Problems

Ted Eisenberg, Section Editor

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This section of the Journal offers readers an opportunity to exchange interesting mathematical problems and solutions. Please send them to Ted Eisenberg, Department of Mathematics, Ben-Gurion University, Beer-Sheva, Israel or fax to: 972-86-477-648. Questions concerning proposals and/or solutions can be sent e-mail to <eisenbt@013.net>. Solutions to previously stated problems can be seen at <http://ssmj.tamu.edu>.

Solutions to the problems stated in this issue should be posted before March 15, 2009

- 5044: *Proposed by Kenneth Korbin, New York, NY.*

  Let N be a positive integer and let

  $$\begin{cases} x = 9N^2 + 24N + 14 \text{ and} \\ y = 9(N+1)^2 + 24(N+1) + 14. \end{cases}$$

  Express the value of y in terms of x, and express the value of x in terms of y.

- 5045: *Proposed by Kenneth Korbin, New York, NY.*

  Given convex cyclic hexagon ABCDEF with sides

  $$\begin{array}{rcl} \overline{AB} & = & \overline{BC} = 85 \\ \overline{CD} & = & \overline{DE} = 104, \text{ and} \\ \overline{EF} & = & \overline{FA} = 140. \end{array}$$

  Find the area of $\triangle BDF$ and the perimeter of $\triangle ACE$.

- 5046: *Proposed by R.M. Welukar of Nashik, India and K.S. Bhanu, and M.N. Deshpande of Nagpur, India.*

  Let $4n$ successive Lucas numbers $L_k, L_{k+1}, \cdots, L_{k+4n-1}$ be arranged in a $2 \times 2n$ matrix as shown below:

  $$\begin{array}{cccccc} 1 & 2 & 3 & 4 & \cdots & 2n \end{array}$$
  $$\begin{pmatrix} L_k & L_{k+3} & L_{k+4} & L_{k+7} & \cdots & L_{k+4n-1} \\ & & & & & \\ L_{k+1} & L_{k+2} & L_{k+5} & L_{k+6} & \cdots & L_{k+4n-2} \end{pmatrix}$$

  Show that the sum of the elements of the first and second row denoted by R_1 and R_2 respectively can be expressed as

  $$R_1 = 2F_{2n}L_{2n+k}$$

  $$R_2 = F_{2n}L_{2n+k+1}$$

  where $\{L_n, n \geq 1\}$ denotes the Lucas sequence with $L_1 = 1, L_2 = 3$ and $L_{i+2} = L_i + L_{i+1}$ for $i \geq 1$ and $\{F_n, n \geq 1\}$ denotes the Fibonacci sequence,
  $F_1 = 1, F_2 = 1, F_{n+2} = F_n + F_{n+1}$.

- 5047: *Proposed by David C. Wilson, Winston-Salem, N.C.*

  Find a procedure for continuing the following pattern:

$$\begin{aligned}
S(n,0) &= \sum_{k=0}^{n}\binom{n}{k} = 2^n\\
S(n,1) &= \sum_{k=0}^{n}\binom{n}{k}k = 2^{n-1}n\\
S(n,2) &= \sum_{k=0}^{n}\binom{n}{k}k^2 = 2^{n-2}n(n+1)\\
S(n,3) &= \sum_{k=0}^{n}\binom{n}{k}k^3 = 2^{n-3}n^2(n+3)\\
&\vdots
\end{aligned}$$

- 5048: *Proposed by Paolo Perfetti, Mathematics Department, University "Tor Vergata," Rome, Italy.*

  Let $a, b, c,$ be positive real numbers. Prove that

$$\sqrt{c^2(a^2+b^2)^2+b^2(c^2+a^2)^2+a^2(b^2+c^2)^2} \;\geq\; \frac{54}{(a+b+c)^2}\frac{(abc)^3}{\sqrt{(ab)^4+(bc)^4+(ca)^4}}.$$

- 5049: *Proposed by José Luis Díaz-Barrero, Barcelona, Spain.*

  Find a function $f:\Re\to\Re$ such that

$$2f(x)+f(-x)=\begin{cases}-x^3-3, & x\leq 1,\\ 3-7x^3, & x>1.\end{cases}$$

Solutions

- 5026: *Proposed by Kenneth Korbin, New York, NY.*

  Given quadrilateral $ABCD$ with coordinates $A(-3,0),\ B(12,0),\ C(4,15)$, and $D(0,4)$. Point P has coordinates $(x,3)$. Find the value of x if

$$\text{area }\triangle\text{PAD}+\text{area }\triangle\text{PBC}=\text{area }\triangle\text{PAB}+\text{area }\triangle\text{PCD}. \qquad (1)$$

