Problems

Ted Eisenberg, Section Editor

****

This section of the Journal offers readers an opportunity to exchange interesting mathematical problems and solutions. Please send them to Ted Eisenberg, Department of Mathematics, Ben-Gurion University, Beer-Sheva, Israel or fax to: 972-86-477-648. Questions concerning proposals and/or solutions can be sent e-mail to <eisenbt@013.net>. Solutions to previously stated problems can be seen at <http://www.ssma.org/publications>.

Solutions to the problems stated in this issue should be posted before April 15, 2019

5529: *Proposed by Kenneth Korbin, New York, NY*

Convex cyclic quadrilateral $ABCD$ has integer length sides and integer area. The distance from the incenter to the circumcenter is 91. Find the length of the sides.

5530: *Proposed by Arsalan Wares, Valdosta State University, Valdosta, GA*

Polygon $ABCD$ is an 11 by 12 rectangle ($AB > AD$). Points P, Q, R, and S are on sides AB, BC, CD, and DA, respectively, such that PR and SQ are parallel to AD and AB, respectively. Moreover, $X = PR \cap QS$. If the perimeter of rectangle $PBQX$ is 5/7 the perimeter of rectangle $SAPX$, and the perimeter of rectangle $RCQX$ is 9/10 the perimeter of rectangle $PBQX$, find the area of rectangle $SDRX$.

5531: *Proposed by Daniel Sitaru, National Economic College "Theodor Costescu," Drobeta Turnu-Severin, Mehedinti, Romania*

For real numbers x, y, z prove that if $x, y, z > 1$ and $xyz = 2\sqrt{2}$, then

$$x^y + y^z + z^x + y^x + z^y + x^z > 9.$$

5532: *Proposed by Arkady Alt, San Jose, CA*

Let a, b, c be positive real numbers and let $a_n = \dfrac{an+b}{an+c}, n \in N$. For any natural number m find $\lim\limits_{n\to\infty} \prod\limits_{k=n}^{nm} a_k$.

5533: *Proposed by José Luis Díaz-Barrero, Barcelona Tech, Barcelona, Spain*

Find the value of the sum

$$\sum_{n=1}^{+\infty} \frac{n^2\alpha^n}{(n-1)!}$$

for any real number $\alpha > 0$. (Here, $0! = 1! = 1$).

5534: *Proposed by Ovidiu Furdui, Technical University of Cluj-Napoca, Cluj-Napoca, Romania*

Calculate $\int_0^1 \int_0^1 (x+y)\ln(x-xy+y)dxdy$.

Solutions

5511: *Proposed by Kenneth Korbin, New York, NY*

A trapezoid with perimeter $58+14\sqrt{11}$ is inscribed in a circle with diameter $17+7\sqrt{11}$. Find its dimensions if each of its sides is of the form $a+b\sqrt{11}$ where a and b are positive integers.

Solution 1 by Anthony J. Bevelacqua, University of North Dakota, Grand Forks, ND

We define four points

$$\begin{aligned} P &= \left(\frac{b}{2}, \frac{\sqrt{d^2-b^2}}{2}\right) \\ Q &= \left(-\frac{b}{2}, \frac{\sqrt{d^2-b^2}}{2}\right) \\ R &= \left(-\frac{a}{2}, -\frac{\sqrt{d^2-a^2}}{2}\right) \\ S &= \left(\frac{a}{2}, -\frac{\sqrt{d^2-a^2}}{2}\right) \end{aligned}$$

where $a=20+6\sqrt{11}$, $b=12+2\sqrt{11}$, $c=13+3\sqrt{11}$, and $d=17+7\sqrt{11}$. Note that $\overline{PQ}$ and $\overline{SR}$ are parallel and that $OP=OQ=OR=OS=d/2$. Thus $PQRS$ is a trapezoid inscribed in the circle with center $O=(0,0)$ and diameter d.
We have $PQ=b$, $SR=a$, and $PS=QR$. Now

$$(PS)^2 = \left(\frac{b}{2}-\frac{a}{2}\right)^2 + \left(\frac{\sqrt{d^2-b^2}}{2}+\frac{\sqrt{d^2-a^2}}{2}\right)^2.$$

Thus

$$\begin{aligned} 4(PS)^2 &= (b-a)^2 + (\sqrt{d^2-b^2}+\sqrt{d^2-a^2})^2 \\ &= 2d^2 - 2ab + 2\sqrt{d^2-b^2}\sqrt{d^2-a^2} \end{aligned}$$

so

$$2(PS)^2 = d^2 - ab + \sqrt{d^2-b^2}\sqrt{d^2-a^2}. \qquad (1)$$

We have

$$\begin{aligned} (d^2-b^2)(d^2-a^2) &= 16300+4800\sqrt{11} \\ &= (80+30\sqrt{11})^2 \end{aligned}$$

and

$$d^2 - ab = 456 + 126\sqrt{11}.$$

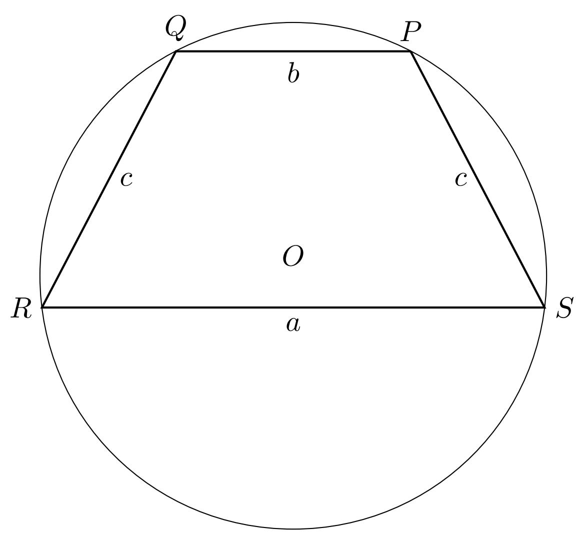
Therefore, by (1), we have

$$\begin{aligned}(PS)^2 &= 268 + 78\sqrt{11} \\ &= c^2.\end{aligned}$$

Finally, the perimeter of $PQRS$ is

$$\begin{aligned}PQ + QR + RS + SP &= b + c + a + c \\ &= 58 + 14\sqrt{11}.\end{aligned}$$

Thus the desired trapezoid has parallel sides of lengths $a = 20 + 6\sqrt{11}$ and $b = 12 + 2\sqrt{11}$ and the other two sides of length $c = 13 + 3\sqrt{11}$.
Here it is.



