# Problems

Ted Eisenberg, Section Editor

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This section of the Journal offers readers an opportunity to exchange interesting mathematical problems and solutions. Please send them to Ted Eisenberg, Department of Mathematics, Ben-Gurion University, Beer-Sheva, Israel or fax to: 972-86-477-648. Questions concerning proposals and/or solutions can be sent e-mail to <eisenbt@013.net>. Solutions to previously stated problems can be seen at <http://www.ssma.org/publications>.

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*Solutions to the problems stated in this issue should be posted before March 15, 2016*

- **5379:** *Proposed by Kenneth Korbin, New York, NY*

  Solve:
$$\frac{(x+1)^4}{(x-1)^2} = 17x.$$

- **5380:** *Proposed by Arkady Alt, San Jose, CA*

  Let $\Delta(x,y,z) = 2(xy+yz+xz) - (x^2+y^2+z^2)$ and $a,b,c$ be the side-lengths of a triangle $ABC$. Prove that
$$F^2 \geq \frac{3}{16} \cdot \frac{\Delta(a^3,b^3,c^3)}{\Delta(a,b,c)},$$
  where $F$ is the area of $\triangle ABC$.

- **5381:** *Proposed by D.M. Batinetu-Giurgiu,"Matei Basarab" National College, Bucharest, and Neculai Stanciu "George Emil Palade" School, Buzău, Romania*

  Prove: In any acute triangle ABC, with the usual notations, holds:
$$\sum_{cyclic} \left(\frac{\cos A \cos B}{\cos C}\right)^{m+1} \geq \frac{3}{2^{m+1}},$$
  where $m \geq 0$ is an integer number.

- **5382:** *Proposed by* Ángel Plaza, University of Las Palmas de Gran Canaria, Spain

  Prove that if $a,b,c$ are positive real numbers, then
$$\left(\sum_{cyclic}\frac{a}{b} + 8\sum_{cyclic}\frac{b}{a}\right)\left(\sum_{cyclic}\frac{b}{a} + 8\sum_{cyclic}\frac{a}{b}\right) \geq 9^3.$$

- **5383:** *Proposed by José Luis Díaz-Barrero, Barcelona Tech, Barcelona, Spain*

Let $n$ be a positive integer. Find $\gcd(a_n, b_n)$, where $a_n$ and $b_n$ are the positive integers for which $(1-\sqrt{5})^n = a_n - b_n\sqrt{5}$.

- **5384:** *Proposed by Ovidiu Furdui, Technical University of Cluj-Napoca, Cluj-Napoca, Romania*

  Find all differentiable functions $f : \Re \to \Re$ which verify the functional equation

$$xf'(x) + f(-x) = x^2, \quad \text{for all} \quad x \in \Re.$$

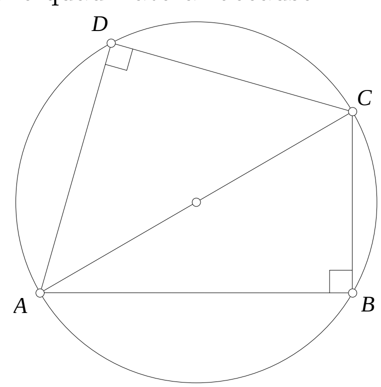
*Solutions*

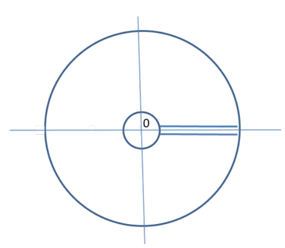
- **5361:** *Proposed by Kenneth Korbin, New York, NY*

  Convex quadrilateral $ABCD$ has perimeter $P = 75 + 61\sqrt{15}$ and has $\angle B = \angle D = 90°$. The lengths of the diagonals are 112 and 128. Find the lengths of the sides.

  **Solution by Ercole Suppa, Teramo, Italy**

  Observe that $ABCD$ is a cyclic quadrilateral because $\angle B = \angle D = 90°$.




Denote $AB = a$, $BC = b$, $CD = c$, $DA = d$. By the Pythagorean theorem applied to triangles $ABC$, $ACD$ and Ptolemy's theorem applied to the quadrilateral $ABCD$ we have

$$a^2+b^2=c^2+d^2=128^2 \tag{1}$$
$$ac+bd=112\cdot 128 \tag{2}$$

Taking into account of (1) and (2) we obtain

$$\begin{gathered}
(a+b+c+d)^2=\left(75+61\sqrt{15}\right)^2 \qquad \Leftrightarrow \\
a^2+b^2+c^2+d^2+2(ab+ac+ad+bc+bd+cd)=61440+9150\sqrt{15} \qquad \Leftrightarrow \\
2\cdot 128^2+2(ab+ac+ad+bc+bd+cd)=61440+9150\sqrt{15} \qquad \Leftrightarrow \\
ac+bd+(a+c)(b+d)=14336+4575\sqrt{15} \qquad \Leftrightarrow \\
(a+c)(b+d)=4575\sqrt{15}
\end{gathered}$$

Putting $a+c=x$, $b+d=y$ we have

$$\begin{cases} x+y=75+61\sqrt{15} \\ xy=4575\sqrt{15} \end{cases}$$

from which, after some algebra, we find $(x,y)=\left(75,61\sqrt{15}\right)$ or $(x,y)=\left(61\sqrt{15},75\right)$. Finally, solving the system

$$\begin{cases} a+c=75 \\ b+d=61\sqrt{15} \\ a^2+b^2=128^2 \\ c^2+d^2=128^2 \end{cases}$$

we get $(a,b,c,d)=\left(7,33\sqrt{15},68,28\sqrt{15}\right)$, $\left(33\sqrt{15},7,28\sqrt{15},68\right)$, $\left(68,28\sqrt{15},7,33\sqrt{15},\right)$ or $\left(28\sqrt{15},68,33\sqrt{15},7\right)$, and the proof is completed.

