrt{3}}{4}s+k\right)=\frac{\sqrt{3}}{4}s-k.$$

Substituting this value of y in the equation for L, one obtains $x=\frac{s}{4}+\frac{\sqrt{3}}{3}k$. Hence, the point $P=\left(\frac{s}{4}+\frac{\sqrt{3}}{3}k,\frac{\sqrt{3}}{4}s-k\right)$ is a base vertex for the black triangle and an apex for the white triangle with side PC. Therefore, the side length of the black triangle is $2\left(\frac{s}{4}+\frac{\sqrt{3}}{3}k\right)=\frac{s}{2}+\frac{2\sqrt{3}}{3}k$ so that its area is $A_B=\frac{\sqrt{3}}{4}\left(\frac{s}{2}+\frac{2\sqrt{3}}{3}k\right)^2$. Moreover, for the white triangle

$$PC=\sqrt{\left(\frac{\sqrt{3}}{3}k-\frac{s}{4}\right)^2+\left(\frac{\sqrt{3}}{4}s-k\right)^2}=\frac{s}{2}-\frac{2\sqrt{3}}{3}k.$$

Therefore, the sum of the areas of the two white triangles is $A_W=\frac{\sqrt{3}}{2}\left(\frac{s}{2}-\frac{2\sqrt{3}}{3}k\right)^2$. Setting $A_W=A_B$, we get

$$\frac{\sqrt{3}}{2}\left(\frac{s^2}{4}-\frac{2\sqrt{3}}{3}sk+\frac{4}{3}k^2\right)=\frac{\sqrt{3}}{4}\left(\frac{s^2}{4}+\frac{2\sqrt{3}}{3}sk+\frac{4}{3}k^2\right).$$

This equation simplifies to the following quadratic equation in k:

$$\frac{4}{3}k^2-2\sqrt{3}sk+\frac{s^2}{4}=0$$

with roots $k = \frac{\sqrt{3}}{2}\left(\frac{3}{2} \pm \sqrt{2}\right)s$. The positive square root of 2 yields a value of $k > \frac{\sqrt{3}}{4}s$ and so is inadmissable. Therefore, $k = \frac{\sqrt{3}}{2}\left(\frac{3}{2} - \sqrt{2}\right)s$, whence

$$t = \frac{\sqrt{3}}{4} + k = \frac{\sqrt{3}}{2}(2 - \sqrt{2})s.$$

Observe that using this value of t, the area of the black triangle as well as the sum of the areas of the two white triangles is $\sqrt{3}\left(\frac{3}{2} - \sqrt{2}\right)s^2$.

Solution 2 by Kee-Wai Lau, Hong Kong, China

When B reaches the midpoint of AC, the black part is only 1/3 of the existing figure, which is a trapezium. So we need to push B downwards further. The black part is then an equilateral triangle with base $2x \tan 30°$, height x and hence area $\frac{x^2}{\sqrt{3}}$. The distance between the fold and AC equals $\frac{\sqrt{3}s}{2} - x$. The white part now consists of two congruent equilateral triangles of lengths $\left(\frac{\sqrt{3}s}{2} - x\right)\sec 30° = s - \frac{2x}{\sqrt{3}}$. Since the area of the white part equals the area of the black part, we have $\frac{\sqrt{3}}{2}\left(s - \frac{2x}{\sqrt{3}}\right)^2 = \frac{x^2}{\sqrt{3}}$. Solving, we obtain $x = \frac{\sqrt{3}\left(2 - \sqrt{2}\right)s}{2}$.

Editor's Comment: **David Stone and John Hawkins both Georgia Southern University in Statesboro, GA** generalized the statement as follows: "The problem asked for the configuration in which the black triangle covered half of the final figure. We could just as well determine when the black triangle covers any given portion of the final figure; say one fourth or nine tenths."

