Problems

Ted Eisenberg, Section Editor

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This section of the Journal offers readers an opportunity to exchange interesting mathematical problems and solutions. Please send them to Ted Eisenberg, Department of Mathematics, Ben-Gurion University, Beer-Sheva, Israel or fax to: 972-86-477-648. Questions concerning proposals and/or solutions can be sent e-mail to <eisenbt@013.net>. Solutions to previously stated problems can be seen at <http://www.ssma.org/publications>.

Solutions to the problems stated in this issue should be posted before
May 15, 2017

- **5439:** *Proposed by Kenneth Korbin, New York, NY*

  Express the roots of the equation $\dfrac{(x+1)^4}{(x-1)^2} = 20x$ in closed form.
  "Closed form" means that the roots cannot be expressed in their approximate decimal equivalents.

- **5440:** *Proposed by Roger Izard, Dallas, TX*

  The vertices of rectangle ABCD are labeled in clockwise order, and point F lines on line segment AB. Prove that $AD + AC > DF + FC$.

- **5441:** *Proposed by Larry G. Meyer, Fremont, OH*

  In triangle ABC draw a line through the ex-center corresponding to side c so that it is parallel to side c. Extend the angle bisectors of A and B to meet the constructed lines at points A' and B' respectively. Find the length of $\overline{A'B'}$ if given either

  (1) Angles A, B, C and the circumradius R

  (2) Sides a, b, c

  (3) The semiperimeter $s,$ the inradius r and the exradius r_c

  (4) Semiperimeter s and side c.

- **5442:** *Proposed by José Luis Díaz-Barrero, Barcelona Tech, Barcelona, Spain.*

  Let L_n be the n^{th} Lucas number defined by $L_0 = 2, L_1 = 1$ and for $n \geq 2, L_n = L_{n-1} + L_{n-2}$. Prove that for all $n \geq 0$,

$$\frac{1}{2}\begin{vmatrix} (L_n + 2L_{n+1})^2 & L_{n+2}^2 & L_{n+1}^2 \\ L_{n+2}^2 & (2L_n + L_{n+1})^2 & L_n^2 \\ L_{n+1}^2 & L_n^2 & L_{n+2}^2 \end{vmatrix}$$

  is the cube of a positive integer and determine its value.

- **5443:** *Proposed by D.M. Băinetu– Giurgiu, "Matei Basarab" National College, Bucharest, Romania and Neculai Stanciu "Geroge Emil Palade" General School, Buzău, Romania*

  Compute $\displaystyle\int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \frac{x}{\sin 2x} dx$.

- **5444:** *Proposed by Ovidiu Furdui and Alina Sîntămărian, Technical University of Cluj-Napoca, Cluj-Napoca, Romania*

  Solve in $\Re$ the equation $\{(x+1)^2\} = 2x^2$, where $\{a\}$ denotes the fractional part of a.

Solutions

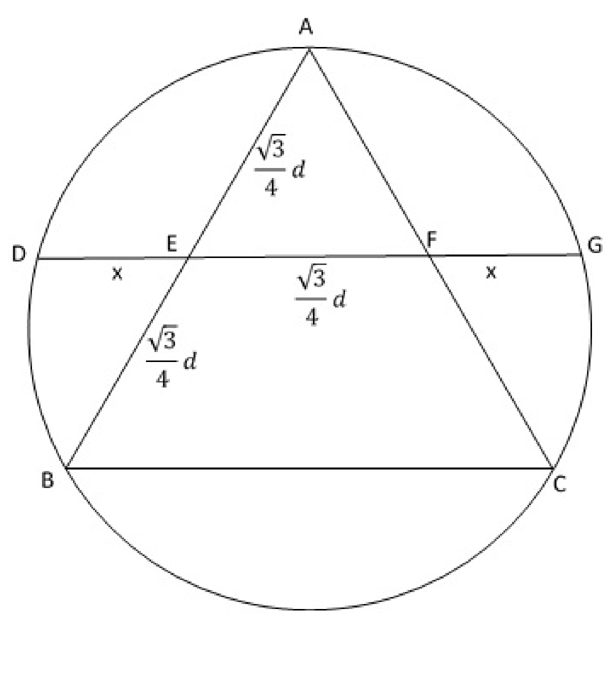
- **5421:** *Proposed by Kenneth Korbin, New York, NY*

  An equilateral triangle is inscribed in a circle with diameter d. Find the perimeter of the triangle if a chord with length $1-d$ bisects two of its sides.

  Solution by Michael N. Fried, Ben-Gurion University, Beer-Sheva, Israel

  Let the triangle be ABC and the chord, $DEFG$. Since the diameter is d, the side of the triangle is $\frac{\sqrt{3}d}{2}$ so that $AE = EB = EF = \frac{\sqrt{3}d}{4}$. Let $DE = FG = x$. So that by the theorem on intersecting chords, we have:

$$x\left(x + \frac{\sqrt{3}d}{4}\right) = \frac{3}{16}d^2$$



On the other hand, since $2x + \frac{\sqrt{3}d}{4} = 1 - d$, we have, $x = \frac{1}{2}\left(1 - d - \frac{\sqrt{3}d}{4}\right)$.
Substituting into the equation from the intersecting chords theorem and simplifying, we obtain a quadratic equation for d:

$$d^2 - 32d + 16 = 0$$

whose solutions are $d = 16 \pm 4\sqrt{15}$. But since $1 - d$ is the length of the chord DG, $d < 1$, so that we have the single solution $d = 16 - 4\sqrt{15}$.
Thus the perimeter of the triangle is:

