Problems

Ted Eisenberg, Section Editor

****

This section of the Journal offers readers an opportunity to exchange interesting mathematical problems and solutions. Please send them to Ted Eisenberg, Department of Mathematics, Ben-Gurion University, Beer-Sheva, Israel or fax to: 972-86-477-648. Questions concerning proposals and/or solutions can be sent e-mail to <eisenbt@013.net>. Solutions to previously stated problems can be seen at <http://www.ssma.org/publications>.

Solutions to the problems stated in this issue should be posted before
May 15, 2019

5535: *Proposed by Kenneth Korbin, New York, NY*

Given positive angles A and B with $A + B = 180^o$. A circle with radius 3 and a circle of radius 4 are each tangent to both sides of $\angle A$. The circles are also tangent to each other Find $\sin A$.

5536: *Proposed by D.M. Bătinetu-Giurgiu, "Matei Basarab" National College, Bucharest, Romania and Neculai Stanciu, "George Emil Palade" School, Buzău, Romania*

If $a \in (0,1)$ then calculate $\lim\limits_{n\to\infty} \sqrt[n]{(2n-1)!!}\left(\sin\left(\dfrac{a\cdot \sqrt[n+1]{(n+1)!}}{\sqrt[n]{n!}}\right) - \sin a\right)$.

5537: *Proposed by Mohsen Soltanifar, Dalla Lana School of Public Health, University of Toronto, Canada*

Let X, Y be two real-valued continuous random variables on the real line with associated mean, median and mode $\overline{x}, \tilde{x}, \hat{x}$, and $\overline{y}, \tilde{y}, \hat{y}$, respectively. For each of the following conditions, show that there are variables X,Y satisfying them or prove such random variables do not exist.

$$\begin{array}{llll} (i) & \overline{x}\le\overline{y}, \quad \tilde{x}\le\tilde{y}, \quad \hat{x}\le\hat{y}, & (v) & \overline{x}>\overline{y}, \quad \tilde{x}\le\tilde{y}, \quad \hat{x}\le\hat{y} \\ (ii) & \overline{x}\le\overline{y}, \quad \tilde{x}\le\tilde{y}, \quad \hat{x}>\hat{y}, & (vi) & \overline{x}>\overline{y}, \quad \tilde{x}\le\tilde{y}, \quad \hat{x}>\hat{y} \\ (iii) & \overline{x}\le\overline{y}, \quad \tilde{x}>\tilde{y}, \quad \hat{x}\le\hat{y}, & (vii) & \overline{x}>\overline{y}, \quad \tilde{x}>\tilde{y}, \quad \hat{x}\le\hat{y} \\ iv) & \overline{x}\le\overline{y}, \quad \tilde{x}>\tilde{y}, \quad \hat{x}>\hat{y}, & (viii) & \overline{x}>\overline{y}, \quad \tilde{x}>\tilde{y}, \quad \hat{x}>\hat{y} \end{array}$$

5538: *Proposed by Seyran Brahimov, Baku State University, Masalli, Azerbaijan*

Solve for all real numbers $x \neq \dfrac{\pi}{2}(2k+1)$, $k \in Z$.

$$2 - 2019x = e^{\tan x} + 3^{\sin x} + \tan^{-1} x.$$

5539: *Proposed by José Luis Díaz-Barrero, Barcelona Tech, Barcelona, Spain*

Let α, β, γ be nonzero real numbers. Find the minimum value of

$$\left(\sum_{cyclic}\left(\frac{1+\sin^2\alpha\sin^2\beta}{\sin^2\alpha}\right)^3\right)^{1/3}$$

5540: *Proposed by Ovidiu Furdui, Technical University of Cluj-Napoca, Cluj-Napoca, Romania*

Let $A \in M_2(\Re)$ be a matrix which has real eigenvalues. Prove that if $\sin A$ is similar to A then $\sin A = A$.

Solutions

5517: *Proposed by Kenneth Korbin, New York, NY*

Find positive integers (a, b, c) such that
$\arccos\left(\frac{a}{1331}\right) = \arccos\left(\frac{b}{1331}\right) + \arccos\left(\frac{c}{1331}\right)$ with $a < b < c$.

Solution 1 by David E. Manes, Oneonta, NY

If $(a, b, c) = (370, 869, 1210)$, then

$$\arccos\left(\frac{370}{1331}\right) = \arccos\left(\frac{869}{1331}\right) + \arccos\left(\frac{1210}{1331}\right) \approx 1.28\,909\,899\,845.$$

Writing the arccosine equation in terms of the cosine function and using the identity $\sin(\arccos x) = \sqrt{1-x^2}$, one obtains

$$\cos\left(\arccos\left(\frac{a}{1331}\right)\right) = \cos\left[\arccos\left(\frac{b}{1331}\right) + \arccos\left(\frac{c}{1331}\right)\right].$$

Therefore,

$$\begin{aligned}
\frac{a}{1331} &= \cos\left(\arccos\left(\frac{b}{1331}\right)\right)\cos\left(\arccos\left(\frac{c}{1331}\right)\right) - \sin\left(\arccos\left(\frac{b}{1331}\right)\right)\sin\left(\arccos\left(\frac{c}{1331}\right)\right)\\
&= \left(\frac{b}{1331}\right)\left(\frac{c}{1331}\right) - \sqrt{1-\left(\frac{b}{1331}\right)^2}\sqrt{1-\left(\frac{c}{1331}\right)^2}\\
&= \frac{bc}{(1331)^2} - \sqrt{1 - \frac{b^2}{(1331)^2} - \frac{c^2}{(1331)^2} + \frac{b^2c^2}{(1331)^4}}\\
&= \frac{bc}{(1331)^2} - \frac{\sqrt{(1331)^4 - (1331)^2b^2 - (1331)^2c^2 + b^2c^2}}{(1331)^2}.
\end{aligned}$$

Thus,

$$\frac{\sqrt{(1331)^4 - (1331)^2b^2 - (1331)^2c^2 + b^2c^2}}{1331} = \frac{bc}{1331} - a.$$

Squaring both sides of this equation yields

$$\frac{(1331)^4-(1331)^2b^2-(1331)^2c^2+b^2c^2}{(1331)^2}=a^2+\frac{b^2c^2}{(1331)^2}-\frac{2abc}{1331},$$
$$(1331)^4-(1331)^2b^2-(1331)^2c^2=(1331)^2a^2-2abc(1331),$$
$$(1331)^3-1331b-1331c=1331a^2-2abc.$$

Writing this equation as a quadratic in a, we get

$$1331a^2-(2bc)a+1331\left(b^2+c^2-(1331)^2\right)=0.$$

Therefore, by the quadratic formula

$$a=\frac{2bc\pm\sqrt{4b^2c^2-4(1331)^2\left(b^2+c^2-(1331)^2\right)}}{2(1331)}.$$

This equation reduces to

$$a=\frac{bc}{1331}\pm\sqrt{\left(\frac{bc}{1331}\right)^2-(b^2+c^2-(1331)^2)}.$$

Noting that $1331=11^3$, we choose values for b and c that are divisible by powers of 11. We summarize the results.

1. If $b=79\cdot 11=869$ and $c=10\cdot 11^2=1210$, then

$$a=\left(\frac{(79\cdot 11)\left(10\cdot 11^2\right)}{1331}\right)\pm\sqrt{((79\cdot 10)^2-((79\cdot 11)^2+(10\cdot 11^2)^2-(1331)^2))}$$
$$=790\pm 420.$$

If $a=790+420=1210=c$, then this root is extraneous. If $a=790-420=370$, then $a<b<c$ and

$$\arccos\left(\frac{370}{1331}\right)=\arccos\left(\frac{869}{1331}\right)+\arccos\left(\frac{1210}{1331}\right)\approx 1.28\,909\,899\,845.$$

All of the following solutions are obtained from integer values for b' and c' such that a' is also an integer that satisfies the above equation for a with b' and c' substituted for b and c, respectively. The integer values for a', b' and c' do not satisfy the parameters of the problem. The values for a, b and c are then obtained as a permutation of a', b' and c' such that $a<b<c$ and the inverse cosine equation is satisfied.

2. If $b'=49\cdot 11=539$ and $c'=6\cdot 11^2=726$, then $a'=294\pm 1020$. Therefore, define a, b and c so that $a=539$, $b=726$ and $c=294+1020=1314$. Then $a<b<c$ and

$$\arccos\left(\frac{539}{1331}\right)=\arccos\left(\frac{726}{1331}\right)+\arccos\left(\frac{1314}{1331}\right)\approx 1.15\,386\,269\,047.$$

3. If $b'=89\cdot 11=979$ and $c'=4\cdot 11^2=484$, then $a'=356\pm 840$. Define the values of a, b and c so that $a=484$, $b=979$ and $c=356+840=1196$. Then $a<b<c$ and

$$\arccos\left(\frac{484}{1331}\right)=\arccos\left(\frac{979}{1331}\right)+\arccos\left(\frac{1196}{1331}\right)\approx 1.19\,862\,779\,283.$$

4. If $b' = 103 \cdot 11 = 1133$ and $c' = 3 \cdot 11^2 = 363$, then $a' = 309 \pm 672$. The values of a, b and c are then $a = 363$, $b = 309 + 672 = 981$ and $c = 1133$. Then $a < b < c$ and

$$\arccos\left(\frac{363}{1331}\right) = \arccos\left(\frac{981}{1331}\right) + \arccos\left(\frac{1133}{1331}\right) \approx 1.29\,456\,969\,603.$$

5. If $b' = 113 \cdot 11 = 1243$ and $c' = 2 \cdot 11^2 = 242$, then $a' = 226 \pm 468$. The values of a, b and c are now $a = 242$, $b = 226 + 468 = 694$ and $c = 1243$. Then $a < b < c$ and

$$\arccos\left(\frac{242}{1331}\right) = \arccos\left(\frac{694}{1331}\right) + \arccos\left(\frac{1243}{1331}\right) \approx 1.38\,796\,118\,98.$$

Solution 2 by Ulrich Abel, Technische Hochschule Mittelhessen, Friedberg, Germany

A computer program found the following 8 integer solutions (a, b, c) of

$$\arccos\frac{a}{1331} = \arccos\frac{b}{1331} + \arccos\frac{c}{1331}$$

with $0 < a < b < c$:

$$\begin{aligned}&(121, 359, 1309)\\&(242, 694, 1243)\\&(253, 847, 1169)\\&(363, 981, 1133)\\&(370, 869, 1210)\\&(484, 979, 1196)\\&(539, 726, 1314)\\&(605, 781, 1315)\end{aligned}$$

<u>Remark:</u> Unfortunately, I don't know a systematic way to find these solutions without a computer.

