Problems Ted Eisenberg, Section Editor

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This section of the Journal offers readers an opportunity to exchange interesting mathematical problems and solutions. Please send them to Ted Eisenberg, Department of Mathematics, Ben-Gurion University, Beer-Sheva, Israel or fax to: 972-86-477-648. Questions concerning proposals and/or solutions can be sent e-mail to <eisenbt@013.net>. Solutions to previously stated problems can be seen at <http://www.ssma.org/publications>.

Solutions to the problems stated in this issue should be posted before
March 15, 2017

- **5427:** *Proposed by Kenneth Korbin, New York, NY*

  Rationalize and simplify the fraction

$$\frac{(x+1)^4}{x(2016x^2-2x+2016)} \quad \text{if} \quad x = \frac{2017+\sqrt{2017-\sqrt{2017}}}{2017-\sqrt{2017-\sqrt{2017}}}.$$

- **5428:** *Proposed by Nicusor Zlota, "Traian Vuia" Technical College, Focsani, Romania*

  If $x > 0$, then $\dfrac{[x]}{\sqrt[4]{[x]^4+([x]+2\{x\})^4}} + \dfrac{\{x\}}{\sqrt[4]{\{x\}^4+([x]+2\{x\})^4}} \geq 1 - \dfrac{1}{\sqrt[4]{2}}$, where $[.]$ and $\{.\}$ respectively denote the integer part and the fractional part of x.

- **5429:** *Proposed by Titu Zvonaru, Comănesti, Romania and Neculai Stanciu, "George Emil Palade" School, Buzău, Romania*

  Prove that there are infinitely many positive integers a, b such that
  $18a^2 - b^2 - 6a - b = 0$.

- **5430:** *Proposed by Oleh Faynshteyn, Leipzig, Germany*

  Let a, b, c be the side-lengths, α, β, γ the angles, and R, r the radii respectively of the circumcircle and incircle of a triangle. Show that

$$\frac{a^3\cdot\cos(\beta-\gamma)+b^3\cdot\cos(\gamma-\alpha)+c^3\cdot\cos(\alpha-\beta)}{(b+c)\cos\alpha+(c+a)\cos\beta+(a+b)\cos\gamma} = 6Rr.$$

- **5431:** *Proposed by José Luis Díaz-Barrero, Barcelona Tech, Barcelona, Spain*

  Let F_n be the n^{th} Fibonacci number defined by $F_1 = 1, F_2 = 1$ and for all $n \geq 3$, $F_n = F_{n-1} + F_{n-2}$. Prove that

$$\sum_{n=1}^{\infty} \left(\frac{1}{11}\right)^{F_n F_{n+1}}$$

  is an irrational number and determine it (*).

The asterisk $(*)$ indicates that neither the author of the problem nor the editor are aware of a closed form for the irrational number.

- **5432:** *Proposed by Ovidiu Furdui and Alina Sîntămărian, Technical University of Cluj-Napoca, Cluj-Napoca, Romania*

Find all differentiable functions $f : (0, \infty) \to (0, \infty)$, with $f(1) = \sqrt{2}$, such that

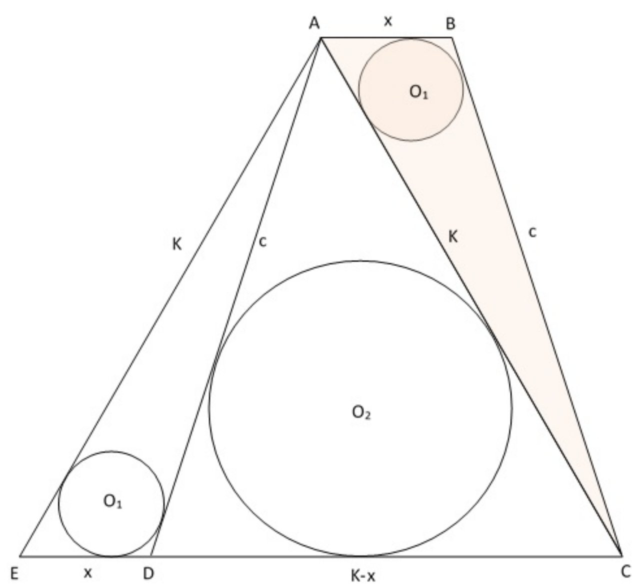
$$f'\left(\frac{1}{x}\right) = \frac{1}{f(x)}, \ \forall x > 0.$$

Solutions

- **5409:** *Proposed by Kenneth Korbin, New York, NY*

Given isosceles trapezoid $ABCD$ with $\overline{AB} < \overline{CD}$, and with diagonal $\overline{AC} = \overline{AB} + \overline{CD}$. Find the perimeter of the trapezoid if $\triangle ABC$ has inradius 12 and if $\triangle ACD$ has inradius 35.

Solution 1 by Michael N. Fried, Ben-Gurion University, Beer-Sheva, Israel



Let $|AB| = x, |AC| = |AD| = c, |AC| = K$ so that, since $|AC| = |AB| + |CD|$, we can write $|DC| = K - x$.
The key observation is that *if the triangle ABC is reflected and transposed so that BC coincides with AD, the resulting figure $AEDC$ is an equilateral triangle.* This is so because:
1) The trapezoid is isosceles, so that $EDA = \pi - ADC$, and, therefore, EDC is a straight line
2) By the given, $|AB| + |DC| = |ED| + |DC| = x + K - x = K$, and, therefore, $|EA| = |AC| = |CE| = K$.