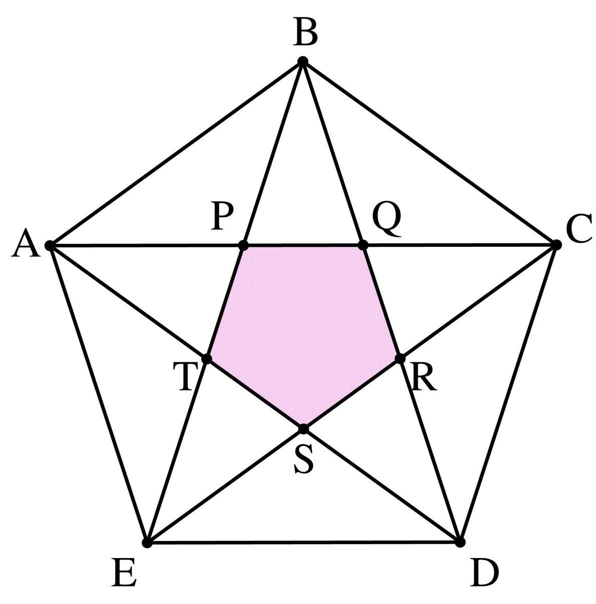
$$3\frac{\sqrt{3}d}{2} = 3\left(\frac{\sqrt{3}}{2}\right)\left(16 - 4\sqrt{15}\right) = 6\left(4\sqrt{3} - 3\sqrt{5}\right)$$

Also solved by Jeremiah Bartz, University of North Dakota, Grand Forks, ND; Bruno Salgueiro Fanego, Viveiro, Spain; Ed Gray, Highland Beach, FL; Paul M. Harms, North Newton, KS; David A. Huckaby, Angelo State University, San Angelo, TX; Kee-Wai Lau, Hong Kong, China; Charles McCracken, Dayton, OH; David Stone and John Hawkins, Georgia Southern University, Statesboro, GA; Titu Zvonaru, Comămesto, Romania and Neculai Stanciu, "George Emil Palade" School, Buzău, Romania, and the proposer.

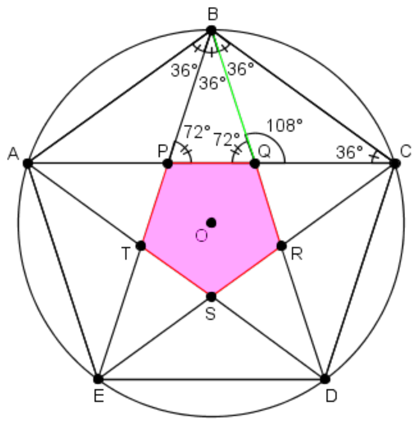
- **5422:** *Proposed by Arsalan Wares, Valdosta State University, Valdosta, GA*

Polygon $ABCDE$ is a regular pentagon. Pentagon $PQRST$ is bounded by diagonals of pentagon $ABCDE$ as shown. Find the following:

$$\frac{\text{the area of pentagon } PQRST}{\text{the area of pentagon } ABCDE}.$$



Solution 1 by Nikos Kalapodis, Patras, Greece



It can be easily checked that pentagon $PQRST$ is regular (since it is equiangular and equilateral). Therefore it is similar to pentagon $ABCDE$. Since the ratio of the areas of two similar polygons is equal to the square of the ratio λ of the corresponding sides, it follows that

$$\frac{\text{the area of pentagon } PQRST}{\text{the area of pentagon } ABCDE} = \lambda^2.$$

By the law of sines in triangles BPQ and QBC we have

$$\lambda = \frac{PQ}{BC} = \frac{\dfrac{PQ}{BQ}}{\dfrac{BC}{BQ}} = \frac{\dfrac{\sin 36^\circ}{\sin 72^\circ}}{\dfrac{\sin 108^\circ}{\sin 36^\circ}} = \frac{\sin^2 36^\circ}{\sin 72^\circ \sin 108^\circ} = \left(\frac{\sin 36^\circ}{\sin 72^\circ}\right)^2 = \left(\frac{\sin 36^\circ}{2\sin 36^\circ \cos 36^\circ}\right)^2$$

$$= \frac{1}{4\cos^2 36^\circ} = \frac{1}{4\left(\dfrac{\sqrt{5}+1}{4}\right)^2} = \frac{4}{6+2\sqrt{5}} = \frac{2}{3+\sqrt{5}} = \frac{3-\sqrt{5}}{2}.$$

Therefore $\lambda^2 = \left(\dfrac{3-\sqrt{5}}{2}\right)^2 = \dfrac{14-6\sqrt{5}}{4} = \dfrac{7-3\sqrt{5}}{2}$.

Solution 2 by Bruno Salgueiro Fanego, Viveiro, Spain

Let $a = \overline{AB}$ and $d = \overline{AC}$ be the lengths of the side and the diagonal of the regular pentagon $ABCDE$.

It is a known result that $d = \varphi a$, where $\phi = \dfrac{1+\sqrt{5}}{2}$ is the golden ratio. (This can be shown, for example by taking into account that triangle ACS and DES have their sides respectively parallel, so they are similar, from where $\dfrac{\overline{AC}}{\overline{DE}} = \dfrac{\overline{CS}}{\overline{ES}}$. Since $ABCS$ is a rhombus, itis a parallelogram and, thus $\overline{CS} = \overline{AB}$ and since $\overline{ES} = \overline{CE} - \overline{CS} = \overline{CE} - \overline{AB}$, we conclude that $\dfrac{\overline{AC}}{\overline{DE}} = \dfrac{\overline{AB}}{\overline{CE} - \overline{AB}}$, or equivalently, $\dfrac{d}{a} = \dfrac{a}{d-a}$, which impies $d^2 - ad - a^2 = 0$ and, hence, $d = \dfrac{a \pm \sqrt{a^2+4a^2}}{2}$, so $d = \phi a$ because $a > 0$ and $d > 0$.)

