Problems

Ted Eisenberg, Section Editor

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This section of the Journal offers readers an opportunity to exchange interesting mathematical problems and solutions. Please send them to Ted Eisenberg, Department of Mathematics, Ben-Gurion University, Beer-Sheva, Israel or fax to: 972-86-477-648. Questions concerning proposals and/or solutions can be sent e-mail to <eisenbt@013.net>. Solutions to previously stated problems can be seen at <http://www.ssma.org/publications>.

Solutions to the problems stated in this issue should be posted before
October 15, 2015

- **5355:** *Proposed by Kenneth Korbin, New York, NY*

  Find the area of the convex cyclic pentagon with sides

$$(13,\ 13,\ 12\sqrt{3}+5,\ 20\sqrt{3},\ 12\sqrt{3}-5).$$

- **5356:** *Proposed by Kenneth Korbin, New York, NY*

  For every prime number p there is a circle with diameter $4p^4+1$. In each of these circles, it is possible to inscribe a triangle with integer length sides and with area

$$(8p^3)(p+1)(p-1)(2p^2-1).$$

  Find the sides of the triangles if $p=2$ and if $p=3$.

- **5357:** *Proposed by Neculai Stanciu, "George Emil Palade" School, Buzău, Romania and Titu Zvonaru, Comănesti, Romania*

  Determine all triangles whose side-lengths are positive integers (of which at least one is a prime number), whose semiperimeter is a positive integer, and whose area is equal to its perimeter.

- **5358:** *Proposed by Arkady Alt, San Jose, CA*

  Prove the identity $\displaystyle\sum_{k=1}^{m} k\binom{m+1}{k+1}r^{k+1} = (r+1)^m(mr-1)+1.$

5359: *Proposed by José Luis Díaz-Barrero, Barcelona Tech, Barcelona, Spain.*

Let a,b,c be positive real numbers. Prove that

$$\sqrt[4]{15a^3b+1}+\sqrt[4]{15b^3c+1}+\sqrt[4]{15c^3a+1}\le\frac{63}{32}(a+b+c)+\frac{1}{32}\left(\frac{1}{a^3}+\frac{1}{b^3}+\frac{1}{c^3}\right).$$

- **5360:** *Proposed by Ovidiu Furdui, Technical University of Cluj-Napoca, Cluj-Napoca, Romania*

Let $n \geq 1$ be an integer and let

$$I_n = \int_0^\infty \frac{\arctan x}{(1+x^2)^n} dx.$$

Prove that

(a) $\displaystyle\sum_{n=1}^{\infty} \frac{I_n}{n} = \zeta(2)$;

(b) $\displaystyle\int_0^\infty \arctan x \ln\left(1 + \frac{1}{x^2}\right) dx = \zeta(2)$.

Solutions

- **5337:** *Proposed by Kenneth Korbin, New York, NY*

Given convex quadrilateral $ABCD$ with sides,

$$\begin{aligned} \overline{AB} &= 1 + 3\sqrt{2} \\ \overline{BC} &= 6 + 4\sqrt{2} \text{ and} \\ \overline{CD} &= -14 + 12\sqrt{2}. \end{aligned}$$

Find side $\overline{AD}$ so that the area of the quadrilateral is maximum.

Solution 1 by Bruno Salgueiro Fanego, Viveiro, Spain

In the published solution to part (b) of problem 787 *Journal Crux Mathematicorum*, $1984, 10(2), 56-58$, it is proved that given three sides $\overline{AB}, \overline{BC}$, and $\overline{CD}$, the area of the quadrilateral $ABCD$ is maximum if, and only if, the length of the fourth side, $\overline{AD}$ is the diameter of the circle passing through B and C, and a root of the polynomial

$$\begin{aligned} x^3 - \left(\overline{AB}^2 + \overline{BC}^2 + \overline{CD}^2\right) - 2\overline{AB} \cdot \overline{BC} \cdot \overline{CD} &= 0. \text{ That is} \\ x^3 - \left(571 - 282\sqrt{2}\right) x - 206 - 104\sqrt{2} &= 0, \end{aligned}$$

whose only real positive root is $x = 7 + 5\sqrt{2}$; so $\overline{AD} = 7 + 5\sqrt{2}$.

