Problems   Ted Eisenberg, Section Editor

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This section of the Journal offers readers an opportunity to exchange interesting mathematical problems and solutions. Please send them to Ted Eisenberg, Department of Mathematics, Ben-Gurion University, Beer-Sheva, Israel or fax to: 972-86-477-648. Questions concerning proposals and/or solutions can be sent e-mail to <eisenbt@013.net>. Solutions to previously stated problems can be seen at <http://www.ssma.org/publications>.

Solutions to the problems stated in this issue should be posted before May 15, 2013

- **5248:** *Proposed by Kenneth Korbin, New York, NY*

  A triangle with sides (a, a, b) has the same area and the same perimeter as a triangle with sides (c, c, d) where a, b, c and d are positive integers and with

  $$\frac{b^2+bd+d^2}{b+d}=7^6.$$

  Find the sides of the triangles.

- **5249:** *Proposed by Tom Moore, Bridgewater State University, Bridgewater, MA*

  (a) Let n be an odd positive integer. Prove that a^n+b^n is the square of an integer for infinitely many integers a and b.

  (b) Prove that a^2+b^3 is the square of an integer for infinitely many integers a and b.

- **5250:** *Proposed by D. M. Bătinetu-Giurgiu, "Matei Basarab" National College, Bucharest, Romania and Neculai Stanciu, "George Emil Palade" Secondary School, Buzău, Romania*

  Let $a \in \left(0, \frac{\pi}{2}\right)$ and $b, c \in (1, \infty)$. Calculate:

  $$\int_{-a}^{a} \ln\left(b^{\sin^3 x}+c^{\sin^3 x}\right)\cdot \sin x \cdot dx.$$

- **5251:** *Proposed by Enkel Hysnelaj, University of Technology, Sydney, Australia and Elton Bojaxhiu, Kriftel, Germany*

  Compute the following sum:

  $$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}(-1)^{m+n}\frac{\cos(m+n)}{(m+n)^2}.$$

- **5252:** *Proposed by José Luis Díaz-Barrero, Polytechnical University of Catalonia, Barcelona, Spain*

Let $\{a_n\}_{n\geq 1}$ be the sequence of real numbers defined by $a_1 = 3, a_2 = 5$ and for all $n \geq 2, a_{n+1} = \frac{1}{2}\left(a_n^2 + 1\right)$. Prove that

$$1 + 2\left(\sum_{k=1}^{n}\sqrt{\frac{F_k}{1+a_k}}\right)^2 < F_{n+2},$$

where F_n represents the n^{th} Fibonacci number defined by $F_1 = F_2 = 1$ and for $n \geq 3, F_n = F_{n-1} + F_{n-2}$.

- **5253:** *Proposed by Ovidiu Furdui, Technical University of Cluj-Napoca, Cluj-Napoca, Romania*

  Calculate
  $$\int_0^1\int_0^1 \frac{\ln x \cdot \ln(xy)}{1-xy}dxdy.$$

Solutions

- **5230:** *Proposed by Kenneth Korbin, New York, NY*

  Given positive numbers x, y, z such that

$$\begin{aligned} x^2 + xy + \frac{y^2}{3} &= 41, \\ \frac{y^2}{3} + z^2 &= 16, \\ x^2 + xz + z^2 &= 25. \end{aligned}$$

Find the value of $xy + 2yz + 3xz$.

Solution 1 by Bruno Salgueiro Fanego, Viveiro, Spain

Note that the given system is equivalent to

$$\begin{aligned} x^2 - 2x\frac{y}{\sqrt{3}}\cos 150^\circ + \left(\frac{y}{\sqrt{3}}\right)^2 &= \left(\sqrt{41}\right)^2, \\ \left(\frac{y}{\sqrt{3}}^2\right) + z^2 &= 4^2, \\ x^2 + 2xz\cos 120^\circ + z^2 &= 5^2. \end{aligned}$$

Let us take the right triangle ABC with $\angle B = 90^\circ, AB = 4$ and $BC = 5$ and let P be the interior point of ABC obtained as the intersection of the semicircle with diameter AB and the spanning arc of angle 120° (this is the locus of the points from which the segment BC is seen from an angle of 120°. Note that $\angle APB = 90^\circ$, $\angle BPC = 120^\circ$ and $\angle CPA = 150^\circ$. If we denote $x = CP, y = \sqrt{3}AP, z = BP$, we obtain the equations in the given system by applying the law of cosines to triangles ACP, ABP, and BCP.

Denoting the area of a triangle by $[\cdots]$ we have:

$$[ACP]+[ABP]+[BCP]=[ABC], \text{ or equivalently,}$$

$$\left(\frac{1}{2}\cdot PC\cdot PA\sin 150^\circ\right)+\left(\frac{1}{2}\cdot PA\cdot PB\right)+\left(\frac{1}{2}\cdot PC\cdot PB\cdot\sin 120\right)=\frac{1}{2}\cdot AB\cdot BC.$$

That is,

$$\left(\frac{1}{2}\cdot x\cdot\frac{y}{\sqrt{3}}\cdot\frac{1}{2}\right)+\left(\frac{1}{2}\cdot\frac{y}{\sqrt{3}}z\right)+\left(\frac{1}{2}\cdot x\cdot z\cdot\frac{\sqrt{3}}{2}\right)=\frac{1}{2}\cdot 4\cdot 5.$$

Multiplying by $4\sqrt{3}$, gives us that

$$xy+2yz+3xz=40\sqrt{3}.$$

Comment by Bruno: Very similar problems to this one are problems #12 of the 1984 All-Soviet Union Mathematical Olympiad and problem # E1 in *Problem Solving Strategies* by Arthur Engel (Springer-Verlag), 1998, pp. 380-381

Solution 2 by Arkady Alt, San Jose, California, USA

Let $S=xy+2yz+3xz$. By replacing y in the original problem with $y\sqrt{3}$ we obtain:

$$\begin{aligned}
x^2+xy\sqrt{3}+y^2 &= a^2+b^2,\\
y^2+z^2 &= a^2, \text{ and}\\
x^2+xz+z^2 &= b^2, \text{where } a=4,\ b=5, \text{ and}\\
S &= xy\sqrt{3}+2\sqrt{3}yz+3xz, \text{ or}\\
x^2+y^2-2\cos\frac{5\pi}{6}xy &= a^2+b^2,\\
y^2+z^2-2\cos\frac{\pi}{2}yz &= a^2,\\
x^2+z^2-2\cos\frac{2\pi}{3}xz &= b^2,\\
\frac{S}{2\sqrt{3}} &= xy\sin\frac{5\pi}{6}+yz\sin\frac{\pi}{2}+zx\sin\frac{2\pi}{3}.
\end{aligned}$$

Consider four points A,B,C,P on a plane such that $PA=x, PB=y, PM=z$ and $\angle APB=\frac{5\pi}{6}, \angle BPC=\frac{\pi}{2}, \angle CPA=\frac{2\pi}{3}$.