We have that $\overline{ES} = \overline{CE} - \overline{AB} = d - a = (\varphi - 1)a$ and

$$\overline{SR} = \overline{ER} - \overline{ES} = \overline{AB} - \overline{ES} = a - (\varphi - 1)a = (2 - \varphi)a$$

and since the ratio of the areas of $PQRST$ and $ABCDE$ equals the square of their similarity ratio $\dfrac{\overline{SR}}{\overline{AB}} = \dfrac{(2-\varphi)a}{a} = 2 - \varphi$, we conclude that

$$\frac{\text{area}(PQRST)}{\text{area}(ABCDE)} = (2-\varphi)^2 \approx 0.145898$$

Solution 3 by Brian Bradie, Christopher Newport University, Newport News, VA

The area of a regular pentagon is proportional to the square of its side length, so

$$\frac{\text{the area of pentagon } PQRST}{\text{the area of pentagon } ABCDE} = \left(\frac{\overline{PQ}}{\overline{DE}}\right)^2. \qquad (1)$$

Because triangle BPQ is similar to triangle BED,

$$\left(\frac{\overline{PQ}}{\overline{DE}}\right)^2 = \left(\frac{\overline{BQ}}{\overline{BD}}\right)^2. \qquad (2)$$

Without loss of generality, suppose that pentagon $ABCDE$ has sides of length 1. By the Law of Cosines,

$$\overline{BD}^2 = 2 - 2\cos 108^\circ = 4\sin^2 54^\circ. \qquad (3)$$

Moreover, triangle BQC is isosceles with $\overline{BQ} = \overline{QC}$; thus, by the Law of Cosines,

$$1 = \overline{BQ}^2(2 - 2\cos 108^\circ) = 4\overline{BQ}^2 \sin^2 54^\circ,$$

so that

$$\overline{BQ}^2 = \frac{1}{4\sin^2 54^\circ}. \qquad (4)$$

Combining equations (1) – (4), it follows that

$$\frac{\text{the area of pentagon } PQRST}{\text{the area of pentagon } ABCDE} = \frac{1}{16\sin^4 54^\circ}.$$

Now,

$$\sin 54^\circ = \frac{1+\sqrt{5}}{4} = \frac{1}{2}\varphi,$$

where φ denotes the Golden Ratio, so

$$\frac{\text{the area of pentagon } PQRST}{\text{the area of pentagon } ABCDE} = \frac{1}{\varphi^4}.$$

Solution 4 by David E. Manes, SUNY at Oneonta, Oneonta, NY

Let $[X]$ denote the area of polygon X. Then

$$\frac{[PQRST]}{[ABCDE]} = \frac{7-3\sqrt{5}}{2} \approx \frac{3}{20},$$

where $ABCDE$ is a regular pentagon.

Assume that $ABCDE$ is inscribed in the unit circle $x^2+y^2=1$. Then the vertices of the pentagon can be chosen as follows:

$B=(0,1), C=(s_1,c_1), D=(s_2,-c_2), E=(-s_2,-c_2)$ and $A=(-s_1,c_1)$, where

$$\begin{aligned}
c_1 &= \cos\left(\frac{2\pi}{5}\right) = \frac{1}{4}(\sqrt{5}-1), \\
c_2 &= \cos\left(\frac{\pi}{5}\right) = \frac{1}{4}(\sqrt{5}+1), \\
s_1 &= \sin\left(\frac{2\pi}{5}\right) = \frac{1}{4}\sqrt{10+2\sqrt{5}}, \\
s_2 &= \sin\left(\frac{4\pi}{5}\right) = \frac{1}{4}\sqrt{10-2\sqrt{5}}.
\end{aligned}$$

Furthermore, the pentagon is symmetric with respect to the y-axis and the pentagon $PQRST$ is also regular since its sides are the bases of five congruent isosceles triangles. If t is the side length of a regular pentagon T, then its area is given by

$[T] = \frac{1}{4}\sqrt{25+10\sqrt{5}}\cdot t^2$.

Let a and b be the side lengths of pentagons $ABCDE$ and $PQRST$, respectively. Then

$$\begin{aligned}
a = BC &= \sqrt{s_1^2+(1-c_1)^2} = \sqrt{\frac{1}{16}(10+2\sqrt{5})+(1-\frac{1}{4}(\sqrt{5}-1)^2} \\
&= \frac{\sqrt{10-2\sqrt{5}}}{2}.
\end{aligned}$$

Therefore,

$$[ABCDE] = \frac{1}{4}\sqrt{25+10\sqrt{5}}\cdot a^2 = \frac{1}{4}\sqrt{25+10\sqrt{5}}\cdot\frac{1}{4}(10-2\sqrt{5}).$$