**Also solved by Hatef I. Arshagi, Guilford Technical Community College, Jamestown, NC; Ed Gray, Highland Beach, FL; Kee-Wai Lau, Hong Kong, China; Gail Nord, Gonzaga University, Spokane, WA; Prishtina Math Gymnasium Problem Solving Group, Republic of Kosova; Toshihiro Shimizu, Kawasaki, Japan; Neculai Stanciu, "George Emil Palade" General School, Buzău, Romania and Titu Zvonaru, Comănesti, Romania; David Stone and John Hawkins, Georgia Southern University, Statesboro, GA.**

- **5362:** *Proposed by Michael Brozinsky, Central Islip, NY*

Two thousand forty seven death row prisoners were arranged from left to right with the numbers 1 through 2047 on their backs in this left to right order. Prisoner 1 was given a gun and shoots prisoner number 2 dead, and then gives the gun to prisoner number 3 who shoots prisoner number 4 and then gives the gun to number 5 and so on, so that every second originally numbered prisoner is shot dead.

This process is then repeated from right to left, starting with the person (in this case number 2047) who last received the gun and then continues to proceed from right to left, and then the direction switches again, and then again until only one prisoner remains standing. What is the number of the prisoner who survives the left to right, right to left shootout? Note that if there had been 2048 prisoners, number 2047 would have no one to whom to hand the gun in the left to right direction after shooting number 2048, and so he would then start the gun in its opposite direction shooting the living prisoner to his immediate left i.e.,number 2045. In this case, number 2047 gets to shoot two prisoners before he hands the gun off to another prisoner.

**Solution 1 by Ashland University Undergraduate Problem Solving Group, Ashland, OH**

Let $a(n) =$ the number of the prisoner who survives when $n$ prisoners are in line. It is given in the problem that $a(2048) = a(2047)$, and from the explanation given, we can similarly conclude that $a(2k) = a(2k-1)$. We can also see that the prisoner left standing for $a(2k+1)$ is the $a(k+1)^{st}$ odd-numbered prisoner from the right end of the line since only odd numbers survived the first gun pass through the line. this gives the relation

$$a(2k+1) = 2k+1-2\left[a(k+1)-1\right] = 2k+3-2a(k+1).$$

From this we can see that

$$a(2^m) = a(2^m-1) = (2^m-1)-2\left[a(2^{m-1}-1+1)-1\right] = 2^m+1-2a(2^{m-1}).$$

We can then solve for an explicit formula using iteration.

$$\begin{aligned}
a(2^m) &= a(2^m-1) = 2^m+1-2a(2^{m-1}) \\
&= 2^m+1-2\left(2^{m-1}+1-2a(2^{m-2})\right) \\
&= 2^m+1-2\left(2^{m-1}+1-2\left[2^{m-2}+1-2a(2^{m-3})\right]\right) \\
&= 2^m+1-2\left(2^{m-1}+1-2\left[2^{m-2}+1-2\left(2^{m-3}+1-2a\left(2^{m-4}\right)\right)\right]\right).
\end{aligned}$$

So if we regroup these equations,

$$\begin{aligned}
a(2^m) &= (a^{2m}-1) = 2^m+1-2a(2^{m-1}) \\
&= (2^m-2^m)+(1-2)+2^2a(2^{m-2})a \\
&= (2^m-2^m+2^m)+(1-2+2^2)-2^3a(2^{m-3}) \\
&= (2^m-2^m+2^m-2\ ^m)+(1-2+2^2-2^3)+2^4a(2^{m-4}).
\end{aligned}$$

We can see that

$$\begin{aligned}
a(2^{2k}) &= 1\left(\frac{1-(-2)^{2k}}{1-(-2)}\right)+2^{2k}a(a^{2k-2k}) \\
&= \left(\frac{1-(-2)^{2k}}{3}\right)+2^{2k}a(1) \\
&= 2^{2k}+\left(\frac{1-2^{2k}}{3}\right)
\end{aligned}$$

$$
\begin{aligned}
&= \frac{3(2^{2k}) + (1 - 2^{2k})}{3} \\
&= \frac{2^{2k} + 1}{3} \\
&= \frac{2^{2k+1} + 1}{3}
\end{aligned}
$$

And

$$
\begin{aligned}
a(2^{2k+1}) &= 2^{2k+1} + 1 + 1\left(\frac{1 - (-2)^{2k+1}}{1 - (-2)}\right) - 2^{2k+1} a\left(2^{[2k+1]-[2k+1]}\right) \\
&= 2^{2k+1} + \left(\frac{1 + 2^{2k+1}}{3}\right) - 2^{2k+1} a(1) \\
&= 2^{2k+1} - 2^{2k+1} + \frac{1 + 2^{2k+1}}{3} \\
&= \frac{2^{2k+1} + 1}{3}
\end{aligned}
$$

So $a\left(2^{2k}\right) = a\left(2^{2k+1}\right) = \dfrac{2^{2k+1} + 1}{3}$, and $a(2047) = a(2048) = a\left(2^{11}\right) = \dfrac{2^{11} + 1}{3} = 683$.

**Solution 2 by Toshihiro Shimizu, Kawasaki, Japan**

We consider the case of $2^n$ prisoner. Let $f(n)$ be the index of the prisoner who remains alive. It's obvious that $f(0) = 1$. In the first left-to-right shootout, $2^{n-1}$ prisoners who were originally indexed as $1, 3, 5, \ldots, 2^n - 1$ are alive. Then, we reindex these prisoner as $2^{n-1}, 2^{n-1} - 1, \ldots, 2, 1$. So the prisoner with new-index $f(n-1)$ is alive. This prisoner is also the prisoner of original-index $f(n)$. Since the prisoner of new-index $i$ corresponds to original index $2^n + 1 - 2i$, it follows that $f(n) = 2^n + 1 - 2f(n-1)$. This relation is equivalent to

$$
f(n) - 2^{n-1} - \frac{1}{3} = -2\left(f(n-1) - 2^{n-2} - \frac{1}{3}\right).
$$

Therefore, $f(n) - 2^{n-1} - 1/3 = (-2)^n(f(0) - 2^{-1} - 1/3) = (-2)^n/6$ or
$f(n) = (-2)^n/6 + 2^{n-1} + 1/3$.