Solution 3 by Albert Stadler, Herrliberg, Switzerland

We note that

$$\frac{a}{11^3} = \cos\left(\arccos\left(\frac{a}{11^3}\right)\right) = \cos\left(\arccos\left(\frac{b}{11^3}\right) + \arccos\left(\frac{c}{11^3}\right)\right) =$$

$$= \frac{b}{11^3}\cdot\frac{c}{11^3} - \sqrt{1 - \frac{b^2}{11^6}}\cdot\sqrt{1 - \frac{c^2}{11^6}},$$

which implies

$$\left(\frac{a}{11^3} - \frac{bc}{11^6}\right)^2 = \left(1 - \frac{b^2}{11^6}\right)\left(1 - \frac{c^2}{11^6}\right),$$

or equivalently,

$$11^3\left(a^2 + b^2 + c^2\right) = 11^9 + 2abc. \qquad (*)$$

An exhaustive computer search in the range $0 < a < b < c \le 1331$ reveals that $(*)$ implies $(a, b, c) \in \{(121, 359, 1309), (242, 694, 1243), (253, 847, 1169), (363, 981, 1133), (370, 869, 1210), (484, 979, 1196), (539, 726, 1314), (605, 781, 1315)\}$.

Note: (*) is equivalent to $\left(\frac{a}{11^3}\right)^2 + \left(\frac{b}{11^3}\right)^2 + \left(\frac{a}{11^3}\right)^2 = 1 + 2\frac{a}{11^3} \cdot \frac{b}{11^3} \cdot \frac{c}{11^3}$.

The Diophantine equation

$$x^2 + y^3 = 2 + z^2 = 1 + 2xyz$$

has been extensively studied in the literature, see the references

[1] L. J. Mordell, On the Integer Solutions of the Equation $x^2 + y^2 + z^2 + 2xyz = n$ Journal of the London Mathematical Society, Volumes 1-28, Issue 4, 1 October 1953, Pages 500-510, https://doi.org/10.1112/jlms/s1-28.4.500

[2] A. Oppenheim, "On the Diophantine Equation $x^2 + y^2 + z^2 + 2xyz = 1$." The American Mathematical Monthly, vol. 64, no. 2, 1957, pp. 101-103. DOI: 10.2307/2310390.

A. Oppenheim provides the general solution of $x^2 + y^2 + z^2 + 2xyz = 1$ in rational integers. The given problem asks for the solutions in rational numbers whereby the denominators equal 11^3.

Editor′s comment: Ken Korbin, proposer of the problem, included in his solution some algebraic expressions and a geometric interpretation of the problem that gives us some insight into how he constructed the problem.

The expressions he listed are:

$$\begin{array}{ll} a) & 121N \\ b) & \left|\left(22N^2 - 1331\right)\right| \\ c) & \left|\left(363N - 4N^3\right)\right| \quad \text{with} \end{array}$$

$$0 < N < \frac{11\sqrt{2}}{2} \quad \text{or} \quad \frac{11\sqrt{3}}{2} < \text{N} < 11.$$

So if $N = 19$, then $(a, b, c) = (370, 869, 1210)$, and if $N = 7$, then $(a, b, c) = (253, 847, 1169)$, and if $N = 6$, then $(a, b, c) = (539, 726, 1314)$, etc.

Suppose $(a, b, c) = (370, 869, 1210)$. Arrange four points A, B, C, D in a circular arrangement with the vertices being in a clockwise direction. Connecting the segments $\overline{AB}, \overline{BD}, \overline{AC}$, and $\overline{CD}$ gives us a diagram that resembles a butterfly.

Ken then stated that for this triplet, (a, b, c), there is a convex cyclic quadrilateral A, B, C, D with

$$\begin{aligned} \overline{AC} &= \text{Diameter} = 1331, \\ \overline{AB} &= 1210, \\ \text{Diagonal } \overline{\text{BD}} &= 869, \text{ and } \overline{\text{CD}} = 370. \end{aligned}$$

Also solved by, Brian D. Beasley, Presbyterian College, Clinton, SC; Anthony J. Bevelacqua, University of North Dakota, Grand Forks, ND; Ed Gray, Highland Beach, FL; David Stone and John Hawkins of Georgia Southern University, Statesboro GA, and the proposer.

5518: *Proposed by Roger Izard, Dallas, TX*

Let triangle PQR be equilateral and let it intersect another triangle ABC at points U, U', W, W', V, V' such that WU', UV', VW' are equal in length, and triangles $AU'W, BV'U, CW'V$ are equal in area (see Figure 1). Show that triangle ABC must then also be equilateral

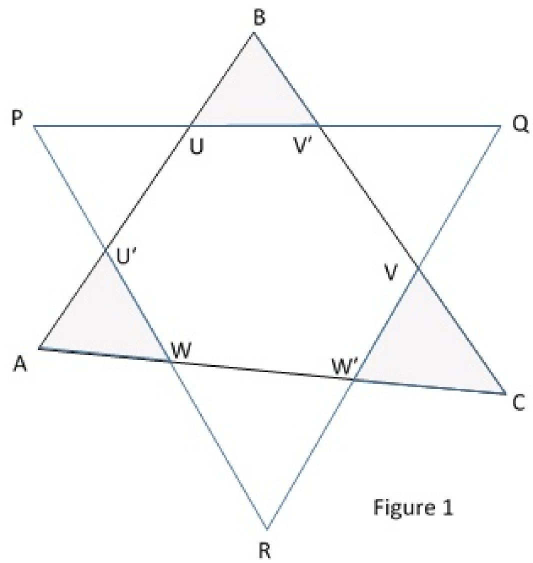

Figure 1

Solution 1 by Kee-Wai Lau, Hong Kong, China

Without loss of generality let $WU' = UV' = VW' = 1$. Let $\angle PU'U = \alpha, \angle RW'W = \beta, \angle QV'V = \gamma$. It is easy to check that $\angle BUV' = \dfrac{2\pi}{3} - \alpha,\ \angle BV'U = \gamma,\ \angle UBV' = \dfrac{\pi}{3} - \gamma + \alpha$. Applying the sine formula to triangle $BV'U$, we have $BU = \dfrac{\sin\gamma}{\sin\left(\frac{\pi}{3} - \gamma + \alpha\right)},\ BV' = \dfrac{\sin\left(\frac{\pi}{3} + \alpha\right)}{\sin\left(\frac{\pi}{3} - \gamma + \alpha\right)}$. Hence the area of triangle $BV'U$ equals $\dfrac{\sin\gamma \sin\left(\frac{\pi}{3} + \alpha\right)}{2\sin\left(\frac{\pi}{3} - \gamma + \alpha\right)} = \dfrac{1}{2\left(\cot\gamma - \cot\left(\frac{\pi}{3} + \alpha\right)\right)}$, using the formula $\sin x - y) = \sin x \cos y - \cos x \sin y$. Similarly the areas of triangles $AU'W$ and $CW'V$ are respectively $\dfrac{1}{2\left(\cot\alpha - \cot\left(\frac{\pi}{3} + \beta\right)\right)}$ and $\dfrac{1}{2\left(\cot\beta - \cot\left(\frac{\pi}{3} + \gamma\right)\right)}$. Given that these areas are equal, so,

$$\cot\gamma - \cot\left(\frac{\pi}{3} + \alpha\right) = \cot\alpha - \cot\left(\frac{\pi}{3} + \beta\right) = \cot\beta - \cot\left(\frac{\pi}{3} + \gamma\right).$$

We only consider the case $\alpha \geq \beta$, since the case $\alpha \leq \beta$ can be treated similarly. We have

$$\cot\alpha - \cot\left(\frac{\pi}{3} + \beta\right) = \cot\gamma - \cot\left(\frac{\pi}{3} + \alpha\right) \geq \cot\gamma - \cot\left(\frac{\pi}{3} + \beta\right),$$

so that $\gamma \geq \alpha$. Hence

$$\cot\gamma - \cot\left(\frac{\pi}{3} + \alpha\right) = \cot\beta - \cot\left(\frac{\pi}{3} + \gamma\right) \geq \cot\beta - \cot\left(\frac{\pi}{3} + \alpha\right),$$

implying $\beta \geq \gamma$. It follows that $\beta = \gamma\alpha$. Thus $\angle UBV' = \frac{\pi}{3}$, and similarly $\angle WAU' = \angle VCW' = \frac{\pi}{3}$. This shows that triangle ABC is also equilateral.