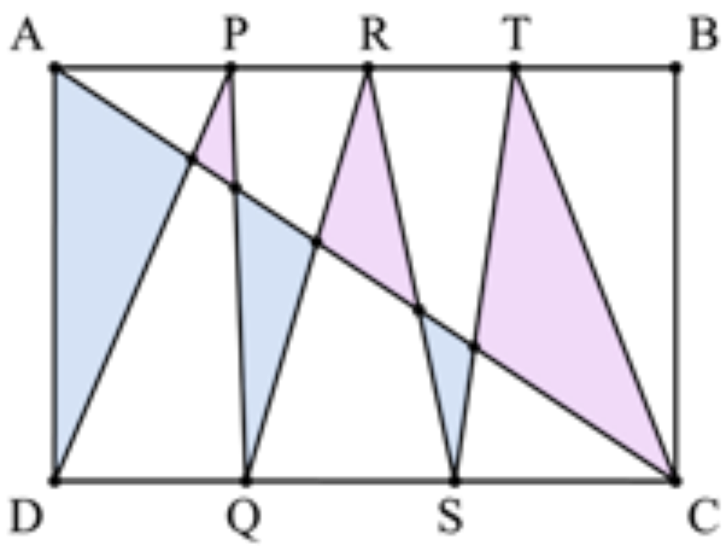
They did this by looking at two cases: 1) $0 < \lambda < \frac{1}{3}$ and 2) $\frac{1}{3} < \lambda < 1$, where the Black Area=$\lambda$Total Area. The constant $\frac{1}{3}$ comes from when the vertex of the Black Triangle lies on the base of the White Triangle. Letting x be the length of the height of the black triangle (measured from its vertex to the fold line) they found that for the first case, where the vertex of the Black Triangle lies in the interior of the White Triangle that: $x = \sqrt{\frac{\lambda}{1+\lambda}} \cdot \frac{\sqrt{3}}{2}s$ and in the second case, where the vertex of the Black Triangle lies in the exterior of the White Triangle that $x = \frac{2\lambda - \sqrt{2\lambda(1-\lambda)}}{3\lambda - 1} \cdot h$, where h is the altitude of the given White Triangle. When $\lambda = \frac{1}{2}$ we obtain the statement of the problem and using their formula reaffirms the above answers.

In concluding their comment they noted, "Another nice example: with $\lambda = \frac{2a^2}{2a^2+1}$, which is very close to 1, we find that $x = \left(1 - \frac{1}{2a+1}\right)h$. That is, in a precisely measurable way, we must fold almost all the way down to get a figure which is almost all black."

Also solved by Brian D. Beasley, Presbyterian College, Clinton, SC; Bruno Salgueiro Fanego (two solutions), Viveiro, Spain; David A. Huckaby, Angelo State University San Angelo, TX; and the proposer.

5465: *Proposed by Arsalan Wares, Valdosta State University, Valdosta, GA*

Quadrilateral $ABCD$ is a rectangle with diagonal AC. Points P, R, T, Q and S are on sides AB and DC and they are connected as shown. Three of the triangles inside the rectangle are shaded pink, and three are shaded blue. Which is larger, the sum of the areas of the pink triangles or the sum of the areas of the blue triangles?



Solution by David A. Huckaby, Angelo State University, San Angelo, TX

Let p be the sum of the areas of the pink triangles, b the sum of the areas of the blue triangles, and w the sum of the areas of the three white polygons below the diagonal.

$$\begin{aligned}
p + w &= \text{the sum of the areas of triangles } DPQ,\ QRS, \text{ and } STC \\
&= \frac{1}{2}(AD \cdot DQ) + \frac{1}{2}(AD \cdot QS) + \frac{1}{2}(AD \cdot SC) \\
&= \frac{1}{2}\left[AD \cdot (DQ + QS + SC)\right] \\
&= \frac{1}{2}(AD \cdot DC) \\
&= \text{the area of triangle } ADC \\
&= b + w
\end{aligned}$$

So $p = b$, that is, the sum of the areas of the pink triangles is equal to the sum of the areas of the blue triangles.

Also solved by Michael N. Fried, Ben-Gurion University of the Negev, Beer-Sheva, Israel; Ed Gray, Highland Beach, FL; Kee-Wai Lau, Hong Kong, China; Albert Stadler, Herrliberg, Switzerland; David Stone and John Hawkins, Georgia Southern University, Statesboro, GA, and the proposer.