Solution by Bruno Salgueiro Fanego, Viveiro, Spain.

$$(1)\quad\Leftrightarrow\quad \frac{1}{2}\left|det\begin{pmatrix}x&3&1\\-3&0&1\\0&4&1\end{pmatrix}\right|+\frac{1}{2}\left|det\begin{pmatrix}x&3&1\\12&0&1\\4&15&1\end{pmatrix}\right|$$

$$+\frac{1}{2}\left|det\begin{pmatrix} x & 3 & 1 \\ -3 & 0 & 1 \\ 12 & 0 & 1 \end{pmatrix}\right| + \frac{1}{2}\left|det\begin{pmatrix} x & 3 & 1 \\ 4 & 15 & 1 \\ 0 & 4 & 1 \end{pmatrix}\right|$$

$$\Leftrightarrow \quad |-4x-3| \ + \ |156-15x| \ = \ 45 \ + \ |11x+4|. \qquad (2)$$

If $x \leq \dfrac{-3}{4}$, then (2) $\Leftrightarrow -4x-3-15x+156 = 45-11x-4 \Leftrightarrow x = 14$, impossible.
If $\dfrac{-3}{4} < x \leq \dfrac{-4}{11}$, then (2) $\Leftrightarrow 4x+3-15x+156 = 45-11x-4 \Leftrightarrow x = 159 = 41$, impossible.
If $\dfrac{-4}{11} < x \leq \dfrac{52}{5}$, then (2) $\Leftrightarrow 4x+3-15x+156 = 45+11x+4 \Leftrightarrow x = 5$.
If $x > \dfrac{52}{5}$, then (2) $\Leftrightarrow 4x+3+15x-156 = 45+11x+4 \Leftrightarrow x = \dfrac{101}{4}$.
Thus, there are two possible values of $x:\ x = 5$ and $\dfrac{101}{4}$.

Also solved by Brian D. Beasley, Clinton, SC; Elsie M. Campbell, Dionne T. Bailey, and Charles Diminnie (jointly), San Angelo, TX; Mark Cassell (student, St. George's School), Spokane, WA; Grant Evans (student, St. George's School), Spokane, WA; John Hawkins and David Stone (jointly), Statesboro, GA; Peter E. Liley, Lafayette, IN; Paul M. Harms, North Newton, KS; Charles, McCracken, Dayton, OH; John Nord, Spokane, WA; Boris Rays, Chesapeake, VA; Britton Stamper, (student, St. George's School), Spokane, WA; Vu Tran (student, Texas A&M University), College Station, TX, and the proposer.

- 5027: *Proposed by Kenneth Korbin, New York, NY.*

Find the x and y intercepts of

$$y = x^7 + x^6 + x^4 + x^3 + 1.$$

Solution by Paolo Perfetti, Mathematics Department, University "Tor Vergata," Rome, Italy.

The point $(0,1)$ is trivial. To find the x intercept we decompose $x^7+x^6+x^4+x^3+1 = (x^4+x^3+x^2+x+1)(x^3-x+1)$ and the value we are looking for is given by $x^3-x+1=0$ since

$$x^4+x^3+x^2+x+1 = (x^2 - x\frac{-1+\sqrt{5}}{2}+1)(x^2 - x\frac{-1-\sqrt{5}}{2}+1) \neq 0.$$

Applying the formula for solving cubic equations, the only real root of $x^3-x+1=0$ is

$$\left(-\frac{1}{2}+\sqrt{\frac{1}{4}-\frac{1}{27}}\right)^{1/3} + \left(-\frac{1}{2}-\sqrt{\frac{1}{4}-\frac{1}{27}}\right)^{1/3} = \left(-\frac{1}{2}+\sqrt{\frac{69}{18}}\right)^{1/3} + \left(-\frac{1}{2}-\sqrt{\frac{69}{18}}\right)^{1/3}$$

whose approximate value is $-1.3247\ldots$

Also solved by Brian D. Beasley, Clinton, SC; Mark Cassell and Britton Stamper (jointly, students at St. George's School), Spokane, WA; Michael

Fried, Kibbutz Revivim, Israel; Paul M. Harms, North Newton, KS; John Hawkins and David Stone (jointly), Statesboro, GA; Charles McCracken, Dayton, OH; John Nord, Spokane, WA; Boris Rays, Chesapeake, VA, and the propser.