Since $\frac{5\pi}{6}+\frac{2\pi}{3}+\frac{\pi}{2}=2\pi$ then, accordingly to the equalities

$$x^2+y^2-2\cos\frac{5\pi}{6}xy = a^2+b^2,$$

$$\begin{aligned} y^2+z^2-2\cos\frac{\pi}{2}yz &= a^2, \\ x^2+z^2-2\cos\frac{2\pi}{3}xz &= b^2, \text{where} \end{aligned}$$

P is the interior point of the right triangle ABC with right angle at C, and sides $BC=a$, $AC=b$.

Then we have $[ABC]=[APB]+[BPC]+[CPA] \iff$

$$\frac{AC\cdot BC}{2}=\frac{PA\cdot PB}{2}\sin\frac{5\pi}{6}+\frac{PB\cdot PC}{2}\sin\frac{\pi}{2}+\frac{PC\cdot PA}{2}\sin\frac{2\pi}{3} \iff$$

$$\begin{aligned} a\cdot b &= xy\sin\frac{5\pi}{6}+yz\sin\frac{\pi}{2}+zx\sin\frac{2\pi}{3} \iff \\ ab &= \frac{S}{2\sqrt{3}} \iff S=2\sqrt{3}ab. \end{aligned}$$

For $a=4$ and $b=5$ we obtain $S=40\sqrt{3}$.

Remark: The original problem is a particular case of a more general problem.

Given positive numbers $x,\ y,\ z,\alpha,\beta,\gamma,a,b,c$ such that $\alpha+\beta+\gamma=2\pi,\ a,b,c$ and

$$\begin{cases} x^2+y^2-2\cos\gamma xy=c^2 \\ y^2+z^2-2\cos\alpha yz=a^2 \\ x^2+z^2-2\cos\beta xz=b^2. \end{cases}$$

Find the value of $|xy\sin\gamma+yz\sin\alpha+zx\sin\beta|$. This problem has a simple vector interpretation.

Indeed, let $\mathbf{x},\mathbf{y},\mathbf{z}$ be three pairwise non-collinear vectors on a plane such that

$$\|\mathbf{x}\|=x,\ \|\mathbf{y}\|=y,\ \|\mathbf{z}\|=z$$

the oriented angles between the pairs of vectors are

$$\angle(\mathbf{x},\mathbf{y})=\gamma, \angle(\mathbf{y},\mathbf{z})=\alpha, \angle(\mathbf{z},\mathbf{x})=\beta.$$

Then according to the conditions of problem, we also have

$$\begin{aligned} \|\mathbf{x}-\mathbf{y}\|^2 &= (\mathbf{x}-\mathbf{y})(\mathbf{x}-\mathbf{y}) \\ &= \|\mathbf{x}\|^2-2(\mathbf{x}\cdot\mathbf{y})+\|\mathbf{y}\|^2 \\ &= x^2+y^2-2\cos\gamma\cdot xy \\ &= c^2 \text{ and similarly,} \\ \|\mathbf{y}-\mathbf{z}\|^2 &= a^2, \\ \|\mathbf{z}-\mathbf{x}\|^2 &= b^2. \end{aligned}$$

It is easy to see that

$$a+b=\|\mathbf{y}-\mathbf{z}\|+\|\mathbf{z}-\mathbf{x}\|\geq\|\mathbf{x}-\mathbf{y}\|=c,$$

and since $\mathbf{y}-\mathbf{z}$ and $\mathbf{z}-\mathbf{x}$ aren't collinear then $a+b>c$.

Similarly, $b+c>a$ and $c+a>b$. Thus the positive numbers a,b,c define a triangle with area with semi-perimeter s and area $F=\sqrt{s(s-a)(s-b)(s-c)}$.

Definition

For any two vectors $\mathbf{x}=(x_1,x_2)\,,\mathbf{y}=(y_1,y_2)$ we define the "exterior product" of two vectors in the plane as follows:

$$\mathbf{x}\wedge\mathbf{y}=x_1y_2-x_2y_1.$$

From this definition we can immediately obtain the following properties of the exterior product:

- 1. $\mathbf{x}\wedge\mathbf{y}=-\mathbf{y}\wedge\mathbf{x}$,
- **2.** $\mathbf{x}\wedge\mathbf{x}=\mathbf{0}$,
- **3.** $\mathbf{x}\wedge(\mathbf{y}+\mathbf{z})=\mathbf{x}\wedge\mathbf{y}+\mathbf{x}\wedge\mathbf{z}$ and $(\mathbf{x}+\mathbf{y})\wedge\mathbf{z}=\mathbf{x}\wedge\mathbf{z}+\mathbf{y}\wedge\mathbf{z}$,
- **4.** $(k\,\mathbf{x})\wedge\mathbf{y}=\mathbf{x}\wedge k\mathbf{y}=k(\mathbf{x}\wedge\mathbf{y})$.

One more property expresses the geometric essence of the exterior product in a plane.

Let

$$\mathbf{e}=(0,1)\,,\varphi=\angle(\mathbf{e},\mathbf{x})\,,\psi=\angle(\mathbf{e},\mathbf{y})\,,\ \angle(\mathbf{x},\mathbf{y})=\psi-\varphi$$

and since

$$\begin{aligned}(x_1,x_2)&=\|\mathbf{x}\|(\cos\varphi,\sin\varphi)\,,\\(y_1,y_2)&=\|\mathbf{y}\|(\cos\psi,\sin\psi)\,,\ \text{then}\\\mathbf{x}\wedge\mathbf{y}&=x_1y_2-x_2y_1\\&=\|\mathbf{x}\|\,\|\mathbf{y}\|(\cos\varphi\sin\psi-\sin\varphi\cos\psi)\\&=\|\mathbf{x}\|\,\|\mathbf{y}\|\sin\angle(\mathbf{x},\mathbf{y})\,.\end{aligned}$$

Hence, $\mathbf{x}\wedge\mathbf{y}$ is the oriented area of the parallelogram defined by $(\mathbf{x},\mathbf{y})$, and $|\mathbf{x}\wedge\mathbf{y}|$ is area of this parallelogram.

Coming back to our problem we obtain

$$\begin{aligned}xy\sin\gamma+yz\sin\alpha+zx\sin\beta&=\|\mathbf{x}\|\,\|\mathbf{y}\|\sin\angle(\mathbf{x},\mathbf{y})+\|\mathbf{y}\|\,\|\mathbf{z}\|\sin\angle(\mathbf{y},\mathbf{z})+\|\mathbf{z}\|\,\|\mathbf{x}\|\sin\angle(\mathbf{z},\mathbf{x})\\&=\mathbf{x}\wedge\mathbf{y}+\mathbf{y}\wedge\mathbf{z}+\mathbf{z}\wedge\mathbf{x}.\end{aligned}$$

Using properties $1-4$ we have

$$(\mathbf{x}-\mathbf{y})\wedge(\mathbf{x}-\mathbf{z})=\mathbf{x}\wedge\mathbf{x}-\mathbf{y}\wedge\mathbf{x}-\mathbf{x}\wedge\mathbf{z}+\mathbf{y}\wedge\mathbf{z}=\mathbf{x}\wedge\mathbf{y}+\mathbf{z}\wedge\mathbf{x}+\mathbf{y}\wedge\mathbf{z}.$$

Thus,

$$|xy\sin\gamma + yz\sin\alpha + zx\sin\beta| = |(\mathbf{x}-\mathbf{y})\wedge(\mathbf{x}-\mathbf{z})| \text{ and since}$$

$|(\mathbf{x}-\mathbf{y})\wedge(\mathbf{x}-\mathbf{z})|$ is the area of the parallelogram defined by vectors $\mathbf{x}-\mathbf{y}, \mathbf{x}-\mathbf{z}$ which is equal to $2F$. So, we obtain finally that

$$|xy\sin\gamma + yz\sin\alpha + zx\sin\beta| = 2\sqrt{s\,(s-a)\,(s-b)\,(s-c)}.$$

Solution 3 by Kee-Wai Lau, Hong Kong, China

We show that $xy + 2yz + 3xz = 40\sqrt{3}$.