5466: *Proposed by D.M. Bătinetu– Giurgiu, "Matei Basarab" National College, Bucharest, Romania and Neculai Stanciu, "Geroge Emil Palade" School, Buzău, Romania*

Let $f:(0,+\infty)\to(0,+\infty)$ be a continuous function. Evaluate

$$\lim_{n\to\infty}\int_{\frac{n^2}{\sqrt[n]{n!}}}^{\frac{(n+1)^2}{\sqrt[n+1]{(n+1)!}}} f\left(\frac{x}{n}\right)dx.$$

Solution 1 by Moti Levy, Rehovot, Israel

The mean value theorem of the integral calculus states:
Let $f(x)$ be continuous function, then

$$\int_a^b f(x)\,dx=(b-a)\,f(\xi)\,,\ a\le\xi\le b.$$

Therefore,

$$\int_{\frac{n^2}{\sqrt[n]{n!}}}^{\frac{(n+1)^2}{\sqrt[n+1]{(n+1)!}}} f\left(\frac{x}{n}\right)dx=\left(\frac{(n+1)^2}{\sqrt[n+1]{(n+1)!}}-\frac{n^2}{\sqrt[n]{n!}}\right)f\left(\frac{\xi}{n}\right),\quad \frac{n^2}{\sqrt[n]{n!}}\le\xi\le\frac{(n+1)^2}{\sqrt[n+1]{(n+1)!}}.$$

Taking limits of both sides,

$$\lim_{n\to\infty}\int_{\frac{n^2}{\sqrt[n]{n!}}}^{\frac{(n+1)^2}{\sqrt[n+1]{(n+1)!}}} f\left(\frac{x}{n}\right)dx=\lim_{n\to\infty}\left(\frac{(n+1)^2}{\sqrt[n+1]{(n+1)!}}-\frac{n^2}{\sqrt[n]{n!}}\right)\lim_{n\to\infty}f\left(\frac{\xi}{n}\right).$$

Since $f(x)$ is continuous then

$$\lim_{n\to\infty}f\left(\frac{\xi}{n}\right)=f\left(\lim_{n\to\infty}\frac{\xi}{n}\right)=f\left(\lim_{n\to\infty}\frac{n}{\sqrt[n]{n!}}\right)$$

Using Stirling's asymptotic formula, we have

$$\sqrt[n]{n!}\sim\frac{n}{e}. \tag{1}$$

By (1),

$$\frac{n}{\sqrt[n]{n!}}\sim e,\quad \frac{n^2}{\sqrt[n]{n!}}\sim e\cdot n,$$

which implies that

$$f\left(\lim_{n\to\infty}\frac{n}{\sqrt[n]{n!}}\right)=f(e)$$

and that

$$\frac{(n+1)^2}{\sqrt[n+1]{(n+1)!}}-\frac{n^2}{\sqrt[n]{n!}}\sim e.$$

We conclude that

$$\lim_{n\to\infty}\int_{\frac{n^2}{\sqrt[n]{n!}}}^{\frac{(n+1)^2}{\sqrt[n+1]{(n+1)!}}} f\left(\frac{x}{n}\right)dx = ef(e).$$

Solution 2 by Bruno Salgueiro Fanego,Viveiro, Spain

Let's proceed as in http://www./oei.es/historico/oim/revistaoim/numero 53/261_Bruno.pdf:

Let $n \in N$; since f is continuous on (x_n, x_{n+1}), by the mean value theorem of integral calculus, we have that $\int_{x_n}^{x^{n+1}} f\left(\frac{x}{n}\right)dx = f\left(\frac{\xi_n}{n}\right)(x_{n+1}-x_n)$for some $\xi_n \in (x_n, x_{n+1})$.

Since$\frac{x_n}{n} < \frac{\xi_n}{n} < \frac{x_{n+1}}{n}$,

$$\lim_{n\to\infty}\frac{x_n}{n} = \lim_{n\to\infty}\sqrt[n]{\frac{n^{2n}}{n^n n!}} = \lim_{n\to\infty}\sqrt[n]{\frac{n^n}{n!}} = \lim_{n\to\infty}\frac{\frac{(n+1)^{n+1}}{(n+1)!}}{\frac{n^n}{n!}} = \lim_{n\to\infty}\frac{(n+1)^n(n+1)n!}{(n+1)!n^n} = \lim_{n\to\infty}\left(\frac{n+1}{n}\right)^n = e$$