We consider the cases with 2047 prisoners and with 2048 prisoners. In the first left-to-right of the later case, the prisoner 2047 shoots 2048, while in the former case, the prisoner 2048 does not initially exist. Thus, in the both two cases, the original index of living prisoners are identical after first left-to-right movement. Thus, the prisoner who, in the end, remains alive, is also same. This prisoner is indexed $f(11) = 683$.

**Solution 3 by David E. Manes, SUNY College at Oneonta, NY**

At the end of the bloodbath, number 683 is the only prisoner standing and it takes ten stages to produce him.

Stage 1: Procedure goes from left to right. The odd numbered prisoners are alive and the even numbered ones are not. Therefore 1023 prisoners have been eliminated and 1024 prisoners are still alive.

Stage 2: Procedure goes from right to left starting with prisoner 2047. The prisoners' decreased are numbered $4k+1, 0 \leq k \leq 511$ while the prisoners alive are numbered $4k+3, 0 \leq k \leq 511$. There are now 512 prisoners alive.

Stage 3: Procedure goes from left to right starting with prisoner 3. Prisoners still alive after this stage are numbered $8k+7, 0 \leq k \leq 255$. There are now 256 prisoners alive.

Stage 4: Procedure goes from right to left starting with prisoner 2043. The prisoners dismissed after this stage have numbers $16k+3, 0 \leq k \leq 127$ and the prisoners still standing have numbers $16k+11, 0 \leq k \leq 127$. There are now 128 prisoners alive.

Stage 5 : Procedure goes from left to right starting with prisoner 11. After this stage the lifeless prisoners have numbers $32k+27, 0 \leq k \leq 63$ and the prisoners still alive have numbers $32k+11, 0 \leq k \leq 63$.

Stage 6: Procedure goes from right to left starting with prisoner 2027. Prisoners no longer playing are numbered $64k+11, 0 \leq k \leq 31$ and the prisoners still playing have numbers $64k+43, 0 \leq k31$.

Stage 7: Procedure goes from left to right starting with prisoner 43. After this stage the prisoners asked to leave are numbered $128k+107, 0 \leq k \leq 15$ and the prisoners still living have numbers $128k+43, 0 \leq k \leq 15$.

Stage 8: Procedure goes from right to left starting with prisoner 1963. After this stage the prisoners not breathing have numbers $256k+171, 0 \leq k \leq 7$.

Stage 9: Procedure goes from left to right starting with prisoner 171. After this stage the extinct prisoners have numbers $512k+427, 0 \leq k \leq 3$ and the prisoners still alive have numbers $512k+171,\ 0k \leq 171$. Prisoners no longer playing are numbered and the prisoners still playing have numbers $64k+43, 0 \leq k \leq 3$, that is, prisoners numbered 171, 683, 1195, and 1707,

Stage 10: Procedure goes from right to left starting with prisoner 1707. The deceased prisoners are numbered 1195 and 171. The only prisoners alive are 683 and 1707, but prisoner 683 has the loaded gun, hence the result.

**Solution 4 by Hatef I. Arshagi, Guilford Technical Community College, Jamestown, NC**

| *Order of shootout* | *Direction of shootout* | *Number of surviving inmates* | *Difference between numbers of two surviving inmates* | *Left − end, Right− end Surviving numbers* |
|---|---|---|---|---|
| 1 | $L \to R$ | 1024 | 2 | $1 - 2047$ |
| 2 | $L \leftarrow R$ | 512 | 4 | $3 - 2047$ |
| 3 | $L \to R$ | 256 | 8 | $3 - 2043$ |
| 4 | $L \leftarrow R$ | 128 | 16 | $11 - 2043$ |
| 5 | $L \to R$ | 64 | 32 | $11 - 2027$ |
| 6 | $L \leftarrow R$ | 32 | 64 | $43 - 2027$ |
| 7 | $L \to R$ | 16 | 128 | $43 - 1963$ |
| 8 | $L \leftarrow R$ | 8 | 256 | $171 - 1963$ |
| 9 | $L \to R$ | 4 | 512 | $171 - 1707$ |
| 10 | $L \leftarrow R$ | 2 | 1024 | $683 - 1707$ |
| 11 | $L \to R$ | 1 | | 683 |

The last surviving inmate has the number 683.

**Solution 5 by Carl Libis and Roland Depratti, Eastern Connecticut State University, Willimantic, CT**

Let $f(x)$ =number of the prisoner that survives when there are $x$ prisoners. Observe that

$$f(2^n+1) = \begin{cases} 1, \text{ if } n \ = 2,4,6,\cdots \\ 2^n+1, \text{ if } n, = 1,3,5,\cdots \end{cases}$$

$$f(2^n) \quad = \begin{cases} \dfrac{2^{n+1}}{3}, \text{ if } n \ = 2,4,6,\cdots \\ \dfrac{2^n+1}{3}, \text{ if } n = 1,3,5,\cdots \end{cases}$$

$$f(2^n-1) = \begin{cases} \dfrac{2^{n+1}}{3}, \text{ if } n \ = 2,4,6,\cdots \\ \dfrac{2^n+1}{3}, \text{ if } n = 1,3,5,\cdots \end{cases}$$

Thus $f(2047) = f(2^{11}-1) = 683$, so when there are 2047 prisoners, then prisoner number 683 will survive.

*Editor's comment:* **Ulrich Abel of Technische Hochschule Mittelhessen in Freiberg, Germany,** wrote that "this problem is a variant of the famous Josephus Problem (see; e.g. http://en.wikpedia.org/wiki/Josephusproblem) or the book Concrete Mathematics by Graham, Knuth and Patashnik."

**Also solved by Ed Gray, Highland Beach, FL; Prishtina Math Gymnasium Problem Solving Group, Republic of Kosova; David Stone and John Hawkins, Georgia Southern University, Statesboro, GA, and the proposer.**

- **5363:** *Proposed by D.M. Bătinetu-Giurgiu, "Matei Basarab" National College, Bucharest, Romania and Neculai Stanciu, "George Emil Palade" General School, Buzaău, Romania*

Let $x \in \Re$ and $A(x) = \begin{pmatrix} x+1 & 1 & 1 & 1 \\ 1 & x+1 & 1 & 1 \\ 1 & 1 & x+1 & 1 \\ 1 & 1 & 1 & x+1 \end{pmatrix}$.