Denote the given equations by $(1), (2)$, and (3) in given order. Then $(2)+(3)-(1)$ gives $2z^2 + xz - xy = 0$, so that

$$y = \frac{2z^2}{x} + z. \qquad (4)$$

Substitute y of (4) into (2) and simplifying gives

$$z^4 + xz^3 + x^2z^2 = 12x^2. \qquad (5)$$

From (5) and (3) we have

$$z^2 = \frac{12x^2}{25}. \qquad (6)$$

Substitute z^2 of (6) into (3) and simplifying, we obtain

$$z = \frac{625 - 37x^2}{25x}. \qquad (7)$$

Substitute z of (7) into (6) and simplifying, we obtain

$$1069x^4 - 46250x^2 + 390625 = 0. \qquad (8)$$

Now (8) gives

$$x^2 = \frac{625\left(37 - 10\sqrt{3}\right)}{1069}, \text{ and } \frac{625\left(37 + 10\sqrt{3}\right)}{1069}.$$

If $x^2 = \dfrac{625\left(37 + 10\sqrt{3}\right)}{1069}$, then by (6), we have $z^2 = \dfrac{300\left(37 + 10\sqrt{3}\right)}{1069}$. Then using (3), we see that $xz = -\dfrac{250\left(30 + 37\sqrt{3}\right)}{1069} < 0$, must be rejected. Hence by (6) and (2), we have

$$x^2 = \frac{625\left(37 - 10\sqrt{3}\right)}{1069},\ z^2 = \frac{300\left(37 - 10\sqrt{3}\right)}{1069},\ y^2 = \frac{12\left(1501 + 750\sqrt{3}\right)}{1069}.$$

By (1) and (3) we obtain

$$xy = \frac{50\left(294 + 65\sqrt{3}\right)}{1069},\ xz = \frac{250\left(-30 + 37\sqrt{3}\right)}{1069}.$$

Since $yz = \dfrac{(xy)(xz)}{x^2} = \dfrac{60\left(65+98\sqrt{3}\right)}{1069}$, we have $xy + 2yz + 3xz = 40\sqrt{3}$.

Remark: David Stone and John Hawkins, Georgia Southern University, Statesboro GA noted that "the problem poses three nice cylinders in space and asks for their intersection. In the first quadrant, this consists of exactly one point. Perhaps the desired expression has a geometric significance and it is possible to make use of the geometry and compute its value without actually solving for x, y and z. There are other points that satisfy the three given equations. For instance, negating the x, y and z gives us another solution (which produces the identical value for $xy + 2yz + 3xz$). But there are others which produce $xy + 2yz + 3xz = -69.282$"

Also solved by Brian D. Beasley, Presbyterian College, Clinton, SC; Ed Gray, Highland Beach, FL; Paul M. Harms, North Newton, KS; Enkel Hysnelaj, University of Technology, Sydney Australia and Elton Bojaxhiu, Kriftel, Germany; Jahangeer Kholdi and Farideh Firoozbakht, University of Isfahan, Khansar, Iran; David Stone and John Hawkins, Georgia Southern University, Statesboro, GA, and the proposer.

- **5231:** *Proposed by Panagiote Ligouras, "Leonardo da Vinci" High School, Noci, Italy*

The lengths of the sides of the hexagon $ABCDEF$ satisfy $AB = BC, CD = DE$, and $EF = FA$. Prove that

$$\sqrt{\frac{AF}{CF}} + \sqrt{\frac{CB}{EB}} + \sqrt{\frac{ED}{AD}} > 2.$$

Solution by Enkel Hysnelaj, University of Technology, Sydney, Australia and Elton Bojaxhiu, Kriftel, Germany.

The inequality will be equivalent to $\sqrt{\dfrac{EF}{CF}} + \sqrt{\dfrac{AB}{EB}} + \sqrt{\dfrac{CD}{AD}} > 2$ Using Ptolemy's Inequality (*http://mathworld.wolfram.com/PtolemyInequality.html*) for quadrilateral $ABCE$ we have

$$AB \cdot CE + BC \cdot AE > EB \cdot AC \Rightarrow \frac{AB}{EB} = \frac{AC}{CE + AE}$$

Using the Ptolemy's Inequality for quadrilateral $EFAC$ and quadrilateral $CDEA$ we obtain

$$\begin{aligned} \frac{EF}{CF} &= \frac{AE}{CA + CE} \\ \frac{CD}{AD} &= \frac{CE}{AE + CA} \end{aligned}$$

Now if $CA = a, CE = b, AE = c$, it is enough to prove that

$$\sqrt{\frac{a}{b+c}} + \sqrt{\frac{b}{c+a}} + \sqrt{\frac{c}{a+b}} > 2.$$

Normalizing this we can assume that $a + b + c = 1$, so we require to prove

$$\sqrt{\frac{a}{1-a}}+\sqrt{\frac{b}{1-b}}+\sqrt{\frac{c}{1-c}}>2.$$

It is obvious we just need to prove that

$$\sqrt{\frac{a}{1-a}}>2\frac{a}{a+b+c}=2a.$$

Squaring both sides and doing easy manipulations we have

$$\frac{a}{1-a}>4a^2 \Rightarrow -4a^2+4a-1<0 \Rightarrow -(2a-1)^2<0.$$

which obviously is true for any $a \in (0,1)$.

Finally we have

$$\sqrt{\frac{a}{1-a}}+\sqrt{\frac{b}{1-b}}+\sqrt{\frac{c}{1-c}}>2a+2b+2c=2(a+b+c)=2.$$

Also solved by Bruno Salgueiro Fanego, Viveiro, Spain; Ed Gray, Highland Beach, FL; Kee-Wai Lau, Hong Kong, China, and the proposer

- **5232:** *Proposed by D. M. Bătinetu-Giurgiu, "Matei Basarab" National College, Bucharest and Neculai Stanciu, "George Emil Palade" Secondary School, Buzău, Romania*

Prove that: If $a,b,c>0$, then,

$$2\sqrt{\frac{a^2+b^2+c^2}{3}}\cdot\frac{\sin x}{x}+\frac{a+b+c}{3}\cdot\frac{\tan x}{x}>a+b+c,$$

for any $x \in \left(0,\frac{\pi}{2}\right)$.