With the geometry of the situation in mind, one can now easily see that since the diameter of O_1 is 24 and the diameter of O_2 is 70, the length of the side of the equilateral triangle (i.e. the diagonal of the original trapezoid) cannot be less than 94 units. This will be important later.
Now, since the twice the area of a triangle is the product of its inradius and its perimeter, we find that twice the area of the triangle AED is $12(c+K+x)$ and twice the area of the triangle ADC is $35(c+K+K-x)$. On the other hand, since we have observed that EAC is equilateral, twice the areas of these triangles are also, respectively, $\frac{\sqrt{3}}{2}Kx$ and $\frac{\sqrt{3}}{2}K(K-x)$. Thus, we can write the following two equations:

$$c+K+x=\frac{\sqrt{3}}{24}Kx \tag{1}$$

$$c+2K-x=\frac{\sqrt{3}}{70}K(K-x) \tag{2}$$

Using the law of cosines in the triangle AED and the fact that angle $< AED = \frac{\pi}{3}$, we have a third equation:

$$c^2=K^2+x^2-Kx \tag{3}$$

Thus, we have three quadratic equations in three unknowns, c, K, and x. We will show that this can be reduced to a single quadratic equation in K, from which we will be able to find x and c.
To make the algebra easier to write out, let us use the following notations:
$q=\frac{\sqrt{3}}{24}$
$p=\frac{\sqrt{3}}{70}$
$Q=qK-1$
$P=pK-1$
The reason for the latter two will become clear in a moment.
Eliminating c from equations 1 and 2, we find that $x=\frac{K(pK-1)}{K(p+q)-2}$ or using our notation above:

$$x=\frac{KP}{P+Q} \tag{4}$$

Note, 1 can be written as $K+c=(qK-1)x=Qx$ (similarly, 2 can be written $K+c=P(K-x)$), so substituting 4 into 1 in this form, we find, after some easy manipulations:

$$c=\frac{K}{P+Q}(PQ-(P+Q)) \tag{5}$$

On the other hand, substituting 4 into 3 and rearranging terms we obtain:

$$c^2=\left(\frac{K}{P+Q}\right)^2((P+Q)^2+P^2-P(P+Q)) \tag{6}$$

Squaring 5, equating it with 6, and canceling $\left(\frac{K}{P+Q}\right)^2$, we obtain:

$$(PQ-(P+Q))^2=(P+Q)^2+P^2-P(P+Q)$$

And after some simplification, we have the equation:

$$PQ(PQ-2(P+Q)+1)=0$$

But since $Qx = K + c > 0$ and $P(K - x) = K + c > 0$, neither P nor Q can be 0, so, we have:

$$PQ - 2(P + Q) + 1 = 0$$

At this point, we can substitute $pK - 1 = P$ and $qK - 1 = Q$, to obtain the following quadratic equation in K:

$$pqK^2 - 3(p + q)K + 6 = 0 \tag{7}$$

Substituting $p = \frac{\sqrt{3}}{70}$ and $q = \frac{\sqrt{3}}{24}$ and solving, we obtain two solutions:
$K = 80\sqrt{3} \approx 138.564$ or $K = 14\sqrt{3} \approx 24.245$. As we noted above, K cannot be less than 94, so we have only $K = 80\sqrt{3}$. Using 4 and 5 to find x and c, we find then that the perimeter of ABCD$= 2c + K = 226\sqrt{3} \approx 391.443$

Solution 2 by David E. Manes, SUNY College at Oneonta, Oneonta, NY

Let $x = \overline{AB}$, $y = \overline{CD}$ and $z = \overline{AD} = \overline{BC}$. Then the perimeter of the trapezoid $ABCD$ is $x + y + 2z = 226\sqrt{3}$ when $x = 17\sqrt{3}$, $y = 63\sqrt{3}$ and $z = 73\sqrt{3}$.
Denote the area of polygon X by $[X]$. Then, by Ptolemy's theorem, $\overline{AC} = \sqrt{xy + z^2}$. Therefore, x + y = $\sqrt{xy + z^2}$. Solving for z^2, we get $z^2 = x^2 + xy + y^2$. The height h of the trapezoid, according to the Pythagorean theorem, is given by

$$h = \sqrt{z^2 - (\frac{y - x}{2})^2} = \frac{\sqrt{3}}{2}(x + y).$$

Therefore,

$$[ABC] = \frac{1}{2} \cdot x \cdot \frac{\sqrt{3}}{2}(x + y)$$

and

$$[ACD] = \frac{1}{2} \cdot y \cdot \frac{\sqrt{3}}{2}(x + y).$$

Let r denote the inradius of triangle T. Then $r \cdot s = [T]$ where s is the semiperimeter of T. For each of the triangles ACD and ABC, this formula reduces to

$$\frac{35}{2}(x + 2y + z) = \frac{\sqrt{3}}{4}y(x + y),$$
$$6(2x + y + z) = \frac{\sqrt{3}}{4}x(x + y),$$

respectively. Multiplying the first equation by x, the second by y and then subtracting the second equation from the first yields the following upon simplification:

$$z(\frac{35}{2}x - 6y) = 6y^2 - 23xy - \frac{35}{2}x^2.$$

Since $z = \sqrt{x^2 + xy + y^2}$, we have

$$\sqrt{x^2 + xy + y^2}(\frac{35}{2}x - 6y) = 6y^2 - 23xy - \frac{35}{2}x^2.$$

Squaring both sides of this equation and then simplifying it, one obtains the equation

$$136y^2 - 249xy - 945x^2 = 0.$$

Regarding this equation as a quadratic in y, one obtains the following roots

$$\begin{aligned} y &= \frac{249x \pm \sqrt{(249x)^2 + 4(136)(945x^2)}}{272} \\ &= \frac{249x \pm 759x}{272}. \end{aligned}$$

Since $y > 0$ we disregard the negative root so that

$$y = \frac{1008}{272}x = \frac{63}{17}x.$$

Moreover,

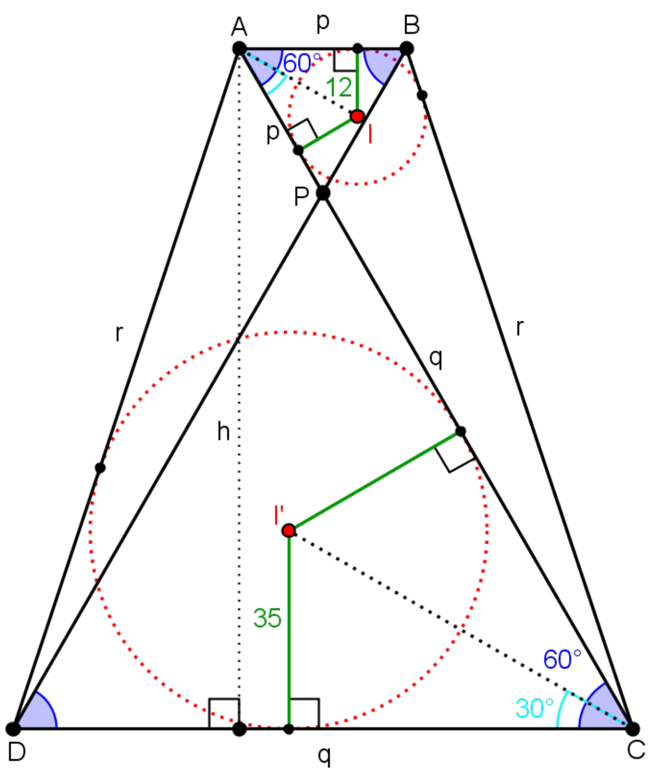
$$z = \sqrt{x^2 + xy + y^2} = \sqrt{x^2 + \frac{63}{17}x^2 + \left(\frac{63}{17}x\right)^2} = \frac{73}{17}x.$$