Solution 2 by Albert Stadler, Herrliberg, Switzerland

The cyclic quadrilateral has the maximal area among all quadrilaterals having the same sequence of side lengths. This is a corollary to Bretschneider's formula (http://en.wikipedia.org/wiki/Bretschneider%27s_formula). It can also be proved using calculus (see([1]). The area of a cyclic quadrilateral with side a, b, c, d is given by Brahmagupta's formula

$$A = \sqrt{(s-a)(s-b)(s-c)(s-d)} \text{ where } s = (a+b+c+d)/2.$$

So if $a = 1 + 3\sqrt{2},\ b = 6 + 4\sqrt{2}$, and $c = -14 + 12\sqrt{2}$ then

$$16A^2 = \left(d - 9 + 13\sqrt{2}\right)\left(d - 19 + 11\sqrt{2}\right)\left(d + 21 - 5\sqrt{2}\right)\left(-d - 7 + 19\sqrt{2}\right).$$

This is a polynomial of degree four whose extremal points are located at the zeros of its derivative. Brute force shows that the extremal points are

$$\begin{aligned} d_1 &= 7 + 5\sqrt{2} > 0, \\ d_2 &= \frac{-7 - 5\sqrt{2} + \sqrt{1987 - 1338\sqrt{2}}}{2} < 0, \\ d_3 &= \frac{-7 - 5\sqrt{2} - \sqrt{1987 - 1338\sqrt{2}}}{2} < 0. \end{aligned}$$

So $\overline{AD} = d_1 = 7 + 5\sqrt{2}$

References: (1) Thomas, Peter, "Maximizing the Area of a Quadrilateral," The College Mathematics Journal, Vol 34. No 4 (September 2003), pp. 315-316.

Solution 3 by Kee-Wai Lau, Hong Kong, China

We show that the area of the quadrilateral is maximum when $\overline{AD} = 7 + 5\sqrt{2}$.

Let $\overline{AD} = x, s$ be the semiperimeter and Δ the area of the quadrilateral. Since the length of any side of a quadrilateral must be less than the sum of the lengths of the other three sides, we have $19 - 112\sqrt{2} < x < -7 + 19\sqrt{2}$. It is well known that

$$\Delta \le \sqrt{\left(s - \overline{AB}\right)\left(s - \overline{BC}\right)\left(s - \overline{AB}\right)\left(s - \overline{AD}\right)},$$

so that $16\Delta^2 \le f(x)$, where

$$f(x) = -x^4 + 2\left(571 - 282\sqrt{2}\right)x^2 + 32(27 + 13\sqrt{2})x - 454337 + 314940\sqrt{2}.$$

It can be checked readily by differentiation that for $19 - 11\sqrt{2} < x < -7 + 19\sqrt{2},\ f(x)$ attains its unique maximum at $x = 7 + 5\sqrt{2}$. Hence

$$\Delta \le \frac{\sqrt{f(7 + 5\sqrt{2})}}{4} = 14\sqrt{-137 + 106\sqrt{2}}.$$

It can also be checked readily that the area of the quadrilateral with sides $\overline{AB} = 1 + 3\sqrt{2},\ \overline{BC} = 6 + 4\sqrt{2},\ \overline{CD} = -14 + 12\sqrt{2},\ \overline{AD} = 7 + 5\sqrt{2}$,

$$\overline{AC} = \sqrt{7\left(-55 + 58\sqrt{2}\right)} \text{ in fact equals } 14\sqrt{-137 + 106\sqrt{2}}.$$

This completes the solution.

Also solved by Arkardy Alt, San Jose, CA; Ed Gray, Highland Beach, FL; Henry Ricardo, New York Math Circle, NY; David Stone and John Hawkins, Georgia Southern University, Statesboro, GA, and the proposer.

- **5338:** *Proposed by Arkady Alt, San Jose, CA*

Determine the maximum value of

$$F\left(x,y,z\right)=\min\left\{\frac{|y-z|}{|x|},\frac{|z-x|}{|y|},\frac{|x-y|}{|z|}\right\},$$

where x,y,z are arbitrary nonzero real numbers.

Solution 1 by Kee-Wai Lau, Hong Kong, China

We show that the maximum value of $F(x,y,z)$ is 1.

We first prove that

$$F(x,y,z)\le 1, \qquad (1)$$

by showing that at least one of the numbers $\frac{|y-z|}{|x|},\frac{|z-x|}{|y|},\frac{|x-y|}{|z|}$ is less than equal to 1.

Suppose, on the contrary, that all of them are greater than 1. From $\frac{|y-z|}{|x|}>1$, we obtain

$$\left(y-z\right)^2>x^2,\ \text{ or } (x+y-z)(x-y+z)<0. \qquad (2)$$

Similarly from $\frac{|z-x|}{|y|}>1$, and $\frac{|x-y|}{|x|}>1$, we obtain respectively

$$(x-y-z)(x+y-z)>0, \qquad (3)$$

and

$$(x-y-z)(x-y+z)>0. \qquad (4)$$

Multiplying (2), (3) and (4) together. we obtain

$$\left(x+y-z\right)^2\left(x-y+z\right)^2\left(x-y-z\right)^2<0,$$

which is false. Thus (1) holds. Since $F(2,-1,1)=1$, we see that the maximum value of $F(x,y,z)$ is 1 indeed.

Solution 2 by Albert Stadler, Herrliberg, Switzerland

We claim that the maximum value equals 1.

Let $x>0$. Then $F(x,x+1,-1)=\min\left\{\frac{x+2}{x},\frac{x+1}{x+1},\frac{1}{1}\right\}=1.$

So the maximum value is ≥ 1.