Solution 2 by Michael Fried, Ben-Gurion University of the Negev, Beer-Sheva, Israel

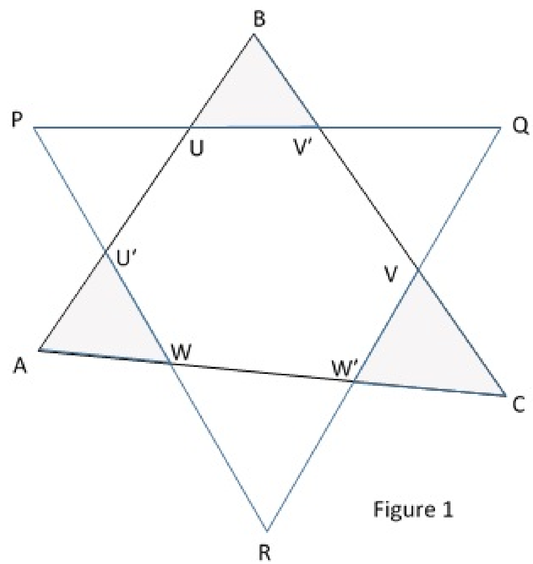

Figure 1

We can turn this into an equivalent problem in the following way. First, slide triangle $AU'W'$ along AB so that U' and U coincide and $CW'V$ along CB so that V' and V coincide (see Figure 2). Since PQR is an equilateral triangle the angles WUV and $W'VU$ are both $60°$. But since also $WU' = UV' = VW'$, the points W' and W must also coincide so that we have an equilateral triangle UVW inscribed in another triangle ABC (the latter is a triangle since AW and $W'C$ are always parallel to AB so that AWC is a straight line, while AUB and CVB are just segments of the original lines $AU'UB$ and $CVV'B$, respectively).

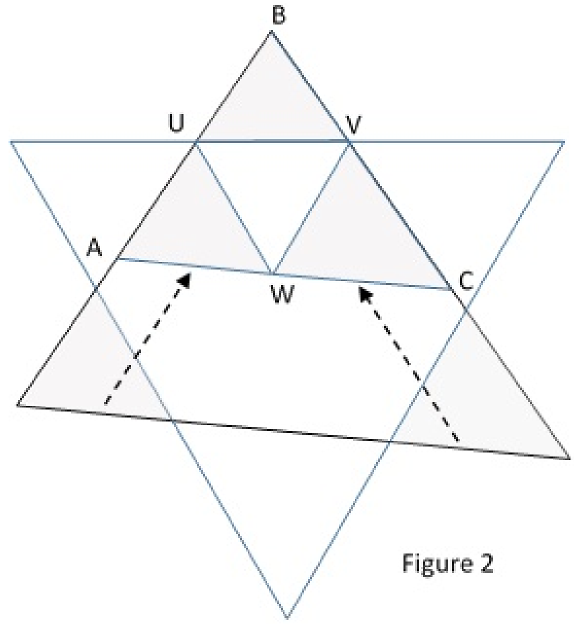

Figure 2

So the new problem can be stated as follows:
If a triangle ABC is circumscribed about an equilateral triangle UVW so that the areas AUW, BVU, CWV are equal, then ABC must also be equilateral.
But we can still do better. Suppose the common area of AUW, BVU, CWV is K, then the locus of all points A such that $AUW = K$ is a line parallel to UW. Similarly, the locus of all points B such that $BVU = K$ is a line parallel to UV. This line is also at the same distance from UV as the previous line is from UW. Finally, the locus of points C such that $CWV = K$ is again a line parallel to VW and at the same distance from VW as the previous line is from UV. These three parallel lines thus form another equilateral triangle XYZ whose sides are parallel to those of UVW and equidistant from them. So, the triangle ABC will circumscribe UVW and be inscribed in XYZ (see Figure 3). As a terminological convention, we will say that ABC is situated *between* UVW and XYZ

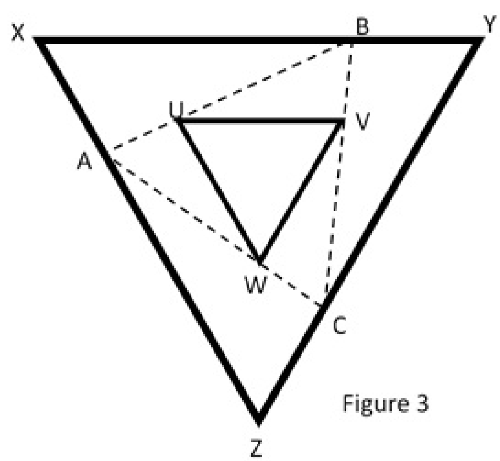

Figure 3

With that, we can formulate yet another problem equivalent to the first:
If UVW and XYZ are concentric equilateral triangles whose respective sides are parallel (UVW inside XYZ), then any triangle ABC situated between UVW and XYZ must be

equilateral.[1]
Now it is easy to show that if there is any such triangle ABC at all, there is at least one which is equilateral. Indeed, there is exactly one such equilateral triangle or there are exactly two: there cannot be more than two, but there can be none.
On each side of UVW draw circular arcs so that each side subtends an angle of $60°$. If the circles do not touch the sides of XYZ then can be no triangle such as ABC, for its angles, such as UAW, would all have to be less than $60°$ since their vertices would have to fall outside the circles (see Figure 4, left).
If the circles are tangent to the sides of XYZ (see Figure 4, right), then joining UA and UB, we that angle $XAU = XBU = 60°$ so that AUB is a straight line parallel to YZ. Similarly BVC and AWC are straight lines, so we obtain in this way one equilateral triangle ABC situated between UVW and XYZ. Moreover, there can be no other such triangle, for the angles of any other triangle, such as the angle $UA'W$ would fall outside the circle and therefore would, again, all be less than $60°$.

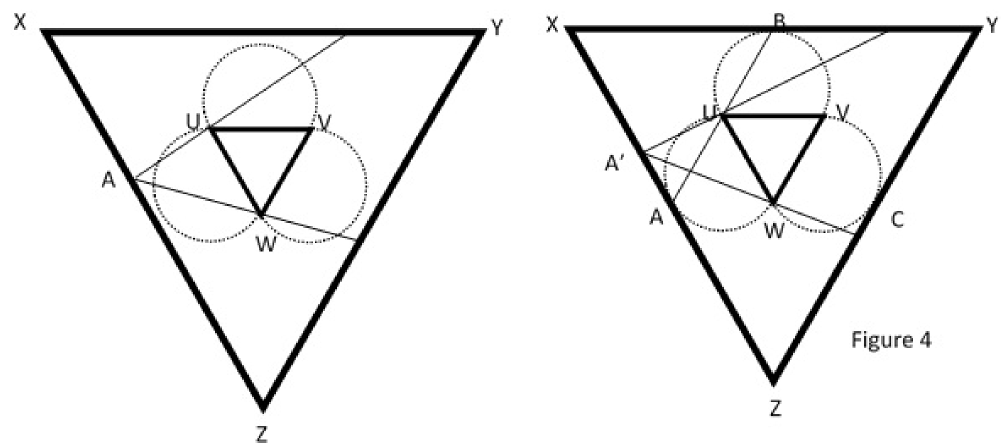

Figure 4

Consider then the last case in which the circles intersect each of the sides of XYZ in two points such as A and A' and B and B' in Figure 5.

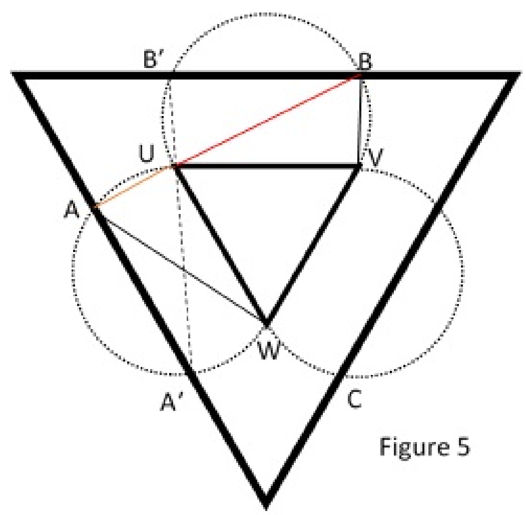

Figure 5

[1]All triangle centers of equilateral triangles coincide, so one can speak about concentric triangles without further ado. The two conditions assure that the corresponding sides of the two triangles are the same distance from one another

Join AU and UB. The angles at B and A are of course, by construction, both $60°$. Then since the distance between the sides of the two equilateral triangles is the same for all three sides (and because all the circles are obviously congruent) the arcs $BV, B'U, AU, A'W$ are all equal, so that also angle $AWU = BUV$. Hence, $AUW = 120° - AWU = 120° - BUV$, and therefore,

$$BUV + VUW + AUW = BUV + 60° + 120° - BUV = 180°$$

so that AUB is a straight line. Joining and extending AW and BV to the point C, we obtain an equilateral triangle situated between UVW and XYZ. A second equilateral triangle can be obtained by repeating the process beginning with points A' and B'. Now, to finish the proof, note that any other line from, say, XY to XZ via U must either begin from $B'B$ and end on $A'A$ or must begin outside $B'B$ and end outside $A'A$ (see Figure 6).