Thus, our solutions are parametrized by x and the problem now is to find the value(s) of x that satisfy the two equations for the inradius. To that end suppose

$$6\left(2x + \frac{63}{17}x + \frac{73}{17}x\right) = \frac{\sqrt{3}}{4}x\left(x + \frac{63}{17}x\right).$$

Then $x = 17\sqrt{3}$. Similarly, the equation $\frac{35}{2}(x + 2y + z) = \frac{\sqrt{3}}{4}y(x + y)$ yields $x = 17\sqrt{3}$. Hence, $y = \frac{63}{17}x = 63\sqrt{3}$ and $z = \frac{73}{17}x = 73\sqrt{3}$ so that the perimeter of the trapezoid $ABCD$ is $x + y + 2z = 226\sqrt{3}$.

Solution 3 by Nikos Kalapodis, Patras, Greece



Let P be the intersection of diagonals AC and BD. Since trapezoid $ABCD$ is isosceles, the triangles ABC and BAD, as well as, the triangles ACD and BDC are congruent, (SAS criterion).

It follows that the triangles PAB and PCD are isosceles with $PA = PB$ and $PC = PD$ (1).
Furthermore, since they are similar (congruent angles) we have
$\dfrac{PA}{AB} = \dfrac{PC}{CD} = \dfrac{PA+PC}{AB+CD} = \dfrac{AC}{AB+CD} = 1$. Thus, $PA = AB$ and $PC = CD$ (2).
From (1) and (2) we conclude that triangles PAB and PCD are equilateral.
Let $p = AB = PA = PB$, $q = CD = PC = PD$, $r = BC = AD$, $t = p+q+r$ and h the height of the trapezoid. Then we have
$\dfrac{p}{q} = \dfrac{ph}{qh} = \dfrac{2[ABC]}{2[ACD]} = \dfrac{12(2p+q+r)}{35(p+2q+r)} = \dfrac{12(p+t)}{35(q+t)}$ or $23pq = t(12q-35p)$ (3)
Since the trapezoid is isosceles, it is cyclic, so by Ptolemy's Theorem we have
$pq + r^2 = (p+q)^2$ (4) or $pq = t(p+q-r)$ (5)
By (3) and (5) we obtain $58p + 11q = 23r$ (6)
Finally, applying the well-known formula $r = (s-a)\tan\dfrac{A}{2}$ in triangles ACD and BAC we have
$23 = 35 - 12 = \left(\dfrac{p+2q+r}{2} - r\right)\dfrac{\sqrt{3}}{3} - \left(\dfrac{2p+q+r}{2} - r\right)\dfrac{\sqrt{3}}{3} = \dfrac{q-p}{2}\cdot\dfrac{\sqrt{3}}{3}$, i.e.
$q - p = 46\sqrt{3}$ (7).
Solving the system of equations (4), (6) and (7) we find $p = 17\sqrt{3}$, $q = 63\sqrt{3}$, and $r = 73\sqrt{3}$.
Therefore the perimeter of trapezoid is $p + q + 2r = 226\sqrt{3}$.

Also solved by Ed Gray, Highland Beach, FL; Kee-Wai Lau, Hong Kong, China; Albert Stadler, Herrliberg, Switzerland; Malik Sheykhov (student at the France-Azerbaijan University in Azerbaijan) and Talman Residli (student at Azerbaijan Medical University in Baku, Azerbaijan); David Stone and John Hawkins, Georgia Southern University, Statesboro, GA, and the proposer

- **5410:** *Proposed by Arkady Alt, San Jose, CA*

For the given integers $a_1, a_2, a_3 \geq 2$ find the largest value of the integer semiperimeter of a triangle with integer side lengths t_1, t_2, t_3 satisfying the inequalities $t_i \leq a_i,\ i = 1, 2, 3$.

Solution 1 by Kee-Wai Lau, Hong Kong, China

Without loss of generality, we assume that $a_1 \geq a_2 \geq a_3$. Let

$$T_1 = \{2, 3, \ldots, a_1\},\ T_2 = \{2, 3, \ldots, a_2\},\ T_3 = \{2, 3, \ldots, a_3\}$$

$$T = \{(t_1, t_2, t_3) : t_1 \in T_1,\ t_2 \in T_2,\ t_3 \in T_3\} \text{ and}$$

$$S = T \cap \{(t_1, t_2 t_3) : t_1, t_2, t_3 \text{ are the side lengths of a triangle}\}.$$

Let $L = \underset{(t_1,t_2,t_3)\in S}{\text{Maximum}} \dfrac{t_1+t_2+t_3}{2}$. We show that $L = \begin{cases} \dfrac{a_1+a_2+a_3}{2}, & \text{if } a_2 + a_3 > a_1 \\ a_2 + a_3 - \dfrac{1}{2}, & \text{if } a_2 + a_3 \leq a_1. \end{cases}$

Case 1: $a_2 + a_3 > a_1$

We have $(a_1, a_2, a_3) \in S$ and clearly $L = \dfrac{a_1 + a_2 + a_3}{2}$.