Compute $A(0) \cdot A(x) \cdot A(y) \cdot A(z), \forall x,y,z \in \Re$.

**Solution 1 by Dionne Bailey, Elsie Campbell, and Charles Diminnie, Angelo State University, San Angelo, TX**

If

$$I_4 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

then $I_4 \cdot M = M \cdot I_4 = M$ for all $4 \times 4$ matrices $M$. Also, for all $t \in R$, it is easily seen that

$$A(t) = A(0) + tI_4$$

and

$$[A(0)]^2 = \begin{pmatrix} 4 & 4 & 4 & 4 \\ 4 & 4 & 4 & 4 \\ 4 & 4 & 4 & 4 \\ 4 & 4 & 4 & 4 \end{pmatrix} = 4A(0).$$

As a result, we have

$$\begin{aligned} A(0) \cdot A(x) &= A(0) \cdot [A(0) + xI_4] \\ &= [A(0)]^2 + xA(0) \\ &= (x+4)A(0) \end{aligned}$$

and

$$\begin{aligned} A(y) \cdot A(z) &= [A(0) + yI_4] \cdot [A(0) + zI_4] \\ &= [A(0)]^2 + (y+z)A(0) + yzI_4 \\ &= (y+z+4)A(0) + yzI_4. \end{aligned}$$

Therefore,

$$\begin{aligned} A(0) \cdot A(x) \cdot A(y) \cdot A(z) &= (x+4)A(0) \cdot [(y+z+4)A(0) + yzI_4] \\ &= (x+4)[4(y+z+4)A(0) + yzA(0)] \\ &= (x+4)(yz+4y+4z+16)A(0) \\ &= (x+4)(y+4)(z+4)A(0). \end{aligned}$$

**Solution 2 by Moti Levy, Rehovot, Israel**

$$A(x) = xI_4 + E_4,$$

where $I_4 := \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$ and $E_4 := \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{pmatrix}$.

$$\begin{aligned} A(0) \cdot A(x) \cdot A(y) \cdot A(z) &= E_4(E_4 + xI_4)(E_4 + yI_4)(E_4 + zI_4) \\ &= E_4^4 + (x+y+z)E_4^3 + (xy+xz+yz)E_4^2 + xyzE_4 \end{aligned}$$

$$\begin{aligned} E_4^2 &= 4E_4, \\ E_4^3 &= 16E_4 \\ E_4^3 &= 64E_4 \end{aligned}$$

$$\begin{aligned} A(0) \cdot A(x) \cdot A(y) \cdot A(z) &= E_4^4 + (x+y+z)E_4^3 + (xy+xz+yz)E_4^2 + xyzE_4 \\ &= (64 + 16(x+y+z) + 4(xy+xz+yz) + xyz)E_4 \\ &= (z+4)(y+4)(x+4)E_4. \end{aligned}$$

**Solution 3 by Paul M. Harms, North Newton, KS**

Computing $A(0), A(x)$, we obtain the value of $(x-4)$ for each element in the product. On the main diagonal of the product $A(x)A(z)$ we have
$(y+1)(z+1)+3 = yz+y+z+4$. The other elements have the value
$(y+1)+(z+1)+2 = y+z+4$. Then the product $A(0)\,[A(y)A(z)]$ has the value
$yz+y+z+4+3(y+z+4)$ for each element. This value is equal to
$yz+4y+4z+16 = (y+4)(z+4)$. The result of the computation requested in the problem is $(x+4)(y+4)(z+4)A(0)$ or a 4 by 4 matrix all of whose elements are $(x+4)(y+4)(z+4)$.

**Solution 4 by David Stone and John Hawkins of Georgia Southern University in Statesboro, GA**

$Editor's\ comment:$ The authors of this solution generalized the problem as follows:

Let $A(x)$ be $m$ instead of $4\times 4$ and we shall compute
$A(0)\cdot A(x_1)\cdot A(x_2)\cdot A(x_3)\cdots A(x_n)$, for $x_i \in \Re$.

Let $A$ be the $m\times m$ matrix $A = A(0)\ A(x) = \begin{vmatrix} 1 & 1 & 1 & . & . & 1 \\ 1 & 1 & 1 & . & . & 1 \\ . & . & . & . & . & . \\ . & . & . & . & . & . \\ . & . & . & . & . & . \\ 1 & 1 & 1 & . & . & 1 \end{vmatrix}$.

Then$A(x) = A + xI$ (where $I$ is the $m\times m$ identity matrix).

Lemma 1: $A^k = m^{k-1}A,\ k \ge 1$.
Proof: Certainly $A^1 = m^{1-1}A$ and an easy computation shows that $A^2 = mA = m^{2-1}A$.
Upon the obvious induction hypothesis,
$A^{k+1} = AA^k = A\left(m^{k-1}A\right) = m^{k-1}A^2 = m^{k-1}(mA) = m^kA,$ as desired.

Lemma 2: For any real $x,\ A\cdot A(x) = (m+x)A$.
Proof:

$$\begin{aligned} A\cdot A(x) = A\cdot(A+xI) &= A^2 + xA \\ &= mA + xA, \text{ by Lemma 1} \\ &= (m+x)A. \end{aligned}$$

Theorem: For $x_1, x_2, x_3, \ldots, x_n, \in \Re$ we have
$A(0)\cdot A(x_1)\cdot A(x_2)\cdot A(x_3)(x_n) = (m+x_1)(m+x_2)(m+x_3)\cdots(m+x_n)A$.

