Problems Ted Eisenberg, Section Editor

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This section of the Journal offers readers an opportunity to exchange interesting mathematical problems and solutions. Please send them to Ted Eisenberg, Department of Mathematics, Ben-Gurion University, Beer-Sheva, Israel or fax to: 972-86-477-648. Questions concerning proposals and/or solutions can be sent e-mail to <eisenbt@013.net>. Solutions to previously stated problems can be seen at <http://www.ssma.org/publications>.

Solutions to the problems stated in this issue should be posted before March 15, 2012

- **5188:** *Proposed by Kenneth Korbin, New York, NY*

  Given $\triangle ABC$ with coordinates $A(-5,0), B(0,12)$ and $C(9,0)$. The triangle has an interior point P such that $\angle APB = \angle APC = 120^\circ$. Find the coordinates of point P.

- **5189:** *Proposed by Kenneth Korbin, New York, NY*

  Given triangle ABC with integer length sides and with $\angle A = 60^\circ$ and with $(a,b,c) = 1$. Find the lengths of b and c if

$$\begin{aligned} i)\ a &= 13, \text{ and if} \\ ii)\ a &= 13^2 = 169, \text{ and if} \\ iii)\ a &= 13^4 = 28561. \end{aligned}$$

- **5190:** *Proposed by Neculai Stanciu, Buzău, Romania*

  Prove: If x, y and z are positive integers such that $\dfrac{x(y+1)}{x-1} \in \mathrm{N}$, $\dfrac{y(z+1)}{y-1} \in \mathrm{N}$, and $\dfrac{z(x+1)}{z-1} \in \mathrm{N}$, then $xyz \le 693$.

- **5191:** *Proposed by José Luis Díaz-Barrero, Barcelona, Spain*

  Let a, b, c be positive real numbers such that $ab + bc + ca = 3$. Prove that

$$\frac{a\sqrt{bc} + b\sqrt{ca} + c\sqrt{ab}}{a^4 + b^4 + c^4} \le 1.$$

- **5192:** *Proposed by G. C. Greubel, Newport News, VA*

  Let $[n] = [n]_q = \dfrac{1-q^n}{1-q}$ be a q number and $\ln_q(x) = \displaystyle\sum_{n=1}^{\infty} \frac{x^n}{[n]}$ be a q-logarithm. Evaluate the following series:

$$i) \qquad \sum_{k=0}^{\infty} \frac{q^{mk}}{[mk+1][mk+m+1]} \text{ and}$$

$$ii) \quad \sum_{k=1}^{\infty} \frac{x^k}{[k][k+m]}.$$

- **5193:** *Proposed by Ovidiu Furdui, Cluj-Napoca, Romania*

  Let f be a function which has a power series expansion at 0 with radius of convergence R.

  a) Prove that $\displaystyle\sum_{n=1}^{\infty} n f^{(n)}(0)\left(e^x - 1 - \frac{x}{1!} - \frac{x^2}{2!} \cdots - \frac{x^n}{n!}\right) = \int_0^x e^{x-t} t f'(t)\,dt, \quad |x| < R.$

  b) Let α be a non-zero real number. Calculate $\displaystyle\sum_{n=1}^{\infty} n\alpha^n \left(e^x - 1 - \frac{x}{1!} - \frac{x^2}{2!} \cdots - \frac{x^n}{n!}\right).$

Solutions

- **5170:** *Proposed by Kenneth Korbin, New York, NY*

  Convex quadrilateral $DEFG$ has coordinates $D(-6,-3)$ and $E(2,12)$. The midpoints of the diagonals are on line l.

  Find the area of the quadrilateral if line l intersects line FG at point $P\left(\dfrac{672}{33}, \dfrac{-9}{11}\right)$.

  Solution 1 by Kee-Wai Lau, Hong Kong, China

  We show that the area of the quadrilateral is 378

  Let H and I be respectively the midpoints of the diagonals DF and EG. Let

$$F = (p,q) \quad \text{and} \quad G = (r,s) \text{ so that}$$

$$H = \left(\frac{p-6}{2}, \frac{q-3}{2}\right) \quad \text{and} \quad I = \left(\frac{r+2}{2}, \frac{s+12}{2}\right).$$

  Using the facts that the points H, I, and P lie on l and that P lies on FG, we obtain respectively the relations

$$\begin{aligned}(150+11s)p + (426-11r)\,q &= 7590 - 15r + 514s \qquad (1)\\ (9+11s)\,p + (224-11r)\,q &= 9r + 224s. \qquad (2)\end{aligned}$$

  By the standard formula, we find the area of the quadrilateral to be

$$\frac{(12-s)p + (r-2)q + 3r - 6s + 66}{2},$$

  which can be written as

$$\frac{\Big((150+11s)p + (426-11r)q\Big) - 2\Big((9+11s)p + (224-11r)q\Big) + 33r - 66s + 726}{22}.$$

By (1) and (2), the last expression equals

$$\frac{(7590 - 15r + 514s) - 2(9r + 224s) + 33r - 66s + 726}{22} = 378,$$

and this completes the solution.