and $\lim_{n\to\infty}\frac{x_{n+1}}{n} = \lim_{n\to\infty}\frac{x_{n+1}}{n+1}\cdot\frac{n+1}{n} = \lim_{n\to\infty}\frac{x_{n+1}}{n+1}\cdot\lim_{n\to\infty}\frac{n+1}{n} = e\cdot 1 = e$, by the sandwich rule we obtain that $\lim_{n\to\infty}\frac{\xi_n}{n} = e$, and, hence,
$\lim_{n\to\infty} f\left(\frac{\xi_n}{n}\right) = \left(\lim_{n\to\infty}\frac{\xi_n}{n}\right) = f(e)$.

Moreover, from Stolz' rule,

$$\lim_{n\to\infty}(x_{n+1}-x_n) = \lim_{n\to\infty}\frac{(x_{n+1}-x_n)}{(n+1)-n} = \lim_{n\to\infty}\frac{x_n}{n} = e.$$

So, the required limit is equal to

$$\lim_{n\to\infty}\int_{x_n}^{x_{n+1}} f\left(\frac{x}{n}\right)dx = \lim_{n\to\infty} f\left(\frac{\xi_n}{n}\right)\lim_{n\to\infty}(x_{n+1}-x_n) = ef(e).$$

Solution 3 by Ioannis D. Sfikas, National and Kapodistrian University of Athens, Greece

This particular problem is similar to Problem 121, which was proposed by D.M. Bătinetu-Giurgiu ("Matei Basarab" National College, Bucharest, Romania) and Neculai Stanciu ("George Emil Palade" School, Buzău, Romania) to the Math Problems Journal, Volume 5, Issue 2 (2015), pp. 420-421. We'll use the following lemma.

Lemma: Let $f : [a, b] \to \Re$ be continuous and $(x_n)_n, (y_n)_n$ two convergent sequences of $[a, b]$ that have the same limit c, then

$$\int_{x_n}^{y_n} f(t)dt = f(c)(y_n - x_n) + O(y_n - x_n).$$

Proof: Let $\epsilon > 0$, then there exists $\delta > 0$ such that $|f(t) - f(c)| < \epsilon$, whenever $|x - c| < \delta$. Since $x_n, y_n \to c$, the there is an $n_0 \in N$ such that $x_n, y_n \in (C - \delta, C + \delta)$, whenever $n > n_0$. Therefore,

$$\left|\int_{x_n}^{y_n} f(t)dt - f(c)(y_n - x_n)\right| \le \int_{x_n}^{y_n} |f(t) - f(c)|\, dt \le \epsilon\, |y_n - x_n|\,.$$

Note that the given integral equals

$$I_n = n\int_{\frac{n}{\sqrt[n]{n!}}}^{n^{\frac{(n+1)^2}{\sqrt[n+1]{(n+1)!}}}} f(t)dt,$$

this comes directly from the substitution $t = \dfrac{x}{n}$. Let x_n, y_n be the lower, upper bound of the last integral respectively then $x_n, y_n \to e$, since $\dfrac{n}{\sqrt[n]{n!}} \to e$, and thus $\dfrac{(n+1)^2}{n\sqrt[n+1]{(n+1)!}} = \dfrac{n+1}{n}\dfrac{(n+1)}{\sqrt[n+1]{(n+1)!}} \longrightarrow e$, as $n \longrightarrow \infty$. Note that by Stolz' theorem

$$n(y_n - x_n) = \frac{(n+1)^2}{\sqrt[n+1]{(n+1)!}} - \frac{n^2}{\sqrt[n]{n!}} \longrightarrow e, \text{ as } n \longrightarrow \infty.$$

By the lemma we have

$$I_n = n\left[f(c)(y_n - x_n) + O(y_n - x_n)\right] = ef(e) + O(1),$$

which proves that the limit equals $ef(e)$.

Also solved by Arkady Alt, San Jose, CA; Kee-Wai Lau, Hong Kong, China; Soumitra Mandal, Scottish Church College, Chandan -Nagar, West Bengal, India; Paolo Perfetti, Department of Mathematics, Tor Vergata University, Rome, Italy; Albert Stadler, Herrliberg, Switzerland; Anna V. Tomova, Varna, Bulgaria, and the proposers.