Proof: We proceed by induction on $n$.
For $n=1,\ A(0)\cdot A(x_1) = A\cdot A(x_1) = (m+x_1)A$ by Lemma 2.
Making the obvious induction hypothesis,

$$\begin{aligned} & A(0)\cdot A(x_1)\cdot A(x_2)\cdot A(x_3)\cdots A(x_{n+1}) \\ & \quad = \{A(0)\cdot A(x_1)\cdot A(x_2)\cdot A(x_3)\cdots A(x_n)\}\cdot A(x_{n+1}) \\ & \quad = \{(m+x_1)(m+x_2)(m+x_3)\cdots(m+x_n)A\}\cdot A(x_{n+1}) \\ & \quad = \{(m+x_1)(m+x_2)(m+x_3)\cdots(m+x_n)\}\cdot\{A\cdot A(x_{n+1})\} \end{aligned}$$

$$= \{(m+x_1)(m+x_2)(m+x_3)\cdots(m+x_n)\}\cdot\{(m+x_{n+1})\,A\} \text{ by Lemma 2}$$

$$= (m+x_1)\,(m+x_2)(m+x_3)\cdots(m+x_n)\cdot(m+x_{n+1})\,A, \text{ as desired.}$$

That is, $A\,(0)\cdot A\,(x_1)\cdot A\,(x_2)\cdot A\,(x_3)\cdots A\,(x_n)$ equals the $m\times m$ matrix

$$(m+x_1)(m+x_2)(m+x_3)\cdots(m+x_n)\begin{vmatrix} 1 & 1 & 1 & . & . & 1 \\ 1 & 1 & 1 & . & . & 1 \\ . & . & . & . & . & . \\ . & . & . & . & . & . \\ . & . & . & . & . & . \\ 1 & 1 & 1 & . & . & 1 \end{vmatrix}.$$

Norte. There are no concerns about non-commutativity in our algebra of matrices, because A commutes with powers of itself and with any scalar matrix c.

Note also that everything above remains true if we let all scalars come from an arbitrary ring with identity (instead of the reals).

**Also solved by Brian D. Beasley, Presbyterian College, Clinton, SC; David Diminnie and Michael Taylor, Texas Instruments Inc., Dallas, TX; Bruno Salgueiro Fanego, Viveiro, Spain; Ed Gray, Highland Beach, FL; Connor Greenhalgh (student, Eastern Kentucky University), Richmond, KY; G. C. Greubel, Newport News, VA; Carl Libis, Columbia Southern University, Orange Beach, AL; David E, Manes, SUNY College at Oneonta, NY; Gail Nord, Gonzaga University, Spokane, WA; Toshihiro Shimizu, Kawasaki, Japan; Morgan Wood (student, Eastern Kentucky University), Richmond, KY, and the proposer.**

- **5364:** *Proposed by Angel Plaza, Universidad de Las Palmas de Gran Canaria, Spain*

Prove that $\displaystyle\sum_{k=0}^{n}\binom{2n-2k}{n-k}\binom{2k}{k}4^{-n}=1.$

**Solution 1 by Henry Ricardo, New York Math Circle, NY**

The generating function of the central binomial coefficient is well known:

$$f(x) \;=\; \frac{1}{\sqrt{1-4x}} \;=\; \sum_{k=0}^{\infty}\binom{2k}{k}x^k.$$

Applying a standard theorem on the Cauchy product of two power series,

$$\left(\sum_{i=0}^{\infty}a_i x^i\right)\cdot\left(\sum_{j=0}^{\infty}b_j x^j\right) \;=\; \sum_{n=0}^{\infty}\left(\sum_{k=0}^{n}a_{n-k}b_k\right)x^n,$$

to $f^2(x)$ yields

$$\begin{aligned}\sum_{k=0}^{n}\binom{2n-2k}{n-k}\binom{2k}{k} &= \text{the coefficient of } x^n \text{ in } \left(\frac{1}{\sqrt{1-4x}}\right)^2 \\ &= \text{the coefficient of } x^n \text{ in } \frac{1}{1-4x} = 4^n,\end{aligned}$$

which proves the given identity.

**Comment**: The identity in the problem has been known since at least the 1930s. In her article "Counting and Recounting: The Aftermath" (*The Mathematical Intelligencer*, Vol. 6, No. 2, 1984), Marta Sved provides some references and describes a number of purely combinatorial proofs of the identity, all based in some way on the count of lattice paths.

**Solution 2 by Arkady Alt, San Jose ,CA**

First note that

$$\begin{aligned}\binom{-1/2}{n} &= \frac{-1/2\,(-1/2-1)\ldots.(-1/2-n+1)}{n!} \\ &= (-1)^n\cdot\frac{1\cdot 3\cdot\ldots\cdot(2n-1)}{2^n n!} \\ &= (-1)^n\cdot\frac{(2n)!}{2^{2n}\,(n!)^2} \\ &= \frac{(-1)^n}{4^n}\binom{2n}{n} \text{ and therefore,} \\ \binom{2n}{n} &= (-4)^n\binom{-1/2}{n}.\end{aligned}$$

Since,

$$\binom{2k}{k}\binom{2n-2k}{n-k} = (-4)^k\binom{-1/2}{k}(-4)^{n-k}\binom{-1/2}{n-k} = (-4)^n\binom{-1/2}{k}\binom{-1/2}{n-k},$$

we have

$$\sum_{k=0}^{n}\binom{2n-2k}{n-k}\binom{2k}{k}4^{-n} = 1 \iff \sum_{k=0}^{n}\binom{-1/2}{k}\binom{-1/2}{n-k} = (-1)^n.$$

Since $\dfrac{1}{\sqrt{1+x}} = (1+x)^{-1/2} = \sum\limits_{n=0}^{\infty}\binom{-1/2}{n}x^n$ and $\sum\limits_{n=0}^{\infty}(-1)^n x^n = \dfrac{1}{1+x}$,

we obtain

$$\left(\frac{1}{\sqrt{1+x}}\right)^2 = \frac{1}{1+x} \iff \left(\sum_{n=0}^{\infty}\binom{-1/2}{n}x^n\right)^2 = \sum_{n=0}^{\infty}(-1)^n x^n$$

$$\iff \quad \sum_{n=0}^{\infty} x^n \sum_{k=0}^{n} \binom{-1/2}{k}\binom{-1/2}{n-k} = \sum_{n=0}^{\infty} n\,(-1)^n\,x^n.$$

Hence, $\sum_{k=0}^{n} \binom{-1/2}{k}\binom{-1/2}{n-k} = (-1)^n$.