Solution 2 by Kee-Wai Lau, Hong Kong, China

We show that the dimensions of the trapezoid are $12 + 2\sqrt{11},\ 20 + 6\sqrt{11},\ 13 + 3\sqrt{11}$, and $13 + 3\sqrt{11}$.

Let $ABCD$ be the trapezoid with $AB||CD$. Since it is inscribed in a circle so it is in fact isosceles. Let $AB = x,\ CD = y,\ AD = BC = z,\ BD = w$, with $x \geq y$.

Since the perimeter of the trapezoid is $58 + 14\sqrt{11}$ so

$$x + y + 2z = 58 + 14\sqrt{11} \qquad (1)$$

Applying the cosine formula respectively to triangles ABD and CDB, we obtain $\cos\angle DAB = \dfrac{x^2 + z^2 - w^2}{2xz}$ and $\cos\angle DCB = \dfrac{y^2 + z^2 - w^2}{2yz}$. From $\angle DAB + \angle DCB = \pi$, we have $\cos\angle DAB = -\cos\angle DCB$, and deduce that

$$w^2 = xy + z^2. \qquad (2)$$

Let h be the length of the perpendicular from D to AB, and d be the diameter of the circumcircle of $\triangle ABD$. By the Pythagorean theorem, we have
$h^2 = z^2 - \left(\frac{x-y}{2}\right)^2 = \frac{(2z+x-y)(2z-x+y)}{4}$. Applying the sine formula to triangle ABD, we have $d = \frac{w}{\sin \angle DAB} = \frac{w}{\left(\frac{h}{z}\right)}$ or $dh = zw$. Since $d = 17 + 7\sqrt{11}$, by (2) we obtain from $d^2h^2 - z^2w^2 = 0$ that

$$(414 + 119\sqrt{11})(2z + x - y)(2z - x + y)(2z - x + y) - 2z^2(xy + z^2) = 0. \qquad (3)$$

Let $x = p + q\sqrt{11}$ and $y = r + s\sqrt{11}$, where p, q, r, s are positive integers. We substitute z of (1) into (3) so that the left side of (3) equals $f + h\sqrt{11}$ where f and g are integers depending only on p, q, r, s. Thus, (3) holds if and only if $f = g = 0$.
From (1), we see that both p and r do not exceed 56 and that both q and s do not exceed 12. By a compute search, we find that (3) holds if and only if
$p = 20, q = 6, r = 12, s = 2$. Hence our solution for the dimensions of the trapezoid.

Solution 3 by Brian D. Beasley, Presbyterian College, Clinton, SC

Since the trapezoid must be isosceles, we denote the lengths of the bases by x and y (with $x \le y$) and the length of each leg by z. Then (see [1]) the radius r of the circle is given by

$$r = z\sqrt{\frac{xy + z^2}{4z^2 - (x-y)^2}}.$$

We write $x = a + b\sqrt{11}$ and $y = c + d\sqrt{11}$, where a, b, c, and d are positive integers. Then $z = (58 + 14\sqrt{11} - x - y)/2 = e + f\sqrt{11}$, with $e = 29 - a/2 - c/2$ and $f = 7 - b/2 - d/2$ also positive integers. Using $r = (17 + 7\sqrt{11})/2$, we eventually obtain $t + u\sqrt{11} = v + w\sqrt{11}$, where:

$$t = 414(4e^2 + 44f^2 - a^2 + 2ac - c^2 - 11b^2 + 22bd - 11d^2) + 1309(8ef - 2ab + 2ad + 2bc - 2cd);$$
$$u = 414(8ef - 2ab + 2ad + 2bc - 2cd) + 119(4e^2 + 44f^2 - a^2 + 2ac - c^2 - 11b^2 + 22bd - 11d^2);$$
$$v = (2e^2 + 22f^2)(ac + 11bd + e^2 + 11f^2) + 44ef(ad + bc + 2ef);$$
$$w = (2e^2 + 22f^2)(ad + bc + 2ef) + 4ef(ac + 11bd + e^2 + 11f^2).$$

Setting $t = v$ and $u = w$, we obtain the following results via computer search:

$$x = 12 + 2\sqrt{11}, \quad y = 20 + 6\sqrt{11}, \quad \text{and} \quad z = 13 + 3\sqrt{11}.$$

[1] https://en.wikipedia.org/wiki/Isosceles_trapezoid

Editor's comments : Computers were called into service on this problem and that seemed to bother some of the solvers. **David Stone and John Hawkins of Georgia Southern University** also obtained the correct result and described their solution as follows: "We do not have an algebraic derivation for this result. Instead, we wrote algebraic conditions on the sides of the trapezoids, used the fact that there are only finitely many chords of the form $a + b\sqrt{11}$, (i.e. possibilities for the sides), then used a BASIC program to test them and find the solution amongst all possibilities." They went on to say that "another approach would be using analytic geometry. This lead us to the same end game−−write a program to check all possible values."

Ken Korbin, proposer of the problem, attached a note to the problem giving us some insights into how he constructed it. He stated: Begin with a circle with diameter K^3 with $K \geq 3$. It is possible to inscribe in this circle a trapezoid with parallel sides of lengths $2K^2$ and $6K^2 - 32$, and with each slant side of length $K^3 - 8K$. He then checked this statement by computing:

$$\text{Arcsin}\left(\frac{2K^2}{K^3}\right)+2\text{Arcsin}\left(\frac{K^3-8K}{K^3}\right)=\text{Arcsin}\left(\frac{6K^2-32}{K^3}\right).$$

For this trapezoid, Perimeter $= 2K^3 + 8K^2 - 16K - 32$.

