Problems

Ted Eisenberg, Section Editor

****

This section of the Journal offers readers an opportunity to exchange interesting mathematical problems and solutions. Please send them to Ted Eisenberg, Department of Mathematics, Ben-Gurion University, Beer-Sheva, Israel or fax to: 972-86-477-648. Questions concerning proposals and/or solutions can be sent e-mail to <eisenbt@013.net>. Solutions to previously stated problems can be seen at <http://www.ssma.org/publications>.

Solutions to the problems stated in this issue should be posted before March 15, 2014

- **5283:** *Proposed by Kenneth Korbin, New York, NY*

  Find the sides of two different isosceles triangles that both have perimeter 162 and area 1008.

- **5284:** *Proposed by Tom Moore, Bridgewater State University, Bridgewater, MA*

  Prove:

  a) $3^{3^n} + 1 \equiv 0 \bmod 28, \ \forall \text{n} \geq 1,$

  b) $3^{3^n} + 1 \equiv 0 \bmod 532, \ \forall \text{n} \geq 2,$

  c) $3^{3^n} + 1 \equiv 0 \bmod 19684, \ \forall \text{n} \geq 3,$

  d) $3^{3^n} + 1 \equiv 0 \bmod 3208492, \ \forall \text{n} \geq 4.$

- **5285:** *Proposed by D.M. Bătinetu–Giurgiu, "Matei Basarab" National College, Bucharest, Romania and Neculai Stanciu, "Geroge Emil Palade" General School, Buzău, Romania*

  Let $\{a_n\}_{n\geq 1}$, and $\{b_n\}_n \geq 1$ be positive sequences of real numbers with $\lim\limits_{n\to\infty} (a_{n+1} - a_n) = a \in \Re_+$ and $\lim\limits_{n\to\infty} \dfrac{b_{n+1}}{nb_n} = b \in \Re_+$.
  For $x \in \Re$, calculate

$$\lim_{n\to\infty} \left(a_n^{\sin^2 x} \left(\left(\sqrt[n+1]{b_{n+1}} \right)^{\cos^2 x} - \left(\sqrt[n]{b_n} \right)^{\cos^2 x} \right) \right).$$

- **5286:** *Proposed by Michael Brozinsky, Central Islip, NY*

  In Cartesianland, where immortal ants live, an ant is assigned a specific equilateral triangle EFG and three distinct positive numbers $0 < a < b < c$. The ant's job is to find a unique point $P(x, y)$ such that the distances from P to the vertices E, F and G of his triangle are proportionate to $a : b : c$ respectively. Some ants are eternally doomed to an impossible search. Find a relationship between a, b and c that guarantees eventual success; i.e., that such a unique point P actually exists.

- **5287:** *Proposed by José Luis Díaz-Barrero, Barcelona Tech, Barcelona, Spain*

  Let u, v, w, x, y, z be complex numbers. Prove that

$$2\text{Re}(ux + vy + zw) \le 3\left(|u|^2 + |v|^2 + |w|^2\right) + \frac{1}{3}\left(|x|^2 + |y|^2 + |z|^2\right).$$

- **5288:** *Proposed by Ovidiu Furdui, Technical University of Cluj-Napoca, Cluj-Napoca, Romania*

  Let $a, b, c \ge 0$ be real numbers. Find the value of

$$\lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} \sum_{j=1}^{n} \frac{1}{\sqrt{i^2 + j^2 + ai + bj + c}}.$$

Solutions

- **5265:** *Proposed by Kenneth Korbin, New York, NY*

  Find positive integers x and y such that

$$2x - y - \sqrt{3x^2 - 3xy + y^2} = 2014,$$

with $(x, y) = 1$.

Solution 1 by G. C. Greubel, Newport News, VA

The process to be considered, for a slightly general class of values, can be seen as follows. Consider the equation

$$2x - y - \sqrt{3x^2 - 3xy + y^2} = a \qquad (1)$$

for which rearranging terms and squaring both sides leads to

$$\begin{aligned} 3x^2 - 3xy + y^2 &= (2x - y - a)^2 \\ 3x^2 - 3xy + y^2 &= 4x^2 + y^2 + a^2 + 2(-2xy - 2ax + ay) \end{aligned}$$

or

$$\begin{aligned} y &= \frac{x^2 - 4ax + a^2}{x - 2a} \\ &= \frac{(x^2 - 4ax + 4a^2) - 3a^2}{x - 2a} \\ &= \frac{(x - 2a)^2 - 3a^2}{x - 2a} \\ y &= x - 2a - \frac{3a^2}{x - 2a}. \end{aligned}$$

This equation provides y in terms of x for a given x. The relations for x and y can be put into a "parametric" form by making the substitution

$$u = \frac{3a^2}{x - 2a} \quad \text{and} \quad v = \frac{3a^2}{u}.$$

From this it can now be seen that

$$\begin{aligned} x &= v + 2a \\ y &= v - u \\ uv &= 3a^2. \end{aligned}$$

It is readily seen that the possible factors of a are the primary values used in u and v. This is to say that if a is a product of four integers, say $\{a_i\}_{1\leq i\leq 4}$, rasied to powers b_i then

$$uv = 3_1^{a\,2b_1} a_2^{2b_2} a_3^{2b_3} a_4^{2b_4}$$

and leads to the forms of u and v being of the form

$$u = 3^{\alpha_1} a_1^{\alpha_2} a_2^{\alpha_3} a_3^{\alpha_4} a_4^{\alpha_5} \quad \text{and} \quad v = 3^{\beta_1} a_1^{\beta_2} a_2^{\beta_3} a_3^{\beta_4} a_4^{\beta_5}, \tag{2}$$

where $\alpha_1 + \beta_1 = 1$ and $\alpha_i + \beta_i = 2b_{i-1}$ for $2 \leq i \leq 5$.

Now, returning to equation (1) it can also be seen in the form

$$(2x - y) - \sqrt{(2x - y)^2 - x(x - y)} = a.$$

Invoking the conditions x and y be positive integers leads to the following conditions. If $x - y = 0$ then this reduces to $0 = a$ which is invalid for all $a \neq 0$. In the case $x - y < 0$ the reduction is seen to be

$$x - |x - y| - \sqrt{(x - |x - y|)^2 + x|x - y|} = a.$$

This equation is also invalid for $a > 0$. The remaining condition $x > y$ is the only option for $a > 0$, $x > 0$ and $y > 0$. In order to be completely valid the statement should be $x > y > 0$ for $a > 0$.

Also by rearranging the equation into the form

$$\sqrt{3x^2 - 3xy + y^2} = 2x - y - a$$

which, for positive integer values x and y, leads to the square root being positive and the condition $2x - y - a \geq 0$ or $2x - y \geq 0$. The conditions $x > y > 0$ and $2x - y \geq 0$ can also be stated as $v > u$ and $u + v + 3a \geq 0$.

Introducing the additional condition $(x, y) = p$ then $p|a$, $p|x$ and $p|y$, or p is the divisor of a, x and y. This condition leads to only relatively prime solutions are considered as solutions of this particular problem.

$\mathbf{a = 2014}$.