Suppose the maximum value is >1. Then there is a triple (x,y,z) with

$$|y-z|>|x|,\ |z-x|>|y|,\ |x-y|>|z|. \qquad (1)$$

By cyclic symmetry, we can assume that $x\le\min(y,z)$.

Assume first that $x\le y\le z$. Then (1) reads as

$z-y>|x|,\ z-x>|y|,\ y-x>|z|$. So $z-x=(z-y)+(y-x)>|x|+|z|\ge z-x$

which is a contradiction.

Assume next that $x \le z \le y$. Then (1) reads as

$y - z > |x|,\ z - x > |y|,\ y - x > |z|$. So $y - x = (y - z) + (z - x) > |x| + |y| \ge y - x$, which is a contradiction.

This concludes the proof.

Solution 3 by Paolo Perfetti, Department of Mathematics, "Tor Vergata" Unversity, Rome Italy

Answer: 1

The symmetry of $F(x, y, z)$ allows us to set $z \le y \le x$. We have two cases:

$$\begin{array}{ll} 1) & 0 < z \le y \le x \text{ and} \\ 2) & z < 0,\ 0 < y \le x. \end{array}$$

Moreover, by observing that $F(x, y, z) = F(-x, -y, -z)$, the case $z \le y < 0$, $x > 0$ is recovered by the case 2) simply changing sign to all the signs and the same happens if $z \le y \le x < 0$.

Now we study the case 1)

$$\frac{|y - z|}{|x|} \le \frac{|x - z|}{|y|} \iff \frac{y - z}{x} \le \frac{x - z}{y} \iff z \le x + y$$

which evidently holds true. Moreover,

$$\frac{|y - z|}{|x|} \le \frac{|x - y|}{|z|} \iff \frac{y - z}{x} \le \frac{x - y}{z} \iff yx + yz \le x^2 + z^2$$

This generates two subcases.

1.1) $0 < z \le y \le x$ and $yx + yz \le x^2 + z^2$. In this case we must find the maximum of the function $\dfrac{y - z}{x}$. We have

$$\frac{y - z}{x} \le \frac{y - z}{y} = 1 - \frac{z}{y} < 1.$$

The value 1 is not attained because $z \ne 0$.

1.2) $0 < z \le y \le x$ and $yx + yz > x^2 + z^2$. In this case we must find the maximum of the function $\dfrac{x - y}{z}$. We have

$$\frac{x - y}{z} < \frac{y - z}{x} \le \frac{y - z}{y} = 1 - \frac{z}{y} < 1.$$

Now we study case 2)

$$F(x, y, z) = \min\left\{\frac{y - z}{x}, \frac{x - z}{y}, \frac{x - y}{-z}\right\}$$

and

$$\frac{y-z}{x} \le \frac{x-z}{y} \iff z \le x+y$$

which evidently holds true.
Moreover,

$$\frac{y-z}{x} \le \frac{z-y}{-z} \iff y \le x+z.$$

This generates two subcases.

2.1) $z < 0$, $0 < y < x$, $y \le x+z$. In this case we must find the maximum of

$$\frac{y-z}{x} \le \frac{x}{x} = 1.$$

The maximum achieved.

2.2) $z < 0$, $0 < y < x$, $y > x+z$. In this case we must find the maximum of

$$\frac{x-y}{-z} \le \frac{x-y}{x-y} = 1.$$

The maximum achieved.

Also solved by Jerry Chu, (student at Saint George's School), Spokane, WA; Ethan Gegner, (student, Taylor University), Upland, IN, and the proposer.

- **5339:** *Proposed by D.M. Bătinetu-Giurgiu, "Matei Basarab" National College, Bucharest, Romania and Neculai Stanciu "George Emil Palade" School, Buzău, Romania*

  Calculate: $\displaystyle\int_0^{\pi/2} \frac{3\sin x + 4\cos x}{3\cos x + 4\sin x} dx.$

  Solution 1 by Haroun Meghaichi (student, University of Science and Technology Houari Boumediene), Algeria

  Consider the general case for $a, b > 0$:

$$I(a,b) = \int_0^{\pi/2} \frac{a\sin x + b\cos x}{b\sin x + a\cos x}\, \mathrm{d}x,$$