In this problem, let

$$K = 1 + \sqrt{11} \approx 4.3166$$

$$\text{Diameter } = K^3 = 34 + 14\sqrt{11} \text{ and the sides of the trapezoid are:}$$

$$24 + 4\sqrt{11},\ 40 + 12\sqrt{11},\ 26 + 6\sqrt{11} \text{ and } 26 + 6\sqrt{11}.$$

Divide each length by 2 to get diameter $17 + 7\sqrt{11}$ and sides $\begin{cases} 12 + 2\sqrt{11} \\ 20 + 6\sqrt{11} \\ 13 + 3\sqrt{11} \\ 13 + 3\sqrt{11} \end{cases}$

Perimeter $= 58 + 14\sqrt{11}$

Check to see if this trapezoid can be inscribed in a circle with diameter $= 17 + 7\sqrt{11}$.

Check:

$$\text{Arcsin}\left(\frac{12+2\sqrt{11}}{17+7\sqrt{11}}\right)+2\text{Arcsin}\left(\frac{13+3\sqrt{11}}{17+7\sqrt{11}}\right)=\text{Arcsin}\left(\frac{20+6\sqrt{11}}{17+7\sqrt{11}}\right).$$

Also solved by David Stone and John Hawkins, Georgia Southern University, Statesboro, GA, and the proposer.

5512: *Proposed by Ángel Plaza, Universidad de Las Palmas de Gran Canaria, Spain*

If $a_k > 0$, $(k = 1, 2, \ldots, n)$ then $\dfrac{n}{\displaystyle\sum_{k=1}^{n} \frac{1}{\frac{1}{k} + a_k}} - \dfrac{n}{\displaystyle\sum_{k=1}^{n} \frac{1}{a_k}} \geq \dfrac{2}{n+1}$.

Solution 1 by Moti Levy, Rehovot, Israel

$$F(x) := \begin{cases} \frac{2}{n+1}, & for \quad x = 0, \\ \frac{n}{\sum_{k=1}^{n} \frac{1}{\frac{1}{k}+a_k x}} - \frac{n}{\sum_{k=1}^{n} \frac{1}{a_k x}}, & for \quad x > 0. \end{cases} \tag{1}$$

Note that $F(x)$ is continuous at $x=0$, since $\lim_{x\to 0} F(x) = F(0)$.

$$\lim_{x\to 0} F(x) = \lim_{x\to 0} \frac{n}{\sum_{k=1}^{n} \frac{1}{\frac{1}{k}+a_k x}} - \lim_{x\to 0} \frac{n}{\sum_{k=1}^{n} \frac{1}{a_k x}}$$
$$= \frac{n}{\sum_{k=1}^{n} \frac{1}{\frac{1}{k}}} - 0 = \frac{2}{n+1}.$$

Thus, in terms of $F(x)$, our original inequality is $F(1) \geq F(0)$.
Now we show that $\frac{dF}{dx} \geq 0$ for $x > 0$ (i.e., $F(x)$ is monotone increasing for $x > 0$),

$$\frac{dF}{dx} = -\frac{n}{\left(\sum_{k=1}^{n} \frac{1}{\frac{1}{k}+a_k x}\right)^2} \frac{d\left(\sum_{k=1}^{n} \frac{1}{\frac{1}{k}+a_k x}\right)}{dx} + \frac{n}{\left(\sum_{k=1}^{n} \frac{1}{a_k x}\right)^2} \frac{d\left(\sum_{k=1}^{n} \frac{1}{a_k x}\right)}{dx} \tag{2}$$
$$= n\left(\frac{\sum_{k=1}^{n} \frac{a_k}{\left(\frac{1}{k}+a_k x\right)^2}}{\left(\sum_{k=1}^{n} \frac{1}{\frac{1}{k}+a_k x}\right)^2} - \frac{\sum_{k=1}^{n} \frac{a_k}{(a_k x)^2}}{\left(\sum_{k=1}^{n} \frac{1}{a_k x}\right)^2} \right). \tag{3}$$

$\frac{dF}{dx} \geq 0$ is equivalent to

$$\frac{\sum_{k=1}^{n} \frac{a_k}{\left(\frac{1}{k}+a_k x\right)^2}}{\left(\sum_{k=1}^{n} \frac{1}{\frac{1}{k}+a_k x}\right)^2} \geq \frac{\sum_{k=1}^{n} \frac{a_k}{(a_k x)^2}}{\left(\sum_{k=1}^{n} \frac{1}{a_k x}\right)^2},$$

or to

$$\left(\sum_{k=1}^{n} \frac{a_k}{\left(\frac{1}{k}+a_k x\right)^2}\right)\left(\sum_{k=1}^{n} \frac{1}{a_k x}\right)^2 \geq \left(\sum_{k=1}^{n} \frac{a_k}{(a_k x)^2}\right)\left(\sum_{k=1}^{n} \frac{1}{\frac{1}{k}+a_k x}\right)^2.$$

We simplify by multiplying both sides by x^2 and obtain,

$$\left(\sum_{k=1}^{n} \frac{a_k}{\left(\frac{1}{k}+a_k x\right)^2}\right)\left(\sum_{k=1}^{n} \frac{1}{a_k}\right) \geq \left(\sum_{k=1}^{n} \frac{1}{\frac{1}{k}+a_k x}\right)^2 \tag{4}$$

But (5) is a direct consequence of the Cauchy-Schwarz inequality.
We conclude that $F(x) \geq F(0)$ in the interval $[0,1]$, hence $F(1) \geq F(0)$.

Solution 2 by Michel Bataille, Rouen, France

Since $\frac{2}{n+1} = \frac{n}{\sum\limits_{k=1}^{n} k}$, it suffices to show that more generally

$$\left(\sum (a_k+b_k)^{-1}\right)^{-1} \geq \left(\sum a_k^{-1}\right)^{-1} + \left(\sum b_k^{-1}\right)^{-1} \qquad (1)$$

holds whenever $a_k, b_k > 0$, $(k = 1, 2, \ldots, n)$. [The problem is the particular case $b_k = \frac{1}{k}$.]
(Here and in what follows $\sum$ means $\sum\limits_{k=1}^{n}$.)
We propose two proofs of (1).