With $a = 2014$ it is quickly seen that the factors are 2, 19, and 53, i.e., $a = 2 \cdot 19 \cdot 53$ and $3a^2 = 3 \cdot 2^2 \cdot 19^2 \cdot 53^2$. The possible factors from this factorable set, in view of equation (2), is seen by:

factors of $uv = 3(2014)^2$

u	v	u	v
$3^0 \cdot 2^0 \cdot 19^0 \cdot 53^0$	$3^1 \cdot 2^2 \cdot 19^2 \cdot 53^2$	$3^1 \cdot 2^0 \cdot 19^0 \cdot 53^0$	$3^0 \cdot 2^2 \cdot 19^2 \cdot 53^2$
$3^0 \cdot 2^2 \cdot 19^0 \cdot 53^0$	$3^1 \cdot 2^0 \cdot 19^2 \cdot 53^2$	$3^1 \cdot 2^2 \cdot 19^0 \cdot 53^0$	$3^0 \cdot 2^0 \cdot 19^2 \cdot 53^2$
$3^0 \cdot 2^0 \cdot 19^2 \cdot 53^0$	$3^1 \cdot 2^2 \cdot 19^0 \cdot 53^2$	$3^1 \cdot 2^0 \cdot 19^2 \cdot 53^0$	$3^0 \cdot 2^2 \cdot 19^0 \cdot 53^2$
$3^0 \cdot 2^0 \cdot 19^0 \cdot 53^2$	$3^1 \cdot 2^2 \cdot 19^2 \cdot 53^0$	$3^1 \cdot 2^0 \cdot 19^0 \cdot 53^2$	$3^0 \cdot 2^2 \cdot 19^2 \cdot 53^0$
$3^0 \cdot 2^2 \cdot 19^2 \cdot 53^0$	$3^1 \cdot 2^0 \cdot 19^0 \cdot 53^2$	$3^1 \cdot 2^2 \cdot 19^2 \cdot 53^0$	$3^0 \cdot 2^0 \cdot 19^0 \cdot 53^2$
$3^0 \cdot 2^2 \cdot 19^0 \cdot 53^2$	$3^1 \cdot 2^0 \cdot 19^2 \cdot 53^0$	$3^1 \cdot 2^2 \cdot 19^0 \cdot 53^2$	$3^0 \cdot 2^0 \cdot 19^2 \cdot 53^0$
$3^0 \cdot 2^2 \cdot 19^2 \cdot 53^2$	$3^1 \cdot 2^0 \cdot 19^0 \cdot 53^0$	$3^1 \cdot 2^2 \cdot 19^2 \cdot 53^2$	$3^0 \cdot 2^0 \cdot 19^0 \cdot 53^0$

Invoking the condition $v > u$ then the possible values are preceded by an asterisk $*$:

factors of $uv = 3(2014)^2$

u	v	u	v
$*3^0 \cdot 2^0 \cdot 19^0 \cdot 53^0$	$*3^1 \cdot 2^2 \cdot 19^2 \cdot 53^2$	$*3^1 \cdot 2^0 \cdot 19^0 \cdot 53^0$	$*3^0 \cdot 2^2 \cdot 19^2 \cdot 53^2$
$*3^0 \cdot 2^2 \cdot 19^0 \cdot 53^0$	$*3^1 \cdot 2^0 \cdot 19^2 \cdot 53^2$	$*3^1 \cdot 2^2 \cdot 19^0 \cdot 53^0$	$*3^0 \cdot 2^0 \cdot 19^2 \cdot 53^2$
$*3^0 \cdot 2^0 \cdot 19^2 \cdot 53^0$	$*3^1 \cdot 2^2 \cdot 19^0 \cdot 53^2$	$*3^1 \cdot 2^0 \cdot 19^2 \cdot 53^0$	$*3^0 \cdot 2^2 \cdot 19^0 \cdot 53^2$
$*3^0 \cdot 2^0 \cdot 19^0 \cdot 53^2$	$*3^1 \cdot 2^2 \cdot 19^2 \cdot 53^0$	$3^1 \cdot 2^0 \cdot 19^0 \cdot 53^2$	$3^0 \cdot 2^2 \cdot 19^2 \cdot 53^0$
$*3^0 \cdot 2^2 \cdot 19^2 \cdot 53^0$	$*3^1 \cdot 2^0 \cdot 19^0 \cdot 53^2$	$3^1 \cdot 2^2 \cdot 19^2 \cdot 53^0$	$3^0 \cdot 2^0 \cdot 19^0 \cdot 53^2$
$3^0 \cdot 2^2 \cdot 19^0 \cdot 53^2$	$3^1 \cdot 2^0 \cdot 19^2 \cdot 53^0$	$3^1 \cdot 2^2 \cdot 19^0 \cdot 53^2$	$3^0 \cdot 2^0 \cdot 19^2 \cdot 53^0$
$3^0 \cdot 2^2 \cdot 19^2 \cdot 53^2$	$3^1 \cdot 2^0 \cdot 19^0 \cdot 53^0$	$3^1 \cdot 2^2 \cdot 19^2 \cdot 53^2$	$3^0 \cdot 2^0 \cdot 19^0 \cdot 53^0$

These eight value pairs for u and v lead to the eight value pairs of x and y, with $(x, y) = 1$, being

x	y
12,172,616	12,168,587
4,060,224	4,056,193
3,046,175	3,042,143
1,018,077	1,014,037
37,736	33,347
15,264	10,153
12,455	6,983
8,360	1,523.

a = 2015

With $a = 2015$ it is quickly seen that the factors are $5, 13$, and 31, i.e., $a = 5 \cdot 13 \cdot 31$ and $3a^2 = 3 \cdot 5^2 \cdot 13^2 \cdot 31^2$. The possible factors from this factorable set, in view of equation (2), is seen by:

factors of $uv = 3(2015)^2$

u	v	u	v
$3^0 \cdot 5^0 \cdot 13^0 \cdot 31^0$	$3^1 \cdot 5^2 \cdot 13^2 \cdot 31^2$	$3^1 \cdot 5^0 \cdot 13^0 \cdot 31^0$	$3^0 \cdot 5^2 \cdot 13^2 \cdot 31^2$
$3^0 \cdot 5^2 \cdot 13^0 \cdot 31^0$	$3^1 \cdot 5^0 \cdot 13^2 \cdot 31^2$	$3^1 \cdot 5^2 \cdot 13^0 \cdot 31^0$	$3^0 \cdot 5^0 \cdot 13^2 \cdot 31^2$
$3^0 \cdot 5^0 \cdot 13^2 \cdot 31^0$	$3^1 \cdot 5^2 \cdot 13^0 \cdot 31^2$	$3^1 \cdot 5^0 \cdot 13^2 \cdot 31^0$	$3^0 \cdot 5^2 \cdot 13^0 \cdot 31^2$
$3^0 \cdot 5^0 \cdot 13^0 \cdot 31^2$	$3^1 \cdot 5^2 \cdot 13^2 \cdot 31^0$	$3^1 \cdot 5^0 \cdot 13^0 \cdot 31^2$	$3^0 \cdot 5^2 \cdot 13^2 \cdot 31^0$
$3^0 \cdot 5^2 \cdot 13^2 \cdot 31^0$	$3^1 \cdot 5^0 \cdot 13^0 \cdot 31^2$	$3^1 \cdot 5^2 \cdot 13^2 \cdot 31^0$	$3^0 \cdot 5^0 \cdot 13^0 \cdot 31^2$
$3^0 \cdot 5^2 \cdot 13^0 \cdot 31^2$	$3^1 \cdot 5^0 \cdot 13^2 \cdot 31^0$	$3^1 \cdot 5^2 \cdot 13^0 \cdot 31^2$	$3^0 \cdot 5^0 \cdot 13^2 \cdot 31^0$
$3^0 \cdot 5^2 \cdot 13^2 \cdot 31^2$	$3^1 \cdot 5^0 \cdot 13^0 \cdot 31^0$	$3^1 \cdot 5^2 \cdot 13^2 \cdot 31^2$	$3^0 \cdot 5^0 \cdot 13^0 \cdot 31^0$