**Solution 3 by Nicusor Zlota, "Traian Vuia" Technical College, Focsani, Romania**

We have $\frac{1}{\sqrt{1-x^2}} = \sum_{n\geq 0} \binom{2n}{n} 2^{-2n} x^{2n}$.

On the other hand, we have $\frac{1}{1-x^2} = \sum_{n\geq 0} x^{2n}$. Squaring the first power series and comparing terms give us $\sum_{k=0}^{n} \binom{2n-2k}{n-k}\binom{2k}{k} 2^{-2n} = 1$, q.e.d.

*Editor's comment* : Several of those who solved this problem also commented on where variations and generalizations of it can be found. E.g., **Ulrich Abel of the Technische Hochschule Mittelhessen in Friedberg, Germany** cited the paper:

Chang, G., Xu, C., "Generalization and probabilistic proof of a combinatorial identity." **American Mathematical Monthly** 118, 175-177, (2011), and also a paper of his which was published in 2015 that further generalizes notions used in the Chang and Xu paper.

Ulrich Abel, Vijay Gupta, and Mircea Ivan, "A generalization of a combinatorial identity by Change and Xu," **Bulletin of Mathematical Sciences,** published by Springer, ISSN 1664-3607. This paper can also be seen at Springer's open line access site < SpringerLink.com > .

Another citation was given by **Moti Levy, of Rehovot Israel.** He mentioned that in <u>Concrete Mathematics</u>, by Graham, Knuth, and Patashnik (second edition) the problem is solved in Section 5.3, "Tricks of the trade," pages 186-187 . **And Carl Libis of Columbia Southern University, Orange Beach, AL** cited http://math.stackexchange.com/questions/687221/proving-sum-k-0n2k-choose-k2n-2k-choose-n-k-4n/688370688370

In addition, **Bruno Salgueirio Fanego of Viveiro, Spain** stated that a probabilistic interpretation of the problem can be found in <http://mathes.pugetsound.edu/~mspivey/AltConvRepring.pdf>. He went on to say that: more generally, it can be demonstrated that, for any real $l$, $\sum_{k=0}^{n} \binom{2n-2k-l}{n-k}\binom{2k+l}{k} 4^{-n} = 1$ (see: http://arxiv.org/pdf/1307.6693.pdf) and that for any integer $m \geq 2$,

$$\sum_{k_1 \cdot k_2 \cdots k_m = n} \binom{2k_1}{k_1}\binom{2k_2}{k_2}\cdots\binom{2k_m}{k_m} 4^{-n} = \frac{\Gamma\left(\frac{m}{2}+n\right)}{n!\,\Gamma\left(\frac{m}{2}\right)},$$

as can be found in <http://129.81.170.14/~vhm/papers_html/prob-bin.pdf>.

**Also solved by Ed Gray, Highland Beach, FL; G. C. Greubel, Newport News, VA; Gail Nord, Gonzaga University, Spokane, WA; Toshihiro Shimizu, Kawasaki, Japan, and the proposer.**

**5365:** *Proposed by José Luis Díaz-Barrero, Barcelona Tech, Barcelona, Spain*

Let $n \geq 3$ be a positive integer. Find all real solutions of the system

$$\left.\begin{array}{c} a_2^3(a_2^2 + a_3^2 + \ldots + a_{j+1}^2) = a_1^2 \\ a_3^3(a_3^2 + a_4^2 + \ldots + a_{j+2}^2) = a_2^2 \\ \cdots\cdots\cdots \\ a_n^3(a_n^2 + a_1^2 + \ldots + a_{j-1}^2) = a_{n-1}^2 \\ a_1^3(a_1^2 + a_2^2 + ... + a_j^2) = a_n^2 \end{array}\right\}$$

for $1 < j < n$.

**Partial solution by the proposer**

Since the RHS of all equations are nonnegative, then the system does not have solutions $(a_1, a_2, \ldots, a_n)$ with negative components. Moreover, $(0, 0, \ldots, 0)$ is a trivial solution. So, it remains to find the positive solutions of the system. To do it, let $m = \min\limits_{1 \leq k \leq n} \{a_k\} = a_p$ and $M = \max\limits_{1 \leq k \leq n} \{a_k\} = a_q$. Then, using the $(q-1)^{th}$ equation yields

$$jM^3m^2 \leq a_q^3(a_q^2 + a_{q+1}^2 + \ldots + a_{q+j-1}^2) = a_{q-1}^2 \leq M^2$$

and from the $(p-1)^{th}$ equation we get

$$jm^3M^2 \geq a_p^3(a_p^2 + a_{p+1}^2 + \ldots + a_{p+j-1}^2) = a_{p-1}^2 \geq m^2$$

Therefore,

$$jM^3m^2 \leq M^2 \Leftrightarrow M \leq \frac{1}{jm^2}$$

and

$$jm^3M^2 \geq m^2 \Leftrightarrow m \geq \frac{1}{jM^2}$$

Since $M \leq \dfrac{1}{jm^2}$, then $j^2m^4 \leq \dfrac{1}{M^2}$ and from $m \geq \dfrac{1}{jM^2}$ follows that

$$m \geq jm^4 \Rightarrow m \leq \sqrt[3]{1/j}$$

Likewise, from $M \leq \dfrac{1}{jm^2}$ and $m \geq \dfrac{1}{jM^2}$ immediately follows

$$M \leq jM^4 \Rightarrow M \geq \sqrt[3]{1/j}$$

So, $m = M = \sqrt[3]{1/j}$ and a positive solution of the given system is

$$\left(\sqrt[3]{1/j}, \sqrt[3]{1/j}, \ldots, \sqrt[3]{1/j}\right)$$

(*) It remains to prove if there exist or not other positive solutions.

*Editor's comment:* When the statement of this problem was published the last line in the system was not there. **Toshihiro Shimizu of Kawasaki, Japan** mentioned that for the sake of symmetry it would be advantageous to add this last line to the system, and the proposer agreed. But as we see, even with this additional condition, a definitive set of solutions was not received.