Solution 1 by Dionne Bailey, Elsie Campbell, and Charles Diminnie, Angelo State University, San Angelo, TX

If $f(x)=2\sin x+\tan x-3x$, then for $x \in \left(0,\frac{\pi}{2}\right)$,

$$\begin{aligned} f'(x) &= 2\cos x+\sec^2 x-3 \\ &= \frac{2\cos^3 x-3\cos^2 x+1}{\cos^2 x} \\ &= \frac{(2\cos x+1)(\cos x-1)^2}{\cos^2 x} \\ &> 0. \end{aligned}$$

Since $f(x)$ is continuous on $\left[0,\frac{\pi}{2}\right)$ and $f(0)=0$, it follows that $f(x)>0$ for all $x \in \left(0,\frac{\pi}{2}\right)$. Therefore, for all $x \in \left(0,\frac{\pi}{2}\right)$,

$$2\sin x+\tan x>3x$$

or

$$\frac{2\sin x+\tan x}{3x}>1. \qquad (1)$$

By the Arithmetic Mean - Root Mean Square Inequality,

$$\sqrt{\frac{a^2+b^2+c^2}{3}}\geq\frac{a+b+c}{3} \qquad (2)$$

when $a,b,c>0$. Since $\sin x>0$ on $\left(0,\frac{\pi}{2}\right)$, we may combine (1) and (2) to get

$$\begin{aligned} & 2\sqrt{\frac{a^2+b^2+c^2}{3}}\cdot\frac{\sin x}{x}+\frac{a+b+c}{3}\cdot\frac{\tan x}{x} \\ \geq\ & \frac{2\sin x+\tan x}{3x}\cdot(a+b+c) \\ >\ & a+b+c \end{aligned}$$

for any $x\in\left(0,\frac{\pi}{2}\right)$.

Solution 2 by Albert Stadler, Herrliberg, Switzerland

By the Cauchy-Schwarz inequality

$$\sqrt{3}\sqrt{a^2+b^2+c^2}\geq a+b+c.$$

So

$$\begin{aligned} & \frac{3}{a+b+c}\left(2\sqrt{\frac{a^2+b^2+c^2}{3}}\cdot\frac{\sin x}{x}+\frac{a+b+c}{3}\cdot\frac{\tan x}{x}-a-b-c\right) \\ \geq\ & 2\frac{\sin x}{x}+\frac{\tan x}{x}-3 \\ =\ & \frac{1}{x}\left(2\sin x+\tan x-3x\right) \\ =\ & \frac{1}{x}\int_0^x\left(2\cos t+\frac{1}{\cos^2 t}-3\right)dt \\ =\ & \frac{1}{x}\int_0^x\frac{2\cos^3 t-3\cos^2 t+1}{\cos^2 t}dt \\ =\ & \frac{1}{x}\int_0^x\frac{(2\cos t+1)(1-\cos t)^2}{\cos^2 t}dt>0,\ \text{for any } x\in\left(0,\frac{\pi}{2}\right). \end{aligned}$$

$$2\sqrt{\frac{a^2+b^2+c^2}{3}}\cdot\frac{\sin x}{x}+\frac{a+b+c}{3}\cdot\frac{\tan x}{x}>a+b+c \text{ for any } x\in\left(0,\frac{\pi}{2}\right).$$

Solution 3 by Paul M. Harms, North Newton, KS

A convergent series for $\dfrac{\sin x}{x}$ is $1-\dfrac{x^2}{3!}+\dfrac{x^4}{5!}-\ldots$, and a convergent series for $\dfrac{\tan x}{x}$ is $1+\dfrac{x^2}{3}+\dfrac{2}{15}x^4+\ldots$ for the interval $\left(0,\dfrac{\pi}{2}\right)$.

In this interval, $\dfrac{\sin x}{x}<1-\dfrac{x^2}{6}$ and $\dfrac{\tan x}{x}<1+\dfrac{x^2}{3}$.

The inequality in the problem holds if we can show that

$$2\sqrt{\frac{a^2+b^2+c^2}{3}}\left(1-\frac{x^2}{6}\right)+\frac{a+b+c}{3}\left(1+\frac{x^2}{3}\right)-(a+b+c)>0.$$

Let the left hand side of the inequality be $f(x)$. Then

$$f'(x)=2\sqrt{\frac{a^2+b^2+c^2}{3}}\left(\frac{-x}{3}\right)+\frac{a+b+c}{3}\left(\frac{2x}{3}\right).$$

The only place where $f'(x)=0$ on the interval $\left[0,\dfrac{\pi}{2}\right]$ is at $x=0$, if $a=b=c$ is not true as in shown below.

To check where $f'(x)<0$ we check where

$$2\sqrt{\frac{a^2+b^2+c^2}{3}}\left(\frac{x}{3}\right)>\frac{a+b+c}{3}\left(\frac{2x}{3}\right).$$

Simplifying we see:

$$\begin{aligned}\frac{a^2+b^2+c^2}{3} &> \left(\frac{a+b+c}{9}\right)^2 \text{ which is equivalent to}\\ 3a^2+3b^2+3c^2-(a+b+c)^2 &= (a-b)^2+(b-c)^2+(c-a)^2>0.\end{aligned}$$

Then $f'(x)<0$ on the interval $\left(0,\dfrac{\pi}{2}\right]$ where $a,b,$ and c are not all the same positive number. If $a=b=c$ is not true, then the inequality will be correct provided $f\left(\dfrac{\pi}{2}\right)>0$. We see that:

$$f\left(\frac{\pi}{2}\right)=\sqrt{\frac{a^2+b^2+c^2}{3}}\left(\frac{24-\pi^2}{12}\right)+\frac{a+b+c}{3}\left(\frac{\pi^2-24}{12}\right).$$

To show that $f\left(\dfrac{\pi}{2}\right)\geq 0$ is suffices to show that

$$\sqrt{\frac{a^2+b^2+c^2}{3}}\geq\frac{a+b+c}{3}.$$

This last inequality was shown previously. The inequality in the problem then is correct when a,b,c are not all the same positive number.

Now consider the case when $a = b = c > 0$. The inequality of the problem is then equivalent to

$$a\left(\frac{2\sin x}{x} + \frac{\tan x}{x} - 3\right) > 0.$$

We have

$$\begin{aligned} \frac{2\sin x}{x} &= 2 - \frac{x^2}{3} + \frac{x^4}{60} - \ldots \text{ and} \\ \frac{\tan x}{x} &= 1 + \frac{x^2}{3} + \frac{2x^4}{15} + \frac{17x^6}{315} + \ldots. \end{aligned}$$

Then the left side of the inequality is

$$a\left[\left(2 - \frac{x^2}{3} + \frac{x^4}{60} - \ldots\right) + \left(1 + \frac{x^2}{3} + \frac{2x^4}{15} + \frac{17x^6}{315} + \ldots\right) - 3\right],$$

and the inequality of the problem can be written as

$$a\left[\left(\frac{x^4}{60} - \frac{2x^6}{7!} + \ldots\right) + \left(\frac{2x^4}{15} + \frac{17x^6}{315} + \ldots\right)\right] > 0.$$

On the interval $\left(0, \frac{\pi}{2}\right)$, the alternating series part is a convergent series whose terms in absolute value are decreasing and whose first term is positive. Thus both series inside the brackets are positive and the inequality of the problem is correct for positive numbers a, b, and c for x in the interval $\left(0, \frac{\pi}{2}\right)$.