Invoking the condition $v > u$ then the possible values are preceded by an asterisk $*$:,

factors of $uv = 3(2015)^2$

u	v	u	v
$*3^0 \cdot 5^0 \cdot 13^0 \cdot 31^0$	$*3^1 \cdot 5^2 \cdot 13^2 \cdot 31^2$	$*3^1 \cdot 5^0 \cdot 13^0 \cdot 31^0$	$*3^0 \cdot 5^2 \cdot 13^2 \cdot 31^2$
$3^0 \cdot 5^2 \cdot 13^0 \cdot 31^0$	$*3^1 \cdot 5^0 \cdot 13^2 \cdot 31^2$	$*3^1 \cdot 5^2 \cdot 13^0 \cdot 31^0$	$*3^0 \cdot 5^0 \cdot 13^2 \cdot 31^2$
$3^0 \cdot 5^0 \cdot 13^2 \cdot 31^0$	$*3^1 \cdot 5^2 \cdot 13^0 \cdot 31^2$	$*3^1 \cdot 5^0 \cdot 13^2 \cdot 31^0$	$*3^0 \cdot 5^2 \cdot 13^0 \cdot 31^2$
$3^0 \cdot 5^0 \cdot 13^0 \cdot 31^2$	$3^1 \cdot 5^2 \cdot 13^2 \cdot 31^0$	$*3^1 \cdot 5^0 \cdot 13^0 \cdot 31^2$	$*3^0 \cdot 5^2 \cdot 13^2 \cdot 31^0$
$3^0 \cdot 5^2 \cdot 13^2 \cdot 31^0$	$3^1 \cdot 5^0 \cdot 13^0 \cdot 31^2$	$3^1 \cdot 5^2 \cdot 13^2 \cdot 31^0$	$3^0 \cdot 5^0 \cdot 13^0 \cdot 31^2$
$3^0 \cdot 5^2 \cdot 13^0 \cdot 31^2$	$3^1 \cdot 5^0 \cdot 13^2 \cdot 31^0$	$3^1 \cdot 5^2 \cdot 13^0 \cdot 31^2$	$3^0 \cdot 5^0 \cdot 13^2 \cdot 31^0$
$3^0 \cdot 5^2 \cdot 13^2 \cdot 31^2$	$3^1 \cdot 5^0 \cdot 13^0 \cdot 31^0$	$3^1 \cdot 5^2 \cdot 13^2 \cdot 31^2$	$3^0 \cdot 5^0 \cdot 13^0 \cdot 31^0$

These eight value pairs for u and v lead to the eight value pairs of x and y, with $(x, y) = 1$, being

x	y
12,184,705	12,180,674
4,064,255	4,060,222
491,257	487,202
166,439	162,334
76,105	71,906
28,055	23,518
8,255	1,342

Solution 2 by Ercole Suppa, Teramo, Italy

The given equation is equivalent to

$$\begin{aligned}
(2x - y - 2014)^2 &= 3x^2 - 3xy + y^2 \qquad \Leftrightarrow \\
x^2 - xy - 8056x + 4028y + 2014^2 &= 0 \qquad \Leftrightarrow \\
y &= x - 4028 - \frac{3 \cdot 2014^2}{x - 4028} \qquad (1)
\end{aligned}$$

where x, y are positive integers such that $\gcd(x, y) = 1$ and $2x - y \geq 2014$.

Since y is integer, we have that $x = 4028 + d$ where d is a divisor of $3 \cdot 2014^2$

Furthermore, since $y > 0$ we have

$$(x - 4028)^2 > 3 \cdot 2014^2 \quad \Leftrightarrow \quad x > 4028 + 2014\sqrt{3}.$$

Therefore $d > 2014\sqrt{3}$ and the possible values of x are:

$$\begin{aligned}
x \in \{&8056, 8360, 9646, 10070, 12455, 15264, 16112, 20882, 23161, 37736, \\
&42294, 57399, 61427, 80560, 110770, 118826, 164141, 217512, 233624, 324254, \\
&644480, 1018077, 2032126, 3046175, 4060224, 6088322, 12172616\}. \qquad (2)
\end{aligned}$$

By using (1) and (2), a simple check shows that the only pairs (x, y) such that $2x - y \geq 2014$ and $\gcd(x, y) = 1$ are:

$$\begin{aligned}
&\{(8360, 1523), (12455, 6983), (15264, 10153), \\
&(37736, 33347), (1018077, 1014037), (3046175, 3042143) \\
&(4060224, 4056193), (12172616, 12168587)\}.
\end{aligned}$$

Solution 3 by Brian D. Beasley Presbyterian College, Clinton, SC

We seek to solve the equation $2x - y - \sqrt{3x^2 - 3xy + y^2} = c$ for any positive integer c. Examining this equation for various values of c, we note the following two patterns of solutions:

(1) Let $x = 3c^2 + 2c$ and $y = 3c^2 - 1$. It is then straightforward to verify that

$$2x - y - \sqrt{3x^2 - 3xy + y^2} = 3c^2 + 4c + 1 - \sqrt{(3c^2 + 3c + 1)^2} = c.$$

Next, let $d = \gcd(x, y)$. If $d > 1$, then there is a prime p such that p divides d. Thus p divides $c(3c + 2)$, so either p divides c or p divides $3c + 2$. But p also divides $3c^2 - 1$, so p cannot divide c. Hence p divides $3c + 2$, but p also divides $x - y = 2c + 1$ and thus divides $2(3c + 2) - 3(2c + 1) = 1$, a contradiction. We therefore conclude that $\gcd(x, y) = 1$.

(2) Let $x = c^2 + 2c$ and $y = c^2 - 3$. (To keep $y > 0$, we assume $c > 1$ here.) It is then straightforward to verify that

$$2x - y - \sqrt{3x^2 - 3xy + y^2} = c^2 + 4c + 3 - \sqrt{(c^2 + 3c + 3)^2} = c.$$

Next, we note that if 3 divides c, then $\gcd(x, y) \geq 3$, so we assume that 3 does not divide c in this case. Let $d = \gcd(x, y)$. If $d > 1$, then there is a prime p such that p divides d. Thus p divides $c(c + 2)$, so either p divides c or p divides $c + 2$. But p also divides $c^2 - 3$, so p cannot divide c, since $p \neq 3$ in this case. Hence p divides $c + 2$, but p also divides $x - y = 2c + 3$ and thus divides $2(c + 2) - (2c + 3) = 1$, a contradiction. We therefore conclude that $\gcd(x, y) = 1$.