To find b, note that the equation of the line containing B and E is

$$y-1 = \Big(\frac{5+\sqrt{5}}{\sqrt{10-2\sqrt{5}}}\Big)x.$$

If $y=c_1$, then $x=\frac{(c_1-1)\sqrt{10-2\sqrt{5}}}{5+\sqrt{5}}$ so that the coordinates for point P are

$$P = \Big(\frac{(c_1-1)\sqrt{10-2\sqrt{5}}}{5+\sqrt{5}}, c_1\Big).$$

By symmetry,

$$Q = \Big(\frac{-(c_1-1)\sqrt{10-2\sqrt{5}}}{5+\sqrt{5}}, c_1\Big).$$

Therefore,

$$b = PQ = \frac{-2(c_1-1)\sqrt{10-2\sqrt{5}}}{5+\sqrt{5}}$$

so that

$$[PQRST] = \frac{\sqrt{25+10\sqrt{5}}(c_1-1)^2(10-2\sqrt{5})}{(5+\sqrt{5})^2}$$
$$= \frac{\frac{1}{16}\sqrt{25+10\sqrt{5}}(\sqrt{5}-5)^2(10-2\sqrt{5})}{(5+\sqrt{5})^2)}.$$

Hence,

$$\frac{[PQRST]}{[ABCDE]} = \frac{\left(\frac{\sqrt{25+10\sqrt{5}}(\sqrt{5}-5)^2)(10-2\sqrt{5})}{16(5+\sqrt{5})^2}\right)}{\left(\frac{\sqrt{25+10\sqrt{5}}(10-2\sqrt{5})}{16}\right)}$$
$$= \frac{(\sqrt{5}-5)^2}{(5+\sqrt{5})^2}$$
$$= \frac{7-3\sqrt{5}}{2} \approx 0.145\,898\,033\,75 \approx \frac{3}{20}.$$

Solution 5 by Albert Stadler, Herrliberg, Switzerland

PQRST is similar to pentagon ABCDE. Therefore,

$$\frac{\text{the area of Pentagon PQRST}}{\text{the area of Pentagon ABCDE}} = \left(\frac{SR}{CD}\right)^2 = \left(\frac{SR}{CS}\right)^2 = \left(\frac{CS-CR}{CS}\right)^2$$

$$= \quad \left(1-\frac{CR}{CS}\right)^2 = \left(1-\frac{\sqrt{5}-1}{2}\right)^2\left(\frac{3-\sqrt{5}}{2}\right)^2 = \left(\frac{6-2\sqrt{5}}{4}\right)^2 = \left(\frac{1-\sqrt{5}}{2}\right)^4,$$

where we have used the fact that in a regular pentagon diagonals are cut in sections whose proportions follow the golden ratio (https://en.wikipedia.org/wiki/Pentagon).

$Editor's\ comment$: At first glance it appears that different answers were obtained for this problem. But letting φ equal the golden ratio, and using the equation $\varphi^2 - \varphi + 1 = 0$ it can be shown that the answers are equivalent to one another.

Scott Brown of Auburn University at Montgomery noted that: The material regarding the area of both pentagons can be found on pp. 308-315 in Tom Koshy's book "Fibonacci and Lucas Numbers with Applications". He went on to state that "evidently the problem is not new," to which I add, but it is still very interesting.

Also solved by Dionne Bailey, Elsie Campbell, and Charles Diminnie, Angelo State University, San Angelo, TX; Jeremiah Bartz, University of North Dakota Grand Forks, ND; Michael N. Fried, Ben-Gurion Univesity, Beer-Sheva, Israel; Ed Gray, Highland Beach, FL; Paul M. Harms, North Newton, KS: David Huckaby, Angelo State University, San Angelo, TX; Ken Korbin (two solutions), NewYork, NY; Kee-Wai Lau, Hong Kong, China; Charles McCracken, Dayton, OH; David Stone and John Hawkins, Georgia Southern University, Statesboro, GA, and the proposer.

- **5423:** *Proposed by Oleh Faynshteyn, Leipzig, Germany*

Let a, b, c be the side-lengths, r_a, r_b, r_c be the radii of the ex-circles and R, r the radii of the circumcircle and incircle respectively of $\triangle ABC$. Show that

$$\frac{(r_a-r)^2+r_br_c}{(s-b)(s-c)}+\frac{(r_b-r)^2+r_cr_a}{(s-c)(s-a)}+\frac{(r_c-r)^2+r_ar_b}{(s-a)(s-b)}\geq 13.$$

Solution 1 by Albert Stadler, Herrliberg, Switzerland

It is well known (https://en.wikipedia.org/wiki/Incircle-and-excircles-of-a-triangle) that

$$\Delta=\sqrt{s(s-a)(s-b)(s-c)},\ r=\frac{\Delta}{s},\ r_a=\frac{\Delta}{s-a},\ r_b=\frac{\Delta}{s-b},\ r_c=\frac{\Delta}{s-c}.$$

The stated inequality is therefore equivalent to

$$\sum_{cycl}\frac{(r_a-r)^2+r_br_c}{(s-b)(s-c)}=\sum_{cycl}s(s-a)\left(\frac{1}{s-a}-\frac{1}{s}\right)^2+\sum_{cycl}\frac{s(s-a)}{(s-b)(s-c)}\geq 13. \qquad (1)$$