Proof 1:
If p is a negative real number, then

$$\left(\sum (a_k+b_k)^p\right)^{1/p} \geq \left(\sum a_k^p\right)^{1/p} + \left(\sum b_k^p\right)^{1/p}.$$

(see G.H. Hardy, J.E. Littlewood, G. Polya, *Inequalities*, C.U.P., 1934, p. 30). Taking $p = -1$ gives (1).

Proof 2:
Let $x_k = \frac{1}{a_k},\ y_k = \frac{1}{b_k}$ and $X_n = \sum x_k,\ Y_n = \sum y_k$. It is readily seen that (1) rewrites as $L_n \le R_n$ where $L_n = \sum \frac{x_k y_k}{x_k + y_k}$ and $R_n = \frac{X_n Y_n}{X_n + Y_n}$.
Since $4L_n = \sum \frac{(x_k+y_k)^2 - (x_k-y_k)^2}{x_k+y_k} = X_n + Y_n - Z_n$ where $Z_n = \sum \frac{(x_k-y_k)^2}{x_k+y_k}$, the inequality $L_n \le R_n$ is successively equivalent to

$$X_n + Y_n - Z_n \le \frac{4X_nY_n}{X_n+Y_n}$$

$$(X_n+Y_n)^2 - 4X_nY_n \le (X_n+Y_n)Z_n$$
$$(X_n - Y_n)^2 \le (X_n+Y_n)Z_n$$
$$\left(\sum(x_k-y_k)\right)^2 \le \left(\sum(x_k+y_k)\right)\left(\sum \frac{(x_k-y_k)^2}{x_k+y_k}\right).$$

With $u_k = \sqrt{x_k+y_k},\ v_k = \frac{x_k-y_k}{\sqrt{x_k+y_k}}$, the latter is just $(\sum u_kv_k)^2 \le (\sum u_k^2)(\sum v_k^2)$, which holds by the Cauchy-Schwarz inequality. Thus $L_n \le R_n$ holds as well and we are done.

Also solved by Ed Gray, Highland Beach, FL; Kee-Wai Lau, Hong Kong, China; Adrian Naco, Polytechnic University of Tirana, Albania; Albert Stadler, Herrliberg, Switzerland, and the proposer.

5513: *Proposed by Michael Brozinsky, Central Islip, NY*

In an $n \times n \times n$ cube partitioned into n^3 congruent cubes by $n-1$ equally spaced planes parallel to each pair of parallel faces, there are 20 times as many non-cubic rectangular parallelepipeds that could be formed as were cubic parallelepipeds. What is n?

Solution 1 by Albert Stadler, Herrliberg, Switzerland

The number of cubic parallelepipeds equals

$$\sum_{r=1}^{n}(n+1-r)^3 = \sum_{r=1}^{n} r^3 = \left(\frac{n(n+1)}{2}\right)^2,$$

while the number of non-cubic rectangular parallelepipeds equals

$$\sum_{\substack{1\le r,s,t\le n \\ r,s,t \text{ not all equal}}} (n+1-r)(n+1-s)(n+1-t) = \left(\sum_{r=1}^{n}(n+1-r)\right)^3 - \sum_{r=1}^{n}(n+1-r)^3 =$$

$$= \left(\sum_{r=1}^{n} r\right)^3 - \sum_{r=1}^{n} r^3 = \left(\frac{n(n+1)}{2}\right)^3 - \left(\frac{n(n+1)}{2}\right)^2.$$

Therefore

$$\left(\frac{n(n+1)}{2}\right)^3 - \left(\frac{n(n+1)}{2}\right)^2 = 20\left(\frac{n(n+1)}{2}\right)^2,$$

which implies that $\frac{n(n+1)}{2} = 21$ and finally, $n = 6$.

Solution 2 by Kee-Wai Lau, Hong Kong, China

Clearly the total number of parallelepipeds is $\binom{n+1}{2}^3 = \frac{n^3(n+1)^3}{8}$.

It can be counted readily that the number of $(n-k)\times(n-k)\times(n-k)$ cubic parallelepipeds equals $(k+1)^3$ for $k = 0, 1, 2, \cdots, n-1$. Hence the total number of cubic parallelepipeds equals $\sum_{k=0}^{n-1}(k+1)^3 = \frac{n^2(n+1)^2}{4}$. So according to the given conditions of the problem we have

$$\frac{n^3(n+1)^3}{8} - \frac{n^2(n+1)^2}{4} = 20\left(\frac{n^2(n+1)^2}{4}\right),$$

which reduces to the equation $n^2 + n - 42 = 0$. It follows that $n = 6$.

Also solved by the proposer.

5514: *Proposed by D. M. Batinetu-Giurgiu, "Matei Basarab" National College, Bucharest, Romania and Neculai Stanciu, "George Emil Palade" School, Buzău, Romania*

If $a \in \left(0, \frac{\pi}{2}\right)$ and $b = \arcsin a$, then calculate $\lim_{n\to\infty} \sqrt[n]{n!}\left(\sin\left(\frac{b\cdot \sqrt[n+1]{(2n+1)!!}}{\sqrt[n]{(2n-1)!!}}\right) - a\right)$.

Solution 1 by Paolo Perfetti, Department of Mathematics, Tor Vergata University, Rome, Italy

Let $\frac{\sqrt[n+1]{(2n+1)!!}}{\sqrt[n]{(2n-1)!!}} \doteq q_n$.

Result I. $\lim_{n\to\infty} q_n = 1$.