Solution 2 by the proposer

Area of Quadrilateral DEFG

$$\begin{aligned}
&= 2\left[\text{Area } \triangle\text{DEP}\right] \\
&= \begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} \\
&= \begin{vmatrix} 2 & 12 & 1 \\ -6 & -3 & 1 \\ \dfrac{224}{11} & -\dfrac{9}{11} & 1 \end{vmatrix} = 378.
\end{aligned}$$

Reference, problem number 5033.

Comment by editor: **David Stone and John Hawkins of Statesboro, GA** showed in their solution that there are infinitely many quadrilaterals satisfying the given conditions of the problem, and that each has an area of 378. Their solution started off by showing that the simplest configuration occurs when the quadrilateral is a parallelogram so that the diagonals coincide. They then exhibited all such parallelograms and showed that each one has the stated area. Their solution of nine pages is too lengthy to reproduce here, but if you would like to see it, please contact me and I will send their solution to you in pdf format.

- **5171:** *Proposed by Kenneth Korbin, New York, NY*

  A triangle has integer length sides $x, x+y$, and $x+2y$.

  Part I: Find x and y if the inradius $r = 2011$.

  Part II: Find x and y if $r = \sqrt{2011}$.

Solution by Dionne Bailey, Elsie Campbell, and Charles Diminnie, San Angelo, TX

For convenience, let $a = x$, $b = x + y$, $c = x + 2y$ be the sides of the triangle. Then, since a, b, c are positive integers, it follows that x is a positive integer and y is an integer (which is not necessarily positive). The semiperimeter s is given by

$$s = \frac{a+b+c}{2} = \frac{3}{2}(x+y)$$

and we have

$$\begin{aligned} s-a &= s-x=\frac{x+3y}{2} \\ s-b &= s-(x+y)=\frac{x+y}{2} \qquad (1) \\ s-c &= s-(x+2y)=\frac{x-y}{2}. \end{aligned}$$

If K is the area of the triangle, then

$$sr=K=\sqrt{s\left(s-a\right)\left(s-b\right)\left(s-c\right)}$$

which reduces to

$$\frac{3}{2}\left(x+y\right)r^2=sr^2=\left(s-a\right)\left(s-b\right)\left(s-c\right)=\frac{x+3y}{2}\frac{x+y}{2}\frac{x-y}{2},$$

i.e.,

$$\left(x+3y\right)\left(x-y\right)=12r^2.$$

Note also that (1) implies that $x+3y$ and $x-y$ are positive integers since x and y are integers and $s-a$ and $s-c$ are positive. Further, if

$$\begin{aligned} x+3y &= k_1 \\ x-y &= k_2 \end{aligned}$$

for positive integers k_1 and k_2 such that $k_1k_2=12r^2$, then at least one of k_1, k_2 is even. Finally, since $4y=k_1-k_2$, it follows that k_1 and k_2 must both be even.

Part I: If $r=2011$, then $12r^2=12\left(2011\right)^2$ and the possibilities for k_1 and k_2 are

$$\begin{aligned} \left(k_1,k_2\right) \in \{&\left(2,6\cdot 2011^2\right),\left(6\cdot 2011^2,2\right),\left(4022,12066\right),\left(12066,4022\right), \\ &\left(6,2\cdot 2011^2\right),\left(2\cdot 2011^2,6\right)\}. \end{aligned}$$

If

$$\begin{aligned} x-y &= 2 \\ x+3y &= 6\cdot 2011^2 \end{aligned}$$

then $x=6,066,183$, $y=6,066,181$, while if

$$\begin{aligned} x-y &= 6\cdot 2011^2 \\ x+3y &= 2 \end{aligned}$$

then $x=18,198,545$, $y=-6,066,181$. The steps in the remaining cases are similar and the results are summarized in the following table:

x	y	a	b	c
6,066,183	6,066,181	6,066,183	12,132,364	18,198,545
18,198,545	−6,066,181	18,198,545	12,132,364	6,066,183
2,022,065	2,022,059	2,022,065	4,044,124	6,066,183
6,066,183	−2,022,059	6,066,183	4,044,124	2,022,065
6,033	2,011	6,033	8,044	10,055
10,055	−2,011	10,055	8,044	6,033