Case 2: $a_2 + a_3 \le a_1$

We have $(a_2 + a_3 - 1, a_2, a_3) \in S$ so that $L \ge a_2 + a_3 - \dfrac{1}{2}$. If $(t_1, t_2, t_3) \in T$ and $t_1 > a_2 + a_3 - 1$, then $(t_1, t_2, t_3) \notin S$. If $(t_1, t_2, t_3) \in T$ then $t_1 < a_2 + a_3 - 1$, then $\dfrac{t_1 + t_2 + t_3}{2} < \dfrac{(a_2 + a_3 - 1) + a_2 + a_3}{2} = a_2 + a_3 - \dfrac{1}{2}$. Hence, $L = a_2 + a_3 - \dfrac{1}{2}$ in this case.

This completes the solution.

Solution 2 by proposer

Let $s = \dfrac{t_1 + t_2 + t_3}{2}$. Since $t_i < s, i = 1, 2, 3$ then by the triangle inequality our problem becomes the following: Find the maximum of s for which there are positive integer numbers t_1, t_2, t_3 satisfying $\mathrm{t}_i \le \min\{a_i, s - 1), i = 1, 2, 3, t_1 + t_2 + t_3 = 2s$.

First note that $s \ge 3, t_i \ge 2, i = 1, 2, 3$. Indeed, since $t_i \le s - 1$ then $1 \le s - t_i, i = 1, 2, 3$ and therefore $t_1 = 2s - t_2 - t_3 = (s - t_2) + (s - t_3) \ge 2$. Cyclicly we obtian $t_2, t_3 \ge 2$ Hence, $2s \ge 6 \iff s \ge 3$.

Since $t_3 = 2s - t_1 - t_2,\ 2 \le t_3 \le \min\{a_3, s - 1\}$, then
$1 \le 2s - t_1 - t_2 \le \min\{a_3, s - 1\} \iff \max\{2s - t_1 - a_3, s + 1 - t_1\} \le t_2 \le 2s - 1 - t_1$, and therefore, we obtain the inequality for t_2, namely that

$$(\mathbf{1})\quad \max\{2s - t_1 - a_3, s + 1 - t_1, 2\} \le t_2 \le \min\{2s - 1 - t_1, a_2, s - 1\}$$

with the conditions of solvability being:

$$(\mathbf{2}) \begin{cases} 2s - t_1 - a_3 & \le & s - 1 \\ 2s - t_1 - a_3 & \le & a_2 \\ s + 1 - t_1 & \le & a_2 \\ 2 & \le & 2s - 1 - t_1 \end{cases} \iff \begin{cases} s + 1 - a_3 & \le & t_1 \\ 2s - a_2 - a_3 & \le & t_1 \\ s + 1 - a_2 & \le & t_1 \\ t_1 & \le & 2s - 3 \end{cases}$$

Since $s - 1 \le 2s - 3$, then $(\mathbf{2})$ together with $2 \le t_1 \le \min\{a_1, s - 1\}$ gives us the bounds for t_1

$(\mathbf{3})\, \max\{s + 1 - a_3, 2s - a_2 - a_3, s + 1 - a_2, 2\} \le t_1 \le= \min\{a_1, s - 1\}$.

Since $2 \le a_i, i = 1, 2, 3$ then $s + 1 - a_2 \le s - 1, s + 1 - a_3 \le s - 1$ and the solvability condition for $(\mathbf{3})$ becomes

$$s + 1 - a_3 \le a_1 \iff s \le a_1 + a_3 - 1, 2s - a_2 - a_3 \le a_1 \iff s \le \left\lfloor \frac{a_1 + a_2 + a_3}{2} \right\rfloor,$$

$$s + 1 - a_2 \le a_1 \iff s \le a_1 + a_2 - 1, 2s - a_2 - a_3 \le s - 1 \iff s \le +a_2 + a_3 - 1.$$

Thus, $s^* = \min\left\{\left\lfloor \dfrac{a_1 + a_2 + a_3}{2} \right\rfloor, a_1 + a_2 - 1, a_2 + a_3 - 1, a_3 + a_1 - 1\right\}$ is the largest integer value of the semiperimeter.

Solution 3 by Ed Gray, Highland Beach, FL

We consider several special cases:

a) If $a_1 = a_2 = a_3 = 2k$, we can equate $t_i = a_i$ for each i. The perimeter is then $6k$ and the semiperimeter is $3k$.

b) Suppose $a_1 = a_2 = a_3 = 2k+1$. We note that $a_1 + a_2 = 4k+2$ and $a_3 - 1 = 2k$. We define $t_1 = a_1,\ t_2 = a_2$ and $t_3 = a_3 - 1$.

c) Suppose that $a_1 = a_2$ and a_3 is larger than either one. In this case we set $t_1 = a_1$ and $t_2 = a_2$. It doesn't matter if a_1, a_2 are both even or both odd, $t_1 + t_2$ is even. We now have to avoid a potential problem. It must be true that $t_1 + t_2 \geq t_3$. Therefore, since if a_3 is large, we need to define $t_3 = a_3 - x$, where x is the integer which is the smallest such that $a_3 - x$ is even and $t_1 + t_2 > t_3$. Since $t_1 + t_2 + t_3$ is even, the semiperimeter is integral.

d) Suppose that $a_1 = a_2$ and a_3 is smaller than either one, in this case set $t_1 = a_1, t_2 = a_2$, so that $t_1 + t_2$ is even. If a_3=2, we let $t_3 = 2$. If $a_3 > 2$, but odd, we we set $t_3 = a_3 - 1$. Then $t_1 + t_2 + t_3$ equals the perimeter which is even and with an integer semiperimeter, and the triangle inequalities hold.

e) Finally, we have the general case: $a_1 < a_2 < a_3$. We set $t_1 = a_1, t_2 = a_2$. If $t_1 + t_2$ is even we need t_3 to be even. If a_3 is very far so that $a_1 + a_2 < a_3$, we let $t_3 = a_3 - x$, where x is the smallest integer which simultaneously makes $t_1 + t_2 + t_3$ even and $t_1 + t_2 > t_3$. If $t_1 + t_2$ is odd, we employ a similar calculation.

Solution 4 by Paul M. Harms, North Newton, KS

Suppose $a_1 \leq a_2 \leq a_3$. The largest perimeter would be $a_1 + a_2 + a_3$ where $t_i = a_i,\ i = 1,2,3$ provided that we have a triangle, i.e., $a_1 + a_2 > a_3$.