It is equivalent to

$$\lim_{n\to\infty} \frac{\ln(2n+1)!!}{n+1} - \frac{\ln(2n-1)!!}{n} = 0 \tag{1}$$

that is

$$\lim_{n\to\infty} \frac{1}{n+1}\ln\frac{(2n+2)!}{2^{n+1}(n+1)!} - \frac{1}{n}\ln\frac{(2n)!}{2^n n!} = 0$$

Let's break the above limit as

$$\lim_{n\to\infty}\left(\frac{\ln(2n+2)!}{n+1} - \frac{\ln(2n)!}{n}\right) + \lim_{n\to\infty}\left(\frac{\ln(2^{-n-1})}{n+1} - \frac{\ln(2^{-n})}{n}\right) +$$
$$+\lim_{n\to\infty}\left(\frac{-\ln((n+1)!)}{n+1} + \frac{\ln n!}{n}\right)$$

$$\lim_{n\to\infty}\left[\frac{\ln(2n+2)!}{n+1} - \frac{\ln(2n)!}{n}\right] = \lim_{n\to\infty}\left[\frac{\ln(2n)!}{n+1} - \frac{\ln(2n)!}{n} + \frac{\ln(2n+2)}{n+1} + \frac{\ln(2n+1)}{n+1}\right] =$$
$$= \lim_{n\to\infty} -\frac{\ln(2n)!}{n(n+1)} \underbrace{=}_{C.S.} \lim_{n\to\infty} -\frac{\ln(2n+2)! - \ln(2n)!}{(n+1)(n+2) - n(n+1)} =$$
$$= \lim_{n\to\infty} -\frac{\ln((2n+2)(2n+1))}{2n+2} = 0$$

C.S. stands for Cesàro–Stolz.

$$\frac{\ln(2^{-n-1})}{n+1} - \frac{\ln(2^{-n})}{n} = 0$$

$$\lim_{n\to\infty} \frac{-\ln((n+1)!)}{n+1} + \frac{\ln n!}{n} = \lim_{n\to\infty} \frac{\ln n!}{n} - \frac{-\ln((n)!)}{n+1} - \lim_{n\to\infty} \frac{-\ln(n+1)}{n+1} =$$
$$= \lim_{n\to\infty} \frac{\ln n!}{n(n+1)} \underbrace{=}_{C.S.} \lim_{n\to\infty} \frac{\ln((n+1)!) - \ln n!}{(n+1)(n+2) - n(n+1)} = \lim_{n\to\infty} \frac{\ln(n+1)}{2n+2} = 0$$

Result II.

$$\lim_{n\to\infty} \frac{\sqrt[n]{n!}}{n} \underbrace{=}_{C.S.} \lim_{n\to\infty} \left(\frac{n!}{n^n}\right)^n = \lim_{n\to\infty} \frac{(n+1)!}{(n+1)^{n+1}} \frac{n^n}{n!} = \frac{1}{e}$$

Result III.

$$\lim_{n\to\infty} n(q_n - 1) = \lim_{n\to\infty} n \ln q_n \cdot \frac{q_n - 1}{\ln q_n} = \lim_{n\to\infty} n \ln q_n =$$
$$= \lim_{n\to\infty} n \left[\frac{\ln(2n+1)!!}{n+1} - \frac{\ln(2n-1)!!}{n}\right] = \lim_{n\to\infty} \ln \frac{(2n+1)!!}{(2n-1)!!} - \frac{\ln(2n+1)!!}{n+1} =$$
$$= \lim_{n\to\infty} \frac{(n+1)\ln(2n+1) - \ln(2n+1)!!}{n+1} \underbrace{=}_{C.S.}$$
$$= \lim_{n\to\infty} \frac{(n+2)\ln(2n+3) - (n+1)\ln(2n+1) - \ln((2n+1)!!) + \ln((2n-1)!!)}{n+2-n-1} =$$
$$= \lim_{n\to\infty} (n+2) \ln\left(1 + \frac{2}{2n+1}\right) = 1$$

The limit we are searching is

$$\lim_{n\to\infty} \frac{\sqrt[n]{n!}}{n} n \left[\sin(bq_n) - \sin b\right] = \lim_{n\to\infty} \frac{n}{e} 2 \left[\sin \frac{b(q_n - 1)}{2} \underbrace{\cos \frac{b(q_n + 1)}{2}}_{\to\sqrt{1-a^2}}\right] =$$
$$= \lim_{n\to\infty} \frac{2}{e}\sqrt{1-a^2} n(q_n - 1) \frac{\sin \frac{b(q_n-1)}{2}}{q_n - 1} = \frac{b}{e}\sqrt{1-a^2} \lim_{n\to\infty} n(q_n - 1) =$$
$$= \frac{b}{e}\sqrt{1-a^2}$$

$|a| \leq 1$ should have been in the statement.

Solution by 2 Moti Levy, Rehovot, Israel

$$(2n-1)!! = \frac{(2n)!}{2^n n!}. \qquad (1)$$

Using the Stirling 's asymptotic formula, we have

$$n! \sim \frac{n^n}{e^n}. \qquad (2)$$

Applying (2) to (1) yields

$$\sqrt[n]{n!} \sim \frac{n}{e},$$

$$\sqrt[n]{(2n-1)!!} \sim \frac{2n}{e}, \qquad \sqrt[n+1]{(2n+1)!!} \sim \frac{2n+2}{e},$$

$$\frac{\sqrt[n+1]{(2n+1)!!}}{\sqrt[n]{(2n-1)!!}} \sim \frac{2n+2}{e}\frac{e}{2n} = 1 + \frac{1}{n}.$$

$$\sqrt[n]{n!}\left(\sin\left(b\frac{\sqrt[n+1]{(2n+1)!!}}{\sqrt[n]{(2n-1)!!}}\right) - a\right) \sim \frac{n}{e}\left(\sin\left(b\left(1+\frac{1}{n}\right)\right) - \sin b\right)$$

$$= 2\frac{n}{e}\sin\frac{b\left(1+\frac{1}{n}\right)-b}{2}\cos\frac{b\left(1+\frac{1}{n}\right)+b}{2}$$

$$= \frac{b}{e}\left(\frac{\sin\frac{b}{2n}}{\frac{b}{2n}}\right)\cos\left(b\left(1+\frac{1}{2n}\right)\right) \to \frac{b\cos b}{e}.$$

We conclude that $\lim_{n\to\infty}\sqrt[n]{n!}\left(\sin\left(b\frac{\sqrt[n+1]{(2n+1)!!}}{\sqrt[n]{(2n-1)!!}}\right) - a\right) = \dfrac{b\cos b}{e}$.

Also solved by Michel Bataille, Rouen, France; Kee-Wai Lau, Hong Kong, China; Albert Stadler, Herrliberg, Switzerland, and the proposer.