Since $c = 2014$ for the given equation and 3 does not divide 2014, this approach produces two solutions:

$$x = 12,172,616 \quad \text{and} \quad y = 12,168,587;$$

$$x = 4,060,224 \quad \text{and} \quad y = 4,056,193.$$

Addendum. This approach generates at least one solution for each value of c, with at least two solutions when 3 does not divide c (and when $c > 1$). However, it does not find all solutions, and it does not necessarily find the smallest solution.

Also solved by Dionne Bailey, Elsie Campbell, and Charles Diminnie, Angelo State University, San Angelo, TX; Bruno Salgueiro Fanego, Viveiro, Lugo, Spain; Ed Gray, Highland Beach, FL; Paul M. Harms, North Newton, KS; Kee-Wai Lau, Hong Kong, China; David E. Manes, SUNY College at Oneonta, NY; David Stone and John Hawkins, Southern Georgia University, Statesborogh,GA, and the proposer.

- **5266:** *Proposed by Tom Moore, Bridgewater State University, Bridgewater, MA*

The pentagonal numbers begin $1, 5, 12, 22, \cdots$ and in general satisfy $P_n = \dfrac{n(3n-1)}{2}$, $\forall n \geq 1$. The positive Jacobsthal numbers, which have applications in tiling and graph matching problems, begin $1, 1, 3, 5, 11, 21, \cdots$ with general term $J_n = \dfrac{2^n - (-1)^n}{3}$, $\forall n \geq 1$. Prove that there exists infinitely many pentagonal numbers that are the sum of three Jacobsthal numbers.

Solution 1 by Solution by Dionne Bailey, Elsie Campbell, and Charles Diminnie,

Angelo State University, San Angelo, TX

For $n \geq 1$, let $k_n = \frac{2}{3}\left(2^{2n-1}+1\right) = 2J_{2n-1}$. Then,

$$\begin{aligned}
P_{k_n} &= \frac{k_n\left(3k_n-1\right)}{2} \\
&= \frac{1}{2}\cdot\frac{2}{3}\left(2^{2n-1}+1\right)\left[2\left(2^{2n-1}+1\right)-1\right] \\
&= \frac{\left(2^{2n-1}+1\right)\left(2^{2n}+1\right)}{3}, \quad \text{while} \\
J_{2n-1}+J_{2n}+J_{4n-1} &= \frac{1}{3}\left[\left(2^{2n-1}+1\right)+\left(2^{2n}-1\right)+\left(2^{4n-1}+1\right)\right] \\
&= \frac{2^{2n-1}+1+2^{4n-1}+2^{2n}}{3} \\
&= \frac{\left(2^{2n-1}+1\right)+2^{2n}\left(2^{2n-1}+1\right)}{} \\
&= \frac{\left(2^{2n-1}+1\right)\left(2^{2n}+1\right)}{3}.
\end{aligned}$$

Therefore, for all $n \geq 1$,

$$J_{2n-1}+J_{2n}+J_{4n-1} = P_{2J_{2n-1}}.$$

Solution 2 by Ed Gray, Highland Beach, FL

The sum of two consecutive Jacobsthal numbers is a power of two since

$$\frac{2^x-(-1)^x}{3}+\frac{2^{x+1}-(-1)^{x+1}}{3} = \frac{1}{3}\left(2^x+2^{x+1}\right) = \frac{1}{3}\left(2^x\right)(1+2) = 2^x.$$

Therefore we need to prove that

$$(1) \quad 2^x + \frac{\left(2^a-(-1)^a\right)}{3} = \frac{n(3n-1)}{2}$$

has infinitely many solutions.

Let a be odd so that $a+1=2L$
Multiplying (1) by 6 gives us

$$(2) \qquad 6(2^x)+2^{a+1}+2 = 3n(3n-1), \text{ or}$$

$$(3) \qquad 9n^2-3n-2^{a+1}-2-6(2^x) = 0.$$

This is a quadratic in n whose solution is by the quadratic formula :

$$(4) \qquad 18n = 3+\sqrt{9+36\left(6(2^x)+2^{a+1}+2\right)}$$

The discriminate D is given by

(5) $D^2 = 81 + 36(2^{a+1}) + 216(2^x))$

(6) Consider$D = 9 + 6(2^L)$. Recall that $a + 1 = 2L$

(7) $D^2 = 81 + 108(2^L) + 36(2^{2L})$

(8) Let $108(2^L) = 216(2^x)$

(9) $2^L = 2^{(x+1)}$

(10) $L = x + 1, 2L = 2x + 2 = a + 1$

Then (4) becomes :

(11) $18n = 3 + 9 + 6(2^L) = 12 + 6(2^L)$

Dividing by 6,

(12) $3n = 2 + 2^L$

Since $2 \equiv -1 \pmod 3$

$2^L \equiv -1^L \equiv\ 1$ if L is even.

Letting $L = 2y$ we obtain $n = \dfrac{1}{3}(2 + 2^{2y})$.

Solution 3 by David E. Manes, SUNY at Oneonta, Oneonta, NY

We will show if $k \geq 0$ and $n\dfrac{2\left(2^{2k+1}+1\right)}{3}$, then

$$P_n = J_{4k+3} + J_{2k+2} + J_{2k+1},$$

from which the result follows.

Observe that if k is a nonnegative integer, the modulo 3

$$2\left(2^{2k+1}+1\right) \equiv 2\left((-1)^{2k+1}+1\right) \equiv 0 \pmod 3.$$

Therefore, $n = \dfrac{2\left(2^{2k+1}+1\right)}{3}$ is a positive integer for each $k \geq 0$. Moreover,

$$\begin{aligned} P_n &= \frac{\frac{2(2^{2k+1}+1)}{3}\left[2(2^{2k+1}+1)-1\right]}{2} \\ &= \left(\frac{2^{2k+1}+1}{3}\right)\left(2^{2k+2}+1\right). \end{aligned}$$

If $k \geq 0$, then

$$\begin{aligned}
J_{4k+3} + J_{2k+2} + J_{2k+1} &= \frac{\left[(2^{4k+3}+1) + (2^{2k+2}-1)\right] + (2^{2k+1}+1)}{3} \\
&= \frac{2^{2k+2}(2^{2k+1}+1) + (2^{2k+1}+1)}{3} \\
&= \left(\frac{2^{2k+1}+1}{3}\right)\left(2^{2k+2}+1\right) \\
&= P_{\frac{2(2^{2k+1}+1)}{3}} \quad = P_n.
\end{aligned}$$

Hence, there exists infinitely many pentagonal numbers $P_{\frac{2(2^{2k+1}+1)}{3}}$ (k$\geq$ 0), that are the sum of three Jacobsthal numbers; namely

$$J_{4k+3} + J_{2k+2} + J_{2k+1}.$$

Also solved by Brian D. Beasley Presbyterian College, Clinton, SC; Kee-Wai Lau, Hong Kong, China; David Stone and John Hawkins, Southern Georgia University, Statesborogh, GA, and the proposer.