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Part II: If $r = \sqrt{2011}$, then $12r^2 = 12 \cdot 2011$ and the possibilities for k_1, k_2 are

$$(k_1, k_2) \in \{(2, 12066), (12066, 2), (6, 4022), (4022, 6)\}.$$

If we solve the system

$$\begin{aligned} x - y &= k_1 \\ x + 3y &= k_2 \end{aligned}$$

for each of these possibilities, the results are:

$\underline{x}$	$\underline{y}$	$\underline{a}$	$\underline{b}$	$\underline{c}$
3,018	3,016	3,018	6,034	9,050
9,050	−3,016	9,050	6,034	3,018
1,010	1,004	1,010	2,014	3,018
3,018	−1,004	3,018	2,014	1,010

Remark: In each situation where the assignments for k_1 and k_2 were reversed, we obtained different values for x and y but the triangle was essentially the same (with the values of a and c reversed).

Comment by editor: **David Stone and John Hawkins of Statesboro, GA** solved the more general problem for a triangle having its sides in the arithmetic progression of $x, x+y$, and $x+2y$ by finding x and y if the inradius $r = p^{m/2}$ where p is an odd prime and $m \geq 1$. For $p \geq 5$ they showed that there are $m+1$ solutions and they described them. For $p = 3$ they showed that there are $\lfloor \frac{m+2}{2} \rfloor$ and also described them.

Also solved by Brian D. Beasley, Clinton, SC; Valmir Bucaj (student, Texas Lutheran University), Seguin, TX; Bruno Salgueiro Fanego, Viveiro, Spain; Paul M. Harms, North Newton, KS; Enkel Hysnelaj, University of Technology, Sydney, Australia jointly with Elton Bojaxhiu, Kriftel, Germany; Kee-Wai Lau, Hong Kong, China; David E. Manes, Oneonta, NY; Boris Rays, Brooklyn, NY; David Stone and John Hawkins (jointly), Statesboro, GA; Titu Zvonaru, Comănesti, Romania jointly with Neculai Stanciu, Buzău, Romania, and the proposer.

- **5172:** *Proposed by Neculai Stanciu, Buzău, Romania*

If a, b and c are positive real numbers, then prove that,

$$\frac{a(b-c)}{c(a+b)} + \frac{b(c-a)}{a(b+c)} + \frac{c(a-b)}{b(c+a)} \geq 0.$$

Solution 1 by Albert Stadler, Herrliberg, Switzerland

We have

$$\frac{a(b-c)}{c(a+b)} + \frac{b(c-a)}{a(b+c)} + \frac{c(a-b)}{b(c+a)} = \frac{a^3b^3 + b^3c^3 + c^3a^3 - a^3b^2c - b^3c^2a - c^3a^2b}{abc(a+b)(b+c)(c+a)}. \qquad (1)$$

By the weighted AM-GM inequality,

$$\frac{2}{3}a^3b^3 + \frac{1}{3}c^3a^3 \geq a^3b^2c,$$

$$\begin{aligned}\frac{2}{3}b^3c^3+\frac{1}{3}a^3b^3 &\geq b^3c^2a,\\ \frac{2}{3}c^3a^3+\frac{1}{3}b^3c^3 &\geq c^3a^2b.\end{aligned}$$

If we these inequalities up we that the numerator of (1) is nonnegative, and the problem statement follows.

Solution 2 by Kee-Wai Lau, Hong Kong, China

Since

$$\frac{a(b-c)}{c(a+b)}+\frac{b(c-a)}{a(b+c)}+\frac{c(a-b)}{b(c+a)}$$

$$= \frac{a^3b^3+b^3c^3+c^3a^3-a^3b^3c-b^3c^2a-c^3a^2b}{abc(a+b)(b+c)(c+a)}$$

$$= \frac{a^3(b-c)^2(2b+c)+b^3(c-a)^2(2c+a)+c^3(a-b)^2(2a+b)}{3abc(a+b)(b+c)(c+a)},$$

the inequality of the problem follows.