- 5028: *Proposed by Michael Brozinsky, Central Islip, NY .*

If the ratio of the area of the square inscribed in an isosceles triangle with one side on the base to the area of the triangle uniquely determine the base angles, find the base angles.

Solution 1 by Brian D. Beasley, Clinton, SC.

Let θ be the measure of each base angle in the triangle, and let y be the length of each side opposite a base angle. Let x be the side length of the inscribed square. We first consider the right triangle formed with θ as an angle and x as a leg, denoting its hypotenuse by z. Then $x = z\sin\theta$. Next, we consider the isosceles triangle formed with the top of the inscribed square as its base; taking the right half of the top of the square as a leg, we form another right triangle with angle θ and hypotenuse $y - z$. Then $\frac{1}{2}x = (y-z)\cos\theta$, so $y = x(\csc\theta + \frac{1}{2}\sec\theta)$. Denoting the area of the square by S and the area of the original triangle by T, we have

$$\frac{T}{S} = \frac{\frac{1}{2}y^2\sin(\pi - 2\theta)}{x^2} = \frac{1}{2}\sin(2\theta)\left(\csc\theta + \frac{1}{2}\sec\theta\right)^2 = \frac{1}{4}\tan\theta + \cot\theta + 1.$$

Let $f(\theta) = \frac{1}{4}\tan\theta + \cot\theta + 1$ for $0 < \theta < \pi/2$. Then it is straightforward to verify that

$$\lim_{\theta\to 0^+} f(\theta) = \lim_{\theta\to \frac{\pi}{2}^-} f(\theta) = \infty$$

and that f attains an absolute minimum value of 2 at $\theta = \arctan(2)$. Hence the ratio T/S (and thus S/T) is uniquely determined when $\theta = \arctan(2) \approx 63.435°$.

Solution 2 by J. W. Wilson, Athens, GA.

With no loss of generality, let the base of the isosceles triangle b be a fixed value and vary the height h of the triangle. Then if $f(h)$ is a function giving the ratio for the compared areas, in order for it to uniquely determine the base angles, there must be either a minimum or maximum value of the function. Let $f(h)$ represent the ratio of the area of the triangle to the area of the square.

It is generally known (and easy to show) that side s of an inscribed square on base b of a triangle is on-half of the harmonic mean of the base b and the altitude h to that base. Thus

$$\begin{aligned} s &= \frac{hb}{h+b}. \text{ So,} \\ f(h) &= \frac{bh}{2s^2}. \text{ Substituting and simplifying this gives :} \\ f(h) &= \frac{h^2+2bh+b^2}{2bh}. \end{aligned}$$

For $h > 0$ it can be shown, by using the arithmetic mean−geometric mean inequality, that this function has a minimum value of 2 when $h = b$.

$$f(h) = \frac{h^2+2bh+b^2}{2bh}$$

$$= \frac{h + 2b + \dfrac{b^2}{h}}{2b}.$$

Since b is fixed, and using the arithmetic mean−geometric mean inequaltiy, we may write:

$$h + \frac{b^2}{h} \geq 2\sqrt{h\frac{b^2}{h}} = 2b, \text{ with equality holding if, and only if,}$$

$$h = \frac{b^2}{h}.$$

Therefore $f(h)$ reaches a maximum if, and only if, $h = b$. This means the base angles can be uniquely determined when the altitude and the base are the same length. Thus, by considering the right triangle formed by the altitude and the base, the base angle would be arctan 2.

Also solved by Bruno Salgueiro Fanego, Viveiro, Spain; John Hawkins and David Stone (jointly; two solutions), Statesboro, GA; Peter E. Liley, Lafayette, IN; Kenneth Korbin, New York, NY; John Nord, Spokane, WA; Boris Rays, Chesapeake, VA, and the proposer.