Note that the derivative of the denominator (with respect to x) is $b\cos x - a\sin x$, and $\{b\sin x + a\cos x, b\cos x - a\sin x\}$ form a base on $R[\cos x, \sin x]$, then there are $\alpha, \beta \in R$ such that

$$a\sin x + b\cos x = \alpha\left(b\sin x + a\cos x\right) + \beta\left(b\cos x - a\sin x\right), \quad \forall\, x \in R$$

$$\Leftrightarrow \quad b - a\alpha - b\beta = a - b\alpha + a\beta = 0.$$

We can easily solve the system to get $(\alpha, \beta) = \left(\dfrac{2ab}{a^2+b^2}, \dfrac{b^2-a^2}{a^2+b^2}\right)$, then

$$I(a,b) = \frac{1}{a^2+b^2}\int_0^{\pi/2} 2ab + (b^2-a^2)\frac{b\cos x - a\sin x}{b\sin x + a\cos x}\, \mathrm{d}x$$

$$= \frac{1}{a^2+b^2}\left[2abx + (b^2-a^2)\ln|a\cos x + b\sin x|\right]_0^{\pi/2}$$

$$= \frac{1}{a^2+b^2}\left(ab\pi + (b^2-a^2)\ln\frac{b}{a}\right).$$

The proposed integral equals $I(4,3) = I(3,4) = \dfrac{1}{25}\left(12\pi + 7\ln\dfrac{4}{3}\right)$.

Solution 2 by Dionne Bailey, Elsie Campbell, Charles Diminnie, and Andrew Siefker, Angelo State University, San Angelo, TX

We attack the problem by using the classical technique for converting a rational function of $\sin x$ and $\cos x$ into an ordinary rational function. If we set

$$u = \tan\left(\frac{x}{2}\right),$$

then the "half-angle" formulas imply that

$$u^2 = \frac{\sin^2\left(\frac{x}{2}\right)}{\cos^2\left(\frac{x}{2}\right)} = \frac{1-\cos x}{1+\cos x}$$

and hence,

$$\cos x = \frac{1-u^2}{1+u^2}. \tag{1}$$

Also, using (1) and the known identity

$$u = \tan\left(\frac{x}{2}\right) = \frac{\sin x}{1+\cos x},$$

we get

$$\sin x = \frac{2u}{1+u^2}. \tag{2}$$

Finally,

$$du = \sec^2\left(\frac{x}{2}\right)\cdot\frac{1}{2}\,dx = \frac{1}{2}\left[1+\tan^2\left(\frac{x}{2}\right)\right]\,dx = \frac{1+u^2}{2}\,dx,$$

i. e.,

$$dx = \frac{2}{1+u^2}\,du. \tag{3}$$

Since $u=0$ when $x=0$ and $u=1$ when $x=\dfrac{\pi}{2}$, (1), (2), and (3) yield (upon simplification)

$$\begin{aligned}\int_0^{\frac{\pi}{2}} \frac{3\sin x + 4\cos x}{3\cos x + 4\sin x}\,dx &= 4\int_0^1 \frac{2u^2-3u-2}{(3u^2-8u-3)(1+u^2)}\,du \\ &= 4\int_0^1 \frac{2u^2-3u-2}{(3u+1)(u-3)(1+u^2)}\,du. \end{aligned} \tag{4}$$

Then, (4) and the partial fraction expansion

$$\frac{2u^2-3u-2}{(3u+1)(u-3)(1+u^2)} = \frac{12}{25}\cdot\frac{1}{1+u^2} - \frac{7}{50}\cdot\frac{u}{1+u^2} + \frac{21}{100}\cdot\frac{1}{3u+1} + \frac{7}{100}\cdot\frac{1}{u-3}$$

imply that

$$\int_0^{\frac{\pi}{2}} \frac{3\sin x+4\cos x}{3\cos x+4\sin x}\,dx = 4\int_0^1 \frac{2u^2-3u-2}{(3u+1)(u-3)(1+u^2)}\,du$$

$$= \frac{48}{25}\tan^{-1}u\Big]_0^1 - \frac{7}{25}\ln\left(1+u^2\right)\Big]_0^1 + \frac{7}{25}\ln|3u+1|\Big]_0^1 + \frac{7}{25}\ln|u-3|\Big]_0^1$$

$$= \frac{12\pi}{25} - \frac{7}{25}\ln 2 + \frac{7}{25}\ln 4 + \frac{7}{25}\ln 2 - \frac{7}{25}\ln 3$$

$$= \frac{12\pi}{25} + \frac{7}{25}\ln\left(\frac{4}{3}\right)$$

Solution 3 by Ethan Gegner, (student, Taylor University), Upland, IN

The value of the integral is $\dfrac{1}{25}\left(12\pi + 7\log(4/3)\right)$.

Define

$$\begin{aligned} I &= \int_0^{\pi/2} \frac{3\sin x+4\cos x}{3\cos x+4\sin x}dx \\ A &= \int_0^{\pi/2} \frac{\sin x}{3\cos x+4\sin x}dx \\ B &= \int_0^{\pi/2} \frac{\cos x}{3\cos x+4\sin x}dx. \end{aligned}$$

Then

$$I = 3A+4B$$

$$\begin{aligned} I + A - B &= \int_0^{\pi/2} \frac{3\cos x + 4\sin x}{3\cos x + 4\sin x} dx = \frac{\pi}{2} \\ I - 6A &= \int_0^{\pi/2} \frac{-3\sin x + 4\cos x}{3\cos x + 4\sin x} dx = \int_3^4 \frac{1}{u} du = \log(4/3) \end{aligned}$$