If $a_1 + a_2 > a_3$, and the perimeter is an even integer, then the largest value of an integer semiperimeter is $\dfrac{a_1 +_2 + a_3}{2}$.

If the perimeter is an odd integer, then a_3 must be at least 3 and we could use sides $t_1 = a_1, t_2 = a_2$ and $t_3 = a_3 - 1$. The largest integer semiperimeter for this case is $\dfrac{a_1 + a_2 + a_3 - 1}{2}$.

Now consider the case where $a_1 + a_2 \leq a_3$. A triangle with a maximum perimeter is when $t_1 = a_1,\ a_2 = t_2$, and $t_3 = a_1 + a_2 - 1$. Here $t_3 > a_1, a_2$ and the perimeter is the odd integer $2a_1 + 2a_2 - 1$. To get the largest integer semiperimeter we could use $t_1 = a_1,\ t_2 = a_2$ and $t_3 = a_1 + a_2 - 2$ which has $a_1 + a_2 - 1$ as the largest integer semiperimeter.

Also solved by Jeremiah Bartz and Timothy Prescott, University of North Dakota, Grand Forks, ND; David Stone and John Hawkins, Georgia Southern University, Statesboro, GA

- **5411:** *Proposed by D.M. Bătinetu-Giurgiu, "Matei Basarab" National College, Bucharest, Romania and Neculai Stanciu, "George Emil Palade" General School, Buzău, Romania*

Let $(a_n)_{n\geq 1}, (b_n)_{n\geq 1}$ be real valued positive sequences with $\lim\limits_{n\to\infty} a_n = \lim\limits_{n\to\infty} b_n = a \in R_+^*$

If $\lim\limits_{n\to\infty} (n(a_n - a)) = b \in R$ and $\lim\limits_{n\to\infty} (n(b_n - a)) = c \in R$ compute

$$\lim_{n\to\infty} \left(a_{n+1} \sqrt[n+1]{(n+1)!} - b_n \sqrt[n]{n!} \right).$$

Note: R_+^* means the positive real numbers without zero.

Solution 1 by Brian Bradie, Christopher Newport University, Newport News, VA

By Stirling's approximation,

$$n! \sim \sqrt{2\pi}n^{n+1/2}e^{-n},$$

so

$$\sqrt[n]{n!} \sim \frac{n}{e} \quad \text{and} \quad \sqrt[n+1]{(n+1)!} \sim \frac{n+1}{e}.$$

It then follows that

$$\begin{aligned} a_{n+1}\sqrt[n+1]{(n+1)!} - b_n\sqrt[n]{n!} &\sim \frac{(n+1)a_{n+1}}{e} - \frac{nb_n}{e} \\ &= \frac{1}{e}\left[(n+1)(a_{n+1}-a) - n(b_n-a) + a\right] \end{aligned}$$

and

$$\begin{aligned} \lim_{n\to\infty}\left(a_{n+1}\sqrt[n+1]{(n+1)!} - b_n\sqrt[n]{n!}\right) &= \lim_{n\to\infty}\frac{1}{e}\left[(n+1)(a_{n+1}-a) - n(b_n-a) + a\right] \\ &= \frac{1}{e}(b-c+a). \end{aligned}$$

Solution 2: by Moti Levy, Rehovot, Israel.

$$\begin{aligned} &\lim_{n\to\infty}\left(((n+1)!)^{\frac{1}{n+1}}a_{n+1} - (n!)^{\frac{1}{n}}b_n\right) \\ &= \lim_{n\to\infty}\left(\frac{((n+1)!)^{\frac{1}{n+1}}}{n+1}\left((n+1)\left(a_{n+1}-a\right)\right) + ((n+1)!)^{\frac{1}{n+1}}a - \frac{(n!)^{\frac{1}{n}}}{n}\left(n\left(b_n-a\right)\right) - (n!)^{\frac{1}{n}}a\right) \\ &= \lim_{n\to\infty}\frac{((n+1)!)^{\frac{1}{n+1}}}{n+1}\left((n+1)\left(a_{n+1}-a\right)\right) - \lim_{n\to\infty}\frac{(n!)^{\frac{1}{n}}}{n}\left(n\left(b_n-a\right)\right) + a\lim_{n\to\infty}\left(((n+1)!)^{\frac{1}{n+1}} - (n!)^{\frac{1}{n}}\right) \\ &= \lim_{n\to\infty}\frac{((n+1)!)^{\frac{1}{n+1}}}{n+1}\lim_{n\to\infty}\left((n+1)\left(a_{n+1}-a\right)\right) \\ &- \lim_{n\to\infty}\frac{(n!)^{\frac{1}{n}}}{n}\lim_{n\to\infty}\left(n\left(b_n-a\right)\right) + a\lim_{n\to\infty}\left(((n+1)!)^{\frac{1}{n+1}} - (n!)^{\frac{1}{n}}\right). \end{aligned}$$

So we are challenged with two limits: $\lim_{n\to\infty}\frac{(n!)^{\frac{1}{n}}}{n}$ and $\lim_{n\to\infty}\left(((n+1)!)^{\frac{1}{n+1}} - (n!)^{\frac{1}{n}}\right)$. We will show that both limits equal to $\frac{1}{e}$.
We begin by stating the well-known asymptotic expansion of the Gamma function:

$$\frac{e^x}{x^x\sqrt{2\pi x}}\Gamma(x+1) \sim 1 + \frac{1}{12x}, \qquad x\to\infty.$$

For positive integer n,

$$\left(\frac{e}{n}\right)^n\frac{n!}{\sqrt{2\pi n}} \sim 1 + \frac{1}{12n}, \qquad n\to\infty.$$