- 5029: *Proposed by José Luis Díaz-Barrero, Barcelona, Spain.*

  Let $x > 1$ be a non-integer number. Prove that

$$\left(\frac{x + \{x\}}{[x]} - \frac{[x]}{x + \{x\}}\right) + \left(\frac{x + [x]}{\{x\}} - \frac{\{x\}}{x + [x]}\right) > \frac{9}{2},$$

where $[x]$ and $\{x\}$ represents the entire and fractional part of x.

Solution by John Hawkins and David Stone, Statesboro, GA.

We improve the lower bound by verifying the more accurate inequality

$$\# \quad \left(\frac{x + \{x\}}{[x]} - \frac{[x]}{x + \{x\}}\right) + \left(\frac{x + [x]}{\{x\}} - \frac{\{x\}}{x + [x]}\right) > \frac{16}{3}.$$

In fact, $\frac{16}{3}$ is a sharp lower bound for $\left(\frac{x + \{x\}}{[x]} - \frac{[x]}{x + \{x\}}\right) + \left(\frac{x + [x]}{\{x\}} - \frac{\{x\}}{x + [x]}\right)$ for x in the interval $(1, 2)$, while this expression becomes much larger for larger x.

For convenience, we let

$$f(x) = \left(\frac{x + \{x\}}{[x]} - \frac{[x]}{x + \{x\}}\right) + \left(\frac{x + [x]}{\{x\}} - \frac{\{x\}}{x + [x]}\right).$$

The function f, defined for $x > 1$, x not an integer, has a"repetitive" behavior. Its graph has a vertical asymptote at each positive integer. On each interval $(n, n+1)$, $f(x)$ decreases(strictly) from infinity to a specific limit, h_n (which we will specify), then repeats the behavior on the next interval, but does not drop down as far, because $h_n < h_{n+1}$ (so $f(x)$ never comes close to $h_1 = \frac{16}{3}$ again.)

We verify these statements by fixing n and examining the behavior on $f(x)$ on the interval $(n, n+1)$. In this case, we let $x = n + t$, where $0 < t < 1$; therefore, $[x] = n$ and $|x| = t$. Thus

$$\begin{aligned} f(x) &= \left(\frac{n+t+t}{n} - \frac{n}{n+t+t}\right) + \left(\frac{n+t+n}{t} - \frac{t}{n+t+n}\right) \\ &= \frac{n+2t}{n} - \frac{n}{n+2t} + \frac{2n+t}{t} - \frac{t}{2n+t}. \end{aligned}$$

We handle the above claims in order:
(1) $\lim_{t\to 0^+} f(x) = \lim_{t\to 0^+} \frac{n+2t}{n} - \frac{n}{n+2t} + \frac{2n+t}{t} - \frac{t}{2n+t} = +\infty.$
(2) Because $f(x)$ has been expressed in terms of t, say

$$g(t) = \frac{n+2t}{n} - \frac{n}{n+2t} + \frac{2n+t}{t} - \frac{t}{2n+t},$$

we can show that $g(t)$ is decreasing by showing its derivative is negative.

We compute the derivative with respect to t:

$$g'(t) = \frac{2}{n} + \frac{2n}{(2t+n)^2} - \frac{2n}{t^2} - \frac{2n}{(t+2n)^2}.$$

Basically, this is negative because of the dominant term $\frac{-2n}{t^2}$, but we can make this more precise:

$$\begin{aligned} g'(t) &< 0 \\ \Leftrightarrow\ & \frac{2}{n} + \frac{2n}{(2t+n)^2} - \frac{2n}{t^2} - \frac{2n}{(t+2n)^2} < 0 \\ \Leftrightarrow\ & \frac{1}{n} + \frac{n}{(2t+n)^2} < \frac{n}{t^2} + \frac{n}{(t+2n)^2} \\ \Leftrightarrow\ & \frac{(2t+n)^2 + n^2}{n(2t+n)^2} < \frac{n(t+2n)^2 + nt^2}{t^2(t+2n)^2} \\ \Leftrightarrow\ & t^2(t+2n)^2\left[(2t+n)^2 + n^2\right] < n(2t+n)^2\left[n(t+2n)^2 + nt^2\right] \\ \Leftrightarrow\ & t^2(t+2n)^2\left[(2t^2 + 2tn + n^2\right] < n^2(2t+n)^2\left[t^2 + 2tn + 2n^2\right] \\ \Leftrightarrow\ & 2t^6 + 10t^5n + 17t^4n^2 + 12t^3n^3 + 4t^2n^4 < 2n^6 + 10n^5t + 17n^4t^2 + 12n^3t^3 + 4n^2t^4 \\ \Leftrightarrow\ & 0 < 2\left(n^6 - t^6\right) + 10nt\left(n^4 - t^4\right) + 17n^2t^2\left(n^2 - t^2\right) - 4n^2t^2\left(n^2 - t^2\right) \\ \Leftrightarrow\ & 0 < 2\left(n^6 - t^6\right) + 10nt\left(n^4 - t^4\right) + 13n^2t^2\left(n^2 - t^2\right), \end{aligned}$$