Also solved by Arkady Alt, San Jose, CA; Michael Brozinsky, Central Islip, NY; Enkel Hysnelaj, University of Technology, Sydney, Australia jointly with Elton Bojaxhiu, Kriftel, Germany; Paul M. Harms, North Newton, KS; David E. Manes, Oneonta, NY; Paolo Perfetti, Department of Mathematics, "Tor Vergatta" University, Rome, Italy, and the proposer.

- **5173:** *Proposed by Pedro H. O. Pantoja, UFRN, Brazil*

Find all triples x, y, z of non-negative real numbers that satisfy the system of equations,

$$\begin{cases} x^2(2x^2+x+2)=xy(3x+3y-z)\\ y^2(2y^2+y+2)=yz(3y+3z-x)\\ z^2(2z^2+z+2)=xz(3z+3x-y)\end{cases}$$

Solution by Enkel Hysnelaj, University of Technology, Sydney, Australia and Elton Bojaxhiu, Kriftel, Germany

Assume for the moment that $x\neq 0, y\neq 0, z\neq 0$.
Without loss of generality, we may assume that $x \geq y$. Looking at equations (1) and (2) in the statement of the problem and using the fact that x, y, z are non-negative real numbers, we observe

$$\begin{aligned} x^2(2x^2+x+2)\geq y^2(2y^2+y+2) \quad &\Rightarrow \quad xy(3x+3y-z)\geq yz(3y+3z-x)\\ &\Rightarrow \quad x(3x+3y-z)\geq z(3y+3z-x)\end{aligned}$$

$$\begin{aligned}
&\Rightarrow \quad 3x^2+3xy-xz \geq 3yz+3z^2-xz \\
&\Rightarrow \quad 3(x-z)(x+y+z) \geq 0 \\
&\Rightarrow \quad x \geq z
\end{aligned}$$

Looking at equations (1) and (3) and using the fact that x, y, z are non-negative real numbers, we observe

$$\begin{aligned}
x^2(2x^2+x+2) \geq z^2(2z^2+z+2) \quad &\Rightarrow \quad xy(3x+3y-z) \geq zx(3z+3x-y) \\
&\Rightarrow \quad y(3x+3y-z) \geq z(3z+3x-y) \\
&\Rightarrow \quad 3xy+3y^2-yz \geq 3z^2+3xz-yz \\
&\Rightarrow \quad 3(y-z)(x+y+z) \geq 0 \\
&\Rightarrow \quad y \geq z
\end{aligned}$$

Similarly, focusing on equations (2) and (3) and using the fact that x, y, z are non-negative real numbers, we observe

$$\begin{aligned}
y^2(2y^2+y+2) \geq z^2(2z^2+z+2) \quad &\Rightarrow \quad yz(3y+3z-x) \geq zx(3z+3x-y) \\
&\Rightarrow \quad y(3y+3z-x) \geq x(3z+3x-y) \\
&\Rightarrow \quad 3y^2+3yz-xy \geq 3xz+3x^2-xy \\
&\Rightarrow \quad 3(y-x)(x+y+z) \geq 0 \\
&\Rightarrow \quad y \geq x.
\end{aligned}$$

This implies that $x = y$. In a similar manner we can prove that $y = z$ and substituting this into equation (1) we obtain

$$x^2(2x^2+x+2) = x^2(3x+3x-x) = 0 \Rightarrow 2(x-1)^2 = 0 \Rightarrow x = 1.$$

So a solution will be $(x, y, z) = (1, 1, 1)$.

Substituting $x = 0$ into equation (3) implies that $z^2(2z^2+z+2) = 0$, so either $z = 0$ or $2z^2+z+2 = 0$. It is easy to see that $2z^2+z+2 = 0$ does not have real roots, so we are left with the option that $z = 0$. Similarly, substituting $z = 0$ into equation (2) gives $y = 0$.

Therefore the set of real valued solutions for the given system is $(x, y, z) = \{(0, 0, 0), (1, 1, 1)\}$.

Also solved by Arkady Alt, San Jose, CA; Kee-Wai Lau, Hong Kong, China; David E. Manes, Oneonta, NY; Paolo Perfetti, Department of Mathematics, "Tor Vergatta" University, Rome, Italy; Albert Stadler, Herrliberg, Switzerland; Neculai Stanciu with Titu Zvonaru (jointly), from Buzău and Comănesti, Romania respectively, and the proposer.