5467: *Proposed by José Luis Díaz-Barrero, Barcelona Tech, Barcelona, Spain*

In an arbitrary triangle $\triangle ABC$, let a, b, c denote the lengths of the sides, R its circumradius, and let h_a, h_b, h_c respectively, denote the lengths of the corresponding altitudes. Prove the inequality

$$\frac{a^2 + bc}{b + c} + \frac{b^2 + ca}{c + a} + \frac{c^2 + ab}{a + b} \ge \frac{3abc}{2R}\sqrt[3]{\frac{1}{h_a \cdot h_b \cdot h_c}},$$

and give the conditions under which equality holds.

Solution 1 by Paolo Perfetti, Department of Mathematics, Tor Vergata University, Rome, Italy

We know that $h_a = (bc)/(2R)$ and cyclic so the inequality actually is

$$\frac{a^2 + bc}{b + c} + \frac{b^2 + ca}{c + a} + \frac{c^2 + ab}{a + b} \ge \frac{3abc}{2R}\left(\frac{8R^3}{(abc)^2}\right)^{\frac{1}{3}} = 3(abc)^{\frac{1}{3}}.$$

We prove the stronger one

$$\frac{a^2+bc}{b+c}+\frac{b^2+ca}{c+a}+\frac{c^2+ab}{a+b}\geq a+b+c,$$

that is

$$\left(\frac{a^2+bc}{b+c}-a\right)+\left(\frac{b^2+ca}{c+a}-b\right)+\left(\frac{c^2+ab}{a+b}-c\right)\geq 0,$$

or

$$\frac{(a-b)(a-c)}{b+c}+\frac{(b-c)(b-a)}{a+c}+\frac{(c-a)(c-b)}{a+b}\geq 0.$$

We can suppose $a\geq b\geq c$ by symmetry so we come to

$$\frac{(a-b)(a-c)}{b+c}+\frac{(a-c)(b-c)}{a+b}\geq\frac{(a-b)(b-c)}{a+c}.$$

This is implied by

$$\frac{(a-b)(a-c)}{b+c}+\underbrace{\frac{(a-b)(b-c)}{a+b}}_{a-c\geq a-b}\geq\frac{(a-b)(b-c)}{a+c},$$

or

$$\frac{a-c}{b+c}+\frac{b-c}{a+b}\geq\frac{b-c}{a+c}.$$

This is in turn implied by

$$\underbrace{\frac{a-c}{a+c}}_{a\geq b}+\frac{b-c}{a+b}\geq\frac{b-c}{a+c}$$

and this evidently holds true by $a-c\geq b-c\geq 0$. The equality case is $a=b=c$.

Solution 2 by Dionne Bailey, Elsie Campbell, and Charles Diminnie, Angelo State University, San Angelo, TX

We will prove the following slight improvement:

$$\begin{aligned}\frac{a^2+bc}{b+c}+\frac{b^2+ca}{c+a}+\frac{c^2+ab}{a+b}&\geq a+b+c\\&\geq 3\sqrt[3]{abc}\\&=\frac{3abc}{2R}\sqrt[3]{\frac{1}{h_a\cdot h_b\cdot h_c}},\end{aligned}\tag{1}$$

with equality if and only if $a=b=c$.

To begin, we note that

$$\begin{aligned}a^4+b^4+c^4-a^2b^2-b^2c^2-c^2a^2&=\frac{\left(a^2-b^2\right)^2+\left(b^2-c^2\right)^2+\left(c^2-a^2\right)^2}{2}\\&\geq 0,\end{aligned}\tag{2}$$

with equality if and only if $a^2 = b^2 = c^2$. Since $a, b, c > 0$, it follows that equality is attained in (2) if and only if $a = b = c$.