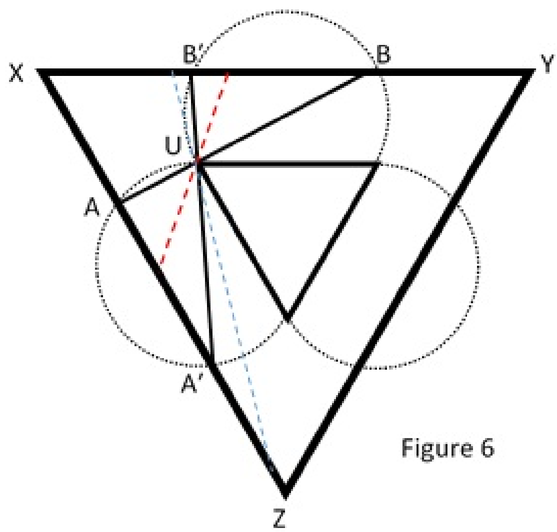

Figure 6

But, in the first case, a triangle ABC situated between UVW andXYZ would have all of its angles greater than $60°$ while, in the second case, all of the angles would be less than $60°$, which is impossible in either case (see Figure 7)

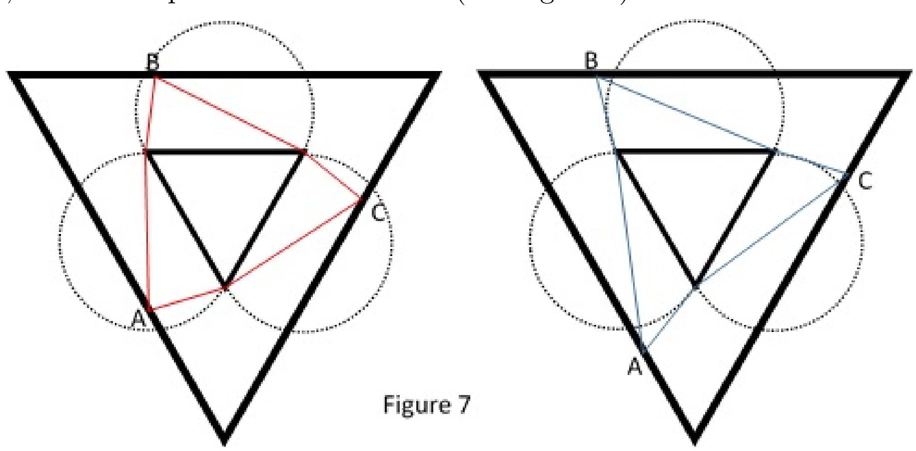

Figure 7

Thus, we can state the final version of the theorem as follows:
Given two concentric equilateral triangles whose sides are parallel there can only be either 0, 1, or 2 triangles situated between the two equilateral triangles, and in every case the triangles are themselves equilateral

Solution 3 by Ed Gray of Highland Beach, FL

Editor′s comment: Following is a heuristic argument using the contrapositive of the statement.

Given equilateral triangle PQR, we have a triangle erected on each side of PQR such that each of the three triangles has equal area and equal base. As a consequence, each of the three must have equal altitudes. (We will attempt the proof going the other way). Let A, B, and C be the apex of each triangle and D, E, and F be points on PQ, QR, and RP respectively which are the feet of the three altitudes. (That is, the altitudes are BD, CF, and AD). If we connect A and B, this line intersects PQ at point U and PR at point U'. If we connect A with C, this line intersects PR at W and QR at W'. Finally, if we connect B with C, this line intersects PQ at V' and QR at V. There is a severe restriction that the lengths WU' on PR, UV' on PQ, and VW' on RQ must all be of the same length. The plan is to show that this can only happen if the points D, E, and F are the mid-points of PQ, QR, and RP respectively. First, we show that indeed, if D, E, and F are the mid-points, then the triangle ABC will be equilateral.

If we picture a coordinate system with $R = (0, 0), P = \left(\frac{-s}{2}, \frac{\sqrt{3}s}{2}\right)$, and $Q = \left(\frac{s}{2}, \frac{\sqrt{3}s}{2}\right)$, s = side length of triangle PQR, the slope of PR is $-\sqrt{3}$, so the slope of AF is $\frac{1}{\sqrt{3}}$; similarly, the slope of QR is $+\sqrt{3}$, so the slope of EC is $\frac{-1}{\sqrt{3}}$. Since AF and EC are the same length, the difference in coordinates between C and E and A and F would be the same. It follows that AC is parallel to PQ. Similarly, BC is parallel to PR and AB is parallel to RQ. It follows that the angles A, B, and C are the same as P, Q and $R = 60°$ and ABC is equilateral. Clearly, $UV' = VW' = WU'$.

Suppose now, at least one altitude foot is not the mid-point of an equilateral triangle side. For instance, if D is closer to P than Q, it is clear that BA is less than BC. If we try to compensate, say, by moving F closer to R, then AC will be the smaller side.

Also solved by the proposer.

5519: *Proposed by Titu Zvonaru, Comănesti, Romania*

Let a, b, c be positive real numbers. Prove that

$$\frac{a^2}{b^2} + \frac{b^2}{c^2} + \frac{c^2}{a^2} + \frac{2abc}{a^3 + b^3 + c^3} \geq \frac{11}{3}. \qquad (1)$$

Solution 1 by Albert Stadler, Herrliberg, Switzerland

Let

$$\frac{a^2}{b^2} + \frac{b^2}{c^2} + \frac{c^2}{a^2} + \frac{2abc}{a^3 + b^3 + c^3} - \frac{11}{3} = \frac{f(a, b, c)}{3a^2b^2c^2(a^3 + b^3 + c^3)}.$$

Then

$f(a, b, c) = 3a^5b^4 + 3a^2b^7 + 3a^7c^2 - 11a^5b^2c^2 + 3a^4b^3c^2 - 11a^2b^5c^2 + 6a^3b^3c^3 +$
$3a^2b^4c^3 + 3a^3b^2c^4 + 3b^5c^4 + 3a^4c^5 - 11a^2b^2c^5 + 3b^2c^7$,

and $f(a,b,c)$ is a cyclic polynomial in the variables a, b, c. We may assume without loss of generality that $a = \min(a,b,c)$. Therefore there are numbers $u \geq 0, v \geq 0, w \geq 0$ such that either $a = u, b = u+v, c = u+v+w$ or $a = u, c = u+v, b = u+v+w$.

A brute force calculation reveals that

$f(u, u+v, u+v+w)$
$= 30u^7v^2 + 152u^6v^3 + 353u^5v^4 + 480u^4v^5 + 401u^3v^6 + 200u^2v^7 + 54uv^8 + 6v^9 + 30u^7vw$
$+264u^6v^2w + 850u^5v^3w + 1461u^4v^4w + 1470u^3v^5w + 856u^2v^6w + 264uv^7w + 33v^8w$
$+30u^7w^2 + 228u^6vw^2 + 885u^5v^2w^2 + 1892u^4v^3w^2 + 2311u^3v^4w^2 + 14589u^2v^5w^2 + 567uv^6w^2$
$+81v^7w^2 + 58u^6w^3 + 388u^5vw^3 + 1197u^4v^2w^3 + 1930u^3v^3w^3 + 1648u^2v^4w^3 + 702uv^5w^3$
$+117v^6w^3 + 71u^5w^4 + 396u^4vw^4 + 918u^3v^2w^4 + 1025u^2v^3w^4 + 540uv^4w^4 + 108v^5w^4$
$+55u^4w^5 + 230u^3vw^5 + 367u^2v^2w^5 + 252uv^2w^5 + 63v^4w^5 + 21u^3w^6 + 63u^2vw^6 +$
$63u^2w^6 + 21v^3w^6 + 3u^2w^7 + 6uvw^7 + 3v^2w^7$

and

$f(u, u+v, u+v+w)$
$= 30u^7v^2 + 152u^6v^3 + 353u^5v^4 + 480u^4v^5 + 401u^3v^6 + 200u^2v^7 + 54uv^8 + 6v^9 + 30u^7vw$
$+192u^6v^2w + 562u^5v^3w + 939u^4v^4w + 936u^3v^5w + 544u^2v^6w + 168uv^7w + 21v^8w$
$+30u^7w^2 + 156u^6vw^2 + 453u^5v^2w^2 + 848u^4v^3w^2 + 976u^3v^4w^2 + 653u^2v^5w^2 + 231uv^6w^2$
$+33v^7w^2 + 58u^6w^3 + 244u^5vw^3 + 513u^4v^2w^3 + 634u^3v^3w^3 + 457u^2v^4w^3 + 180uv^5w^3$
$+30v^6w^3 + 71u^5w^4 + 234u^4vw^4 + 309u^3v^2w^4 + 203u^2v^3w^4 + 75uv^4w^4 + 15v^5w^4$
$+55u^4w^5 + 116u^3vw^5 + 70u^2v^2w^5 + 12uv^2w^5 + 3v^4w^5 + 21u^3w^6 + 21u^2vw^6 + 3u^2w^7.$

All coefficients are positive. Therefore $f(a,b,c) \geq 0$, if $a \geq 0, b \geq 0, c \geq 0$.

Note: Let $f(a,b,c)$ be a cyclic real polynomial in the variables a, b, c (that is $f(a,b,c) = f(b,c,a) = f(c,a,b)$), which is claimed to be nonnegative for $a \geq 0, b \geq 0, c \geq 0$. It has happened to me multiple times that I was unable to apply the AM-GM inequality directly to prove that $f(a,b,c) \geq 0$ (assuming that $a \geq 0, b \geq, c \geq 0$). However the following brute force approach was mostly successful: due to the fact that $f(a,b,c)$ is cyclic one may assume that $a = \min(a,b,c)$. Then there are nonnegative variables u, v, w such that either $(a,b,c) = (u, u+v, u+v+w)$ or $(a,b,c) = (u, u+v+w, u+v)$. Then $f(u, u+v, u+v+w)$ and $f(u, u+v+w, u+v)$ are polynomials in u, v, w, and when multiplied out one sees (very often) that all coefficients are positive (as in the case above), showing that $f(a,b,c) \geq 0$ if $a \geq 0, b \geq 0, c \geq 0$. Multiplying out requires a computational effort, no doubt about that, but it is a purely mechanical task and does not require any creativity. Computer algebra systems are a very useful assistant for this specific computation.