**5366:** *Proposed by Ovidiu Furdui and Alina Sintămărian, Technical University of Cluj-Napoca, Cluj-Napoca, Romania*

Find all nonconstant, differentiable functions $f : \Re \to \Re$ which verify the functional equation $f(x+y) - f(x-y) = 2f'(x)f(y)$, for all $x, y \in \Re$.

**Solution 1 by Moti Levy, Rehovot, Israel**

We will show that all the solutions of the *functional* equation (1) must satisfy the *differential* equation (2):

$$f(x+y) - f(x-y) = 2f^{'}(x)f(y), \quad \text{for all } x, y \in R, \tag{1}$$

$$f^{''}(x)f(x) - \left(f^{'}(x)\right)^2 + 1 = 0, \quad f(0) = 0, f^{'}(0) = 1. \tag{2}$$

We divide both sides of (1) by $y$ and take the limit $y \to 0$.

$$\frac{f(x+y) - f(x-y)}{y} = 2f^{'}(x)\frac{f(y)}{y} \tag{3}$$

The left hand side approaches the derivative $f^{'}(x)$

$$\lim_{y\to 0} \frac{f(x+y) - f(x-y)}{2y} = f^{'}(x),$$

and the right hand side is equal to $f^{'}(x)\lim_{y\to 0}\frac{f(y)}{y}$.
It follows that

$$\lim_{y\to 0} \frac{f(y)}{y} = 1 \quad \Longrightarrow \quad f(0) = 0. \tag{4}$$

By Taylor's theorem,

$$f(y) = f(0) + f^{'}(\theta)\, y, \qquad 0 \le \theta \le y.$$

$$\lim_{y\to 0} \frac{f(y)}{y} = 1 = \lim_{y\to 0} \frac{f^{'}(\theta)\, y}{y} \Longrightarrow f^{'}(0) = 1.$$

Thus we have derived the initial conditions,

$$f(0) = 0, \; f^{'}(0) = 1. \tag{5}$$

Differentiation of (1) with respect to the variable $y$, gives

$$f^{'}(x+y) + f^{'}(x-y) = 2f^{'}(x)f^{'}(y)\,. \tag{6}$$

Setting $x = y$ in (1) and in (6), we obtain

$$f(2x) = 2f^{'}(x)f(x)\,, \tag{7}$$

$$f^{'}(2x) + 1 = 2\left(f^{'}(x)\right)^2. \tag{8}$$

Now, $f^{'}(x)=\frac{f(2x)}{2f(x)}$ from (7), implies that $f^{'}(x)$ it is differentiable function (for $f(x)\neq 0)$, (actually, by this argument $f(x)$ is infinitely differentiable). Differentiating (7) gives

$$f^{'}(2x)=f^{''}(x)f(x)+\left(f^{'}(x)\right)^{2}. \tag{9}$$

By equating $f^{'}(2x)$ in (8) and (9), we obtain the differential equation,

$$f^{''}(x)f(x)-\left(f^{'}(x)\right)^{2}+1=0 \tag{10}$$

Now we differentiate (10),

$$f^{(3)}(x)f(x)+f^{''}(x)f^{'}(x)-2f^{'}(x)f^{''}(x)=0$$

or

$$\frac{f^{(3)}(x)}{f^{''}(x)}=\frac{f^{'}(x)}{f(x)} \tag{11}$$

$$\begin{aligned}\ln f^{''}(x)&=\ln f(x)+c\\ f^{''}(x)&=k^{2}f(x)\end{aligned}$$

$$f(x)=\alpha e^{kx}+\beta e^{-kx}$$

$$\begin{aligned}f(0)&=0, &&\Longrightarrow &&\alpha+\beta=0\\ f^{'}(0)&=1, &&\Longrightarrow &&k\alpha-k\beta=1\end{aligned}$$

$$f(x)=\frac{e^{kx}-e^{-kx}}{2k},\qquad k\in C.$$

Let $k=\sigma+i\tau$, $\sigma,\tau\in R$, then

$$\begin{aligned}f(x)&=\frac{e^{\sigma x}(\cos\tau x+i\sin\tau x)-e^{-\sigma x}(\cos\tau x-i\sin\tau x)}{2(\sigma+i\tau)}\\ &=\frac{(\sigma-i\tau)\left(e^{\sigma x}(\cos\tau x+i\sin\tau x)-e^{-\sigma x}(\cos\tau x-i\sin\tau x)\right)}{2(\sigma^{2}+\tau^{2})}\\ &=\frac{e^{\sigma x}(\sigma\cos(\tau x)+\tau\sin(\tau x))-e^{-\sigma x}(\sigma\cos(\tau x)-\tau\sin(\tau x))}{2(\sigma^{2}+\tau^{2})}\\ &+i\frac{e^{\sigma x}(\sigma\sin\tau x-\tau\cos\tau x)-e^{-\sigma x}(\sigma\sin\tau x-\tau\cos\tau x)}{2(\sigma^{2}+\tau^{2})}.\end{aligned}$$

Since we are requested to find only the real functions, then $\sigma$ must be equal to 0 or $\tau$ must be equal to 0.
When $\sigma=0$ then

$$f(x)=\frac{\sin(\tau x)}{\tau},\quad \tau\in R\backslash\{0\}.$$

When $\tau=0$ then

$$f(x)=\frac{\sin(i\sigma x)}{i\sigma}=\frac{\sinh(\sigma x)}{\sigma},\quad \tau\in R\backslash\{0\}.$$

One can check that $f(x)=\lim_{\tau\to 0}\frac{\sin(\tau x)}{\tau}=x$ is also a solution of the differential equation (2).

It is easy to check that $f(x)=\begin{cases}\frac{\sin(\tau x)}{\tau}, & \tau\in R\backslash\{0\}\\ \frac{\sinh(\sigma x)}{\sigma}, & \sigma\in R\backslash\{0\}\\ x\end{cases}$, the family of solution of (2), are indeed solution of (1).

**Solution 2 by Toshihiro Shimizu, Kawasaki, Japan**

Let $(F)$ be the functional equation in the problem statement.