Next, we use (2) to obtain

$$\begin{aligned}
&\frac{a^2+bc}{b+c} + \frac{b^2+ca}{c+a} + \frac{c^2+ab}{a+b} \\
&= \frac{\left(a^2-b^2\right)+\left(b^2+bc\right)}{b+c} + \frac{\left(b^2-c^2\right)+\left(c^2+ca\right)}{c+a} + \frac{\left(c^2-a^2\right)+\left(a^2+ab\right)}{a+b} \\
&= \frac{a^2-b^2}{b+c} + \frac{b^2-c^2}{c+a} + \frac{c^2-a^2}{a+b} + a+b+c \\
&= \frac{\left(a^2-c^2\right)+\left(c^2-b^2\right)}{b+c} + \frac{b^2-c^2}{c+a} + \frac{c^2-a^2}{a+b} + a+b+c \\
&= \left(a^2-c^2\right)\left(\frac{1}{b+c} - \frac{1}{a+b}\right) + \left(b^2-c^2\right)\left(\frac{1}{c+a} - \frac{1}{b+c}\right) + a+b+c \\
&= \left(a^2-c^2\right)\frac{a-c}{(a+b)(b+c)} + \left(b^2-c^2\right)\frac{b-a}{(b+c)(c+a)} + a+b+c \\
&= \frac{\left(a^2-c^2\right)^2 + \left(b^2-c^2\right)\left(b^2-a^2\right)}{(a+b)(b+c)(c+a)} + a+b+c \\
&= \frac{a^4+b^4+c^4-a^2b^2-b^2c^2-c^2a^2}{(a+b)(b+c)(c+a)} + a+b+c \\
&\geq a+b+c, \qquad (3)
\end{aligned}$$

with equality if and only if $a = b = c$.

Also, the Arithmetic - Geometric Mean Inequality implies that

$$a+b+c \geq 3\sqrt[3]{abc}, \qquad (4)$$

with equality if and only if $a = b = c$.

For the final step, let $K = area\,(\triangle ABC)$. Then,

$$K = \frac{1}{2}ah_a = \frac{1}{2}bh_b = \frac{1}{2}ch_c$$

and hence,

$$h_a = \frac{2K}{a}, \quad h_b = \frac{2K}{b}, \quad \text{and} \quad h_c = \frac{2K}{c}.$$

Since $R = \dfrac{abc}{4K}$, we have

$$\begin{aligned}
\frac{3abc}{2R}\sqrt[3]{\frac{1}{h_a \cdot h_b \cdot h_c}} &= \frac{12KR}{2R}\sqrt[3]{\frac{abc}{8K^3}} \\
&= \frac{6K}{2K}\sqrt[3]{abc} \\
&= 3\sqrt[3]{abc}. \qquad (5)
\end{aligned}$$

If we combine (3), (4), and (5), statement (1) follows and equality is attained throughout if and only if $a = b = c$.

Solution 3 by Arkady Alt, San Jose, CA

Let $F = [ABC]$ (area) and let s be its semi-perimeter.
Since $h_a = \dfrac{2F}{a}, h_b = \dfrac{2F}{b}, h_c = \dfrac{2F}{c}$ and $abc = 4RF$ then

$$\begin{aligned}\sqrt[3]{\frac{1}{h_a h_b h_c}} &= \sqrt[3]{\frac{abc}{8F^3}} = \frac{1}{2F}\sqrt[3]{abc} \text{ and}\\ \frac{3abc}{2R}\sqrt[3]{\frac{1}{h_a h_b h_c}} &= 3\sqrt[3]{abc}.\end{aligned}$$

Thus, original inequality becomes

(1) $\quad \dfrac{a^2+bc}{b+c} + \dfrac{b^2+ca}{c+a} + \dfrac{c^2+ab}{a+b} \geq 3\sqrt[3]{abc}.$

Since $\dfrac{4a^2}{b+c} \geq 4a - b - c \iff (2a-b-c)^2 \geq 0$ we have

$$\begin{aligned}\sum_{cyc}\frac{a^2+bc}{b+c} &= \sum_{cyc}\frac{a^2}{b+c} + \sum_{cyc}\frac{bc}{b+c} \geq \sum_{cyc}\frac{4a-b-c}{4} + \sum_{cyc}\frac{bc}{b+c}\\ &= \frac{a+b+c}{2} + \sum_{cyc}\frac{bc}{b+c} = \sum_{cyc}\left(\frac{b+c}{4} + \frac{bc}{b+c}\right) \geq \sum_{cyc} 2\sqrt{\frac{b+c}{4}\cdot\frac{bc}{b+c}}\\ &= \sum_{cyc}\sqrt{bc} \geq 3\sqrt[3]{\sqrt{bc}\cdot\sqrt{ca}\cdot\sqrt{ab}} = 3\sqrt[3]{abc}.\end{aligned}$$

Solution 4 by Nicusor Zlota,"Traian Vuia" Technical College, Focsani, Romania, and Corneliu-Manescu Avram, Ploiesti, Romania

Assume that $a \geq b \geq c$.