Using $\left(1+\frac{1}{12n}\right)^{\frac{1}{n}} \sim 1 + \frac{1}{12n^2}$ and $\left(\sqrt{2\pi n}\right)^{\frac{1}{n}} \sim 1$, we get

$$\frac{e}{n}(n!)^{\frac{1}{n}} \sim 1 + \frac{1}{12n^2}, \qquad n\to\infty,$$

or

$$\frac{(n!)^{\frac{1}{n}}}{n} \sim \frac{1}{e}\left(1+\frac{1}{12n^2}\right), \quad n \to \infty,$$

which implies

$$\lim_{n\to\infty} \frac{(n!)^{\frac{1}{n}}}{n} = \frac{1}{e}.$$

$$\lim_{n\to\infty}\left(((n+1)!)^{\frac{1}{n+1}} - (n!)^{\frac{1}{n}}\right) = \lim_{n\to\infty}(n!)^{\frac{1}{n}}\left(\frac{((n+1)!)^{\frac{1}{n+1}}}{(n!)^{\frac{1}{n}}} - 1\right)$$

$$\frac{(n!)^{\frac{1}{n}}}{n} \sim \frac{1}{e}\left(1+\frac{1}{12n^2}\right); \quad \frac{((n+1)!)^{\frac{1}{n+1}}}{n+1} \sim \frac{1}{e}\left(1+\frac{1}{12n^2}\right)$$

$$\frac{\frac{((n+1)!)^{\frac{1}{n+1}}}{n+1}}{\frac{(n!)^{\frac{1}{n}}}{n}} \sim 1 \quad \Rightarrow \quad \frac{((n+1)!)^{\frac{1}{n+1}}}{(n!)^{\frac{1}{n}}} \sim \frac{n+1}{n} = 1+\frac{1}{n}$$

$$\frac{((n+1)!)^{\frac{1}{n+1}}}{(n!)^{\frac{1}{n}}} - 1 \sim \frac{1}{n}$$

$$(n!)^{\frac{1}{n}}\left(\frac{((n+1)!)^{\frac{1}{n+1}}}{(n!)^{\frac{1}{n}}} - 1\right) \sim \frac{n}{e}\left(1+\frac{1}{12n^2}\right)\frac{1}{n} \sim \frac{1}{e}.$$

We conclude that

$$\lim_{n\to\infty}\left(((n+1)!)^{\frac{1}{n+1}} - (n!)^{\frac{1}{n}}\right) = \frac{1}{e}.$$

Now back to the original limit

$$\begin{aligned}
&\lim_{n\to\infty}\frac{((n+1)!)^{\frac{1}{n+1}}}{n+1}\lim_{n\to\infty}\left((n+1)\left(a_{n+1}-a\right)\right)\\
&-\lim_{n\to\infty}\frac{(n!)^{\frac{1}{n}}}{n}\lim_{n\to\infty}\left(n\left(b_n-a\right)\right) + a\lim_{n\to\infty}\left(((n+1)!)^{\frac{1}{n+1}} - (n!)^{\frac{1}{n}}\right)\\
&= \frac{1}{e}b - \frac{1}{e}c + a\frac{1}{e} = \frac{a+b-c}{e}.
\end{aligned}$$

Also solved by Arkady Alt, San Jose, CA; Paul M. Harms, North Newton, KS; Paolo Perfetti, Department of Mathematics, Tor Vergata, Rome, Italy, and the proposers.

- **5412:** *Proposed by Michal Kremzer, Gliwice, Silesia, Poland*

Given positive integer M. Find a continuous, non-constant function $f: R \to R$ such that $f(f(x)) = f([x])$, for all real x, and for which the maximum value of $f(x)$ is M.

Note: $[x]$ is the greatest integer function.

Solution 1 by Tommy Dreyfus, Tel Aviv University, Israel

Let $f(x) = 0$ except for $M < x < M+1$, where $f(x) = M - 2M\left|x - \left(M+\frac{1}{2}\right)\right|$.

Then f is continuous, attains its maximum at $f\left(M+\dfrac{1}{2}\right)=M$, and $f(f(x))=f([x])=0$ for all x.

Solution 2 by Albert Stadler, Herrliberg Switzerland

Let $f(x)=\begin{cases} M\sin^2(\pi x), & \text{if} \quad x<0 \text{ or } x>M \\ 0, & \text{if} \quad 0\le x\le M. \end{cases}$

$f(x)$ is continuous and non-constant. In addition $f(n)=0$ for all integers n.
$0\le f(x)\le M$ and the maximum M is assumed.

$f([x])=0$ for all real x since $[x]$ is an integer. $f(f(x))=0$ for all real x, since $0\le (x)\le M$ and $f(y)=0$ for $0\le y\le M$.

Solution 3 by Moti Levy, Rehovot, Israel
Let $f:R\to R$ be defined as follows (M is positive integer):

$$f(x)=\begin{cases} M\left[\frac{x}{M+1}\right]\sin^2(\pi x), & for \quad M+2\ge x\ge M+1 \\ 0, & otherwise. \end{cases}$$

The function $f(x)$ is continuous and its maximum value over R is M.
Clearly (since $\sin^2(\pi[x])=0$) ,

$$f([x])=0.$$

By its definition $0\le f(x)\le M$. Hence, $f(f(x))=0$, since $\left[\frac{f(x)}{M+1}\right]=0$.

$$f(f(x))=\begin{cases} M\left[\frac{f(x)}{M+1}\right]\sin^2(\pi f(x))=0, & for \quad M+2\ge x\ge M+1 \\ 0, & otherwise. \end{cases}$$

We conclude that $f(x)$ is continuous and non-constant function with maximum value M, which satisfies $f(f(x))=f([x])=0$.

Solution 4 by The Ashland University Undergraduate Problem Solving Group, Ashland, OH

The following function satisfies the given conditions;

$$f(x)=\begin{cases} -x+2M-2 & \text{if } M-2\le x\le M-3/2 \\ x+1 & \text{if } M-3/2<x\le M-1 \\ M & \text{otherwise.} \end{cases}$$

We can easliy check that f is continuous by noting that:

$$f(M-2)=M,\ f\left(M-\frac{3}{2}\right)=M-\frac{1}{2},\ \text{and } f(M-1)=M.$$

We now show f satisfies $f(f(x))=f([x])$.
When $x\le M-2,\ [x]\le M-2$ and $f(f(x))=f(M)=M=f([x])$.
When $x\ge M-1,\ [x]\ge M-1$ and $f(f(x))=f(M)=M=f([x]$.
Finally, when $M-2<x<M-1,\ [x]=M-2$ and $M-\dfrac{1}{2}\le f(x)<M$.
Thus, $f(f(x))=M=f([x])$.

Also solved by Michael N. Fried, Ben-Gurion University, Beer-Sheva, Israel, and the proposer.