Put $u:=s-a,\ v:=s-b,\ w:=s-c$. Then $s=u+v+w$. By the triangle inequality, $u\geq 0,\ v\geq 0,\ w\geq 0$. So (1) is equivalent to

$$\begin{aligned}&\sum_{cycl}(u+v+w)u\left(\frac{1}{u}-\frac{1}{u+v+w}\right)^2+\sum_{cycl}\frac{(u+v+w)}{vw}\\ =\ & -6+\sum_{cycl}\frac{u+v+w}{u}+\sum_{cycl}\frac{u}{u+v+w}+\sum_{cycl}\frac{u^2}{vw}+\sum_{cycl}\frac{u}{w}+\sum_{cycl}\frac{u}{v}\\ =\ & -2+\sum_{cycl}\frac{v}{u}+\sum_{cycl}\frac{w}{u}+\sum_{cycl}\frac{u^2}{vw}+\sum_{cycl}\frac{u}{w}+\sum_{cycl}\frac{u}{v}\\ =\ & -2+2\sum_{cycl}\frac{v}{u}+2\sum_{cycl}\frac{w}{u}+\sum_{cycl}\frac{u^2}{vu}\geq 13. \qquad (2)\end{aligned}$$

By the AM-GM inequality,

$$\sum_{cycl}\frac{v}{u}\geq 3\sqrt[3]{\frac{v}{u}\cdot\frac{w}{v}\cdot\frac{u}{w}}=3,\ \sum_{cycl}\frac{w}{u}\geq 3\sqrt[3]{\frac{w}{u}\cdot\frac{u}{v}\cdot\frac{v}{w}}=3,\ \sum_{cycl}\frac{u^2}{vw}\geq 3\sqrt[3]{\frac{u^2}{vw}\cdot\frac{v^2}{wu}\cdot\frac{w^2}{uv}}=3.$$

So (2) holds true.

Solution 2 by Arkady Alt, San Jose, CA

Let F be area of the triangle. Since $r_a=\dfrac{F}{s-a},\ r_b=\dfrac{F}{s-b},\ r_c=\dfrac{F}{s-c},\ r=\dfrac{F}{s}$ then

$$\frac{(r_a-r)^2+r_br_c}{(s-b)(s-c)} = \frac{\left(\frac{F}{s-a}-\frac{F}{s}\right)^2+\frac{F}{s-b}\cdot\frac{F}{s-c}}{(s-b)(s-c)}$$

$$= \frac{F^2\left(\dfrac{a^2}{s^2\left(s-a\right)^2}+\dfrac{1}{\left(s-b\right)\left(s-c\right)}\right)}{(s-b)(s-c)}$$

$$= \frac{F^2\left(a^2\left(s-b\right)\left(s-c\right)+s^2\left(s-a\right)^2\right)}{s^2\left(s-a\right)^2\left(s-b\right)^2\left(s-c\right)^2}$$

$$= \frac{a^2\left(s-b\right)\left(s-c\right)+s^2\left(s-a\right)^2}{F^2}$$

$$= \frac{4a^2\left(a+c-b\right)\left(a+b-c\right)+\left(a+b+c\right)^2\left(b+c-a\right)^2}{16F^2}$$

$$= \frac{\left(4\left(bc^3+b^3c\right)-6\left(a^2b^2+6a^2c^2-b^2c^2\right)+5a^4+b^4+c^4+4a^2bc\right)}{16F^2} \text{ and, therefore,}$$

$$\sum_{cyc}\frac{(r_a-r)^2+r_br_c}{(s-b)(s-c)} = \frac{1}{F^2}\sum_{cyc}\left(4bc\left(a^2+b^2+c^2\right)-6\left(a^2b^2+6a^2c^2-b^2c^2\right)+5a^4+b^4+c^4\right)$$

$$= \frac{4\left(a^2+b^2+c^2\right)\left(ab+bc+ca\right)-6\left(a^2b^2+b^2c^2+a^2c^2\right)+7\left(a^4+b^4+c^4\right)}{16F^2}$$

$$= \frac{4\left(a^2+b^2+c^2\right)\left(ab+bc+ca\right)-20\left(a^2b^2+b^2c^2+a^2c^2\right)+7\left(a^2+b^2+c^2\right)^2}{16F^2}.$$

Let $x:=s-a,\ y:=s-b,\ z:=s-c,\ p:=xy+yz+zx,\ q:=xyz$. Due to the homogeneity of the original inequality we can assume that $s=1$. Then $a=1-x, b=1-y, c=1-z$,

$$x,y,z>0,\ x+y+z=1,\ a+b+c=2,\ abc=p-q,\ F=\sqrt{xyz}=\sqrt{q},$$

$$ab+bc+ca=1+p, a^2+b^2+c^2=2\left(1-p\right),$$

$$a^2b^2+b^2c^2+a^2c^2=\left(ab+bc+ca\right)^2-2abc\left(a+b+c\right)$$

$$= \left(1+p\right)^2-4\left(p-q\right)=\left(1-p\right)^2+4q, \text{ and original inequality becomes}$$

$$\frac{8\left(1-p^2\right)-20\left(\left(1-p\right)^2+4q\right)+28\left(1-p\right)^2}{16q}\geq 13 \iff \frac{1-p-5q}{q}\geq 13 \iff 1-p\geq 18q.$$

Since $q = xyz \leq \frac{x+y+z}{3} \cdot \frac{xy+yz+zx}{3} = \frac{p}{9}$ and
$p = xy + yz + zx \leq \frac{(x+y+z)^2}{3} = \frac{1}{3}$, then
$1 - p - 18q \geq 1 - p - 18 \cdot \frac{p}{9} = 1 - 3p \geq 0.$

Solution 3 and 4 by Nikos Kalapodis, Patras, Greece

Using the well-known formulas $S = sr$, $S = r_a(s-a)$ and $S = \sqrt{s(s-a)(s-b)(s-c)}$ we have

$(r_a - r)^2 = \left(\frac{S}{s-a} - \frac{S}{s}\right)^2 = \frac{a^2S^2}{s^2(s-a)^2} = \frac{a^2(s-b)(s-c)}{s(s-a)}$ and

$r_b rc = \frac{S}{s-b} \cdot \frac{S}{s-c} = \frac{S^2}{(s-b)(s-c)} = s(s-a).$

It follows that $\frac{(r_a-r)^2}{(s-b)(s-c)} = \frac{a^2}{s(s-a)}$, and $\frac{r_b rc}{(s-b)(s-c)} = \frac{s(s-a)}{(s-b)(s-c)} = \frac{(s-a)^2}{r^2}$.