Solution 2 by Moti Levy, Rehovot, Israel

Since the inequality is homogenous, then we may assume without loss of generality that $a + b + c = 1$.
By Titu's lemma,

$$\frac{a^2}{b^2} + \frac{b^2}{c^2} + \frac{c^2}{a^2} \geq \frac{(a+b+c)^2}{a^2+b^2+c^2}.$$

Hence it is enough to prove that

$$\frac{(a+b+c)^2}{a^2+b^2+c^2} + \frac{2abc}{a^3+b^3+c^3} \geq \frac{11}{3}. \tag{1}$$

Now we use the p, q, r notation: $p = a+b+c, \quad q = ab+bc+ca, \quad r = abc$.

The following equations and inequalities are well known:

$$a^2+b^2+c^2 = p^2-2q, \tag{2}$$
$$a^3+b^3+c^3 = p^3-3pq+3r, \tag{3}$$
$$r \le \frac{pq}{9}, \tag{4}$$
$$r \le \frac{p^3}{27}. \tag{5}$$

Using (2), (3) and setting $p=1$, inequality (1) can be rewritten as follows,

$$\frac{1}{1-2q}+\frac{2r}{1-3q+3r}-\frac{11}{3} \ge 0.$$

By inequality (4) $\ r \le \frac{q}{9}$, so that

$$\begin{aligned}\frac{1}{1-2q}+\frac{2r}{1-3q+3r}-\frac{11}{3} &\ge \frac{1}{1-18r}+\frac{2r}{1-24r}-\frac{11}{3}\\ &= \frac{8\left(r-\frac{1}{27}\right)\left(r-\frac{2}{45}\right)}{3\left(r-\frac{1}{18}\right)\left(r-\frac{1}{24}\right)}.\end{aligned}$$

By inequality (5), $r \le \frac{1}{27}$ and it follows immediately that $\frac{8\left(r-\frac{1}{27}\right)\left(r-\frac{2}{45}\right)}{3\left(r-\frac{1}{18}\right)\left(r-\frac{1}{24}\right)} \ge 0.$

Solution 3 by Michel Bataille, Rouen France

Let L denote the left-hand side of the inequality and let $x=\frac{a}{b},\ y=\frac{b}{c},\ z=\frac{c}{a}$. Then, $x,y,z>0,\ xyz=1$ and

$$L = x^2+y^2+z^2+\frac{2}{\frac{x}{z}+\frac{y}{x}+\frac{z}{y}}.$$

By the Cauchy-Schwarz inequality, we have

$$\left(\frac{x}{z}+\frac{y}{x}+\frac{z}{y}\right)^2 \le (x^2+y^2+z^2)\left(\frac{1}{z^2}+\frac{1}{x^2}+\frac{1}{y^2}\right) = (x^2+y^2+z^2)(x^2y^2+y^2z^2+z^2x^2).$$

It follows that

$$L \ge x^2+y^2+z^2+\frac{2}{\sqrt{(x^2+y^2+z^2)(x^2y^2+y^2z^2+z^2x^2)}}$$

and it is sufficient to prove that

$$u+v+w+\frac{2}{\sqrt{(u+v+w)(uv+vw+wu)}} \ge \frac{11}{3} \tag{1}$$

whenever $u,v,w>0$ and $uvw=1$.
Now, from $(u+v+w)^2 = u^2+v^2+w^2+2(uv+vw+wu) \ge 3(uv+vw+wu)$ we deduce that $(u+v+w)^3 \ge 3(u+v+w)(uv+vw+wu)$ so that (1) will certainly hold if

$$u+v+w+\frac{2\sqrt{3}}{(u+v+w)^{3/2}} \ge \frac{11}{3}. \tag{2}$$

To prove (2), we consider the function f defined on $(0,\infty)$ by $f(x)=x+2\sqrt{3}x^{-3/2}$. From the derivative $f'(x)=x^{-5/2}(x^{5/2}-3^{3/2})$ we deduce that f is increasing on

$[3^{3/5}, \infty)$. Since $u+v+w \geq 3\sqrt[3]{uvw} = 3 > 3^{3/5}$, the inequality $f(u+v+w) \geq f(3) = \frac{11}{3}$ holds and (2) follows.

Solution 4 by Paolo Perfetti, Department of Mathematics, Tor Vergata University, Rome, Italy

Two inequalities will be used.

$$\frac{a^2}{b^2} + \frac{b^2}{c^2} + \frac{c^2}{a^2} \geq \frac{a^2+b^2+c^2}{\sqrt[2]{abc}} \geq 3\frac{a^2+b^2+c^2}{ab+bc+ca}$$

The L.H.S. is

$$a^2c + b^2a + c^2b \geq (a+b+c)(abc)^{2/3}$$

and

$$\frac{a^2c+a^2c+b^2a}{3} \geq a^{\frac{5}{3}}(bc)^{\frac{2}{3}}, \qquad \frac{b^2a+b^2a+c^2b}{3} \geq (ac)^{\frac{2}{3}}b^{\frac{5}{3}}, \qquad \frac{a^2c+b^2a+c^2b}{3} \geq (ab)^{\frac{2}{3}}c^{\frac{5}{3}}$$

Moreover

$$a^{\frac{5}{3}}(bc)^{\frac{2}{3}} + (ac)^{\frac{2}{3}}b^{\frac{5}{3}} + (ab)^{\frac{2}{3}}c^{\frac{5}{3}} = (a+b+c)(abc)^{2/3}$$

The R.H.S. follows trivially by $ab+bc+ca \geq 3(abc)^{\frac{2}{3}}$

It suffices to prove

$$3\frac{a^2+b^2+c^2}{ab+bc+ca} + \frac{2abc}{a^3+b^3+c^3} \geq \frac{11}{3} \tag{1}$$

Let's change variables $a+b+c = 3u$, $ab+bc+ca = 3v^2$, $abc = w^3$. The inequality (1) is

$$3\frac{9u^2-6v^2}{3v^2} + \frac{2w^3}{27u^3-27uv^2+3w^3} \geq \frac{11}{3}$$

$$\frac{3(-3v^2w^3+3u^2w^3-56u^3v^2+29v^4u+27u^5)}{v^2(w^3+9u^3-9uv^2)} \geq 0$$

This is a linear increasing function of w^3 because $u^2 \geq v^2$ by the AGM thus the inequality holds true if and only if it holds true for the minimum value of the variable w^3. Once fixed the values of (u,v), the minimum value occurs when $c=0$ (or cyclic) or $c=b$ (or cyclic).

Let $c=0$. The inequality becomes

$$\frac{9a^5+9a^2b^3+9b^2a^3+9b^5-11ab^4-11a^4b}{3ab(a^3+b^3)} \geq 0$$

$$9a^5+9b^2a^3 \geq 18a^4b, \qquad 9b^5+9b^3a^2 \geq 18b^4a$$

If $c=b$ the inequality becomes

$$\frac{(9a^3-4a^2b-10ab^2+14b^3)(-b+a)^2}{3b(2a+b)(a^3+2b^3)}$$

$$\frac{10}{3}b^3 + \frac{10}{3}b^3 + \frac{10}{3}a^3 \geq 10ab^2$$

$$\frac{4}{3}a^3 + \frac{4}{3}a^3 + \frac{4}{3}b^3 \geq 4a^2b$$

and the proof is complete.

Solution 5 by Adrian Naco, Polytechnic University, Tirana, Albania

Firstly let us prove the following inequality, using the well known ABC method:

$$\frac{2abc}{a^3+b^3+c^3}+\frac{2}{3}\geq\frac{ab+bc+ca}{a^2+b^2+c^2}, \qquad (2)$$

The last inequality is equivalent to the following one,

$$f(a,b,c)=abc\left(\sum_{cyc}^{a,b,c}a^2\right)+\frac{2}{3}\left(\sum_{cyc}^{a,b,c}a^3\right)\left(\sum_{cyc}^{a,b,c}a^2\right)-\left(\sum_{cyc}^{a,b,c}a^3\right)\left(\sum_{cyc}^{a,b,c}ab\right)\geq 0$$

Since the expression on the left of the last inequality is of the third degree and is a symmetrical one in terms of a, b, c, based on the ABC method for solving inequalities in three variables, the minimum value is attainable when,

$$(a-b)(b-c)(c-a)=0 \quad \text{or/and} \quad abc=0.$$

WLOG let check first the case $a=b$ and then the case $a=0$.
Considering $a=b$ and doing easy manipulations, the inequality (2) is transformed equivalently to the following inequalities,

$$\frac{2a^2c}{2a^3+c^3}+\frac{2}{3}\geq\frac{a^2+2ac}{2a^2+c^2} \quad \Leftrightarrow \quad \frac{3a^2c+4a^3+2c^3}{6a^3+3c^3}\geq\frac{a^2+2ac}{2a^2+c^2}$$

$$\Leftrightarrow \quad (3a^2c+4a^3+2c^3)(2a^2+c^2)\geq(6a^3+3c^3)(a^2+2ac)$$
$$\Leftrightarrow \quad 2(a-c)^4(a+c)\geq 0$$

The last inequality is true since a and b are positive real numbers.
If $a=0$ and doing easy manipulations the inequality (2) is transformed equivalently to the following inequalities,

$$\frac{2}{3}\geq\frac{bc}{b^2+c^2} \quad \Leftrightarrow \quad 2(b^2+c^2)\geq\frac{3}{b}c \quad \Leftrightarrow \quad \frac{3}{2}(b+c)^2+\frac{1}{2}(b^2+c^2)\geq 0.$$

The last inequality is true, since b and c are positive real numbers.
Referring to the book, *Secrets in Inequalities*,
Vol.1, Pham Kim Hung, 2007, BIL Publishing House, Page 193-195,
it has been proved the following inequality,

$$\frac{a^2}{b^2}+\frac{b^2}{c^2}+\frac{c^2}{a^2}+\frac{9(ab+bc+ca)}{a^2+b^2+c^2}\geq 12, \qquad (3).$$

Using the inequalities (2) and (3), the given inequality (1), the statement of the problem, is transformed to the following inequalities,

$$\frac{a^2}{b^2}+\frac{b^2}{c^2}+\frac{c^2}{a^2}+\frac{2abc}{a^3+b^3+c^3} \geq \frac{a^2}{b^2}+\frac{b^2}{c^2}+\frac{c^2}{a^2}+\frac{2(ab+bc+ca)}{a^2+b^2+c^2}-\frac{4}{3}=$$

$$=\frac{7}{9}\left(\frac{a^2}{b^2}+\frac{b^2}{c^2}+\frac{c^2}{a^2}\right)+\frac{2}{9}\left[\frac{a^2}{b^2}+\frac{b^2}{c^2}+\frac{c^2}{a^2}+\frac{9(ab+bc+ca)}{a^2+b^2+c^2}\right]-\frac{4}{3} \geq$$

$$\geq \frac{7}{9}\cdot 3\cdot\sqrt[3]{\frac{a^2}{b^2}\cdot\frac{b^2}{c^2}\cdot\frac{c^2}{a^2}}+\frac{8}{3}-\frac{4}{3}=\frac{11}{3}.$$

Equality is obtained for $a=b=c$.