5515: *Proposed by José Luis Díaz-Barrero, Barcelona Tech, Barcelona, Spain*

Let n be a positive integer. Prove that

$$\frac{1}{2^n}\left(\sum_{k=1}^{n}\sqrt{\frac{1}{n^2}+\binom{n-1}{k-1}^2}\right)^2 \geq 1.$$

Solution 1 by Dionne Bailey, Elsie Campbell, Charles Diminnie, and Trey Smith, Angelo State University, San Angelo, TX

Let $n \geq 1$ and $1 \leq k \leq n$. Then, since $f(x) = \sqrt{x}$ is concave on $(0, \infty)$, Jensen's

Theorem implies that

$$\begin{aligned}
\sqrt{\frac{1}{n^2}+\left(\begin{array}{c} n-1 \\ k-1 \end{array}\right)^2} &= \sqrt{2}\sqrt{\frac{\frac{1}{n^2}+\left(\begin{array}{c} n-1 \\ k-1 \end{array}\right)^2}{2}} \\
&= \sqrt{2} f\left(\frac{\frac{1}{n^2}+\left(\begin{array}{c} n-1 \\ k-1 \end{array}\right)^2}{2}\right) \\
&\geq \sqrt{2}\frac{f\left(\frac{1}{n^2}\right)+f\left[\left(\begin{array}{c} n-1 \\ k-1 \end{array}\right)^2\right]}{2} \\
&= \frac{\frac{1}{n}+\left(\begin{array}{c} n-1 \\ k-1 \end{array}\right)}{\sqrt{2}}.
\end{aligned}$$

If we let $i = k-1$ for $k = 1, \ldots, n$ and use the known result that

$$\sum_{i=0}^{m}\left(\begin{array}{c} m \\ i \end{array}\right) = 2^m$$

for $m \geq 0$, it follows that

$$\begin{aligned}
\sum_{k=1}^{n}\sqrt{\frac{1}{n^2}+\left(\begin{array}{c} n-1 \\ k-1 \end{array}\right)^2} &\geq \frac{1}{\sqrt{2}}\left[\sum_{k=1}^{n}\frac{1}{n}+\sum_{k=1}^{n}\left(\begin{array}{c} n-1 \\ k-1 \end{array}\right)\right] \\
&= \frac{1}{\sqrt{2}}\left[1+\sum_{i=0}^{n-1}\left(\begin{array}{c} n-1 \\ i \end{array}\right)\right] \\
&= \frac{1}{\sqrt{2}}\left(1+2^{n-1}\right).
\end{aligned}$$

Further, the Arithmetic - Geometric Mean Inequality yields

$$\begin{aligned}
1+2^{n-1} &\geq 2\sqrt{2^{n-1}} \\
&= 2\cdot 2^{\frac{n-1}{2}} \\
&= 2^{\frac{n+1}{2}}.
\end{aligned}$$

Hence,

$$\begin{aligned}
\sum_{k=1}^{n}\sqrt{\frac{1}{n^2}+\left(\begin{array}{c} n-1 \\ k-1 \end{array}\right)^2} &\geq \frac{1}{\sqrt{2}}\left(1+2^{n-1}\right) \\
&\geq \frac{1}{\sqrt{2}}2^{\frac{n+1}{2}}. \\
&= 2^{\frac{n}{2}},
\end{aligned}$$

and we have

$$\begin{aligned}
\frac{1}{2^n}\left(\sum_{k=1}^{n}\sqrt{\frac{1}{n^2}+\left(\begin{array}{c} n-1 \\ k-1 \end{array}\right)^2}\right)^2 &\geq \frac{1}{2^n}\left(2^{\frac{n}{2}}\right)^2 \\
&= 1.
\end{aligned}$$

Solution 2 by Albert Stadler, Herrliberg, Switzerland

The statement hold true for $n=1$. Let $n>1$. Then

$$\frac{1}{2^n}\left(\sum_{k=1}^{n}\sqrt{\frac{1}{n^2}+\binom{n-1}{k-1}^2}\right)^2 \geq \frac{1}{2^n}\left(\sum_{k=1}^{n}\binom{n-1}{k-1}\right)^2=\frac{1}{2^n}\left(2^{n-1}\right)^2=2^{n-2}\geq 1,$$

as claimed.

Solution 3 by Angel Plaza, University of Las Palmas de Gran Canaria, Spain

For $n=1$ the equality holds. For $n>1$, we have

$$\begin{aligned}
\frac{1}{2^n}\left(\sum_{k=1}^{n}\sqrt{\frac{1}{n^2}+\binom{n-1}{k-1}^2}\right)^2 &> \frac{1}{2^n}\left(\sum_{k=1}^{n}\binom{n-1}{k-1}\right)^2\\
&= \frac{1}{2^n}\left(\sum_{k=0}^{n-1}\binom{n-1}{k}\right)^2\\
&= \frac{1}{2^n}\left(2^{n-1}\right)^2\\
&= \frac{2^{2n-2}}{2^n}\\
&= 2^{n-2}\\
&\geq 1.
\end{aligned}$$

Also solved by Michel Bataille, Rouen, France; Bruno Salgueiro Fanego, Viveiro, Spain; Ed Gray, Highland Beach, FL; Henry W. Gould, West Virginia University, Morgantown, WV with Scott H. Brown, Auburn University, Montgomery, AL; Kee-Wai Lau, Hong Kong, China; Moti Levy, Rehovot, Israel; Adrian Naco, Polytechnic University of Tirana, Albania; Paolo Perfetti, Department of Mathematics, Tor Vergata University, Rome, Italy Ioannis D.Sfikas, National and Kapodistrian University of Athens, Greece; Ramiz Valizada (student of Yagub N. Aliyev), ADA University, Baku, Azerbaijan, and the proposer.

5516: *Proposed by Ovidiu Furdui and Alina Sîntămărian both at the Technical University of Cluj-Napoca, Cluj-Napoca, Romania*

Calculate $\displaystyle\sum_{n=1}^{\infty} n\left(\zeta(3)-1-\frac{1}{2^3}-\cdots-\frac{1}{n^3}-\frac{1}{2n^2}\right)$.

Solution 1 by Michel Bataille, Rouen, France

Let $S=\sum\limits_{n=1}^{\infty} n\left(\zeta(3)-1-\frac{1}{2^3}-\cdots-\frac{1}{n^3}-\frac{1}{2n^2}\right)$. We claim that $S=\frac{1}{4}-\frac{\pi^2}{12}$.