By Cauchy-Schwarz inequality and the well-known inequality $s \geq 3\sqrt{3}r$ we have

$$\sum_{cyc} \frac{(r_a-r)^2 + r_b r_c}{(s-b)(s-c)} = \sum_{cyc} \frac{(r_a-r)^2}{(s-b)(s-c)} + \sum_{cyc} \frac{r_b r_c}{(s-b)(s-c)} = \sum_{cyc} \frac{a^2}{s(s-a)} + \sum_{cyc} \frac{(s-a)^2}{r^2} \geq$$

$$\frac{(a+b+c)^2}{s(s-a+s-b+s-c)} + \frac{(s-a+s-b+s-c)^2}{3r^2} = \frac{4s^2}{s^2} + \frac{s^2}{3r^2} \geq 4 + \frac{1}{3} \cdot 27 = 4 + 9 = 13.$$

Solution 4

Using the well-known formulas $(s-b)(s-c) = rr_a$, $r_a + r_b + r_c = r + 4R$, $\frac{1}{r_a} + \frac{1}{r_b} + \frac{1}{r_c} = \frac{1}{r}$ and Euler's inequality ($R \geq 2r$) we have

$$\sum_{cyc} \frac{(r_a-r)^2}{(s-b)(s-c)} = \sum_{cyc} \frac{r_a^2 + r^2 - 2rr_a}{rr_a} = \sum_{cyc} \left(\frac{r_a}{r} + \frac{r}{r_a} - 2\right) = \frac{1}{r}\sum_{cyc} r_a + r\sum_{cyc} \frac{1}{r_a} - 6$$

$$= \frac{1}{r}(r+4R) + r \cdot \frac{1}{r} - 6 = \frac{4R}{r} - 4 \geq 4, \text{ and}$$

$$\sum_{cyc} \frac{r_b r_c}{(s-b)(s-c)} = \sum_{cyc} \frac{r_b r_c}{rr_a} = \frac{r_a r_b r_c}{r} \sum_{cyc} \frac{1}{r_a^2} = \frac{r_a r_b r_c}{r} \sum_{cyc} \left(\frac{1}{r_a}\right)^2 \geq \frac{r_a r_b r_c}{r} \sum_{cyc} \frac{1}{r_a r_b}$$

$$= \frac{1}{r}\sum_{cyc} r_a = \frac{1}{r}(r+4R) = 1 + \frac{4R}{r} \geq 9.$$

Therefore $\sum_{cyc} \frac{(r_a-r)^2 + r_b r_c}{(s-b)(s-c)} = \sum_{cyc} \frac{(r_a-r)^2}{(s-b)(s-c)} + \sum_{cyc} \frac{r_b r_c}{(s-b)(s-c)} \geq 4 + 9 = 13.$

Also solved by Brian Bradie, Christopher Newport University, Newport News, VA; Bruno Salgueiro Fanego, Viveiro, Spain; Ed Gray, Highland Beach, FL; Moti Levy, Rehovot, Israel; Albert Stadler Herrliberg, Switzerland; Titu Zvonaru,Comănesti, Romania and Neculai Stanciu "Geroge Emil Palade" School, Buză, Romania, and the proposer.

- **5424:** *Proposed by Nicusor Zlota, "Traian Vuia" Technical College, Forcsani, Romania*

Let a, b, c and d be positive real numbers such that $abc + bcd + cda + dab = 4$. Prove that $(a^8 - a^4 + 4)(b^7 - b^3 + 4)(c^6 - c^2 + 4)(d^5 - d + 4) \geq 256$.

Solution 1 by Ed Gray, Highland Beach, FL

Like most of these inequality problems, I find there is generally a solution evident by inspection that actually makes the inequality a strict equality. Then, by choosing numbers with the correct orientation, one can show that the numbers seen by inspection actually provide an extremum. Let's put labels on the givens in the statement of the problem.

(1) $abc + bcd + cda + dab = 4$. We wish to prove:
(2) $(a^8 - a^4 + 4)(b^7 - b^3 + 4)(c^6 - c^2 + 4)(d^5 - d + 4) \geq 256$.
Clearly, if $a = b = c = d = 1$, the following relations hold:
(3) $abc = 1$,
(4) $bcd = 1$,
(5) $cda = 1$,
(6) $dab = 1$, and
(7) $abc + bcd + cda + dab = 4$. Also,
(8) $a^8 - a^4 = 0$,
(9) $b^7 - b^3 = 0$,
(10) $c^6 - c^2 = 0$,
(11) $d^5 - d = 0$, so that the product in (2) becomes $4^4 = 256$.
Therefore, if we show that choices for a, b, c, d with at least one < 1 all makes the product of $(a^8 - a^4 + 4)(b^7 - b^3 + 4)(c^6 - c^2 + 4)(d^5 - d + 4) > 256$, the conjecture would be true.