Solution 6 by Nicusor Zlota, "Traian Vuia" Technical College, Focsani, Romania

The equality given in the statement of the problem is equivalent to

$$\frac{a^2}{b^2}+\frac{b^2}{c^2}+\frac{c^2}{a^2}-3 \geq \frac{2\left(a+b^3+c^3\right)-2abc}{3(a^3+b^3+c^3)}, \text{ or}$$

$$\frac{(a^2-b^2)^2}{a^2b^2}+\frac{(a^2-b^2)(c^2-b^2)}{a^2c^2} \geq \frac{2(a+b+c)((a-b)^2+(c-a)(c-b))}{3(a^3+b^3+c^3)}, \text{ or}$$

$$(a-b)^2\left(\frac{(a+b)^2}{a^2b^2}-\frac{2(a+b+c)}{3(a^3+b^3+c^3)}\right)+(c-a)(c-b)\left(\frac{(a+c)(b+c)}{a^cc^2}-\frac{2(a+b+c)}{3(a^3+b^3+c^3)}\right) \geq 0.$$

Finally, we only need to prove that

$$\frac{(a+b)^2}{a^2b^2}-\frac{2(a+b+c)}{3(a^3+b^3+c^3)} \geq \frac{(a+b)^2}{a^2b^2}-\frac{2(a+b+\frac{a+b}{2})}{3(a^3+b^3)}=\frac{(a+b)^2}{a^2b^2}-\frac{1}{a^2-ab+b^2}>0$$

and

$$\frac{(a+c)(b+c)}{a^2c^2}-\frac{2(a+b+c)}{3(a^3+b^3+c^3)} \geq \frac{ab}{a^2b^2}-\frac{2(a+b+\frac{a+b}{2})}{3(a^3+b^3)}=\frac{1}{ab}-\frac{1}{a^2-ab+b^2} \geq 0.$$

Also solved by Ed Gray, Highland Beach, FL; Kee-Wai Lau, Hong Kong, China; Daniel Văcaru, Pitesti, Romania, and the proposer.

5520: *Proposed by Raquel León (student) and Angel Plaza, University of Las Palmas de Gran Canaria, Spain*

Let n be a positive integer. Prove that

$$\sum_{k=0}^{2n}\binom{2n+k}{k}\binom{2n}{k}\frac{(-1)^k}{2^k}\frac{1}{k+1}=0.$$

Solution 1 by Kee-Wai Lau, Hong Kong, China

We first need the following well-known indentities:

$$\sum_{k=0}^{\infty}\binom{2k}{k}t^k = \frac{1}{\sqrt{1-4t}}, \tag{1}$$

$$(1-t)^{-n-1} = \sum_{m=0}^{\infty}\binom{n+m}{n}t^m, \tag{2}$$

for $|t| < \frac{1}{4}$.

Let x, y be real numbees satisfying $|x| \le \frac{1}{2}$ and $|y| < \frac{1}{4}$. By substituting $t = xy(1-y)^{-2}$ into (1) and then using (2), we have

$$\begin{aligned}
(1-2(1+2x)y+y^2)^{1/2} &= (1-y)^{-1}\left(1-4xy(1-y)^{-2}\right)^{-1/2} \\
&= \sum_{k=0}^{\infty}\binom{2k}{k}x^k y^k\,(1-y)^{-2k-1} \\
&= \sum_{k=0}^{\infty}\sum_{m=0}^{\infty}\binom{2k}{k}\binom{2k+m}{2k}x^k y^{m+k}
\end{aligned}$$

Replacing y by $-y$ we have

$$(1+2(1+2x)y+y^2)^{-1/2} = \sum_{k=0}^{\infty}\sum_{m=0}^{\infty}(-1)^{m+k}\binom{2k}{k}\binom{2k+m}{2k}x^k y^{m+k}.$$

Hence,

$$\begin{aligned}
(1-2(1+2x)y+y^2)^{-1/2} &+ (1+2(1+2x)y+y^2)^{-1/2} \\
&= 2\sum_{k=0}^{\infty}\sum_{n=0}^{\infty}\binom{2n+k}{2k}\binom{2k}{k}x^k y^{2n} \\
&= 2\sum_{k=0}^{\infty}\sum_{n=0}^{\infty}\binom{2n+k}{2k}\binom{2n}{k}x^k y^{2n}
\end{aligned}$$

Thus,

$$\begin{aligned}
&\sum_{n=0}^{\infty}\left(\sum_{k=0}^{2n}\binom{2n+k}{k}\binom{2n}{k}\frac{(-1)^k}{2^k}\frac{1}{k+1}\right)y^{2n} \\
= \;& 2\int_{-1/2}^{0}\sum_{k=0}^{2n}\sum_{n=0}^{\infty}\binom{2n+k}{k}\binom{2n}{k}x^k y^{2n}dx \\
= \;& \int_{-1/2}^{0}\left(1-2(1+2x)y+y^2\right)^{-1/2}dx + \int_{-1/2}^{0}\left(1+2(1+2x)y+y^2\right)^{-1/2}dx
\end{aligned}$$

$$= \frac{\sqrt{1+y^2}-1+y}{2y} + \frac{1+y-\sqrt{1+y^2}}{2y}$$

$$= 1,$$

whenever $|y| < \frac{1}{4}$. It follows that $\sum_{k=0}^{2n} \binom{2n+k}{k}\binom{2n}{k}\frac{(-1)^k}{2^k}\frac{1}{k+1} = 0$ as desired.

Solution 2 by G.C. Greubel, Newport News, VA

Consider the series

$$S_n(x) = \sum_{k=0}^{n} \binom{n+k}{k}\binom{n}{k}\frac{x^k}{k+1}.$$

Now consider the generating function of this series. This will be determined in the following. One component that will be required is the use of Catalan numbers, C_n, and the generating function

$$\sum_{n=0}^{\infty} C_n\, t^n = \frac{1-\sqrt{1-4t}}{2t}. \tag{6}$$

With in mind, then:

$$\begin{aligned}
\sum_{n=0}^{\infty} S_n(x)\, t^n &= \sum_{n=0}^{\infty}\sum_{k=0}^{n} \binom{n+k}{k}\binom{n}{k}\frac{x^k\, t^n}{k+1} \\
&= \sum_{n,k=0}^{\infty} \binom{n+2k}{k}\binom{n+k}{k}\frac{(xt)^k\, t^n}{k+1} \\
&= \sum_{n,k=0}^{\infty} \frac{(n+2k)!\,(xt)^k\, t^n}{n!\,k!\,(k+1)!} \\
&= \sum_{k=0}^{\infty} \frac{(2k)!\,(xt)^k}{k!\,(k+1)!}\cdot\sum_{n=0}^{\infty}\frac{(2k+1)_n\, t^n}{n!} \\
&= \sum_{k=0}^{\infty} \frac{(2k)!\,(xt)^k}{k!\,(k+1)!}\,(1-t)^{-2k-1} \\
&= \frac{1}{1-t}\sum_{k=0}^{\infty} C_k \left(\frac{xt}{(1-t)^2}\right)^k \\
&= \frac{1}{1-t}\,\frac{(1-t)^2}{2xt}\left(1-\sqrt{1-\frac{4xt}{(1-t)^2}}\right) \\
&= \frac{1}{2xt}\left(1-t-\sqrt{1-2(1+2x)t+t^2}\right).
\end{aligned}$$

Since a generating function of the series has been established consider the reduction when $x=-1/2$. This reduces to

$$\sum_{n=0}^{\infty} S_n\left(-\frac{1}{2}\right) t^n = 1-\frac{1-\sqrt{1+t^2}}{t}. \tag{7}$$

which, by use of (1), becomes

$$\sum_{n=0}^{\infty} S_n\left(-\frac{1}{2}\right) t^n = 1+\sum_{n=0}^{\infty}\frac{(-1)^n\,C_n}{2^{2n+1}}\,t^{2n+1}.$$

By considering even and odd terms it can be stated that

$$S_m\left(-\frac{1}{2}\right) = \begin{cases} 1 & m=0 \\ \frac{(-1)^n\,C_n}{2^{2n+1}} & m=2n+1 \\ 0 & m=2n, n\geq 1 \end{cases}.$$

This leads to

$$\sum_{k=0}^{2n}\binom{2n+k}{k}\binom{2n}{k}\frac{(-1)^k}{2^k\,(k+1)} = 0$$
$$\sum_{k=0}^{2n+1}\binom{2n+k+1}{k}\binom{2n+1}{k}\frac{(-1)^k}{2^k\,(k+1)} = \frac{(-1)^n\,C_n}{2^{2n+1}}.$$

Editor's comment: The proposers of this problem also used the notion of a generating function They stated: "We will show that the generating function of the sequence $(a_m)_{m\geq 0}$ with $a_m=\sum_{k=0}^{m}\binom{m+k}{k}\binom{m}{k}\left(\frac{-1}{2}\right)^k\frac{1}{k+1}$ only has odd terms, from where the result follows."