We have $\zeta(3)-1-\frac{1}{2^3}-\cdots-\frac{1}{n^3}=\sum\limits_{k=n+1}^{\infty}\frac{1}{k^3}$ and

$$\frac{1}{n^2}=\sum_{k=n+1}^{\infty}\left(\frac{1}{(k-1)^2}-\frac{1}{k^2}\right)=\sum_{k=n+1}^{\infty}\frac{2k-1}{k^2(k-1)^2},$$

hence

$$\zeta(3) - 1 - \frac{1}{2^3} - \cdots - \frac{1}{n^3} - \frac{1}{2n^2} = \sum_{k=n+1}^{\infty} \left(\frac{1}{k^3} - \frac{2k-1}{2k^2(k-1)^2} \right) = \sum_{k=n+1}^{\infty} \frac{2-3k}{2k^3(k-1)^2}$$

and so $S = \sum_{n=1}^{\infty} na_n$ with $a_n = \sum_{k=n+1}^{\infty} \frac{2-3k}{2k^3(k-1)^2}$.

Now, let $S_N = \sum_{n=1}^{N} na_n$ where N is an integer with $N > 2$. Summing by parts gives

$$S_N = \sum_{n=1}^{N-1} \frac{n(n+1)}{2}(a_n - a_{n+1}) + \frac{N(N+1)}{2}\, a_N. \tag{1}$$

But, as $k \to \infty$, we have

$$\begin{aligned}
\frac{2-3k}{2k^3(k-1)^2} &= \frac{1}{k^3} - \frac{2k(1-\frac{1}{2k})}{2k^4(1-\frac{1}{k})^2} \\
&= \frac{1}{k^3} - \frac{1}{k^3}\left(1 - \frac{1}{2k}\right)\left(1 - \frac{1}{k}\right)^{-2} \\
&= \frac{1}{k^3}\left(1 - \left(1 - \frac{1}{2k}\right)\left(1 + \frac{2}{k} + o(1/k)\right)\right) \\
&= \frac{1}{k^3}\left(-\frac{3}{2k} + o(1/k)\right) \sim \frac{-3}{2k^4}
\end{aligned}$$

and so

$$a_N \sim -\frac{3}{2}\sum_{k=N+1}^{\infty} \frac{1}{k^4} \sim -\frac{3}{2}\cdot\frac{1}{3N^3} = -\frac{1}{2N^3} \qquad (N \to \infty).$$

It readily follows that $\lim\limits_{N\to\infty} \frac{N(N+1)}{2}\, a_N = 0$.

On the other hand, $a_n - a_{n+1} = \frac{2-3(n+1)}{2(n+1)^3n^2} = \frac{1}{(n+1)^3} - \frac{2n+1}{2n^2(n+1)^2}$ and a simple calculation yields

$$\frac{n(n+1)}{2}(a_n - a_{n+1}) = \frac{-(3n+1)}{4n(n+1)^2} = -\frac{1}{4}\cdot\left(\frac{1}{n} - \frac{1}{n+1} + \frac{2}{(n+1)^2}\right).$$

As a result, we obtain

$$\sum_{n=1}^{N-1} \frac{n(n+1)}{2}(a_n - a_{n+1}) = -\frac{1}{4}\left(\sum_{n=1}^{N-1}\left(\frac{1}{n} - \frac{1}{n+1}\right) + \sum_{n=1}^{N-1} \frac{2}{(n+1)^2}\right).$$

Since $\lim\limits_{N\to\infty} \sum_{n=1}^{N-1} \left(\frac{1}{n} - \frac{1}{n+1}\right) = 1$ and $\lim\limits_{N\to\infty} \sum_{n=1}^{N-1} \frac{2}{(n+1)^2} = 2\left(\frac{\pi^2}{6} - 1\right)$, we deduce (using (1)) that

$$S = \lim_{N\to\infty} S_N = -\frac{1}{4}\left(1 + \frac{\pi^2}{3} - 2\right) = \frac{1}{4} - \frac{\pi^2}{12},$$

as claimed.

Solution 2 by Paola Perfetti, Department of Mathematics, Tor Vergata University, Rome, Italy

The sum is

$$\lim_{N\to\infty}\sum_{n=1}^{N} n\left(\sum_{k=n+1}^{\infty}\frac{1}{k^3}-\frac{1}{2n^2}\right)$$

Summation by parts

$$\sum_{k=1}^{n} a_k b_k = A_n b_n + \sum_{k=2}^{n} A_{k-1}(b_{k-1}-b_k), \qquad A_n=\sum_{k=1}^{n} a_k$$

gives

$$\frac{N(N+1)}{2}\left(\sum_{k=N+1}^{\infty}\frac{1}{k^3}-\frac{1}{2N^2}\right)+\sum_{k=2}^{N}\frac{k(k-1)}{2}\left(\frac{1}{2k^2}-\frac{1}{2(k-1)^2}+\frac{1}{k^3}\right)=$$

$$=\frac{N(N+1)}{2}\underbrace{\Big(\sum_{k=N+1}^{\infty}\frac{1}{k^3}-\frac{1}{2N^2}\Big)}_{=A}+\sum_{k=2}^{N}\left(-\frac{1}{2k^2}+\frac{1}{4k}-\frac{1}{4(k-1)}\right)$$

We know that

$$\frac{1}{2(N+1)^2}-\frac{1}{2N^2}=\int_{N+1}^{\infty}\frac{dx}{x^3}-\frac{1}{2N^2}<A<\int_{N}^{\infty}\frac{dx}{x^3}-\frac{1}{2N^2}=0$$

and

$$\lim_{N\to\infty}\frac{N(N+1)}{2}\Big(\frac{1}{2(N+1)^2}-\frac{1}{2N^2}\Big)=\lim_{N\to\infty}\frac{-4N-2}{2N(N+1)}=0$$

thus

$$\lim_{N\to\infty}\frac{N(N+1)}{2}\left(\int_{N+1}^{\infty}\frac{dx}{x^3}-\frac{1}{2N^2}\right)=0$$

while

$$\lim_{N\to\infty}\sum_{k=2}^{N}\left(-\frac{1}{2k^2}+\frac{1}{4k}-\frac{1}{4(k-1)}\right)=-\frac{\pi^2}{12}+\frac{1}{2}-\frac{1}{4}=\frac{3-\pi^2}{12}$$