Also solved by Arkady Alt, San Jose, CA; Ed Gray, Highland Beach, FL; Kee-Wai Lau, Hong Kong, China; Adrian Naco, Polytechnic University, Tirana, Albania; Paolo Perfetti, Department of Mathematics, "Tor Vergata," University,Rome, Italy; Ángel Plaza, University of Las Palmas de Gran Canaria, Spain; Boris Rays, Brooklyn, NY, and the proposer.

- **5233:** *Proposed by Anastasios Kotronis, Athens, Greece*

Let $x \geq \dfrac{1+\ln 2}{2}$ and let $f(x)$ be the function defined by the relations:

$$\begin{aligned} f^2(x) - \ln f(x) &= x \\ f(x) &\geq \frac{\sqrt{2}}{2}. \end{aligned}$$

- 1. Calculate $\displaystyle\lim_{x\to+\infty} \frac{f(x)}{\sqrt{x}}$, if it exists.
- 2. Find the values of $\alpha \in \Re$ for which the series $\displaystyle\sum_{k=1}^{+\infty} k^{\alpha}\left(f(k) - \sqrt{k}\right)$ converges.
- 3. Calculate $\displaystyle\lim_{x\to+\infty} \frac{\sqrt{x}f(x) - x}{\ln x}$, if it exists.

Solution 1 by Arkady Alt, San Jose, CA

1. Since $x \geq \dfrac{1+\ln 2}{2}$ and $f(x) \geq \dfrac{\sqrt{2}}{2}$ then
$\ln f(x) + x \geq x + \ln\left(\dfrac{\sqrt{2}}{2}\right) \geq \dfrac{1+\ln 2}{2} - \dfrac{\ln 2}{2} = \dfrac{1}{2}$
and, therefore, for such x and $f(x)$ we have

$$f^2(x) - \ln f(x) = x \iff$$

$$f(x) = \sqrt{x + \ln f(x)} \text{ and}$$

$$f(x) \geq \sqrt{x + \ln\left(\frac{\sqrt{2}}{2}\right)} = \sqrt{x - \frac{\ln 2}{2}}.$$

Hence, $\lim\limits_{x\to+\infty} f(x) = \infty$

Since $f(x) > 0$ then

$$f^2(x) - \ln f(x) = x \iff f(x) = \frac{x}{f(x)} + \frac{\ln f(x)}{f(x)}$$

and, therefore,

$$f(x) \leq \frac{x}{\sqrt{x - \frac{\ln 2}{2}}} + \frac{\ln f(x)}{f(x)}.$$

Hence,

$$\frac{\sqrt{x - \frac{\ln 2}{2}}}{\sqrt{x}} \leq \frac{f(x)}{\sqrt{x}} \leq \frac{\sqrt{x}}{\sqrt{x - \frac{\ln 2}{2}}} + \frac{\ln f(x)}{\sqrt{x} f(x)}.$$

Since

$$\lim_{x\to+\infty} \frac{\sqrt{x - \frac{\ln 2}{2}}}{\sqrt{x}} = 1, \lim_{x\to+\infty} \frac{\sqrt{x}}{\sqrt{x - \frac{\ln 2}{2}}} = 1 \text{ and } \lim_{x\to+\infty} \frac{\ln f(x)}{f(x)} = 0.$$

Then by the squeeze principle we obtain

$$\lim_{x\to+\infty} \frac{f(x)}{\sqrt{x}} = 1.$$

2. First note that series $\sum\limits_{n=1}^{\infty} \dfrac{\ln n}{n^p}$ is convergent if $p > 1$ and divergent if $p \leq 1$.

(Let $p > 1$ and $\varepsilon = \dfrac{1-p}{2}$. Since $p - \varepsilon = \frac{3p-1}{2} > 1$ then series $\sum\limits_{n=1}^{\infty} \dfrac{1}{n^{p-\varepsilon}}$ is convergent.

There is $n_0 \in N$ such that $\ln n < n^{\varepsilon}$ for all $n > n_0$ (because $\lim\limits_{n\to\infty} \dfrac{\ln n}{n^q} = 0$ for any $q > 0$).

Hence,

$$\sum_{n=1}^{\infty}\frac{\ln n}{n^p}=\sum_{k=1}^{n_0}\frac{\ln k}{k^p}+\sum_{n=n_0+1}^{\infty}\frac{\ln n}{n^p}<\sum_{k=1}^{n_0}\frac{\ln k}{k^p}+\sum_{n=n_0+1}^{\infty}\frac{n^{\varepsilon}}{n^p}=\sum_{k=1}^{n_0}\ln k k^p+\sum_{n=n_0+1}^{\infty}\frac{1}{n^{p-\varepsilon}}.$$

If $p\leq 1$ then $\sum\limits_{n=3}^{\infty}\frac{\ln n}{n^p}>\sum\limits_{n=3}^{\infty}\frac{1}{n^p}$, where by p test $\sum\limits_{n=3}^{\infty}\frac{1}{n^p}$ is divergent series and, therefore, the series $\sum\limits_{n=1}^{\infty}\frac{\ln n}{n^p}$ is divergent.)

Also note that $\lim\limits_{x\to+\infty}\frac{\ln f(x)}{\ln x}=\frac{1}{2}$. Indeed,

$$\begin{aligned}\lim_{x\to+\infty}\left(\frac{2\ln f(x)}{\ln x}-1\right) &= 2\lim_{x\to+\infty}\frac{\ln\left(\frac{f(x)}{\sqrt{x}}\right)}{\ln x}\\ &= 2\lim_{x\to+\infty}\frac{1}{\ln x}\cdot\lim_{x\to+\infty}\ln\left(\frac{f(x)}{\sqrt{x}}\right)\\ &= 2\lim_{x\to+\infty}\frac{1}{\ln x}\cdot\ln\left(\lim_{x\to+\infty}\frac{f(x)}{\sqrt{x}}\right)=2\cdot 0\cdot\ln 1=0.\end{aligned}$$

Since

$$f^2(x)-\ln f(x)=x\iff f(x)-\sqrt{x}=\frac{\ln f(x)}{f(x)+\sqrt{x}},\text{ then}$$

$n^{\alpha}\left(f(n)-\sqrt{n}\right)=\dfrac{n^{\alpha}\ln f(n)}{f(n)+\sqrt{n}}$ and, therefore,

$$\lim_{n\to\infty}\frac{n^{\alpha}\left(f(n)-\sqrt{n}\right)}{n^{\alpha-1/2}\ln n}=\lim_{n\to\infty}\left(\frac{1}{n^{\alpha-1/2}\ln n}\cdot\frac{n^{\alpha}\ln f(n)}{f(n)+\sqrt{n}}\right)=\lim_{n\to\infty}\frac{\sqrt{n}\ln f(n)}{\left(f(n)+\sqrt{n}\right)\ln n}$$

$$=\lim_{n\to\infty}\frac{\ln f(n)}{\left(\frac{f(n)}{\sqrt{n}}+1\right)\ln n}=\lim_{n\to\infty}\frac{\ln f(n)}{\ln n}\cdot\lim_{n\to\infty}\frac{1}{\left(\frac{f(n)}{\sqrt{n}}+1\right)}=\frac{1}{4}.$$

Thus, by the limit convergency test, both series $\sum\limits_{n=1}^{\infty}n^{\alpha}\left(f(n)-\sqrt{n}\right)$ and $\sum\limits_{n=1}^{\infty}\frac{\ln n}{n^{1/2-\alpha}}$ have the same character of convergency.