Solving this system yields $I = \dfrac{1}{25}\left(12\pi + 7\log(4/3)\right).$

Solution 4 by Bruno Salgueiro Fanego, Viveiro, Spain

Since $\dfrac{d}{dx}\left(ax + b\ln(2\cos x + 4\sin x)\right) = \dfrac{(4a-3b)\sin x + (3a+4b)\cos x}{3\cos x + 4\sin x}$ when $3\cos x + 4\sin x > 0$ and $b \in \Re$, if we take $a, b, \in \Re$ such that $4a - 3b = 3$ and $3a + 4b = 4$, that is, $a = \dfrac{24}{25}$ and $b = \dfrac{7}{25}$, we obtain that $\dfrac{1}{25}\left(24x + 7\ln(3\cos x + 4\sin x)\right)$ is a primitive of $\dfrac{3\sin x + 4\cos x}{3\cos x + 4\sin x}$ in $[0, \pi/2]$, so, by Barrow's rule,

$$\begin{aligned} \int_0^{\pi/2} \frac{3\sin x + 4\cos x}{3\cos x + 4\sin x} dx &= \frac{1}{25}\left(24x + 7\ln(3 + 4())\right)\Big|_0^{\pi/2} \\ &= \frac{1}{25}\left(12x + 7\ln(3\cdot 0 + 4\cdot 1)\right) - \frac{1}{25}\left(24\cdot 0 + 7\ln(31 + 4\cdot 0)\right) \\ &= \frac{12\pi}{25} + \frac{7}{25}\ln\left(\frac{4}{3}\right). \end{aligned}$$

Solution 5 by Brian D. Beasely, Presbyterian College, Clinton, SC

We let $f(x) = 3\sin x + 4\cos x$ and $g(x) = 3\cos x + 4\sin x$. Since $g'(x) = -3\sin x + 4\cos x$, we seek constants A and B such that

$$\frac{f(x)}{g(x)} = A\left(\frac{g'(x)}{g(x)}\right) + B.$$

This produces $A = 7/25$ and $B = 24/25$, so

$$\begin{aligned} \int_0^{\pi/2} \frac{f(x)}{g(x)} dx &= \int_0^{\pi/2} \left[A\left(\frac{g'(x)}{g(x)}\right) + B\right] dx \\ &= A\ln(g(x)) + Bx]_0^{\pi/2} \\ &= A\ln\left(\frac{4}{3}\right) + B\left(\frac{\pi}{2}\right) \\ &= \frac{7}{25}\ln\left(\frac{4}{3}\right) + \frac{12\pi}{25}. \end{aligned}$$

Addendum. We may generalize the above technique to show that

$$\int_0^{\pi/2} \frac{m\sin x + n\cos x}{3\cos x + 4\sin x} dx = A\ln\left(\frac{4}{3}\right) + B\left(\frac{\pi}{2}\right),$$

where $A = (-3m+4n)/25$ and $B = (4m+3n)/25$.

We may further generalize to show that

$$\int_0^{\pi/2} \frac{m\sin x + n\cos x}{p\cos x + q\sin x} dx = A\ln\left|\frac{q}{p}\right| + B\left(\frac{\pi}{2}\right),$$

where $A = (-pm+qn)/(p^2+q^2)$ and $B = (qm+pn)/(p^2+q^2)$, provided we place appropriate restrictions on the values of p and q (to keep $p\cos x + q\sin x \neq 0$ for each x in $[0, \pi/2]$, to avoid $p = 0$ or $q = 0$, etc.).

Also solved by Arkady Alt, San Jose, CA; Andrea Fanchini, Gantú, Italy; Paul M. Harms, North Newton, KS; Ed Gray, Highland Beach, FL; G.C. Greubel, Newport News, VA; Kee-Wai Lau, - Hong Kong, China; Daniel López, Center for Mathematical Sciences, UNAM, Morelia, Mexico; Paolo Perfetti, Department of Mathematics, "Tor Vergata" University, Rome, Italy; Henry Ricardo (two solutions), New York Math Circle, NY; Albert Stadler, Herrliberg, Switzerland; Vu Tran (student, Purdue University),West Lafayette, IN; Nicusor Zlota, "Traian Vuia" Technical College, Focsani, Romania; Titu Zvonaru, Comănesti, Romania, and the proposers.