- **5174:** *Proposed by José Luis Díaz-Barrero, Barcelona, Spain*

Let n be a positive integer. Compute:

$$\lim_{n\to\infty} \frac{n^2}{2^n} \sum_{k=0}^{n} \frac{k+4}{(k+1)(k+2)(k+3)} \binom{n}{k}.$$

Solution 1 by Ángel Plaza, University of Las Palmas de Gran Canaria, Spain

$(1+x)^n = \sum_{k=0}^{n} \binom{n}{k} x^k$, and by integration $\dfrac{(1+x)^{n+1}-1}{n+1} = \sum_{k=0}^{n} \binom{n}{k} \dfrac{x^{k+1}}{k+1}$.

Iterating the same technique, it is obtained:

$$\frac{(1+x)^{n+2}-(n+2)x-1}{(n+1)(n+2)} = \sum_{k=0}^{n} \binom{n}{k} \frac{x^{k+2}}{(k+1)(k+2)}.$$

$$\frac{(1+x)^{n+3}-(n+2)(n+3)x^2/2-(n+3)x-1}{(n+1)(n+2)(n+3)} = \sum_{k=0}^{n} \binom{n}{k} \frac{x^{k+3}}{(k+1)(k+2)(k+3)}.$$

Now, multiplying each term of the preceding equation by x, differentiating with respect to x and letting $x=1$, we obtain

$$\frac{(n+5)2^{n+2}-3(n+2)(n+3)/2-2(n+3)-1}{(n+1)(n+2)(n+3)} = \sum_{k=0}^{n} \binom{n}{k} \frac{(k+4)}{(k+1)(k+2)(k+3)}.$$

And therefore, the proposed limit becomes

$$\begin{aligned} L &= \lim_{n\to\infty} \frac{n^2}{2^n} \frac{(n+5)2^{n+2}-3(n+2)(n+3)/2-2(n+3)-1}{(n+1)(n+2)(n+3)} \\ &= \lim_{n\to\infty} \frac{n^2}{2^n} \frac{(n+5)2^{n+2}}{n^3} = 4. \end{aligned}$$

Solution 2 by Anastasios Kotronis, Athens, Greece

For $n \in N$ and and $x \in \Re$ we have $(1+x)^n = \sum_{k=0}^{n} \binom{n}{k} x^k$ so $x^4(1+x)^n = \sum_{k=0}^{n} \binom{n}{k} x^{k+4}$.
Now differentiate to obtain

$$\begin{aligned} 4x^3(1+x)^n + nx^4(1+x)^{n-1} &= \sum_{k=0}^{n} (k+4) \binom{n}{k} x^{k+3}, \quad \text{so} \\ 4(1+x)^n + nx(1+x)^{n-1} &= \sum_{k=0}^{n} (k+4) \binom{n}{k} x^k. \end{aligned}$$

Now integrate on $[0,x]$ to obtain

$$\frac{3(1+x)^{n+1}}{n+1} + x(1+x)^n - \frac{3}{n+1} = \sum_{k=0}^{n} \frac{(k+4)}{k+1} \binom{n}{k} x^{k+1}.$$

Integrating once again gives us

$$\frac{2(1+x)^{n+2}}{(n+1)(n+2)} + \frac{x(1+x)^{n+1}}{n+1} - \frac{3x}{n+1} - \frac{2}{(n+1)(n+2)} = \sum_{k=0}^{n} \frac{(k+4)}{(k+1)(k+2)} \binom{n}{k} x^{k+2}.$$

And by integrating still again gives us

$$\sum_{k=0}^{n} \frac{(k+4)}{(k+1)(k+2)(k+3)} \binom{n}{k} x^{k+3} =$$

$$\frac{(1+x)^{n+3}}{(n+1)(n+2)(n+3)}+\frac{x(1+x)^{n+2}}{(n+1)(n+2)}-\frac{3x^2}{2(n+1)}-\frac{2x}{(n+1)(n+2)}-\frac{1}{(n+1)(n+2)(n+3)}.$$

Setting $x=1$ above, we easily see that

$$\frac{n^2}{2^n}\sum_{k=0}^{n}\frac{(k+4)}{(k+1)(k+2)(k+3)}\binom{n}{k}\overset{n\to+\infty}{\longrightarrow}4.$$

Also solved by Arkady Alt, San Jose, CA; Dionne Bailey, Elsie Campbell, and Charles Diminnie (jointly), San Angelo, TX; Enkel Hysnelaj, University of Technology, Sydney, Australia jointly with Elton Bojaxhiu, Kriftel, Germany; Kee-Wai Lau, Hong Kong, China; David E. Manes, Oneonta, NY; Paolo Perfetti, Department of Mathematics, "Tor Vergatta" University, Rome, Italy; Albert Stadler, Herrliberg, Switzerland; David Stone and John Hawkins (jointly), Statesboro, GA, and the proposer.