- **5267:** *Proposed by D. M. Bătinetu-Giurgiu, "Matei Basarab" National College, Bucharest, Romania, and Neculai Stanciu, "Geroge Emil Palade" General School, Buzău, Romania*

Let n be a positive integer. Prove that

$$\frac{F_n L_{n+2}^2}{F_{n+3}} + \frac{F_{n+1} L_{n+3}^2}{F_n + F_{n+2}} + (L_n + L_{n+2})^2 \geq 2\sqrt{6}\left(\sqrt{L_n L_{n+1}}\right) L_{n+2},$$

where F_n and L_n represents the nth Fibonacci and Lucas Numbers defined by $F_0 = 0, F_1 = 1$, and for all $n \geq 0$, $F_{n+2} = F_{n+1} + F_n$; and $L_0 = 2, L_1 = 1$, and for all $n \geq 0,\ L_{n+2} = L_{n+1} + L_n$, respectively.

Solution by G. C. Greubel, Newport News, VA

The inequality to be shown valid is that of

$$\frac{F_n L_{n+2}^2}{F_{n+3}} + \frac{F_{n+1} L_{n+3}^2}{F_n + F_{n+2}} + (L_n + L_{n+2})^2 \geq 2\sqrt{6}\left(\sqrt{L_n L_{n+1}}\right) L_{n+2}. \qquad (1)$$

Using the AM-GM inequality then it can be seen that

$$\frac{F_n L_{n+2}^2}{F_{n+3}} + \frac{F_{n+1} L_{n+3}^2}{F_n + F_{n+2}} \geq 2\left[\frac{F_n F_{n+1}}{F_{n+3} L_{n+1}}\right]^{1/2} L_{n+2} L_{n+3}. \qquad (2)$$

It can be shown that

$$\frac{1}{3} \geq \left[\frac{F_n F_{n+1}}{F_{n+3} L_{n+1}}\right]^{1/2} \geq \frac{1}{4}, \qquad (3)$$

which is valid for $n \geq 1$, for which its use in equation (2) leads to

$$\frac{F_n L_{n+2}^2}{F_{n+3}} + \frac{F_{n+1} L_{n+3}^2}{F_n + F_{n+2}} \geq \frac{1}{2} L_{n+2} L_{n+3}. \qquad (4)$$

By making use of this on the left-hand side of (1) it is now left to show that

$$\frac{1}{2} L_{n+2} L_{n+3} + (L_n + L_{n+2})^2 \geq 2\sqrt{6 L_n L_{n+2}}\, L_{n+2}. \qquad (5)$$

Multiplying both sides by 2 yields

$$L_{n+2} L_{n+3} + 2\,(L_n + L_{n+2})^2 \geq 4\sqrt{6 L_n L_{n+2}}\, L_{n+2}. \qquad (6)$$

It is with little difficulty to show that

$$L_{n+2} L_{n+3} + 2\,(L_n + L_{n+2})^2 = 2L_{n+3}^2 - 11 L_{n+2} L_{n+3} + 18 L_{n+2}^2 \qquad (7)$$

which, when used in (6), leads to

$$2L_{n+3}^2 - 11 L_{n+2} L_{n+3} + 18 L_{n+2}^2 \geq 4\sqrt{6 L_n L_{n+2}}\, L_{n+2}. \qquad (8)$$

Now consider

$$2L_{n+3}^2 - 11 L_{n+2} L_{n+3} + 8 L_{n+2}^2$$

which, when use of the AM-GM inequality is made,[1] namely $L_{n+2} \geq 2\sqrt{L_n L_{n+1}}$, becomes

$$\begin{aligned}
2L_{n+3}^2 - 11 L_{n+2} L_{n+3} + 8 L_{n+2}^2 \ &\geq\ 8 L_{n+1} L_{n+2} - 11 L_{n+2} L_{n+3} + 32 L_n L_{n+1} \\
&\geq\ 32 L_n L_{n+1} - 22 L_n L_{n+1} - 3 L_{n+1} L_{n+2} \\
&\geq\ 10 L_n L_{n+1} - 3 L_{n+1} L_{n+2} \\
&\geq\ L_{n+1}\,(7 L_n - 3 L_{n+1}) \\
&\geq\ 7 L_n^2 + L_n L_{n-1} \\
&\geq\ L_n\,(L_{n+2} + 5 L_n) \geq 0. \qquad (9)
\end{aligned}$$

From this it is then seen that, when (9) is used in (8),

$$\begin{aligned}
2L_{n+3}^2 - 11 L_{n+2} L_{n+3} + 18 L_{n+2}^2 \ &=\ \left(2L_{n+3}^2 - 11 L_{n+2} L_{n+3} + 8 L_{n+2}^2\right) + 10 L_{n+2}^2 \\
&\geq\ 10 L_{n+2}^2 \\
&\geq\ 20\sqrt{L_n L_{n+1}}\, L_{n+2}. \qquad (10)
\end{aligned}$$

Since this represents the left-hand side of the inequality (8) then it is seen that

$$20\sqrt{L_nL_{n+1}}L_{n+2} \geq 4\sqrt{6L_nL_{n+1}}L_{n+2} \qquad (11)$$

and leads to the result $20 \geq 4\sqrt{6}$ which reduces to $5 \geq \sqrt{6}$. Since this is a valid inequality the original statement holds. For the case $n = 0$ equation (3) can be stated as

$$\frac{1}{3} \geq \left[\frac{F_nF_{n+1}}{F_{n+3}L_{n+1}}\right]^{1/2} \geq 0. \qquad (12)$$

Then by following a similar pattern the statement leads to the same result.
Thus, for $n \geq 0$,

$$\frac{F_nL_{n+2}^2}{F_{n+3}} + \frac{F_{n+1}L_{n+3}^2}{F_n + F_{n+2}} + (L_n + L_{n+2})^2 \geq 2\sqrt{6}\left(\sqrt{L_nL_{n+1}}\right)L_{n+2}. \qquad (13)$$

[1] It is seen that $L_{n+2} = L_{n+1} + L_n \geq 2\sqrt{L_nL_{n+1}}$.

Also solved by Ed Gray, Highland Beach, FL, and the proposers.

- **5268:** *Proposed by Pedro H.O. Pantoja, IMPA, Rio de Janeiro, Brazil*

Let $N = 121^a + a^3 + 24$. Determine all positive integers a for which

a) N is a perfect square.

b) N is a perfect cube.

Solution 1 by Ed Gray, Highland Beach, FL

(a) The answer to the first part of the question is that there are none, other than the trivial solution of $\mathbf{a = 0}$. We will now show why this is the case.

(1) Let $121^a = (11^2)^a = 11^{2a} = (11^a)^2$, so

(2) $N = (11^a)^2 + a^3 + 24$. Suppose $N = m^2$, so,

(3) $m^2 = (11^a)^2 + a^3 + 24$. Clearly, $m > (11)^a$. Let

(4) $m = (11)^a + b$

(5) $m^2 = (11^a)^2 + 2b(11)^a + b^2$. Equating (2) to (5)

(6) $(11^a)^2 + a^3 + 24 = (11^a)^2 + 2b(11^a) + b^2$. Simplifying gives

(7) $a^3 + 24 = 2b(11)^a + b^2$.

Note that for every positive integer a, $(11)^a > a^3$, since $a(\ln(11)) > 3\ln(a)$, dividing by $3a$ gives $\dfrac{\ln(11)}{3} > \dfrac{\ln(a)}{a}$.