Since $\sum\limits_{n=1}^{\infty}\frac{\ln n}{n^{1/2-\alpha}}$ converges if $1/2-\alpha>1\iff\alpha<-1/2$ and diverges if $1/2-\alpha\leq 1\iff -1/2\leq\alpha$ we may conclude that series $\sum\limits_{n=1}^{\infty}n^{\alpha}\left(f(n)-\sqrt{n}\right)$ is convergent if $\alpha<-1/2$ and divergent if $-1/2\leq\alpha$.

3. Since

$$\begin{aligned}
f(x)-\sqrt{x} &= \frac{\ln f(x)}{f(x)+\sqrt{x}} \text{ then} \\
\lim_{x\to+\infty}\frac{\sqrt{x}f(x)-x}{\ln x} &= \lim_{x\to+\infty}\frac{\sqrt{x}\,(f(x)-\sqrt{x})}{\ln x} \\
&= \lim_{x\to+\infty}\frac{\sqrt{x}\ln f(x)}{\ln x\,(f(x)+\sqrt{x})} \\
&= \lim_{x\to+\infty}\frac{\sqrt{x}\ln f(x)}{\ln x\,(f(x)+\sqrt{x})} \\
&= \lim_{x\to+\infty}\frac{\ln f(x)}{\ln x}\cdot\lim_{x\to+\infty}\frac{1}{\frac{f(x)}{\sqrt{x}}+1} = \frac{1}{2}\cdot\frac{1}{2} = \frac{1}{4}.
\end{aligned}$$

Solution 2 by Paolo Perfetti, Department of Mathematics, "Tor Vergata" University, Rome, Italy

• 1. The function $t^2-\ln t$ is strictly increasing for $t\geq 1/\sqrt{2}$ thus the equation $t^2-\ln t = x$ admits a unique solution for any $x\geq(1+\ln 2)/2$. This defines the function $f(x)$ of the problem which is strictly increasing and then it admits the limit L which can be finite of infinite. If L is finite the equation $f^2(x)=\ln f(x)+x$ cannot hold thus $L=+\infty$. Moreover the differentiability of $t^2-\ln t$ assures the differentiability of $f(x)$ and in particular

$$2ff' = \frac{f'}{f}+1 \implies f'(x) = \frac{f}{2f^2-1}$$

whence using l'Höpital

$$\lim_{x\to\infty}\frac{f^2}{x} = \lim_{x\to\infty}2ff' = \lim_{x\to\infty}2f\frac{f}{2f^2-1} = 1 \implies \lim_{x\to\infty}\frac{f}{\sqrt{x}} = 1$$

• 2. We have $f(x)=\sqrt{x}+o(\sqrt{x})$ thus $\ln f(x) = \frac{1}{2}\ln x+\ln(1+o(1)) = \frac{1}{2}\ln x + o(1)$ and

$$f(x) = \sqrt{x+\ln f} = \sqrt{x+\frac{1}{2}\ln x + o(1)} = \sqrt{x}\sqrt{1+\frac{1}{2}\frac{\ln x}{x}+\frac{o(1)}{x}}$$

whence

$$f(x) = \sqrt{x}\left(1+\frac{1}{4}\frac{\ln x}{x}+o\left(\frac{\ln x}{x}\right)\right)$$

and then

$$\sum_{k=1}^{\infty}k^{\alpha}(f(k)-\sqrt{k}) = \sum_{k=1}^{\infty}\left[k^{\alpha-\frac{1}{2}}\frac{\ln k}{4}+k^{\alpha+\frac{1}{2}}o\left(\frac{\ln k}{k}\right)\right] = \sum_{k=1}^{\infty}k^{\alpha-\frac{1}{2}}\frac{\ln k}{4}\left(1+o\left(\frac{1}{k}\right)\right).$$

Thus the series converges if and only if converges the series $\sum_{k=1}^{\infty} k^{\alpha-\frac{1}{2}} \ln k$ and this occurs if and only if $\alpha < -1/2$.
This may be seen for instance by using the Cauchy–condensation test after observing that $k^\alpha \ln k$ decreases definitively in k for $\alpha < 0$. Thus we investigate the convergence of the series

$$\sum_{k=1}^{\infty} 2^k 2^{k(\alpha-\frac{1}{2})} k \frac{\ln 2}{2} = \frac{\ln 2}{2} \sum_{k=1}^{\infty} 2^{k(\alpha+\frac{1}{2})} k$$

Here we can use any of the countless method to study such a series. For instance the ratio test

$$\lim_{n\to\infty} \frac{2^{(k+1)(\alpha+\frac{1}{2})}(k+1)}{2^{k(\alpha+\frac{1}{2})} k} = 2^{\alpha+\frac{1}{2}}$$

If $\alpha + 1/2 < 0$ the series converges. If $\alpha + 1/2 > 0$ the series diverges. If $\alpha = -1/2$ we have the series

$$\frac{\ln 2}{2} \sum_{k=1}^{\infty} 2^{k(\alpha+\frac{1}{2})} k = \frac{\ln 2}{2} \sum_{k=1}^{\infty} k$$

thus diverges.

- 3. By employing $f(x) = \sqrt{x}\left(1 + \frac{1}{4}\frac{\ln x}{x} + o\left(\frac{\ln x}{x}\right)\right)$

$$\lim_{x\to\infty} \frac{\sqrt{x} f(x) - x}{\ln x} = \lim_{x\to\infty} \frac{\frac{1}{4}\ln x + o(\ln x)}{\ln x} = \frac{1}{4}.$$

Solution 3 by Kee-Wai Lau, Hong Kong, China

Firstly we have $f(x) = \sqrt{x + \ln f(x)} \geq \sqrt{x - \frac{\ln 2}{2}}$.

Using the well-known inequality $e^x \geq 1 + x$ for real x, we obtain $\ln f(x) \leq f(x) - 1$. Hence

$$f^2(x) - 1 + f(x), \text{ so that } \mathrm{f}(x) \leq \frac{1 + \sqrt{4x - 3}}{2}.$$

So by the squeezing principle, we have $\lim_{x+\infty} \frac{f(x)}{\sqrt{x}} = 1$. This answers part one.