- **5340:** *Proposed by Oleh Faynshteyn, Leipzig, Germany*

Let a, b and c be the side-lengths, and s the semi-perimeter of a triangle. Show that

$$\frac{a^2+b^2}{(s-c)^2} + \frac{b^2+c^2}{(s-a)^2} + \frac{c^2+a^2}{(s-b)^2} \geq 24.$$

Solution 1 by Ángel Plaza, University of Las Palmas de Gran Canaria, Spain

Changing variables by letting $s-a = x$, $s-b = y$ and $s-c = z$ the proposed inequality is equivalent to the following one, for x, y and z positive real numbers:

$$\sum_{\text{cyclic}} \left(1+\frac{y}{z}\right)^2 + \left(1+\frac{x}{z}\right)^2 \geq 24.$$

The last inequality follows by the power-mean, arithmetic-mean, geometric-mean inequality:

$$\begin{aligned}
\sqrt{\frac{\sum_{\text{cyclic}} \left(1+\frac{y}{z}\right)^2 + \left(1+\frac{x}{z}\right)^2}{6}} &\geq \frac{\sum_{\text{cyclic}} \left(1+\frac{y}{z}\right) + \left(1+\frac{x}{z}\right)}{6} \\
&= 1 + \frac{\sum_{\text{cyclic}} \left(\frac{y}{z}+\frac{x}{z}\right)}{6} \\
&\geq 1 + \sqrt[6]{\prod_{\text{cyclic}} \frac{y}{z}\cdot\frac{x}{z}} \\
&= 2
\end{aligned}$$

from where the result follows, with equality if and only if $x = y = z$, that is if $a = b = c$.

Solution 2 by Nikos Kalapodis, Patras, Greece

$a + b + c = 2s \Longrightarrow a^2 = (s - b + s - c)^2$.
Using the well-known inequality $(x + u)^2 \geq 4xy$ for $x = s - b$ and $y = s - c$ we have

$$(s - b + s - c)^2 \geq 4(s - b)(s - c), \text{ i.e.,}$$

$$a^2 \geq 4(s - b)(s - c) \quad (1)$$

Similarly we obtain,

$$b^2 \geq 4(s - c)(s - a) \qquad (2)$$

$$c^2 \geq 4(s - a)(s - b). \qquad (3)$$

Applying the well known inequality $x^2 + y^2 \geq 2xy$, to (1), (2), and (3) we have

$$\frac{a^2 + b^2}{(s - c)^2} + \frac{b^2 + c^2}{(s - a)^2} + \frac{c^2 + a^2}{(s - b)^2} =$$

$$\left[\left(\frac{a}{s - b}\right)^2 + \left(\frac{a}{s - c}\right)^2\right] + \left[\left(\frac{b}{s - c}\right)^2 + \left(\frac{b}{s - a}\right)^2\right] + \left[\left(\frac{c}{(s - a)}^2 + \left(\frac{c}{(s - b)^2}\right)\right] \geq$$

$$\frac{2a^2}{(s - b)(s - c)} + \frac{2b^2}{(s - c)(s - a)} + \frac{2c^2}{(s - a)(s - b)} \geq 2(4 + 4 + 4) = 24.$$

Solution 3 by Arkady Alt, San Jose, CA

Note that $\sum_{cyc} \frac{a^2 + b^2}{(s - c)^2} \geq 24 \iff \sum_{cyc} \frac{a^2 + b^2}{(a + b - c)^2} \geq 6.$

Since $a^2 \geq a^2 - (b - c)^2 \iff \frac{a^2}{a + b - c} \geq c + a - b$
and

$$b^2 \geq b^2 - (c - a)^2 \iff \frac{b^2}{a + b - c} \geq b + c - a$$

then by AM-GM Inequality we have

$$\sum_{cyc} \frac{a^2}{(a + b - c)^2} \geq \sum_{cyc} \frac{c + a - b}{a + b - c} \geq 3\sqrt[3]{\frac{c + a - b}{a + b - c} \cdot \frac{a + b - c}{b + c - a} \cdot \frac{b + c - a}{c + a - b}} = 3$$

and

$$\sum_{cyc} \frac{b^2}{(a + b - c)^2} \geq \sum_{cyc} \frac{b + c - a}{a + b - c} \geq 3\sqrt[3]{\frac{b + c - a}{a + b - c} \cdot \frac{c + a - b}{b + c - a} \cdot \frac{a + b - c}{c + a - b}} = 3.$$

Thus, $\sum_{cyc} \frac{a^2 + b^2}{(a + b - c)^2} \geq 6.$

Solution 4 by D.M. Bătinetu-Giurgiu, Bucharest, Romania

We shall prove that

$$\frac{xa^m + yb^m}{(s-c)^m} + \frac{xb^m + yc^m}{(s-a)^m} + \frac{xc^m + ya^m}{(s-b)^m} \geq 3\sqrt{xy} \cdot 2^{m+1}, \text{ where } m, x, y > 0.$$

Proof: We denote the area of the triangle by F, its circumradius by R and its inradius by r.