The maximum value of $\dfrac{\ln(a)}{a}$ is when its derivative equals zero, or
$\dfrac{a \cdot (\frac{1}{a}) - \ln(a)}{a^2} = \dfrac{1 - \ln(a)}{a^2} = 0$, which implies that $a = e$.

So the maximum value of $\dfrac{\ln(a)}{a} = \dfrac{\ln(e)}{e} = \dfrac{1}{e} = 0.3678$, and $\dfrac{\ln(a)}{a}$ is monotonically decreasing for $a > e$.

Now $\frac{\ln(11)}{3} = 0.7993$, so $(11)^a > a^3$. We note for $a = 2$, the equation in (7) becomes: $32 = 242b + b^2$, which is clearly impossible, and the situation only gets worse for $a > 2$. For $a = 1$ the equation in (7) becomes:

(8) $25 = 22b + b^2$ which clearly has no interger solution. So, the only solution is the trivial one, i.e., when $a = 0$.

(b) The answer to the second part of the question is no; N can never be a perfect cube. By (2) we have:

(9) $N = (11)^{2a} + a^3 + 24$. First, suppose that a is of the form $3y$ and $N = m^3$. Then,

(10) $m^3 = (11)^{6y} + 27y^3 + 24$, or

(11) $m^3 = (11^{2y})^3 + 27y^3 + 24$. Then,

(12) $m > (11)^{2y}$. Letting $m = 11^{2y} + b$

(13) $m^3 = (11)^{6y} + 3(11)^{4y}b + 3(11)^{2y}b^2 + b^3$. Equating (11) and (13),

(14) $(11)^6y + 27y^3 + 24 = (11)^{6y} + 3b(11)^{4y} + 3b^2(11)^{2y} + b^3$. Canceling the term $(11)^{6y}$,

(15) $27y^3 + 24 = 3b(11)^{4y} + 3b^2(11)^{2y} + b^3$.

As before, we show that $(11)^{4y} > 27y^3$ since $4y(\ln(11)) > \ln(27) + 3\ln(y)$ or $9.591y > 3.2958 + 3\ln(y)$ or $1 > \frac{0.3436}{y} + \frac{0.3128\ln(y)}{y}$.

We have seen the maximum value of $\frac{\ln y}{y} = 0.3678$ when $y = e$.

If $y = e$, $\frac{0.3436}{2.71828} + (0.3128)(0.3678) = 0.1264 + 0.115 = 0.241$.

For $y = 1$, $1 > 0.3436$ and the right hand side is monotonically decreasing. Notice that we have not used the coefficient $3b$, the additional term $3b^2(11)^2y$, or b^3. The smallest we can make the right hand side is for $y = b = 1$, and the value is $(3)(1)(14641) + (3)(10)(121) + 1 = 132133$, while the right hand side has the value of 51.

There was nothing special about the parameter y and we would get these wildly different values on different sides of the equation for $a = 3y, 3(y+1), 3(y+2)\cdots$. By continuity any value of a sandwiched between any of the above numbers will suffer the same fate. In summary, there can never be an integer cube.

Solution 2 by Kee Wai Lau, Hong Kong, China

We show that for all positive integers a, N is neither a perfect square nor a perfect cube.

a) We first show that for $a = 2, 3, 4, \cdots$,

$$a^3 + 24 < 11^a. \qquad (1)$$

Clearly (1) hold for $a = 2$. Suppose (12) hold for $a \geq 2$. Then

$$(k+1)^3 + 24 < 8k^3 + 24 < 8(k^3 + 24) < 8(11^k) < 11^{k+1}.$$

so (1) is true for $a = k + 1$ an so for $a = 2, 3, 4\cdots$. Now suppose, on the contrary, that $N = n^2$, where n is a positive integer. Then

$$a^3 + 24 = (n + 11^a)(n - 11^a > 11^a.$$

By (1), $a = 1$, so that $n = \sqrt{146}$, which is a contradiction. Thus N is never a perfect square.

b) It can be proved readily by induction that for positive integers m

$$\begin{cases} N \equiv 2(\text{mod } 9), & a = 3m-2 \\ N \equiv 3(\text{mod } 9), & a = 3m-1 \\ N \equiv 7(\text{mod } 9), & a = 3 \end{cases}$$

However, the cube of a positive integer is aways congruent either to 0 or 1 or 8(mod 9). It follows that N is never a perfect cube.

Solution 3 by David Stone and John Hawkins of Georgia Southern University, Statesboro, GA and Chuck Garner, Rockdale Magnet School, Conyers, GA.

There are no such integers a in either (a) or (b).

When $a = 1,\ N = 146$, which is neither a square nor a cube. Now assume $a \geq 2$.

For part (a), we can show that N is trapped between consecutive squares, so cannot itself be a square.

$$21^{2a} < N = 11^{2a} + a^3 + 24 < (11^a + 1)^2 = 11^{2a} + 2 \cdot 11^a + 1.$$

The first inequality is clear.

The second, $N = 11^{2a} + a^3 + 24 < (11^a + 1)^2 = 11^{2a} + 2 \cdot 11^a + 1$ is equivalent to $a^3 + 23 < 2 \cdot 11^a$, which can be verified by a straightforward induction argument.

For part (b), we take advantage of the fact that a cube cannot take on many values 9. Namely, only $0, 1$ and 8. note,

$$\text{mod } 9,\ 11^{2\text{a}} \equiv \begin{cases} 1, & \text{if } a \equiv 0 \text{ mod } 3 \\ 4, & \text{if } a \equiv 1 \text{ mod } 3 \\ 7, & \text{if } a \equiv 2 \text{ mod } 3, \quad \text{and} \end{cases}$$

$$\text{mod } 9,\ \text{a}^3 \equiv \begin{cases} 0, & \text{if } a \equiv 0 \text{ mod } 3 \\ 1, & \text{if } a \equiv 1 \text{ mod } 3 \\ 8, & \text{if } a \equiv 2 \text{ mod } 3. \end{cases}$$

$$\text{Thus mod } 9,\ \text{N} = 11^{2\text{a}} + \text{a}^3 + 24 \ \equiv \begin{cases} 1 + 0 + 6 \equiv 7, & \text{if } a \equiv 0 \text{ mod } 3 \\ 4 + 1 + 6 \equiv 2, & \text{if } a \equiv 1 \text{ mod } 3\ . \\ 7 + 8 + 6 \equiv 3, & \text{if } a \equiv 2 \text{ mod } 3 \end{cases}$$

That is, N is congruent to $2,\ 3$ or 7, and never congruent to $0, 1$, or 8, N cannot be a cube.

Comment : Numerical evidence suggests that the power of 11 is so dominant that N also lies between identifiable consecutive cubes $m^3 < N < (m+1)^3$, where m=$\left[11^{2a/3}\right]$.

Also solved by David E. Manes, SUNY College at Oneonta, NY, and the proposer.

- **5269:** *Proposed by José Luis Díaz-Barrero, Barcelona Tech, Barcelona, Spain*

Let $\{a_n\}_{n \geq 1}$ be the sequence defined by

$$a_1 = 1,\ a_2 = 5,\ a_{n-1}^2 - a_n a_{n-2} + 4 = 0.$$

Show that all of the terms of the sequence are integers.