- **5413:** *Proposed by José Luis Díaz-Barrero, Barcelona Tech, Barcelona, Spain*

Compute

$$\lim_{n\to\infty} \frac{1}{n} \sum_{1\le i\le j\le n} \frac{1}{\sqrt{(n^2+(i+j)n+ij)}}.$$

Solution 1 by Brian Bradie, Christopher Newport University, Newport News, VA

$$\begin{aligned}
\lim_{n\to\infty} \frac{1}{n} \sum_{1\le i\le j\le n} \frac{1}{\sqrt{n^2+(i+j)n+ij}} &= \lim_{n\to\infty} \sum_{1\le i\le j\le n} \frac{1}{\sqrt{(1+i/n)(1+j/n)}} \cdot \frac{1}{n^2} \\
&= \int_0^1 \int_0^x \frac{1}{\sqrt{(1+x)(1+y)}}\, dy\, dx \\
&= \int_0^1 \frac{2}{\sqrt{1+x}} \cdot \sqrt{1+y}\Big|_0^x \\
&= \int_0^1 \left(2 - \frac{2}{\sqrt{1+x}}\right) dx \\
&= \left(2x - 4\sqrt{1+x}\right)\Big|_0^1 \\
&= 6 - 4\sqrt{2}.
\end{aligned}$$

Solution 2 by Kee-Wai Lau, Hong Kong, China

Since $n^2+(i+j)n+ij = (n+i)(n+j)$, it is easy to check that

$$\sum_{1\le i\le j\le n} \frac{1}{\sqrt{n^2+(i+j)n+ij}} = \frac{1}{2}\left(\sum_{i=1}^{n} \frac{1}{\sqrt{n+i}}\right)^2 + \frac{1}{2}\sum_{i=1}^{n} \frac{1}{n+i} \qquad (1)$$

Now $\displaystyle\lim_{n\to\infty} \frac{1}{\sqrt{n}} \sum_{i=1}^{n} \frac{1}{\sqrt{n+i}} = \lim_{n\to\infty} \sum_{i=1}^{n} \frac{1}{n\sqrt{n+\frac{1}{n}}} = \int_0^1 \frac{dx}{\sqrt{1+x}} = 2\left(\sqrt{2}-1\right)$, and from $\displaystyle 0 < \sum_{i=1}^{n} \frac{1}{n+i} \le \frac{n}{n+1}$, we have $\displaystyle\lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} \frac{1}{n+i} = 0$, so by (1), we obtain

$$\lim_{n\to\infty} \frac{1}{n} \sum_{1\le i\le j\le n} \frac{1}{\sqrt{n^2+(i+j)n+ij}} = 2(3-2\sqrt{2}).$$

Also solved by Arkady Alt, San Jose, CA; Bruno Salgueiro Fanego, Viveiro, Spain; Moti Levy, Rehovot, Israel; Paolo Perfetti, Department of Mathematics, Tor Vergata, Rome, Italy; Albert Stadler, Herrliberg, Switzerland, and the proposer.

- **5414:** *Proposed by Ovidiu Furdui, Technical University of Cluj-Napoca, Cluj-Napoca, Romania*

Let $A, B \in M_2(C)$ be such that $2015AB - 2016BA = 2017I_2$. Prove that

$$(AB - BA)^2 = O_2.$$

Here, C is the set of complex numbers.

Solution 1 by Anthony J. Bevelacqua, University of North Dakota, Grand Forks, ND

Recall that the characteristic polynomial of a 2x2 matrix M is $p_M(x) = \det(M - xI_2)$. An easy calculation shows that $p_M(x) = x^2 - \text{tr}(M)x + \det(M)$ where $\det(M)$ is the determinate of M and $\text{tr}(M)$ is its trace. By the Cayley-Hamilton Theorem we have $p_M(M) = 0_2$.

We first note that AB and BA have the same characteristic polynomial $p(x)$ because $\det(AB) = \det(BA)$ and $\text{tr}(AB) = \text{tr}(BA)$.

We given $2015AB - 2016BA = 2017I_2$. Adding AB to both sides of this equation yields

$$2016(AB - BA) = AB + 2017I_2.$$

Taking the determinant of this we find

$$2016^2 \det(AB - BA) = \det(AB + 2017I_2) = p(-2017).$$

Similarly adding BA to both sides of the original equation and taking the determinant yields

$$2015^2 \det(AB - BA) = 2015^2 \det(AB - BA) = p(-2017).$$

Thus

$$2016^2 \det(AB - BA) = 2015^2 \det(AB - BA)$$

and so $\det(AB - BA) = 0$.

Since $\text{tr}(AB - BA)$=$\text{tr}(AB)$−$\text{tr}(BA) = 0$ we see that the characteristic polynomial of $AB - BA$ is x^2. Thus, $(AB - BA)^2 = 0_2$.

Essentially the same argument would establish the following mild generalization: Let $A, B \in M_2(K)$ were K is a field. Let $s, t \in K$ with $s \neq \pm 1$ and t$\neq 0$. Then $AB - sBA = tI_2$ implies $(AB - BA)^2 = 0_2$.

Solution 2 by Albert Stadler, Herrliberg, Switzerland

AB and BA have the same eigenvalues, since $\det(xI_2 - AB) = \det(xI_2 - BA)$. Indeed, when A is nonsingular this result follows from the fact that AB and BA are similar: $BA = A^{-1}(AB)A$.

For the case where both A and B are singular, one may remark that the desired identity is an equality between polynomials in x and the coefficients of the matrices. Thus, to prove this equality, it suffices to prove that it is veified on a non-empty open subset (for the usual topology, or, more generally, for the Zariski topology) of the space of all the coefficients. As the non-singular matrices form such an open subset of the space of all matrices, this proves the result.