It would be sufficient to consider three cases:
(A) $a < 1,\ b = 1,\ c = 1,\ d > 1$,
(B) $a < 1,\ b < 1,\ c = 1,\ d > 1$,
(C) $a < 1,\ b < 1,\ c < 1,\ d > 1$,
In each case, we choose a, b, c as necessary and compute d by using (1).
To be explicit, we choose the following numbers:
(A) $a = .99,\ b = 1,\ c = 1$, calculated value of d is 1.010067114
Evaluating Eq.(2) using these numbers, the product is 256.1952096
(B) $a = .99,\ b = .99,\ c = 1$, calculated value of d is 1.02020202
Evaluating Eq.(2) using these numbers, the product is 256.489
(C) $a = .99, b = .99, c = .99$, calculated value of d is 1.030405401
Evaluating Eq.(2) using these numbers, the product is 256.9590651

Solution 2 by Moti Levy, Rehovot, Israel

The inequality can be simplified by applying

$$\begin{aligned} a^8 - a^4 &\geq a^4 - 1, \\ b^7 - b^3 &\geq b^4 - 1, \\ c^6 - c^2 &\geq c^4 - 1, \\ d^5 - d &\geq d^4 - 1. \end{aligned}$$

Hence it suffices to prove that

$$(a^4 + 3)(b^4 + 3)(c^4 + 3)(d^4 + 3) \geq 256.$$

We rewrite the left hand side:

$$\begin{aligned} &(a^4 + 3)(b^4 + 3)(c^4 + 3)(d^4 + 3) \\ &= (a^4 + 1 + 1 + 1)(1 + b^4 + 1 + 1)(1 + 1 + c^4 + 1)(1 + 1 + 1 + d^4). \end{aligned}$$

By Holder's inequality,

$$(a^4 + 1 + 1 + 1)(1 + b^4 + 1 + 1)(1 + 1 + c^4 + 1)(1 + 1 + 1 + d^4) \geq (a + b + c + d)^4 .$$

Thus, what is left is to show that $abc + bcd + cda + dab = 4$ implies that $a + b + c + d \geq 4$.

To this end, we employ elementary symmetric polynomials notation:

$$\begin{aligned} p_1 &= e_1 = a + b + c + d, \\ p_2 &= a^2 + b^2 + c^2 + d^2, \\ p_3 &= a^3 + b^3 + c^3 + d^3, \\ e_3 &= abc + bcd + cda + dab. \end{aligned}$$

It is well known (from Newton's identities) that

$$p_1^3 - 3p_1p_2 + 2p_3 = 6e_3.$$

We also have from the power mean inequality $\sqrt{\frac{p_2}{4}} \geq \frac{p_1}{4}$,

$$p_2 \geq \frac{p_1^2}{4},$$

and from Chebyshev's inequality $a^3 + b^3 + c^3 + d^3 \geq abc + bcd + cda + dab$,

$$p_3 \geq e_3.$$

It follows that,

$$p_1^3 - 3p_1p_2 = 6e_3 - 2p_3 \geq 4e_3.$$

$p_2 \geq \frac{p_1^2}{4}$ implies that

$$p_1^3 - 3p_1\frac{p_1^2}{4} \geq p_1^3 - 3p_1p_2,$$

or that

$$\frac{p_1^3}{4} \geq 4e_3 = 16.$$

We conclude that $p_1 \geq 4$ and this fact completes the solution.

Also solved by Albert Stadler, Herrliberg, Switzerland, and the proposer.

- **5425:** *Proposed by José Luis Díaz-Barrero, Barcelona Tech, Barcelona, Spain*

  Let F_n be the n^{th} Fibonacci number defined by $F_0 = 0, F_1 = 1$, and for all $n \geq 2, F_n = F_{n-1} + F_{n-2}$. If n is an odd positive integer the show that $1 + det(A)$ is the product of two consecutive Fibonacci numbers, where

$$A = \begin{pmatrix} F_1^2 - 1 & F_1F_2 & F_1F_3 & \dots & F_1F_n \\ F_2F_1 & F_2^2 - 1 & F_2F_3 & \cdots & F_2F_n \\ F_3F_1 & F_3F_2 & F_3^2 - 1 & \cdots & F_3F_n \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ F_nF_1 & F_nF_2 & F_nF_3 & \cdots & F_n^2 - 1 \end{pmatrix}$$

Brian Bradie, Christopher Newport University, Newport News, VA

We will establish the more general result that for any positive integer n the quantity $1 + (-1)^{n-1}\det(A)$ is the product of two consecutive Fibonacci numbers. Toward this end, let

$$B = A + I = \begin{bmatrix} F_1 \\ F_2 \\ F_3 \\ \vdots \\ F_n \end{bmatrix} \begin{bmatrix} F_1 & F_2 & F_3 & \cdots & F_n \end{bmatrix}.$$

The matrix B is a rank 1 matrix with eigenvalue

$$\sum_{j=1}^{n} F_j^2 = F_nF_{n+1}$$

of algebraic multiplicity 1 and eigenvalue 0 of algebraic multiplicity $n - 1$. It then follows that the matrix A has eigenvalue $F_nF_{n+1} - 1$ of algebraic multiplicity 1 and eigenvalue -1 of algebraic multiplicity $n - 1$. Thus,

$$\det(A) = (-1)^{n-1}(F_nF_{n+1} - 1),$$

and

$$1 + (-1)^{n-1}\det(A) = 1 + (F_nF_{n+1} - 1) = F_nF_{n+1}.$$

Also solved by Moti Levy, Rehovot, Israel; Albert Stadler, Herrliberg, Switzerland, and the proposer.