Solution 1 by Ercole Suppa, Teramo, Italy

From the given recurrence we get

$$a_n a_{n-2} = a_{n-1}^2 + 4 \qquad (1)$$

$$a_{n+1} a_{n-1} = a_n^2 + 4 \qquad (2)$$

Now subtracting (1) and (2) from each other, we find that for every $n \in N$:

$$\begin{aligned}
a_n a_{n-2} - a_{n+1} a_{n-1} &= (a_{n-1} - a_n)(a_{n-1} + a_n) \quad \Leftrightarrow \\
a_n a_{n-2} - a_{n+1} a_{n-1} &= a_{n-1}^2 - a_n^2 \quad \Leftrightarrow \\
a_n (a_{n-2} + a_n) &= a_{n-1}(a_{n+1} + a_{n-1}) \quad \Leftrightarrow \\
\frac{a_{n-2} + a_n}{a_{n-1}} &= \frac{a_{n+1} + a_{n-1}}{a_n} \qquad (3)
\end{aligned}$$

Therefore the expression $\dfrac{a_{n-2} + a_n}{a_{n-1}}$ is constant. From the initial conditions we obtain

$$\begin{aligned}
\frac{a_{n-2} + a_n}{a_{n-1}} &= \frac{a_3 + a_1}{a_2} = \frac{29 + 1}{5} = 6 \Rightarrow \\
a_n &= 6a_{n-1} - a_{n-2}, \quad \forall n \geq 3. \qquad (4)
\end{aligned}$$

By using (4) a simple induction on n show that all the terms of the sequence are integers.

Solution 2 Dionne Bailey, Elsie Campbell, and Charles Diminnie, Angelo State University, San Angelo, TX

Since

$$a_{n-1}^2 - a_n a_{n-2} + 4 = 0$$

for $n \geq 3$, we have

$$a_n a_{n-2} = a_{n-1}^2 + 4 \geq 4.$$

Therefore, $a_{n-2} \neq 0$ for all $n \geq 3$ and we may write the recursive formula for $\{a_n\}$ in the form

$$a_n = \frac{a_{n-1}^2 + 4}{a_{n-2}}$$

for all $n \geq 3$, or equivalently

$$a_{n+2} = \frac{a_{n+1}^2 + 4}{a_n} \qquad (1)$$

for all $n \geq 1$.

When we evaluate the first six terms using (1) and the initial values $a_1 = 1$ and $a_2 = 5$, we obtain

$$a_1 = 1, \quad a_2 = 5, \quad a_3 = 29, \quad a_4 = 169, \quad a_5 = 985, \quad \text{and} \quad a_6 = 5741.$$

These entries suggest the following alternative recursive definition for $\{a_n\}$:

$$a_1 = 1, \quad a_2 = 5, \quad \text{and} \quad a_{n+2} = 6a_{n+1} - a_n \text{ for } n \geq 1. \ 2 \tag{2}$$

We will establish (2) by Mathematical Induction. Let $P(n)$ be the statement

$$a_{n+2} = 6a_{n+1} - a_n.$$

Then, the conditions

$$a_3 = \frac{a_2^2 + 4}{a_1} = 29 = 6a_2 - a_1$$

and

$$a_4 = \frac{a_3^2 + 4}{a_2} = 169 = 6a_3 - a_2$$

imply that $P(1)$ and $P(2)$ are true. If we assume that $P(1), P(2), \ldots, P(n)$ are true for some $n \geq 2$, then in particular, $a_{n+2} = 6a_{n+1} - a_n$ and
$a_{n+1} = 6a_n - a_{n-1}$. It follows that

$$\begin{aligned}
a_{n+3} &= \frac{a_{n+2}^2 + 4}{a_{n+1}} \\
&= \frac{(6a_{n+1} - a_n)^2 + 4}{a_{n+1}} \\
&= 36a_{n+1} - 12a_n + \frac{a_n^2 + 4}{a_{n+1}} \\
&= 6(6a_{n+1} - a_n) - 6a_n + \frac{a_{n+1}a_{n-1}}{a_{n+1}} \\
&= 6a_{n+2} - (6a_n - a_{n-1}) \\
&= 6a_{n+2} - a_{n+1}
\end{aligned}$$

and hence, $P(n+1)$ is true also. By Mathematical Induction, $P(n)$ is true for all $n \geq 1$, i.e., $a_{n+2} = 6a_{n+1} - a_n$ for all $n \geq 1$.

As a result, the conditions

$$a_1 = 1, \quad a_2 = 5, \quad \text{and} \quad a_{n+2} = 6a_{n+1} - a_n \text{ for } n \geq 1$$

and a trivial Mathematical Induction argument imply that a_n is an integer for all $n \geq 1$. Additionally, this new description affords us a method for finding a formula for the sequence $\{a_n\}$. Using the customary technique for solving homogeneous linear difference equations, we look for solutions of the form $a_n = \lambda^n$, with $\lambda \neq 0$. Then, the formula

$$a_{n+2} = 6a_{n+1} - a_n$$

simplifies to

$$\lambda^2 = 6\lambda - 1$$

whose solutions are $\lambda = 3 \pm 2\sqrt{2}$. The general solution is of the form

$$a_n = c_1 \left(3 + 2\sqrt{2}\right)^n + c_2 \left(3 - 2\sqrt{2}\right)^n$$

for some constants c_1 and c_2. Further, the initial values $a_1 = 1$ and $a_2 = 5$ yield

$$c_1 = \frac{2 - \sqrt{2}}{4} \quad \text{and} \quad c_2 = \frac{2 + \sqrt{2}}{4}.$$

Finally, since

$$3 \pm 2\sqrt{2} = \frac{\left(2 \pm \sqrt{2}\right)^2}{2},$$

we get

$$\begin{aligned}
a_n &= 2 - \sqrt{2}4\left(3 + 2\sqrt{2}\right)^n + \frac{2+\sqrt{2}}{4}\left(3 - 2\sqrt{2}\right)^n \\
&= \frac{1}{4}\left[\left(2-\sqrt{2}\right)\frac{\left(2+\sqrt{2}\right)^{2n}}{2^n} + \left(2+\sqrt{2}\right)\frac{\left(2-\sqrt{2}\right)^{2n}}{2^n}\right] \\
&= \frac{(2)\left(2+\sqrt{2}\right)^{2n-1} + (2)\left(2-\sqrt{2}\right)^{2n-1}}{2^{n+2}} \\
&= \frac{\left(2+\sqrt{2}\right)^{2n-1} + \left(2-\sqrt{2}\right)^{2n-1}}{2^{n+1}}
\end{aligned}$$

for all $n \geq 1$.

Solution 3 by Kee-Wai Lau, Hong Kong, China

For positive integers n, let $b_n = \dfrac{(2-\sqrt{2})(3+2\sqrt{2})^n + (2+\sqrt{2})(3-2\sqrt{2})^n}{4} > 0$. It is easy to check that $b_1 = 1, b_2 = 54$ and for $n \geq 3, b_n = b_{n-1} - b_{n-2}$. Hence b_n are always positive integers.