Solution 3 by Kee-Wai Lau, Hong Kong, China

We show that

$$\sum_{n=1}^{\infty} n(\left(\zeta(3)-1-\frac{1}{2^3}-\cdots\frac{1}{n^3}-\frac{1}{2n^2}\right)=\frac{3-\pi^2}{12} \tag{1}$$

For $x>0$, denote by $f(x)$ the function $\dfrac{1}{x^3}$, so that

$$\zeta(3)-1-\frac{1}{2^3}-\ldots-\frac{1}{n^3}=\sum_{k=n+1}^{\infty} f(k).$$

It can be proved readily by induction that for positive integers M, we have

$$2\sum_{n=1}^{M} n\left(\sum_{k=n+1}^{\infty} f(k)-\frac{1}{2n^2}\right)=M(M+1)\sum_{n=M+2}^{\infty} f(n)+\frac{1}{M+1}-\sum_{n=1}^{M+1}\frac{1}{n^2}. \qquad (2)$$

Since $f(x)$ is strictly decreasing, so

$$\frac{1}{2(M+2)^2}=\int_{M+2}^{\infty} f(x)dx<\sum_{n=M+2}^{\infty} f(n)<\int_{M+1}^{\infty} f(x)dx=\frac{1}{2(M+1)^2}.$$

It follows that

$$\lim_{M\to\infty} M(M+1)\sum_{n=M+2}^{\infty} f(n)=\frac{1}{2} \qquad (3)$$

Now (1) follows from $(2),(3)$ and the facts that $\lim\limits_{M\to\infty}\dfrac{1}{M+1}=0$ and $\sum\limits_{n=1}^{\infty}\dfrac{1}{n^2}=\dfrac{\pi^2}{6}$.

Also solved by Ed Gray, Highland Beach, FL (partial solution); Moti Levy, Rehovot, Israel; Albert Stadler, Herrliberg, Switzerland, and the proposer.

Mea Culpa

Stanley Rabinowitz of Chelmsford, MA should have been credited with having solved 5506. Like several of the other readers, he generalized problem 5506 and I had marked his solution to this generalization for publication. It was inadvertently omitted from the January issue of the column, and so it is being listed here.

Solution to 5506 by Stanley Rabinowitz, Chelmsford, MA

We will find the more general solution: $\Omega_n=\det\left[\begin{pmatrix}1 & c\\ c & c^2\end{pmatrix}^n+\begin{pmatrix}c^2 & -c\\ -c & 1\end{pmatrix}^n\right]$.

Let $\mathbf{A}=\begin{pmatrix}1 & c\\ c & c^2\end{pmatrix}$, $\mathbf{B}=\begin{pmatrix}c^2 & -c\\ -c & 1\end{pmatrix}$, and $\mathbf{S}=\mathbf{A}+\mathbf{B}$. Although matrix multiplication is not commutative, it is associative. In the expansion of $(\mathbf{A}+\mathbf{B})^n$, every term except $\mathbf{A}^n$ and $\mathbf{B}^n$has an $\mathbf{A}$ next to a $\mathbf{B}$. Since $\mathbf{AB}=\mathbf{BA}=\mathbf{0}$, therefore $\mathbf{S}^n=\mathbf{A}^n+\mathbf{B}^n$.

Thus, $\Omega_n=\det[\mathbf{A}^n+\mathbf{B}^n]=\det[\mathbf{S}^n]=(\det[\mathbf{S}])^n=\begin{vmatrix}1+c^2 & 0\\ 0 & c^2+1\end{vmatrix}^n=(c^2+1)^{2n}$.

The solution by Paul M. Harms of North Newton, KS to 5506 was received by yours truly three weeks after he had mailed it. His method of solution is also unique.

Let A be the matrix in the problem with two elements of 5 and let B be the matrix in the problem with two elements of -5. We have

$$A^2=26A,\ A^3=26A^2=26^2A,\ldots,\ A^{100}=26^{99}A.$$

In a similar manner, $B^2=26B,\ B^3=26^2B,\ \ldots,B^{100}=26^{99}B$. Then the matrix $A^{100}+B^{100}$ has the number $26^{99}(1+25)=26^{100}$ along the main diagonal and

$26^{99}(5+(-5))=0$ for the other two elements . The value of the determinate in the problem is then $(26^{100})^2 = 26^{200}$.

G. C. Greubel of Newport News, VA should have been credited for solving 5505 and 5510. His solution to 5505 also developed the generalization stated above, and his solution for 5510 also generalized the problem showing:

$$S(a) = \sum_{n=1}^{\infty} \left[a^{2n} \left(\zeta(2n) - 1\right) - \sum_{k=2}^{a} \left(\frac{a}{k}\right)^{2n} \right],$$

can be reduced to

$$S(a) = \frac{a}{2} \sum_{k=1}^{2a} \frac{1}{k} = \frac{a}{2}\, H_{2a},$$

where H_n is the n^{th} Harmonic number. By setting $a = 2, 3, 4$ he quickly determined that

$$\sum_{k=1}^{\infty} \left[4^n \left(\zeta(2n) - 1\right) - 1\right] = H_4 = \frac{25}{12},$$

$$\sum_{k=1}^{\infty} \left[9^n \left(\zeta(2n) - 1\right) - 1 - \left(\frac{3}{2}\right)^{2n}\right] = H_6 = \frac{49}{20}, \text{ and}$$

$$\sum_{k=1}^{\infty} \left[16^n \left(\zeta(2n) - 1\right) - 1 - 4^n - \left(\frac{4}{3}\right)^{2n}\right] = H_8 = \frac{761}{280}.$$

Editor′s Comments

Late solutions were received to the following problems;

5506: Aydin Javadov, (student of Yagub Aliyev), ADA University, Baku, Azerbaijan.

5508: Rasul Balayev, Ilkin Guluzada, Nuru Nurdil, and Leyla Shamoyeva (students of Yagub Aliyev), ADA University, Baku, Azerbaijan.