- **5426:** *Proposed by Ovidiu Furdui, Technical University of Cluj-Napoca, Cluj-Napoca, Romania*

  Let $(a_n)_{n\geq 1}$ be a strictly increasing sequence of natural numbers. Prove that the series

$$\sum_{n=1}^{\infty} \frac{\sqrt{a_n}}{[a_n, a_{n+1}]} \text{ converges.}$$

Here $[x, y]$ denotes the least common multiple of the natural numbers x and y.

Solution 1 by Moti Levy, Rehovot, Israel

It is known that

$$\operatorname{lcm}(a,b)\gcd(a,b) = ab.$$

Clearly, $a = A\gcd(a,b)$ and $b = B\gcd(a,b)$. If $a > b$ then $A > B$ and $a - b = (A-B)\gcd(a,b) > \gcd(a,b)$.

$$a - b > \gcd(a,b) = \frac{ab}{\operatorname{lcm}(a,b)},$$

or

$$\frac{1}{\operatorname{lcm}(a,b)} < \frac{a-b}{ab} = \frac{1}{b} - \frac{1}{a}, \qquad a > b. \tag{1}$$

It follows from (1) that

$$\frac{\sqrt{a_n}}{\operatorname{lcm}(a_n, a_{n+1})} < \sqrt{a_n}\left(\frac{1}{a_n} - \frac{1}{a_{n+1}}\right) = \frac{1}{\sqrt{a_n}} - \frac{1}{\sqrt{a_{n+1}}}\sqrt{\frac{a_n}{a_{n+1}}},$$

so that

$$\sum_{n=1}^{\infty} \frac{\sqrt{a_n}}{\operatorname{lcm}(a_n, a_{n+1})} < \sum_{n=1}^{\infty}\left(\frac{1}{\sqrt{a_n}} - \frac{1}{\sqrt{a_{n+1}}}\sqrt{\frac{a_n}{a_{n+1}}}\right). \tag{2}$$

Let us define a sequence of positive real numbers $(b_n)_{n\geq 1}$ as follows:

$$b_{2k-1} = \frac{1}{\sqrt{a_k}}, \tag{3}$$

$$b_{2k} = \frac{1}{\sqrt{a_{k+1}}}\sqrt{\frac{a_{k+1}}{a_{k+2}}} \tag{4}$$

By definition (4), $\sum_{n=1}^{\infty}\left(\frac{1}{\sqrt{a_n}} - \frac{1}{\sqrt{a_{n+1}}}\sqrt{\frac{a_n}{a_{n+1}}}\right) = \sum_{n=1}^{\infty}(-1)^{n+1} b_n$.

The terms of the sequence $(b_n)_{n\geq 1}$ satisfy: $b_n > b_{n+1} > 0$ and $\lim_{n\to\infty} b_n = 0$.

The series $\sum_{n=1}^{\infty}(-1)^{n+1} b_n$ converges by the Alternating Series Test (called also Leibniz Criterion).

Inequality (2) implies that the series $\sum_{n=1}^{\infty}\frac{\sqrt{a_n}}{\operatorname{lcm}(a_n,a_{n+1})}$ converges as well.

Remark: The idea for this solution came from the enjoyable short paper by D. Borwein, who solved a conjecture of P. Erdös.

Reference: D. Borwein, "A Sum of Reciprocals of Least Common Multiples", Canadian Mathematical Bulletin, Volume 20 (1), 1978, pp. 117-118.

Solution 2 by Kee-Wai Lau, Hong Kong, China

Denote by (x,y) the greatest common divisor of the natural numbers x and y.
It is well known that $(x,y)[x,y] = xy$. Hence for any natural number $M \geq 2$, we have

$$\sum_{n=1}^{M} \frac{\sqrt{a_n}}{[a_n, a_{n+1}]} = \sum_{n=1}^{M} \frac{(a_n, a_{n+1})}{\sqrt{a_n}a_{n+1}}$$

$$
\begin{aligned}
&= \sum_{n=1}^{M} \frac{(a_n, a_{n+1} - a_n)}{\sqrt{a_n} a_{n+1}} \\
&\leq \sum_{n=1}^{M} \frac{a_{n+1} - a_n}{\sqrt{a_n} a_{n+1}} \\
&= \sum_{n=1}^{M} \frac{1}{\sqrt{a_n}} - \sum_{n=2}^{M+1} \frac{\sqrt{a_{n-1}}}{a_n} \\
&= \frac{1}{\sqrt{a_1}} - \frac{\sqrt{a_M}}{a_{M+1}} + \sum_{n=2}^{M} \frac{\sqrt{a_n} - \sqrt{a_n - 1}}{a_n} \\
&\leq \frac{1}{\sqrt{a_1}} + \sum_{n=2}^{M} \frac{\sqrt{a_n} - \sqrt{a_n - 1}}{\sqrt{a_n}\sqrt{a_{n-1}}} \\
&= \frac{1}{\sqrt{a_1}} + \sum_{n=2}^{M} \frac{1}{\sqrt{a_{n-1}}} - \sum_{n=2}^{M} \frac{1}{\sqrt{a_n}} \\
&= \frac{2}{\sqrt{a_1}} - \frac{1}{\sqrt{a_M}} \\
&\leq \frac{2}{\sqrt{a_1}}.
\end{aligned}
$$

Thus $\displaystyle\sum_{n=1}^{\infty} \frac{\sqrt{a_n}}{[a_n, a_{n+1}]}$ converges.

Also solved by Ed Gray, Highland Beach, FL and the author.