Using the equation $a_n a_{n-2} = a_{n-1}^2 + 4$, we prove readily by induction that
$a_n = \dfrac{(2-\sqrt{2})(3+2\sqrt{2})^n + (2+\sqrt{2})(3-2\sqrt{2})^n}{4}$ as well.
Thus, $a_n = b_n$ are positive integers.

Editor's comment: David Stone and John Hawkins of Georgia Southern University in Statesboro, GA also solved the problem by generating a few terms of the given sequence, and then finding a recursive definition for these initial terms that was different from the given recursion in the statement of the problem. Then, using induction, they showed that the new recursive definition satisfied the recursion in the statement of the problem. Essentially, their solution path was that used in Solution 1 above.
They also commented that the problem can also be solved as it is in Solution 3 above, where one finds an explicit formula for the Fibonacci sequence. They continued on the following way:

Comment 1: Other Fibonacci-like properties can be derived. For instance, the ratio of consecutive terms, $\dfrac{a_{n+1}}{a_n}$ approaches $\alpha = 3 + 2\sqrt{2} \approx 5.8284$.

Comment 2: In the proposed problem the true nature of the $\{an\}_{n\geq 1|}$ was cleverly disguised by an unfamiliar recurrence relation: $a_n = \dfrac{a_{n-1}^2 + 4}{a_{n-2}}$. Perhaps there is a similar relation fot he Fibonacci numbers.

Comment 3: The sequence $\{a_n\}_{n\geq 1|}$ is A001653 at the Online Encyclopedia of Integer Sequences. Several interesting properties and applications are given: the recurrence relation $a_n = 6a_{n-1]-a_{n-2}}$ is given. We do not see (in this encyclopedia) the recurrence relation that was given in the problem statement (so perhaps it is heretofore unknown).

Also solved by Arkady Alt, San Jose, CA; Bruno Salgueiro Fanego, Viveiro, Lugo, Spain; Ed Gray, Highland Beach, FL; G. C. Greubel, Newport News, VA; Kenneth Korbin, New York, NY; Carl Libis and Junhua Wu, Lane College, Jackson, TN; Carl Libis (a second solution), Lane College, Jackson, TN; David E. Manes, SUNY College at Oneonta, NY; Angel Plaza, Universidad de Las Palmas de Gran Canaria, Spain, and the proposer.

- **5270:** *Proposed by Ovidiu Furdui, Technical University of Cluj-Napoca, Cluj-Napoca, Romania*

Let $k \geq 1$ be an integer. Calculate

$$\int_0^1 \int_0^1 (x+y)^k \, (-1)^{\left\lfloor \frac{1}{x} - \frac{1}{y} \right\rfloor} \, dxdy,$$

where$\lfloor x \rfloor$ denotes the floor of x.

Solution 1 by Angel Plaza, Universidad de Las Palmas de Gran Canaria, Spain.

For any point $(x, y) \in [0, 1]$, also $(y, x) \in [0, 1]$. Note that for (x, y) such that $\dfrac{1}{x} - \dfrac{1}{y} \in (m, m+1)$ with $m \in$, then $\left\lfloor \dfrac{1}{x} - \dfrac{1}{y} \right\rfloor = m$, but for the corresponding point (y, x) also in the domain $[0, 1]$ we have that $\dfrac{1}{y} - \dfrac{1}{x} \in (-(m+1), -m)$ and therefore $\left\lfloor \dfrac{1}{y} - \dfrac{1}{x} \right\rfloor = -(m+1)$. Since $(-1)^m = -(-1)^{-(m+1)}$ and $(x+y)^k = (y+x)^k$ the proposed integral is 0.

Solution 2 by Perfetti Paolo, Department of Mathematics, "Tor Vergata" University, Rome, Italy

Let $A = \{(x, y) \colon 0 \leq x \leq 1, 0 \leq y \leq 1,\ y \geq x\}$ and $B = \{(x, y) \colon 0 \leq x \leq 1, 0 \leq y \leq 1,\ y \leq x\}$. By doing $(x, y) \to (y, x)$ we get

$$\int\int_A (x+y)^k (-1)^{\lfloor \frac{1}{x} - \frac{1}{y} \rfloor} dxdy = \int\int_B (y+x)^k (-1)^{\lfloor \frac{1}{y} - \frac{1}{x} \rfloor} dxdy.$$

Moreover because of $\lfloor x \rfloor + \lfloor -x \rfloor = -1$ we get

$$\int\int_B (x+y)^k (-1)^{-1-\lfloor \frac{1}{x} - \frac{1}{y} \rfloor} dxdy = -\int\int_B (x+y)^k (-1)^{\lfloor \frac{1}{x} - \frac{1}{y} \rfloor},$$

and then

$$\begin{aligned}\int_0^1 \int_0^1 (x+y)^k (-1)^{\lfloor \frac{1}{x} - \frac{1}{y} \rfloor} dxdy &= \int\int_A (x+y)^k (-1)^{\lfloor \frac{1}{x} - \frac{1}{y} \rfloor} dxdy + \int\int_B (x+y)^k (-1)^{\lfloor \frac{1}{x} - \frac{1}{y} \rfloor} dxdy \\ &= -\int\int_B (x+y)^k (-1)^{\lfloor \frac{1}{x} - \frac{1}{y} \rfloor} dxdy + \int\int_B (x+y)^k (-1)^{\lfloor \frac{1}{x} - \frac{1}{y} \rfloor} dxdy = 0.\end{aligned}$$

Solution 3 by the proposer

The integral equals 0. We have, based on symmetry reasons, that

$$\int_0^1 \int_0^1 x \, (x+y)^{k-1} \, (-1)^{\left\lfloor \frac{1}{x} - \frac{1}{y} \right\rfloor} \, dxdy = \int_0^1 \int_0^1 y (x+y)^{k-1} \, (-1)^{\left\lfloor \frac{1}{y} - \frac{1}{x} \right\rfloor} \, dxdy.$$

On the other hand, for all real numbers x that are not integers, one has

$$\lfloor x \rfloor + \lfloor -x \rfloor = -1.$$

It follows that,

$$\begin{aligned}
\int_0^1 \int_0^1 x\,(x+y)^{k-1}\,(-1)^{\left\lfloor \frac{1}{x}-\frac{1}{y}\right\rfloor}\,dxdy &= \int_0^1 \int_0^1 y(x+y)^{k-1}\,(-1)^{\left\lfloor \frac{1}{y}-\frac{1}{x}\right\rfloor}\,dxdy\\
&= -\int_0^1 \int_0^1 y(x+y)^{k-1}\,(-1)^{-\left\lfloor \frac{1}{x}-\frac{1}{y}\right\rfloor}\,dxdy\\
&= -\int_0^1 \int_0^1 y(x+y)^{k-1}\,(-1)^{\left\lfloor \frac{1}{x}-\frac{1}{y}\right\rfloor}\,dxdy,
\end{aligned}$$

and the result follows.

Also solved by Paul M. Harms, North Newton, KS, and by Ed Gray, Highland Beach, FL.

Mea Culpa; *once again*

My sincerest apologies to David Stone and to John Hawkins of Georgia Southern University, for inadvertently forgetting to mention that they had correctly solved problems 5260, 5261, and 5262; and also to Brian D. Beasley, of Presbyterian College in Clinton, South Carolina for his solution to 5262.