Suppose that $f(x) = \sqrt{x} + g(x)$, where $\lim_{x\to+\infty} \frac{(x)}{\sqrt{x}} - 0$. From

$$\left(\sqrt{x} + g(x)\right) - \ln\sqrt{x} - \ln\left(1 + \frac{g(x)}{\sqrt{x}}\right) = x,$$

we see that $g(x) \sim \frac{\ln x}{4\sqrt{x}}$ as $x \to +\infty$.

Thus $\sum_{k=1}^{+\infty} k^\alpha \left(f(k) = \sqrt{k}\right)$ converges for $\alpha < \frac{-1}{2}$, diverges for $\alpha \geq \frac{-1}{2}$ and that

$$\lim_{x\to+\infty} \frac{\sqrt{x} f(x) - x}{\ln x} = \lim_{x\to+\infty} \frac{\sqrt{x} g(x)}{\ln x} = \frac{1}{4}.$$

These answer parts two and three.

Also solved by Adrian Naco, Polytechnic University, Tirana, Albania; Albert Stadler, Herrliberg, Switzerland, and the proposer.

- **5234:** *Proposed by José Luis Díaz-Barrero, Polytechnical University of Catalonia, Barcelona, Spain*

Let $a < b$ be positive real numbers and let $f_i : [a, b] \to \Re$ $(i = 1, 2)$ be continuous functions in $[a, b]$ and differentiable in (a, b). If f_2 is strictly decreasing then prove that there exists an $\alpha \in (a, b)$ such that

$$f_2(b) < f_2(\alpha) + 2\left(\frac{f_2'(\alpha)}{f_1'(\alpha)}\right) < f_2(a).$$

Comment by Editor: **Paolo Perfetti of the Department of Mathematics at Tor Vergata University in Rome, Italy** provided a counter-example to the above statement. The incompleteness of the statement was acknowledged by José Luis and he revised the problem. Following is his solution to the revised statement.

5234 (Revised:) *Proposed by José Luis Díaz-Barrero, BARCELONA TECH, Barcelona, Spain.*

Let $a < b$ be positive real numbers and let $f_i : [a, b] \to \Re$ $(i = 1, 2)$ be continuous functions in $[a, b]$ and differentiable in (a, b). (1) If f_1 and f_2 are strictly decreasing, then prove that there exists $\alpha \in (a, b)$ such that

$$f_2(b) < f_2(\alpha) + 2\left(\frac{f_2'(\alpha)}{f_1'(\alpha)}\right)$$

(2) If f_1 is strictly increasing and f_2 is strictly decreasing, then prove that there exists $\alpha \in (a, b)$ such that

$$f_2(\alpha) + 2\left(\frac{f_2'(\alpha)}{f_1'(\alpha)}\right) < f_2(a)$$

Solution by the proposer

Consider the function $f : [a, b] \to \Re$ defined by

$$f(x) = \Big(f_2(a) - f_2(x)\Big)\Big(f_2(b) - f_2(x)\Big)e^{f_1(x)}$$

Since f is continuous in $[a, b]$, differentiable in (a, b), and $f(a) = f(b) = 0$, then on account of Rolle's theorem there exists $\alpha \in (a, b)$ such that $f'(\alpha) = 0$. That is,

$$f_1'(\alpha)e^{f_1(\alpha)}\Big(f_2(a) - f_2(\alpha)\Big)\Big(f_2(b) - f_2(\alpha)\Big) - f_2'(\alpha)\Big(f_2(a) + f_2(b) - 2f_2(\alpha)\Big)e^{f_1(\alpha)} = 0$$

from which follows

$$\frac{f_2'(\alpha)}{f_1'(\alpha)} = \frac{\Big(f_2(a) - f_2(\alpha)\Big)\Big(f_2(b) - f_2(\alpha)\Big)}{f_2(a) + f_2(b) - 2f_2(\alpha)}$$

(1) Now we prove the first part of the statement. Indeed, we have that

$$f_2(b) < f_2(\alpha) + 2\left(\frac{f_2'(\alpha)}{f_1'(\alpha)}\right)$$

is equivalent to

$$(f_2(b) - f_2(\alpha) < 2\,\frac{\Big(f_2(a) - f_2(\alpha)\Big)\Big(f_2(b) - f_2(\alpha)\Big)}{f_2(a) + f_2(b) - 2f_2(\alpha)},$$

where $f_2(b) - f_2(\alpha) < 0$, $f_2(a) - f_2(\alpha) > 0$ and $f_2(a) + f_2(b) - 2f_2(\alpha) < 0$ because the RHS of the preceding inequality is positive. Then, after division by $f_2(b) - f_2(\alpha) < 0$ and multiplication by $\dfrac{1}{f_2(a) + f_2(b) - 2f_2(\alpha)} < 0$ yields

$$f_2(a) + f_2(b) - 2f_2(\alpha) < 2\left(f_2(a) - f_2(\alpha)\right)$$

or $f_2(b) < f_2(a)$. The preceding trivially holds because f_2 is strictly decreasing.

(2) To prove the second part of the statement, we have

$$f_2(\alpha) + 2\left(\frac{f_2'(\alpha)}{f_1'(\alpha)}\right) < f_2(a)$$

is equivalent to

$$f_2(a) - f_2(\alpha) < 2\,\frac{\Big(f_2(a) - f_2(\alpha)\Big)\Big(f_2(b) - f_2(\alpha)\Big)}{f_2(a) + f_2(b) - 2f_2(\alpha)},$$

where $f_2(a) - f_2(\alpha) > 0$, $f_2(b) - f_2(\alpha) < 0$ and $f_2(a) + f_2(b) - 2f_2(\alpha) > 0$ because the RHS of the preceding inequality is negative. Then, after rearranging terms we get

$$2\left(f_2(b) - f_2(\alpha)\right) < f_2(a) + f_2(b) - 2f_2(\alpha)$$

from which follows $f_2(b) < f_2(a)$ that again holds on account that f_2 is strictly decreasing. This completes the proof.

- **5235:** *Proposed by Albert Stadler, Herrliberg, Switzerland*

On December 21, 2012 ("12-21-12") the Mayan Calendar's 13th Baktun cycle will end. On this date the world as we know it will also change. Since every end is a new beginning we are looking for natural numbers n such that the decimal representation of 2^n starts and ends with the digit sequence 122112. Let S be the set of natural numbers n such that $2^n = 122112\ldots 122112$. Let $s(x)$ be the number of elements of S that are $\leq x$.

Prove that $\displaystyle\lim_{x\to\infty} \frac{s(x)}{x}$ exists and is positive. Calculate the limit.

Solution 1 by Brian D. Beasley, Presbyterian College, Clinton, SC

First, we determine the probability that a power of 2 begins with 122112. As noted in [1] and [2], Benford's Law may be generalized as follows: The probability that the decimal representation of a number begins with the string of digits n is $\log_{10}(1 + 1/n)$. Since the sequence of the powers of 2 satisfies Benford's Law (see [1]), we conclude that the probability that a power of 2 begins with 122112 is $\log_{10}(1 + 1/122112)$.