First, we will prove that $\dfrac{a^2+bc}{b+c} + \dfrac{b^2+ca}{c+a} + \dfrac{c^2+ab}{a+b} \geq a+b+c \iff$

$\dfrac{a^2+bc}{b+c} - a + \dfrac{b^2+ca}{c+a} - b + \dfrac{c^2+ab}{a+b} - c \geq 0 \iff$

$\dfrac{(a-b)(a-c)}{b+c} + \dfrac{(b-c)(b-a)}{c+a} + \dfrac{(c-a)(c-b)}{a+b} \geq 0 \iff$

$(a-b)\left(\dfrac{a-c}{b+c} - \dfrac{b-c}{c+a}\right) + (b-a)\left(\dfrac{b-a}{c+a} - \dfrac{c-a}{a+b}\right) + (c-a)\left(\dfrac{a-c}{b+c} - \dfrac{b-c}{c+a}\right) \geq 0$

$(a-b)^2\dfrac{a+b}{(b+c)(c+a)} + (b-c)^2\dfrac{b+c}{(a+b)(c+a)} + (c-a)^2\dfrac{c+a}{(a+b)(b+c)} \geq 0.$

Then, it suffices to prove that

$$a = b = -c \geq \frac{3abc}{2R}\sqrt[3]{\frac{1}{h_a h_b h_c}} = \frac{3abc}{2R}\sqrt[3]{\frac{abc}{8S^3}} = \frac{3abc}{2R}\frac{1}{2S}\sqrt[3]{abc} = \sqrt[3]{abc},$$

which is the AM-GM inequality.

Equality holds for $a = b = c$.

Also solved by Hatef I. Arshagi, Guilford Technical Community College, Jamestown, NC; Bruno Salgueiro Fanego, Viveiro, Spain; Ed Gray, Highland Beach, FL; Kee-Wai Lau, Hong Kong, China; Moti Levy Rehovot, Israel; Ioannis D. Sfikas, National and Kapodistrian University of Athens, Greece; Albert Stadler, Herrliberg, Switzerland; Neculai Stanciu, "George Emil Palade" School, Buzău, Romania and Titu Zvonaru, Comănesti, Romania; David Stone and John Hawkins, Georgia Southern University, Statesboro,GA, and the proposer.

Editor′s note: **Hatef I. Arshagi's solution was dedicated to the memory of Mrs. Alieh Ataee**.

5468: *Proposed by Ovidiu Furdui and Alina Sîntămărian, both at the Technical University of Cluj-Napoca, Cluj-Napoca, Romania*

Find all differentiable functions $f : \Re \to \Re$ with $f(0) = 1$ such that $f'(x) = f^2(-x)f(x)$, for all $x \in \Re$.

Solution 1 by Moti Levy, Rehovot, Israel

Let us differentiate both sides of the given differential equation,

$$f''(x) = -2f(-x)f'(-x)\,f(x) + f^2(-x)f'(x). \tag{1}$$

The following two equation are direct consequence of the original equation.

$$f'(-x) = f^2(x)f(-x), \tag{2}$$

$$f^2(-x) = \frac{f'(x)}{f(x)}. \tag{3}$$

After substitution of (2) and (3) in (1), we get differential equation (4) with initial conditions at $x = 0$,

$$f''(x) + 2f^2(x)f'(x) - \frac{\left(f'(x)\right)^2}{f(x)} = 0, \quad f(0) = f'(0) = 1. \tag{4}$$