Also solved by Michele Bataille, Rouen, France; Ed Gray, Highland Beach, FL; Moti Levi, Rehovot, Israel; Albert Stadler, Herrliberg, Switzerland, and the proposer.

5521: *Proposed by José Luis Díaz-Barrero, Barcelona Tech, Barcelona, Spain*

Let $a>0$ be a real number. If f is an odd non-constant real function having second derivative in the interval $[-a,a]$ and $f'(-a)=f'(a)=0$, then prove that there exists a point $c\in(-a,a)$ such that

$$\frac{1}{2}\,f''(c)\geq\frac{|f(a)|}{a^2}$$

Solution 1 by Michel Bataille, Rouen, France

We shall use the following lemma: Let u,v be real numbers with $u<v$ and let $g:[u,v]\to R$. If g has a second derivative in $[u,v]$ and $g'(u)=g'(v)=0$, then for some $c_0\in(u,v)$,

$$|g''(c_0)|\geq\frac{4|g(v)-g(u)|}{(v-u)^2}.$$

To deduce the required result, we take $u=-a, v=a, g=f$; because f is odd, this yields $|f''(c_0)| \geq \dfrac{4|f(a)-f(-a)|}{(a+a)^2} = \dfrac{2|f(a)|}{a^2}$ for some $c_0 \in (u,v)$. If $f''(c_0) \geq 0$, we take $c=c_0$. Otherwise, we have $f''(-c_0) = -f''(c_0) = |f''(c_0)|$ (note that f'' is odd) and we take $c=-c_0$.

Proof of the lemma. Let $m = \dfrac{u+v}{2}$; for some c_1, c_2 in the interval (u,v), we have

$$g(m)-g(u) = (m-u)g'(u) + \frac{(m-u)^2}{2}g''(c_1) = \frac{(v-u)^2}{8}g''(c_1)$$

and

$$g(m)-g(v) = (m-v)g'(v) + \frac{(m-v)^2}{2}g''(c_2) = \frac{(v-u)^2}{8}g''(c_2)$$

(from the Taylor-Lagrange formula), hence

$$\begin{aligned} |g(v)-g(u)| \leq |g(m)-g(v)| + |g(m)-g(u)| &\leq \frac{(v-u)^2}{8}(|g''(c_1)| + |g''(c_2)|) \\ &\leq \frac{(v-u)^2}{4}\max(|g''(c_1)|, |g''(c_2)|). \end{aligned}$$

Now, taking $c_0 = c_1$ if $|g''(c_1)| \geq |g''(c_2)|$ and $c_0 = c_2$ otherwise, we obtain $|g''(c_0)| \geq \dfrac{4|g(v)-g(u)|}{(v-u)^2}$.

Note. The hypothesis f non constant is not necessary: if f is odd and constant, then f is the zero function and the result remains true.

Solution 2 by Moti Levy, Rehovot, Israel

Let us assume that the statement $\exists c \in (-a,a)$ such that $f''(c) \geq \frac{2}{a^2}|f(a)|$ is false, then we have

$$f''(x) < \frac{2}{a^2}|f(a)| \quad for\ all\ x \in (-a,a). \tag{1}$$

The second derivative of an odd function is odd, i.e., $f''(x)$ is an odd function,

$$-f''(x) = f''(-x). \tag{2}$$

Inequality (1) is valid if we replace x by $-x$, hence

$$f''(-x) < \frac{2}{a^2}|f(a)| \quad for\ all\ x \in (-a,a). \tag{3}$$

By (2) and (3) we have

$$f''(x) > -\frac{2}{a^2}|f(a)|. \tag{4}$$

Equations (1) and (4) imply that that

$$\left|f''(x)\right| < \frac{2}{a^2}|f(a)| \quad for\ all\ x \in (-a,a). \tag{5}$$

By integration by parts,

$$f(t) - f(-a) = \int_{-a}^{t} f'(x)\,dx = xf'(x)\Big]_{-a}^{t} - \int_{-a}^{t} xf''(x)\,dx. \tag{6}$$

$$= tf'(t) - \int_{-a}^{t} xf''(x)\,dx.$$

Setting $t = a$ in (6),

$$f(a) - f(-a) = af'(a) - \int_{-a}^{a} xf''(x)\,dx.$$

Noting that $f(a) = -f(-a)$ and $f'(a) = 0$, we get

$$2f(a) = -\int_{-a}^{a} xf''(x)\,dx$$

and by taking the absolute value of both sides,

$$2|f(a)| = \left|\int_{-a}^{a} xf''(x)\,dx\right| \le \int_{-a}^{a} |x|\left|f''(x)\right| dx.$$

Now we use (5) and $\int_{-a}^{a} |x|\,dx = a^2$ to obtain

$$\int_{-a}^{a} |x|\left|f''(x)\right| dx < \frac{2}{a^2}|f(a)| \int_{-a}^{a} |x|\,dx = 2|f(a)|.$$

We arrived at the absurd $|f(a)| < |f(a)|$, therefore our assumption is false and we deduce that indeed there exists a point $c \in (-a, a)$ such that $f''(c) \ge \dfrac{2}{a^2}|f(a)|$.

Solution 3 by Ioannis D. Sfikas, National and Kapodistrian University of Athens, Greece

We find $c \in (-a, a)$ such that $\dfrac{1}{2}f''(c) = \dfrac{|f(a)|}{a^2}$. By Taylor's theorem, there exists $c_1 \in (0, a)$ such that:

$$0 = f(0) = f(a) - f'(a)a + \frac{f''(c_1)}{2}a^2 = f(a) + \frac{f(c_1)}{2}a^2,$$

which gives: $\dfrac{f''(c_1)}{2} = -\dfrac{f(a)}{a^2}$. Similarly, there exists $c_2 \in (-a, 0)$, such that:

$$0 = f(0) = f(-a) + f'(-a)a + \frac{f''(c_2)}{2}a^2 = -f(-a) + \frac{f''(c_2)}{2}a^2,$$

which gives: $\dfrac{f''(c_2)}{2} = \dfrac{f(a)}{a^2}$. So if $f(a) > 0$, then $\dfrac{f''(c_2)}{2} = \dfrac{|f(a)|}{a^2}$, and if $f(a) < 0$, then $\dfrac{f''(c_1)}{2} = \dfrac{|f(a)|}{a^2}$.

Solution 4 by Ulrich Abel, Technische Hochschule Mittelhessen, Germany

Since f is odd on $[-a, +a]$ we have

$$2f(a) = f(a) - f(-a) = \int_{-a}^{a} f'(t)\,dt.$$

By the condition $f'(a) = f'(-a) = 0$, integration by parts yields

$$2f(a) = -\int_{-a}^{a} t f''(t)\, dt.$$

If we assume that $f''(t) < 2a^{-2}|f(a)|$, for all $t \in (-a, +a)$, it would follow the contradiction

$$2|f(a)| = \left| -\int_{-a}^{a} t f''(t)\, dt \right| < 2a^{-2}|f(a)| \int_{-a}^{a} |t|\, dt < 2|f(a)|.$$

Remark: In my opinion, it is not necessary to propose that f is non-constant. The only constant odd function is the zero function. In this case the inequality $f''(c) \geq 2a^{-2}|f(a)|$ is valid, for all $c \in (-a, +a)$.

Also solved by Kee Wai Lau, Hong Kong, China; Paolo Perfetti, Department of Mathematics, Tor Vegata University, Rome, Italy; Albert Stadler, Herrliberg, Switzerland, and the proposer.

5522: *Proposed by Ovidiu Furdui and Cornel Vălean from Technical University of Cluj-Napoca, Cluj-Napoca, Romania and Timiş, Romania, respectively*

Calculate

$$\int_0^1 \int_0^1 \frac{\log(1-x) - \log(1-y)}{x-y} dxdy.$$

Solution 1 by G.M. Greubel, Newport News, VA

First consider the integrand by expanding the logarithms into power series as follows.

$$\begin{aligned} f(x,y) &= \frac{\ln(1-x) - \ln(1-y)}{x-y} \\ &= -\sum_{n=1}^{\infty} \frac{1}{n} \frac{x^n - y^n}{x-y} \\ &= -\sum_{n=1}^{\infty} \sum_{r=0}^{n-1} \frac{1}{n} x^{n-r-1} y^r. \end{aligned}$$

Now, integration with respect to x and y yields

$$\begin{aligned} I &= \int_0^1 \int_0^1 \frac{\ln(1-x) - \ln(1-y)}{x-y}\, dx\, dy \\ &= -\sum_{n=1}^{\infty} \sum_{r=0}^{n-1} \frac{1}{n} \int_0^1 \int_0^1 x^{n-r-1} y^r\, dx\, dy \\ &= -\sum_{n=1}^{\infty} \sum_{r=0}^{n-1} \frac{1}{n(r+1)(n-r)} \\ &= -\sum_{n=1}^{\infty} \frac{S_n}{n}, \end{aligned}$$

where S_n is the sum, and evaluation, given by

$$\begin{aligned}
S_n &= \sum_{r=0}^{n-1} \frac{1}{(r+1)(n-r)} \\
&= \frac{1}{n+1} \sum_{r=0}^{n-1} \left(\frac{1}{r+1} + \frac{1}{n-r} \right) \\
&= \frac{1}{n+1} \left(\sum_{r=1}^{n} \frac{1}{r} + \sum_{r=0}^{n-1} \frac{1}{n-r} \right) \\
&= \frac{2\,H_n}{n+1}.
\end{aligned}$$