and this last inequality is true because $0 < t < 1 < n$.

(3) Finally, we compute the lower bound at the right-hand endpoint:

$$\lim_{t\to 1^-} f(x) = \lim_{t\to 1^-}\left[\frac{n+2t}{n} - \frac{n}{n+2} + \frac{2n+t}{t} - \frac{t}{2n+t}\right]$$

$$
\begin{aligned}
&= \frac{n+2}{n} - \frac{n}{n+2} + \frac{2n+1}{1} - \frac{1}{2n+1} \\
&= 2n+1 - \frac{1}{2n+1} + \frac{4(n+1)}{n(n+2)}.
\end{aligned}
$$

Thus, we see that $h_n = 2n+1 - \dfrac{1}{2n+1} + \dfrac{4(n+1)}{n(n+2)} \approx 2n+1$, so the intervals' lower bounds increase linearly with n.

Note that $h_1 = 3 + \dfrac{7}{3} = \dfrac{16}{3}$, so $f(x) > \dfrac{16}{3}$ for $1 < x < 2$. So inequality (#) has been verified.

As stated above, the lower bound on x then grows, for instance,

$h_2 = 5 + \dfrac{13}{10} = \dfrac{63}{10}$, so $f(x) > \dfrac{63}{10}$ for $2 < x < 3$,

and

$h_3 = 7 + \dfrac{97}{105} = \dfrac{832}{105}$, so $f(x) > \dfrac{832}{105}$ for $3 < x < 4$.

Comment: The inequality # is sharp in the sense that no value larger than $\dfrac{16}{3}$ can be used. That is, by (3) above, we know that values of x very close to 2 produce values of $f(x)$ just above and arbitrarily close to $\dfrac{16}{3}$. We can see this precisely:

$$
\begin{aligned}
f\left(2 - \frac{1}{m}\right) &= f\left(1 + \frac{m-1}{m}\right) \\
&= \left(\frac{\dfrac{2m-1}{m} + \dfrac{m-1}{m}}{1} - \frac{1}{\dfrac{2m-1}{m} + \dfrac{m-1}{m}} \right) + \left(\frac{\dfrac{2m-1}{m} + 1}{\dfrac{m-1}{m}} - \frac{\dfrac{m-1}{m}}{\dfrac{2m-1}{m} + 1} \right) \\
&= \frac{3m-2}{m} - \frac{m}{3m-2} + \frac{3m-1}{m-1} - \frac{m-1}{3m-1} \\
&= \frac{16}{3} + \frac{2}{3}\left[\frac{3}{m(m-1)} - \frac{1}{3(m-1)(3m-2)}\right].
\end{aligned}
$$

(John and David accompanied their above solution with a graph generated by Maple showing how the lower bounds increase from $\dfrac{16}{3}$ for various values of x.)

Also solved by Bruno Salgueiro Fanego, Viveiro, Spain; Paul M. Harms, North Newton, KS; Paolo Perfetti, Mathematics Department, University "Tor Vergata," Rome, Italy; Vu Tran (student, Texas A&M University), College Station, TX; Boris Rays, Chesapeake, VA, and the proposer.

- 5030: *Proposed by José Luis Díaz-Barrero, Barcelona, Spain.*

  Let $A_1, A_2, \cdots, A_n \in M_2(\mathbf{C}), (n \geq 2)$, be the solutions of the equation $X^n = \begin{pmatrix} 2 & 1 \\ 6 & 3 \end{pmatrix}$.

Prove that $\sum_{k=1}^{n} Tr(A_k) = 0$.

Solution by John Hawkins and David Stone, Statesboro, GA.