By the substitution $f(x) = \sqrt{g(x)}$,

$$\begin{aligned} f &= \sqrt{g}, \\ f' &= \frac{1}{2\sqrt{g}}g', \\ f'' &= \frac{1}{2\sqrt{g}}g'' - \frac{1}{4\left(\sqrt{g}\right)^3}\left(g'\right)^2, \end{aligned}$$

we arrive at the equivalent differential equation

$$g^{''}+2gg^{'}-\frac{1}{g}\left(g^{'}\right)^{2}=0,\quad g\left(0\right)=1,g^{'}\left(0\right)=2. \tag{5}$$

Now we want to lower the order of (5) by the substitution $g^{'}=\frac{dg}{dx}=z,\ \ g^{''}=\frac{d^2g}{dx^2}=z\frac{dz}{dg}$,

$$z\frac{dz}{dg}+2gz-\frac{1}{g}z^2=0,$$

or

$$g\frac{dz}{dg}-z=-2g^2. \tag{6}$$

The solution of (6) is

$$z=cg-2g^2.$$

The initial conditions on g dictate that $c=4$, thus we obtain the following differential equation for g,

$$\frac{dg}{dx}=4g\left(x\right)-2g^2\left(x\right).$$

or

$$\frac{dx}{dg}=\frac{1}{4g-2g^2}.$$

After integration over g, we get

$$x=\frac{1}{4}\ln\frac{g}{2-g}+c$$

or

$$g\left(x\right)=2k\frac{e^{4x}}{ke^{4x}+1}.$$

Again, the initial condition $g\left(0\right)=1$ dictates $k=1$,

$$g\left(x\right)=\frac{2e^{4x}}{e^{4x}+1}.$$

We conclude that the

$$f\left(x\right)=\sqrt{2}\frac{e^{2x}}{\sqrt{e^{4x}+1}}.$$

Solution 2 by Ioannis D. Sfikas, National and Kapodistrian University of Athens, Greece

The defining relation

$$f'(x)=f^2(-x)f(x), \tag{1}$$

implies that f is continuously differentiable. Setting $-x$ in the relation (1), one gets for every $=x,\ f'(-x)=f^2(x)f(-x)$, Multiplying (1) by $f(x)$ yields for every x

$$f^{'}(x)f(x)=f^2(-x)f^2(x)=\left[f^2(x)f(-x)\right]f(-x)=f^{'}(-x)f(-x),$$

that is $x \to f'(x)f(x)$ is even. Therefore, $\int_{-x}^{x} f'(t)f(t)dt = 2\int_0^x f'(t)f(t)dt$, and since an antiderivative of $f'f$ is $\frac{f^2}{2}$, this implies that for every x, $f^2(x) + f^2(-x) = 2$. Replacing $f^2(-x)$ in the defining relation one get for every x

$$f'(x) - 2f(x) - f^2(x).$$

This non-linear differential equation seems to have only one solution, namely

$$x \to \frac{\sqrt{2}e^{2x}}{\sqrt{e^{4x}+1}} \tag{2}$$

Conversely, it is easily checked that (2) is indeed a solution to the equation.

Solution 3 by Kee-Wai Lau, Hong Kong, China

We show that

$$f(x) = \frac{\sqrt{2}e^{2x}}{\sqrt{1+e^{4x}}}. \tag{1}$$

From the given equation , we obtain $f(x)f'(x) = f^2(-x)f^2(x)$, so that

$$f(-x)f'(-x) = f(x)f'(x) \tag{2}$$

Integrating (2) with respect to x, and making use of the fact that $f(0) = 1$, we obtain

$$f^2(-x) = 2 - f^2(x). \tag{3}$$

Substituting (3) into the given equation, we obtain $f'(x) = \left(2 - f^2(x)\right) f(x)$ or $\frac{d(f(x))}{(1-f^2(x))f(x)} = dx$. Integrating boh sides we obtain

$$\frac{\ln\left(f(x)\right)}{2} - \frac{\ln\left(2 - f^2(x)\right)}{4} = x + C,$$

where C is a constant. Since $f(0) = 1$, so $C = 0$. Now (1) follows easily by simple algebra.

Editor's comment: **Anna Tomova of Varna Bulgaria** expressed her solution in terms of a hyperbolic function; $f(x) = \frac{e^x}{\sqrt{\cosh 2x}}$, $f(0) = 1$.

Also solved by Albert Stadler, Herrliberg, Switzerland; Anna V. Tomova, Varna, Bulgaria, and the proposers.