- **5175:** *Proposed by Ovidiu Furdui, Cluj-Napoca, Romania*

Find the value of,

$$\lim_{n\to\infty}\frac{1}{n}\sum_{i,j=1}^{n}\frac{i+j}{i^2+j^2}.$$

Solution 1 by Kee-Wai Lau, Hong Kong, China

We first note by symmetry that

$$\sum_{i=1}^{n}\sum_{j=1}^{n}\frac{i+j}{i^2+j^2}=2\sum_{i=1}^{n}\sum_{j=1}^{n}\frac{i+j}{i^2+j^2}-\sum_{i=1}^{n}\frac{1}{i}. \qquad (1)$$

It is well known that for a sequence $\{a_n\}$ such that $\lim_{n\to\infty}a_n=l$ then $\lim_{n\to\infty}\frac{\sum_{i=1}^{n}a_i}{n}=l$ as well. Hence, it follows from (1) that

$$\begin{aligned}
&\lim_{n\to\infty}\frac{1}{n}\sum_{i=1}^{n}\sum_{j=1}^{n}\frac{i+j}{i^2+j^2}\\
&= 2\lim_{n\to\infty}\sum_{j=1}^{n}\frac{n+j}{n^2+j^2}-\lim_{n\to\infty}\frac{1}{n}\\
&= 2\lim_{n\to\infty}\sum_{j=1}^{n}\frac{1+\frac{j}{n}}{n\left(1+\left(\frac{j}{n}\right)^2\right)}\\
&= 2\int_0^1\frac{1+x}{1+x^2}dx
\end{aligned}$$

$$= \left[2\arctan(x) + \ln(1+x^2)\right]\Big|_0^1$$

$$= \frac{\pi}{2} + \ln 2.$$

Solution 2 by Paolo Perfetti, Department of Mathematics, "Tor Vergatta" University, Rome, Italy

Answer: $\dfrac{\pi}{2} + \ln 2$

Proof: The limit is actually

$$\lim_{n\to\infty} \frac{1}{n^2} \sum_{i,j=1}^{n} \frac{\dfrac{i}{n} + \dfrac{j}{n}}{\dfrac{i^2}{n^2} + \dfrac{j^2}{n^2}}$$

which is the Riemann–sum of

$$\int\int_{[0,1]^2} \frac{x+y}{x^2+y^2} dxdy = 2\int\int_{[0,1]^2} \frac{x}{x^2+y^2} dxdy \;=\; I$$

$$I = \int_0^1 \left[\left(\ln(x^2+y^2)\right)\Big|_0^1\right] dy = \int_0^1 \left(\ln(1+y^2) - 2\ln y\right) dy.$$

Integrating by parts,

$$\int_0^1 \ln(1+y^2)dy = y\ln(1+y^2)\Big|_0^1 - 2\int_0^1 \frac{y^2}{1+y^2} dy$$

$$= \ln 2 - 2\int_0^1 \left(1 - \frac{1}{1+y^2}\right) dy$$

$$= \ln 2 - 2 + 2\arctan y\Big|_0^1 = \ln 2 - 2 + 2\left(\frac{\pi}{4}\right).$$

Moreover,

$$-2\int_0^1 \ln y dy = -2(y\ln y - y)\Big|_0^1 = 2$$

from which the result follows by summing the two integrals.

Comment by editor: Many of the solvers approached the problem in a similar manner as Paolo, by showing that

$$\frac{1}{n^2} \sum_{i,j=1}^{n} \frac{\dfrac{i}{n} + \dfrac{j}{n}}{\left(\dfrac{i}{n}\right)^2 + \left(\dfrac{j}{n}\right)^2} \Longrightarrow \int_0^1\int_0^1 \frac{x+y}{x^2+y^2} dxdy \text{ as } n\to\infty,$$

but they raised the caveat that we must be careful in applying the limit because the function $\phi(x,y) = \dfrac{x+y}{x^2+y^2}$ is not continuous at $(x,y) = (0,0)$. They then showed that in this case, the limit does indeed hold.

Also solved by Arkady Alt, San Jose, CA; Anastasios Kotronis, Athens, Greece; Enkel Hysnelaj, University of Technology, Sydney, Australia jointly with Elton Bojaxhiu, Kriftel, German; David E. Manes, Oneonta, NY; Albert Stadler, Herrliberg, Switzerland; David Stone and John Hawkins (jointly), Statesboro, GA, and the proposer.