Let x be an eigenvalue of AB. Then

$$\begin{aligned} 0 = \det(xI_2 - AB) &= \det\left(xI_2 - \frac{2016}{2015}BA - \frac{2017}{2015}I_2\right) \\ &= \frac{2016^2}{2015^2}\det\left(\frac{x - \frac{2017}{2015}}{\frac{2016}{2015}}I_2 - BA\right) \\ &= \frac{2016^2}{2015^2}\det\left(\frac{x - \frac{2017}{2015}}{\frac{2016}{2015}}I_2 - AB\right) \\ &= \frac{2016^2}{2015^2}\det\left(\frac{2015x - 2017}{2016}I_2 - AB\right). \quad (1) \end{aligned}$$

$\det(xI_2 - AB)$ is a quadratic polynomial in x, let's say $\det(xI_2 - AB) = ax^2 + bx + c$.
(1) then implies that
$ax^2 + bx + c = \frac{2016^2}{2015^2}\left(a\left(\frac{2015x - 2017}{2016}\right)^2 + b\left(\frac{2015 - 2017}{2016}\right) + c\right)$. We compare the coefficients of the polynomials and see that $b = 4034a,\ c = 2017^2a$.
So the quadratic polynomial reads as $ax^2 + 4034ax + 2017^2a = a(x + 2017)^2$ which shows that the characteristic polynomial of AB and BA has -2017 as a double zero, x is an eigenvector of both AB and BA corresponding to the eigenvalue -2017. Therefore there are numbers u and v such that AB is similar to $\begin{pmatrix} -2017 & u \\ 0 & -2017 \end{pmatrix}$
and BA is similar to $\begin{pmatrix} -2017 & v \\ 0 & -2017 \end{pmatrix}$. Therefore, $(AB - BA)^2$ is thus similar to
$\begin{pmatrix} 0 & u - v \\ 0 & 0 \end{pmatrix} = 0_2$, which implies that $(AB - BA)^2 = 0_2$.

Solution 3 by Michael N. Fried, Ben-Gurion University, Beer-Sheva, Israel

Let us write $[A, B]$ for $AB - BA$. Since $traceAB = traceBA$, we have, as is well-known, $trace[A, B] = 0$. Thus, keeping in mind that $[A, B]$ is a 2×2 matrix, the characteristic polynomial of $[A, B]$ is $x^2 + det[A, B] = 0$, so that if its eigenvalues are λ_1 and λ_2, we have $\lambda = \lambda_1 = -\lambda_2$ and $\lambda^2 = -det[A, B]$. Moreover, since every matrix satisfies its own characteristic polynomial,

$$[A, B]^2 = -det[A, B]I$$

Therefore, $[A, B]^2 = 0$, which is what we want to show, if and only if
$\lambda^2 = -det[A, B] = 0$, that is, if and only if $\lambda = 0$. We will show that, indeed, $\lambda = 0$
Consider the given equation $pAB - (p + 1)BA = (p + 2)I$. By adding BA or AB to both sides, we obtain, respectively:

$$p[A, B] = (p + 2)I + BA \quad (8)$$

$$(p + 1)[A, B] = (p + 2)I + AB \quad (9)$$

Let λ be an eigenvalue for $[A,B]$ and v the corresponding eigemvector. Thus, we have by (8):

$$p[A,B]v = p\lambda v = ((p+2)I + BA)v$$

Thus,

$$BAv = (p\lambda - (p+2))v$$

so that, $p\lambda - (p+2)$ is an eigenvalue for BA. Since $-\lambda$ is the other eigenvalue of $[A,B]$, we find that $-(p\lambda + (p+2))$ is the second eigenvalue of BA.
In the same way, using equation (9), we find the eigenvalues of AB to be
$(p+1)\lambda - (p+2)$ and $-((p+1)\lambda + (p+2))$
The determinant of any matrix is of course equal to the product of the eigenvalues. Moreover, $detAB = detBA$. Hence:

$$-(p\lambda + (p+2))(p\lambda - (p+2)) = -((p+1)\lambda + (p+2))((p+1)\lambda - (p+2))$$

From which we have:

$$((p+1)^2 - p^2)\lambda^2 = 0$$

So that $\lambda = 0$, which is what we wished to prove.

Solution 4 by Brian D. Beasley, Presbyterian College, Clinton, SC

Assume $A, B \in M_2(C)$ with $nAB - (n+1)BA = (n+2)I_2$ for some positive integer n. Write $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ and $B = \begin{bmatrix} e & f \\ g & h \end{bmatrix}$.
Then $(AB - BA)^2 = kI_2$, where

$$k = (bg - cf)^2 + (af + bh - be - df)(ce + dg - ag - ch).$$

By hypothesis, we have:

$$n(ae + bg) - (n+1)(ae + cf) = n + 2 \qquad n(af + bh) - (n+1)(be + df) = 0$$
$$n(cf + dh) - (n+1)(bg + dh) = n + 2 \qquad n(ce + dg) - (n+1)(ag + ch) = 0.$$

Thus $n(af + bh - be - df) = be + df$ and $n(ce + dg - ag - ch) = ag + ch$. Also, $(n+1)/n = (af + bh)/(be + df) = (ce + dg)/(ag + ch)$, and $ae - dh = (2n+1)(bg - cf)$.

Substituting yields

$$k = \frac{(ae - dh)(bg - cf)}{2n+1} + \frac{(be + df)(ag + ch)}{n^2}.$$

Then

$$
\begin{aligned}
(2n+1)k &= abeg - acef - bdgh + cdfh + \frac{2n+1}{n^2}(abeg + bceh + adfg + cdfh) \\
&= \left(\frac{n+1}{n}\right)^2 (abeg + cdfh) - (acef + bdgh) + \frac{2n+1}{n^2}(adfg + bceh) \\
&= \left(\frac{n+1}{n}\right)^2 [(ag+ch)(be+df) - adfg - bceh] - (acef + bdgh) + \frac{2n+1}{n^2}(adfg + bceh) \\
&= \left(\frac{n+1}{n}\right)^2 (ag+ch)(be+df) - (adfg + bceh) - (acef + bdgh) \\
&= \left(\frac{af+bh}{be+df}\right)\left(\frac{ce+dg}{ag+ch}\right)(ag+ch)(be+df) - (adfg + bceh) - (acef + bdgh) \\
&= (af+bh)(ce+dg) - (adfg + bceh) - (acef + bdgh) \\
&= 0.
\end{aligned}
$$

Hence $k = 0$ as needed.

Also solved by Moti Levy, Rehovot, Israel, and the proposer.

Mea Culpa

Paul M. Harms of North Newton, KS and Jeremiah Bartz of University of North Dakota, Grand Forks, ND should have each been credited with having solved problem 5403.