By the AM-GM inequality and taking into account that
$F = sr = \sqrt{s(s-a)(s-b)(s-c)}$ we have that

$$\begin{aligned}
\sum_{cyclic} \frac{xa^m + yb^m}{(s-c)^m} \geq 2\sqrt{xy} \sum_{cyclic} \frac{\left(\sqrt{ab}\right)^m}{(s-c)^m} &\geq 2\sqrt{xy} \cdot 3 \cdot \sqrt[3]{\prod_{cyclic} \frac{\left(\sqrt{ab}\right)^m}{(s-c)^m}} \\
&= 6\sqrt{xy} \cdot \sqrt[3]{\left(\frac{abc}{(s-a)(s-b)(s-c)}\right)^m} \\
&= 6\sqrt{xy} \cdot \sqrt[3]{\frac{(4RF)^m s^m}{(s(s-a)(s-b)(s-c))^m}} \\
&= 6\sqrt{xy} \cdot \sqrt[3]{\frac{4^m R^m F^m s^m}{F^{2m}}} \\
&= 6\sqrt{xy} \cdot \sqrt[3]{\frac{4^m R^m s^m}{F^m}} \\
&= 6\sqrt{xy} \cdot \sqrt[3]{\frac{4^m R^m s^m}{s^m r^m}} \\
&= 6\sqrt{xy} \cdot \sqrt[3]{4^m \left(\frac{R}{r}\right)^m} \\
&\geq {}^{Euler(R \geq 2r)} 6\sqrt{xy} \cdot \sqrt[3]{4^m 2^m} \\
&= 6\sqrt{xy} \cdot \sqrt[3]{2^{3m}} = 6\sqrt{xy} \cdot \sqrt[3]{2^{3m}} = 3\sqrt{xy} 2^{m+1}
\end{aligned}$$

If we take $m = 2$ we obtain a solution to problem 5340.

Solution 5 by Paul M. Harms, North Newton, KS

If $x > 0$, then using calculus we can show that the minimum value of both expressions

$$\begin{cases} x + \dfrac{1}{x} \\ x^2 + \dfrac{1}{x^2} \end{cases}$$

is 2 and occurs at $x = 1$. I will use several substitutions to get the left side of the problem inequality into a form easier to use.

First let $t > 0$ and $r > 0$ such that $a = rc$ and $b = tc$. Then $s = \dfrac{c}{2}(r + t + 1)$ and the left side of the problem inequality is

$$\frac{(r^2 + t^2)}{\left(\dfrac{t + r - 1}{2}\right)^2} + \frac{(t^2 + 1)}{\left(\dfrac{t - r + 1}{2}\right)^2} + \frac{(r^2 + 1)}{\left(\dfrac{r - t + 1}{2}\right)^2}.$$

Now let $\begin{cases} 2H = r + t - 1, \\ 2L = t - r + 1 \\ 2K = r - t + 1. \end{cases}$ Then $\begin{cases} r = H + K \\ t = H + L \\ L = 1 - K \end{cases}$ with H, L and K positive since $s - a,\ s - b$ and $s - c$ are positive.

The inequality in terms of the positive numbers H, K and L can be written as

$$\frac{(H + K)^2 + (H + L)^2}{H^2} + \frac{(H + L)^2 + 1}{L^2} + \frac{(H + K)^2 + 1}{K^2} \geq 24.$$

Working with the left side of the inequality we can obtain

$$\left(2 + 2\frac{K}{H} + \left(\frac{K}{H}\right)^2 + 2\frac{L}{H} + \left(\frac{L}{H}\right)^2\right) + \left(\left(\frac{H}{L}\right)^2 + 2\frac{L}{H} + 1 + \frac{1}{L^2}\right) + \left(\left(\frac{H}{K}\right)^2 + 2\frac{H}{K} + 1 + \frac{1}{K^2}\right)$$

$$= 2\left(\frac{K}{H} + \frac{H}{K}\right) + 2\left(\frac{L}{H} + \frac{H}{L}\right) + 2\left(\left(\frac{H}{K}\right)^2 + \left(\frac{K}{H}\right)^2\right) + \left(\left(\frac{L}{H}\right)^2 + \left(\frac{H}{L}\right)^2\right) + 4 + \frac{1}{K^2} + \frac{1}{L^2}.$$

Each of the brackets in the last expression has the form $\left(x + \dfrac{1}{x}\right)$ or $\left(x^2 + \dfrac{1}{x^2}\right)$ so the minimum value of each bracket is 2. Then the left side of the original problem inequality is greater than or equal to $2(2) + 2(2) + 2 + 2 + 4 + \dfrac{1}{K^2} + \dfrac{1}{L^2}$. If we can show that this expression is greater than or equal 24, the original inequality is correct.

We must show that $\dfrac{1}{K^2} + \dfrac{1}{L^2}$ is at least 8. Since K and L are positive numbers such that $L = 1 - K$, the derivative of the two terms is $\dfrac{-2}{K^3} - \dfrac{2}{L^3}(-1)$. Letting the derivative equal to zero, we obtain $K = L = \dfrac{1}{2}$. The value of 8 is clearly a minimium for $\dfrac{1}{K^2} + \dfrac{1}{L^2}$. Thus the problem inequality is correct.