The involvement of the Trace function is a red herring. Actually, for $A_1, A_2, A_3, \ldots, A_n$ as specified in the problem, we have $\sum_{k=1}^{n} A_k = 0$. Therefore, since Tr is linear, $\sum_{k=1}^{n} Tr(A_k) = Tr\left(\sum_{k=1}^{n}(A_k\right) = Tr(0) = 0$. In fact $\sum_{k=1}^{n} T r(A_k) = 0$ for any linear transformation $T : M_2(C) \longrightarrow W$ to any complex vector space W.

Here is our argument. For convenience, let $B = \begin{pmatrix} 2 & 1 \\ 6 & 3 \end{pmatrix}$. Note that $B^2 = 5B$. Thus $B^3 = BB^2 = B5B = 5B^2 = 5^2B$. Inductively, $B^k = 5^{k-1}B$ for $k \geq 1$.

Therefore, $B = \frac{1}{5^{n-1}}B^n = \left[\frac{1}{5^{(n-1)/n}}B\right]^n$, so $\mathrm{A}_1 = \frac{1}{5^{(n-1)/n}}B$ is an n^{th} root of B:

$$A_1^n = \left[\frac{1}{5^{(n-1)/n}}B\right]^n = \frac{1}{5^{n-1}}B^n = \frac{1}{5^{n-1}}5^{n-1}B = B.$$

Now let $\xi = e^{2\pi i/n}$ be the primitive n^{th} root of unity. Then

$$0 = \xi^n - 1 = (\xi - 1)(\xi^{n-1} + \xi^{n-2} + \xi^{n-3} + \cdots + \xi + 1),$$

so,

$$(\#) \qquad (\xi^{n-1} + \xi^{n-2} + \xi^{n-3} + \cdots + \xi + 1) = 0.$$

With $A_1 = \frac{1}{5^{(n-1)/n}}B$ as above, let $A_k = \xi^{k-1}A_1$ for $k = 2, 3, \ldots, n$. These n distinct matrices are the n^{th} roots of B, namely:

$$A_k^n = [\xi^{k-1}A_1]^n = \xi^{(k-1)n}A_1^n = (\xi^n)^{k-1}A_1^n = 1^{k-1}A_1^n = A_1^n = B.$$

Therefore,

$$\begin{aligned} \sum_{k=1}^{n} A_k &= \sum_{k=1}^{n} \xi^{k-1}A_1 = \left(\sum_{k=1}^{n} \xi^{k-1}\right)A_1 \\ &= 0 \cdot A_1 \text{ by } (\#) \\ &= 0. \end{aligned}$$

Comment 1: Implicit in the problem statement is that the given matrix equation has exactly n solutions. This is true for this particular matrix B. But it is not true in general. Gantmacher ("Matrix Theory", page 233) gives an example of a 3×3 matrix with infinitely many square roots: $\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$.

Comment 2: The result would be true for B any 2×2 matrix having determinant zero but trace non-zero. In that case, we would have $B^2 = Tr(B)B$ and we use $A_1 = \dfrac{1}{Tr(B)^{(n-1)/n}}B.$

Comment 3: More generally, let V be a vector space over C and $c_1, c_2, \ldots, c_n$ be complex scalars whose sum is zero. Also let A be any vector in V and let $A_k = c_k A$ for $k = 1, 2, \cdots, n$. Then

$$\sum_{k=1}^{n} A_k = \sum_{k=1}^{n} c_k A = \Big(\sum_{k=1}^{n} c_k\Big) A = 0 \cdot A = 0.$$

Also solved by Bruno Salgueiro Fanego, Viveiro, Spain, and the proposer.

- 5031: *Ovidiu Furdui, Toledo, OH.*

  Let x be a real number. Find the sum

$$\sum_{n=1}^{\infty}(-1)^{n-1}n\Big(e^x - 1 - x - \frac{x^2}{2!} - \cdots - \frac{x^n}{n!}\Big).$$

Solution 1 by Paul M. Harms, North Newton, KS.

We know that $e^x = 1 + x + \dfrac{x^2}{2!} + \dfrac{x^3}{3!} + \cdots + \dfrac{x^n}{n!} + \cdots$.

The expression
$(-1)^{n-1}n\Big(e^x - 1 - x - \dfrac{x^2}{2!} - \cdots - \dfrac{x^n}{n!}\Big) = (-1)^{n-1}n\Big(\dfrac{x^{n+1}}{(n+1)!} + \dfrac{x^{n+2}}{(n+2)!} + \cdots\Big)$.