There exists $y_1$ such that $f'(y_1)\neq 0$, otherwise $f$ would be constant. Taking $x=y_1, y\neq 0$ to $(F)$ we get

$$\frac{f(y_1+y)-f(y_1-y)}{2y}=f'(y_1)\frac{f(y)}{y}.$$

Taking the limit $y\to 0$, we get $f'(y_1)=f'(y_1)f'(0)$ or $f'(0)=1$. Taking $x=0$ to $(F)$, we get $f(y)-f(-y)=2f(y)$ or $f(-y)=-f(y)$. Especially, $f(0)=0$.

Then, we get
$2f'(-x)f(y)=f(-x+y)-f(-x-y)=-f(y-x)+f(x+y)=2f'(x)f(y)$. We take $y=y_0$ such that $f(y_0)\neq 0$, where such $y_0$ exists since $f$ is not constant. Then, we get $f'(-x)=f'(x)$ for all $x\in R$.

We show that $f'$is differentiable. Taking $y=y_0$ to $(F)$,
$f'(x)=\big(f(x+y_0)-f(x-y_0)\big)/\big(2f(y_0)\big)$ for all $x\in R$.

Thus, it follows that

$$\begin{aligned}\frac{f'(x+h)-f'(x)}{h}&=\frac{f(x+h+y_0)-f(x+h-y_0)-f(x+y_0)+f(x-y_0)}{2f(y_0)h}\\&=\frac{1}{2f(y_0)}\left(\frac{f(x+h+y_0)-f(x+y_0)}{h}-\frac{f(x+h-y_0)-f(x-y_0)}{h}\right)\\&\to\frac{1}{2f(y_0)}\big(f'(x+y_0)-f'(x-y_0)\big)\ (h\to 0)\end{aligned}$$

Thus $f'$ is differentiable.

Differentiating with respect to $x$, we get

$$f'(x+y)-f'(x-y)=2f''(x)f(y)$$

Exchanging $x$ and $y$, (l.h.s) is not changed. Thus $f''(x)f(y)=f(x)f''(y)$ for any $x,y\in R$. Especially for $y=y_0$, we get the result that $f''(x)=cf(x)$ for some constant $c\in R$. It's known functional equation and we omit the detail.

If $c>0$, we can write as $f(x)=C_1\exp(Cx)+C_2\exp(-Cx)$. From the fact that $f(0)=0$ and $f'(0)=1$, we get $C_1+C_2=0$ and $C(C_1-C_2)=1$. Thus, we can write as

$f(x) = \big(\exp(Cx) - \exp(-Cx)\big)/(2C) = \sinh(Cx)/(2C)$. It is easy to check that this function satisfies $(F)$.

If $c < 0$, we can write as $f(x) = C_1 \cos(Cx) + C_2 \sin(Cx)$. From the fact that $f(0) = 0$ and $f'(0) = 1$, we get $C_1 = 0$, $CC_2 = 1$. Thus, we can write as $f(x) = \sin(Cx)/C$. Again, it is easy to check that this function satisfies $(F)$.

If $c = 0$, we can write as $f(x) = Cx + D$. From the fact that $f(0) = 0$ and $f'(0) = 1$. We get $f(x) = x$. It also satisfies $(F)$.

Finally, we get $f(x) = \sinh(Cx)/(2C)$ or $f(x) = \sin(Cx)/C$ or $f(x) = x$ where $C \neq 0$ is constant.

**Solution 3 by Kee-Wai Lau, Hong Kong, China**

We show that $f(x) = x$, $\dfrac{e^{ax} - e^{-ax}}{2a}$ or $\dfrac{\sin(bx)}{b}$, where $a$ and $b$ are nonzero numbers.

By putting $y = 0$ into the into the given functional equation

$$f(x+y) - f(x-y) = 2f'(x)f(y) \qquad (1)$$

we obtain we obtain $f'(x)(0) = 0$. Since $f$ is non-constant, so there exists $a \in \Re$ such that $f'(a) \neq 0$. Hence $f(0) = 0$. Differentiate (1) with respect to $y$, we obtain

$$f'(x+y) + f'(x-y) = 2f'(x)f'(y). \qquad (2)$$

By putting $y = 0$ and $x = a$ into (2), we obtain $f'(0) = 1$. By putting $x = 0$ into (1), we obtain $f(-y) = -f(y)$. Hence by interchanging $x$ and $y$ in (1), we obtain

$$f(x+y) + f(x-y) = 2f'(y)f(x), \qquad (3)$$

Adding up (1) and (3), ) we obtain

$$f(x+y) = f'(x)f(y) + f'(y)f(x). \qquad (4)$$

Differentiating (4) with respect to $x$, we obtain

$$f'(x+y) = f''(x)f(y) + f'(y)f'(x). \qquad (5)$$

Differentiating (4) with respect to $y$, we obtain

$$f'(x+y) = f'(x)f'(y) + f''(y)f(x) \qquad (6)$$

From (5) and (6), we obtain $f''(x)f(y) = f''(y)f(x)$ for all $x, y \in \Re$.

It follows that $f''(x) = kf(x)$, where $k$ is a constant.

If $k = 0$ then $f''(x) = 0$, so that $f$ is a linear function. Since $f(0) = f'(0) - 1 = 0$, so $f(x)$. If $k = a^2$, then $f''(x) - a^2 f(x) = 0$.

By standard methods, we obtain $(x) = \dfrac{e^{ax} - e^{-ax}}{2a}$. If $k = -b^2$, then $f''(x) + b^2 f(x) = 0$.

By standard methods, we obtain $f(x) = \dfrac{\sin(bx)}{b}$.

This completes the solution.

**Also solved by Arkady Alt, San Jose, CA; Hatef I. Arshagi, Guilford Technical Community College, Jamestown, NC, and the proposers.**

$$Mea - Culpa$$

**The names of Bruno Salgueiro Fanego of Viveiro, Spain** and **David E. Manes of SUNY College at Oneonta, NY** should have been listed as having solved problem 5358; their names were inadvertently omitted from the list.