Here, H_n denotes the harmonic number. Returning to the integral it is determined that

$$\begin{aligned}
I &= -\sum_{n=1}^{\infty} \frac{2\,H_n}{n\,(n+1)} \\
&= 2 \sum_{n=1}^{\infty} \left(\frac{H_n}{n+1} - \frac{H_n}{n} \right) \\
&= 2 \left(\sum_{n=2}^{\infty} \frac{H_n - \frac{1}{n}}{n} - \sum_{n=1}^{\infty} \frac{H_n}{n} \right) \\
&= -2\,\zeta(2) = -\frac{\pi^2}{3}.
\end{aligned}$$

It can now be stated that

$$\int_0^1 \int_0^1 \frac{\ln(1-x) - \ln(1-y)}{x-y}\, dx\, dy = -2\,\zeta(2) = -\frac{\pi^2}{3}.$$

Solution 2 by Ioannis D. Sfikas, National and Kapodistrian University of Athens, Greece

We have

$$\begin{aligned}
I &= \int_0^1 \int_0^1 \frac{\log(1-x) - \log(1-y)}{x-y} dxdy = -\int_0^1 \int_0^1 \frac{\sum\limits_{n=1}^{+\infty} \frac{x^n}{n} - \sum\limits_{n=1}^{+\infty} \frac{y^n}{n}}{x-y} dxdy \\
&= -\int_0^1 \int_0^1 \frac{1}{x-y} \sum_{n=1}^{+\infty} \frac{x^n - y^n}{n} dxdy \\
&= -\int_0^1 \int_0^1 \sum_{n=1}^{+\infty} \frac{(x-y)\left(x^{n-1} + x^{n-2}y + \cdots + xy^{n-2} + y^{n-1}\right)}{n(x-y)} dxdy \\
&= -\int_0^1 \int_0^1 \sum_{n=1}^{+\infty} \frac{1}{n} \left(x^{n-1} + x^{n-2}y + \cdots + xy^{n-2} + y^{n-1}\right) dxdy
\end{aligned}$$

$$= -\int_0^1 \left[\sum_{n=1}^{+\infty}\frac{1}{n}\int_0^1 \left(x^{n-1}+x^{n-2}y+\cdots+xy^{n-2}+y^{n-1}\right)dx\right]dy$$

$$= -\int_0^1 \sum_{n=1}^{+\infty}\frac{1}{n}\left[\frac{1}{n}+\frac{y}{n-1}+\cdots+\frac{y^{n-2}}{2}+y^{n-1}\right]dy$$

$$= -\sum_{n=1}^{+\infty}\frac{1}{n}\int_0^1 \frac{1}{n}\left[\frac{1}{n}+\frac{y}{n-1}+\cdots+\frac{y^{n-2}}{2}+y^{n-1}\right]dy$$

$$= -\sum_{n=1}^{+\infty}\frac{1}{n}\left[\frac{1}{n}+\frac{1}{2(n-1)}+\frac{1}{3(n-2)}+\cdots+\frac{1}{3(n-2)}+\frac{1}{2(n-1)}+\frac{1}{n}\right]$$

$$= -\sum_{n=1}^{+\infty}\frac{2}{n}\left[\frac{1}{n}+\frac{1}{2(n-1)}+\frac{1}{3(n-2)}+\cdots\right]$$

$$= -\sum_{n=1}^{+\infty}\frac{2}{n^2}-\sum_{n=1}^{+\infty}\frac{1}{n(n-1)}-\sum_{n=1}^{+\infty}\frac{2}{3n(n-2)}-\cdots$$

$$\cong -\frac{\pi^2}{3}.$$

So,

$$I \cong -\frac{\pi^2}{3} \cong -3.28986813.$$

Solution 3 by Bruno Salgueiro Fanego, Viveiro, Spain

$$\int_0^1\int_0^1 \frac{\log(1-x)-\log(1-y)}{x-y}dxdy = \int_0^1\int_0^1 \frac{-\sum_{n=1}^{\infty}\frac{x^n}{n}+\sum_{n=1}^{\infty}\frac{y^n}{n}}{x-y}dxdy$$

$$= \int_0^1\int_0^1 \frac{\sum_{n=1}^{\infty}\frac{y^n-x^n}{n}}{x-y}dxdy$$

$$= -\int_0^1\int_0^1 \sum_{n=1}^{\infty}\frac{1}{n}\frac{y^n-x^n}{y-x}dxdy$$

$$= -\int_0^1\int_0^1 \sum_{n=1}^{\infty}\frac{1}{n}\sum_{k=0}^{n-1}x^k y^{n-k-1}dxdy$$

$$
\begin{aligned}
&= -\sum_{n=1}^{\infty}\frac{1}{n}\sum_{k=0}^{n-1}\int_0^1 x^k dx\int_0^1 y^{n-k-1}dy \\
&= -\sum_{n=1}^{\infty}\frac{1}{n}\sum_{k=0}^{n-1}\frac{1}{k+1}\frac{1}{n-k} \\
&= -\sum_{n=1}^{\infty}\frac{1}{n}\sum_{k=0}^{n-1}\frac{1}{n+1}\left(\frac{1}{k+1}+\frac{1}{n-k}\right) \\
&= -\sum_{k=0}^{\infty}\frac{1}{n(n+1)}\sum_{k=0}^{n-1}\left(\frac{1}{k+1}+\frac{1}{n-k}\right) \\
&= -\sum_{n=1}^{\infty}\frac{1}{n(n+1)}\left(-\sum_{k=0}^{n-1}\frac{1}{k+1}+\sum_{k=0}^{n-1}\frac{1}{n-k}\right) \\
&= -\sum_{n=1}^{\infty}\frac{1}{n(n+1)}\left(\sum_{k=0}^{n-1}\frac{1}{k+1}+\sum_{k=0}^{n-1}\frac{1}{n-k}\right) \\
&= -\sum_{n=1}^{\infty}\frac{1}{n(n+1)}\left(H_n+H_n\right) \\
&= -2\sum_{n=1}^{\infty}\frac{H_n}{n(n+1)} = -\frac{\pi^2}{3},
\end{aligned}
$$

where this last identity follows from entry 55.2.7 on page 361 of the book by E.R. Hansen *A Table of Series and Products*, Prentice-Hall, Englewood Cliffs, NJ, 1975; $\displaystyle\sum_{n=2}^{\infty}\frac{1}{n(x+n)}\left[\psi(x+n)-\psi(x+1)\right]=\frac{1}{x^2}\left[\frac{\pi^2 x}{6}-\gamma-\psi(x+1)\right]$ is valid for all positive integers, and in this particular case, when $x=1$.

Solution 4 by Moti Levy, Rehovot, Israel

Define

$$I(\alpha) := \int_0^1\int_0^1 \frac{\ln(1-\alpha x)-\ln(1-\alpha y)}{x-y}dxdy.$$

We will differentiate under the integral sign.

$$
\begin{aligned}
\frac{dI}{d\alpha} &= \int_0^1\int_0^1 \frac{\partial\left(\frac{\ln(1-\alpha x)-\ln(1-\alpha y)}{x-y}\right)}{\partial\alpha}dxdy = -\int_0^1\int_0^1\frac{1}{(1-\alpha x)(1-\alpha y)}dxdy \\
&= -\left(\int_0^1\frac{1}{1-\alpha x}dx\right)\left(\int_0^1\frac{1}{1-\alpha y}dy\right) = -\left(\frac{\ln(1-\alpha)}{\alpha}\right)^2, \quad \alpha\le 1.
\end{aligned}
$$

$$I(\alpha) = \int_0^{\alpha} \left(-\frac{1}{t^2}\right) \ln^2(1-t)\,dt + \text{constant}$$

But since $I(0) = \int_0^1 \int_0^1 \frac{\ln(1)-\ln(1)}{x-y} dxdy = 0$, then the constant is zero.
Setting $\alpha = 1$, we get

$$I(1) = \int_0^1 \ln^2(1-t) \left(\frac{-1}{t^2}\right) dt$$

By integration by parts,

$$I(1) = -2 \int_0^1 \frac{1}{1-t} \frac{1}{t} \ln(1-t)\,dt.$$

By change of variable $x = -\ln(1-t)$,

$$I(1) = -2 \int_0^{\infty} \frac{x}{e^x - 1} dx = -2\zeta(2) = -\frac{\pi^2}{3}.$$

Excerpt from Richard P. Feynman, the American theoretical physicist, book "Surely You're Joking, Mr. Feynman:"

"One thing I never did learn was contour integration. I had learned to do integrals by various methods show in a book that my high school physics teacher Mr. Bader had given me. The book also showed how to differentiate parameters under the integral sign - It's a certain operation. It turns out that's not taught very much in the universities; they don't emphasize it. But I caught on how to use that method, and I used that one damn tool again and again. So because I was self-taught using that book, I had peculiar methods of doing integrals. The result was that, when guys at MIT or Princeton had trouble doing a certain integral, it was because they couldn't do it with the standard methods they had learned in school. If it was contour integration, they would have found it; if it was a simple series expansion, they would have found it. Then I come along and try differentiating under the integral sign, and often it worked. So I got a great reputation for doing integrals, only because my box of tools was different from everybody else's, and they had tried all their tools on it before giving the problem to me."

Also solved by Michel Bataille, Rouen, France; Ed Gray, Highland Beach, FL; Kee -Wai Lau, Hong Kong, China; Paolo Perfetti, Department of Mathematics, Tor Vergata University, Rome, Italy; Albert Stadler, Herrliberg, Switzerland, and the proposer.

Addendum

A late solution by G.C. Greubel of Newport News, VA was received for problem 5515.