So the sum $\displaystyle\sum_{n=1}^{\infty}(-1)^{n-1}n\Big(\frac{x^{n+1}}{(n+1)!} + \frac{x^{n+2}}{(n+2)!} + \cdots\Big)$ equals

$$\begin{aligned}
& \Big(\frac{x^2}{2!} + \frac{x^3}{3!} + \cdots\Big) - 2\Big(\frac{x^3}{3!} + \frac{x^4}{4!} + \cdots\Big) + 3\Big(\frac{x^4}{4!} + \frac{x^5}{5!} + \cdots\Big) - 4\Big(\frac{x^5}{5!} + \cdots\Big) + \cdots \\
= & \frac{(1)x^2}{2!} + \frac{(1-2)x^3}{2!} + \frac{(1-2+3)x^4}{4!} + \frac{(1-2+3-4)x^5}{5!} + \cdots \\
= & \frac{x^2}{2!} - \frac{x^3}{3!} + \frac{2x^4}{4!} - \frac{2x^5}{5!} + \frac{3x^6}{6!} - \frac{3x^7}{7!} + \frac{4x^8}{8!} - \frac{4x^9}{9!} \cdots
\end{aligned}$$

We need to find the sum of this alternating series..

We have

$$\begin{aligned}
\sinh x &= x + \frac{x^3}{3!} + \frac{x^5}{5!} + \cdots\cdots \\
\frac{x}{2}\sinh x &= \frac{1}{2}x^2 + \frac{\frac{2}{4}x^4}{3!} + \frac{\frac{3}{6}x^6}{5!} + \frac{\frac{4}{8}x^8}{7!} + \cdots \\
&= \frac{x}{2!} + \frac{2x^4}{4!} + \frac{3x^6}{6!} + \frac{4x^8}{8!} + \cdots.
\end{aligned}$$

The positive terms of the alternating series sum to $\dfrac{x}{2}\sinh x$. Each negative term of the alternating series is an antiderivative of the previous term except for the minus sign. The general anitderivative of $\frac{x}{2}\sinh x$ is $\frac{1}{2}\Big[x\cosh x - \sinh x\Big] + C$. Using Taylor series we can show that $\frac{-1}{2}\Big[x\cosh x - \sinh x\Big]$ equals the sum of the negative terms of the alternating series. The sum in the problem is

$$\frac{x}{2}\sinh x - \frac{1}{2}\Big[x\cosh x - \sinh x\Big] = \frac{x+1}{2}\sinh x - \frac{x}{2}\cosh x.$$

Solution 2 by N. J. Kuenzi, Oshkosh, WI.

Let

$$F(x) = \sum_{n=1}^{\infty}(-1)^{n-1}n\left(e^x - 1 - x - \frac{x^2}{2!} - \cdots - \frac{x^n}{n!}\right).$$

Differentiation yields

$$\begin{aligned}
F'(x) &= \sum_{n=1}^{\infty}\left((-1)^{n-1}n(e^x - 1 - x - \cdots - \frac{x^{n-1}}{(n-1)!}\right) \\
&= \sum_{n=1}^{\infty}\left((-1)^{n-1}n(e^x - 1 - x - \cdots - \frac{x^{n-1}}{(n-1)!} - \frac{x^n}{n!} + \frac{x^n}{n!}\right) \\
&= F(x) + \sum_{n=1}^{\infty}(-1)^{n-1}n\frac{x^n}{n!} \\
&= F(x) + x\left(1 - x + \frac{x^2}{2!} - \frac{x^3}{3!} + \cdots + (-1)^m\frac{x^m}{m!} + \cdots\right) \\
&= F(x) + xe^{-x}.
\end{aligned}$$

Solving the differential equation

$$\begin{aligned}
F'(x) &= F(x) + xe^{-x} \text{ with initial conditions F(0) = 0 yields} \\
F(x) &= \frac{e^x - (1+2x)e^{-x}}{4}.
\end{aligned}$$

Also solved by Charles Diminnie and Andrew Siefker (jointly), San Angelo, TX; Bruno Salgueiro Fanego, Viveiro, Spain; Paolo Perfetti, Mathematics Department, University "Tor Vergata," Rome, Italy, and the proposer.