Next, we determine the probability that a power of 2 ends with 122112. We start by noting that $2^{89} \equiv 562112 \pmod{10^6}$, which is the first occurrence of a power of 2 that is congruent to 112 modulo 1000. The next occurrence of such a power of 2 is 2^{189}, with each successive occurrence at $2^{100k+89}$. We find that the first power of 2 that is congruent to 122,112 modulo 10^6 is

$$2^{3089} \equiv 122112 \pmod{10^6},$$

and the sequence becomes periodic modulo 10^6 at

$$2^{12589} \equiv 2^{89} \equiv 562112 \pmod{10^6}.$$

Hence every 12500th term of the sequence of powers of 2 is congruent to 122,112 modulo 10^6, so the probability that a power of 2 ends with 122112 is 1/12500.

Finally, we calculate

$$\lim_{x\to\infty} \frac{s(x)}{x} = \frac{1}{12500} \log_{10}\left(1 + \frac{1}{122112}\right) \approx 2.845 \times 10^{-10}.$$

References.

[1] "Benford's Law," Wikipedia web page, http://en.wikipedia.org/wiki/Benford's_law

[2] Theodore P. Hill, The Significant-Digit Phenomenon, *The American Mathematical Monthly,* Vol. 102, No. 4 (Apr. 1995), pp. 322-327

Solution 2 by proposer

We first claim that $2^n \equiv 122112 \pmod{10^6}$ if and only if $n = 3089 \pmod{12500}$.

We first note that $122112 = 2^8 \cdot 3^2 \cdot 53$. Of course $n \geq 6$. So $2^n = 122112 (\text{mod } 10^6)$ is equivalent to $2^{n-6} \equiv 1908 \pmod{5^6}$.

We note that $2^0 \equiv 1 \text{ mod}(5)$, $2^1 \equiv 2 \text{ mod}(5)$, $2^2 \equiv 4 \text{ mod}(5)$, $2^3 \equiv 3 \text{ mod}(5)$, $2^4 \equiv 1 \text{ mod}(5)$.

So $2^n \equiv 3 \text{ mod}(5)$ if and only if $n \equiv 3 \text{ mod}(4)$.

Then $2^3 \equiv 8 \text{ mod}(25)$, $2^7 \equiv 3 \text{ mod}(25)$, $2^{11} \equiv 23 \text{ mod}(25)$, $2^{15} \equiv 18 \text{ mod}(25)$, $2^{19} \equiv 13 \text{ mod}(25)$, $2^{23} \equiv 8 \text{ mod}(25)$.

So $2^n \equiv 8 \text{ mod}(25)$ if and only if $n \equiv 3 \text{ mod}(20)$

Then $2^3 \equiv 8 \text{ mod}(125)$, $2^{23} \equiv 108 \text{ mod}(125)$, $2^{43} \equiv 83 \text{ mod}(125)$, $2^{63} \equiv 58 \text{ mod}(125)$, $2^{83} \equiv 33 \text{ mod}(125)$, $2^{103} \equiv 8 \text{ mod}(125)$

So $2^n \equiv 33 \text{ mod}(125)$ if and only if $n \equiv 83 \text{ mod}(100)$

Then $2^{83} \equiv 33 \text{ mod}(625)$, $2^{183} \equiv 533 \text{ mod}(625)$, $2^{283} \equiv 408 \text{ mod}(625)$, $2^{383} \equiv 283 \text{ mod}(625)$, $2^{4833} \equiv 158 \text{ mod}(625)$, $2^{583} \equiv 33 \text{ mod}(625)$

So $2^n \equiv 33 \text{ mod}(625)$ if and only if $n \equiv 83 \text{ mod}(500)$.

Then $2^{83} \equiv 2533 \text{ mod}(3125)$, $2^{583} \equiv 1908 \text{ mod}(3125)$, $2^{1083} \equiv 1283 \text{ mod}(3125)$, $2^{1583} \equiv 658 \text{ mod}(3125)$, $2^{2083} \equiv 33 \text{ mod}(3125)$, $2^{2583} \equiv 2533 \text{ mod}(3125)$.

So $2^n \equiv 1908 \text{ mod}(3125)$ if and only if $n \equiv 583 \text{ mod}(2500)$.

Then $2^{583} \equiv 5033 \text{ mod}(5^6)$, $2^{3083} \equiv 1908 \text{ mod}(5^6)$, $2^{5583} \equiv 14408 \text{ mod}(5^6)$, $2^{8083} \equiv 11283 \text{ mod}(5^6)$, $2^{10583} \equiv 8158 \text{ mod}(5^6)$, $2^{13083} \equiv 5033 \text{ mod}(5^6)$.

So $2^n \equiv 1908 \text{ mod}(5^6)$ if and only if $n \equiv 3083 \text{ mod}(12500)$.

$2^{n-6} \equiv 1908 \pmod{5^6}$ if and oinly if $n \equiv 3089 \pmod{12500}$.

$2^n \equiv 122112 \pmod{10^6}$ if and only if $n \equiv 3089 \pmod{12500}$.

Therefore we can assume that $n = 3089 + 12500k$ for some nonnegative integer k.

The fact that $2^{n=3089+12500k}$ starts with the digits 122112 implies that there is an integer m such that

$$1.22112 \cdot 10^m < 2^{3089+12500k} < 1.22113 \cdot 10^m.$$

This is equivalent to saying that

$$\{3089 + 12500k) \log_{10} 2\} \in (\log_{10} 1.22112, \log_{10} 1.22113),$$

where $\{x\}$ denotes the fractional part of the real number x.

$\log_{10} 2$ is irrational, for the assumption that $\log_{10} 2 = p/q$ for some coprime natural numbers $p \geq 1$ and $q \geq 1$ would imply that $10^p = 2^q$, which cannot be due to the uniqueness of the prime number factorization. Therefore the sequence $\{12500k \log_{10} 2\}$ is equidistributed mod 1, and we conclude that the portion of natural numbers that satisfy the condition $2^n = 122112 \ldots 122112$ equals

$$\lim_{x\to\infty} \frac{1}{x} \left(\sum_{\substack{n \leq x \\ 2^n = 122112\ldots122112}} 1 \right) = \lim_{x\to\infty} \frac{1}{x} \left(\sum_{\substack{3089+12500k \leq x \\ \{(3089+12500k)\log_{10} 2\} \in (\log_{10} 1.22112, \log_{10} 1.22113)}} 1 \right)$$

$$= \lim_{x\to\infty} \frac{1}{x} \left(\sum_{\substack{k \leq \frac{x}{12500} \\ \{12500k \log_{10} 2\} \in (\log_{10} 1.22112 \log_{10} 1.22113}} 1 \right)$$

$$= \frac{1}{12500} \lim_{y\to\infty} \frac{1}{y} \left(\sum_{\substack{k \leq y \\ \{12500k \log_{10} 2\} \in (\log_{10} 1.22112, \log_{10} 1.22113)}} 1 \right)$$

$$= \frac{\log_{10} \frac{122113}{122112}}{12500} = \frac{\log\left(1 + \frac{1}{122112}\right)}{12500 \log 10} \approx 2.8 \cdot 10^{-10}.$$