Solution 6 by Henry Ricardo, New York Math Circle, NY

It is a known consequence of the arithmetic-geometric mean inequality that the side-lengths of a triangle satisfy the inequality

$$(b+c-a)(c+a-b)(a+b-c) \ \le \ abc.$$

Using this fact and the arithmetic-geometric mean inequality twice more, we have

$$\begin{aligned}
\frac{a^2+b^2}{(s-c)^2}+\frac{b^2+c^2}{(s-a)^2}+\frac{c^2+a^2}{(s-b)^2} \ &\ge \ 3\left(\frac{(a^2+b^2)(b^2+c^2)(c^2+a^2)}{(s-a)^2(s-b)^2(s-c)^2}\right)^{1/3}\\
&\ge \ 3\left(\frac{(2ab)(2bc)(2ac)}{[(b+c-a)(a+c-b)(a+b-c)]^2/64}\right)^{1/3}\\
&\ge \ 3\left(\frac{8a^2b^2c^2}{(abc)^2/64}\right)^{1/3} \ = \ 24.
\end{aligned}$$

Also solved by Dionne Bailey, Elsie Campbell, and Charles Diminnie, Angelo State University, San Angelo, TX; D. M.Btinetu-Giurgiu, Bucharest, Romania; Bruno Salgueiro Fanego, Viveiro, Spain; Ethan Gegner (student, Taylor University), Upland, IN; Ed Gray, Highland Beach, FL; Nikos Kalapodis (two additional solutions to #2 above), Patras, Greece; Kee-Wai Lau, Hong Kong, China; Haroun Meghaichi (student, University of Science and Technology Houari Boumediene), Algeria; Albert Stadler, Herrliberg, Switzerland; Nicusor Zlota, "Traian Vuia" Technical College, Focsani, Romania; Titu Zvonaru and Neculai Stanciu, Romania, and the proposer.

- **5341:** *Proposed by José Luis Díaz-Barrero, Barcelona Tech, Barcelona, Spain*

Let $z_1, z_2, \cdots, z_n$, and $w_1, w_2, \cdots, w_n$ be sequences of complex numbers. Prove that

$$Re\left(\sum_{k=1}^{n} z_k w_k\right) \le \frac{3}{(n+1)(n+2)}\sum_{k=1}^{n}|z_k|^2+\frac{3n^2+6n+1}{20}\sum_{k=1}^{n}|w_k|^2.$$

Solution 1 by Kee-Wai Lau, Hong Kong, China

We have

$$\begin{aligned}
Re\left(\sum_{k=1}^{n} z_k w_k\right) \le \left|\sum_{k=1}^{n} z_k w_k\right| \ &\le \ \sum_{k=1}^{n}|z_k|\,|w_k|\\
&= \ \sum_{k=1}^{n}\left|\frac{\sqrt{6}z_k}{\sqrt{(n+1)(n+2)}}\right|\left|\frac{\sqrt{(n+1)(n+2)}w_k}{\sqrt{6}}\right|\\
&\le \ \frac{1}{2}\left(\sum_{k=1}^{n}\left(\left|\frac{\sqrt{6}z_k}{\sqrt{(n+1)(n+2)}}\right|^2+\left|\frac{\sqrt{(n+1)(n+2)}w_k}{\sqrt{6}}\right|^2\right)\right)\\
&= \ \frac{3}{(n+1)(n+2)}\sum_{k=1}^{n}|z_k|^2+\frac{(n+1)(n+2)}{12}\sum_{k=1}^{n}|w_k|^2.
\end{aligned}$$

Since

$$\frac{(n+1)(n+2)}{12} = \frac{3n^2+6n+1}{20} - \frac{(n-1)(4n+7)}{60} \leq \frac{3n^2+6n+1}{20},$$

so the inequality of the problem holds.

Solution 2 by Ethan Gegner (student, Taylor University), Upland, IN

For $n \in N$, define

$$f(n) = \left(\frac{3}{(n+1)(n+2)}\right)\left(\frac{3n^2+6n+1}{20}\right)$$

and observe that f is an increasing function of n; thus, $f(n) \geq f(1) = 1/4$ for all $n \in N$.

Applying AM-GM inequality and then Cauchy's inequality, we obtain

$$\begin{aligned}
\frac{3}{(n+1)(n+2)}\sum_{k=1}^{n}|z_k|^2 + \frac{3n^2+6n+1}{20}\sum_{k=1}^{n}|w_k|^2 \quad &\geq \quad 2\sqrt{f(n)\left(\sum_{k=1}^{n}|z_k|^2\right)\left(\sum_{k=1}^{n}|w_k|\right)^2} \\
&\geq \quad \left(\sum_{k=1}^{n}|z_k|^2\right)^{1/2}\left(\sum_{k=1}^{n}|w_k|^2\right)^{1/2} \\
&\geq \quad \sum_{k=1}^{n}|z_k|\,|w_k| \\
&\geq \quad Re\left(\sum_{k=1}^{n}z_k w_k\right).
\end{aligned}$$

Solution 3 by Paolo Perfetti, Department of Mathematics, "Tor Vergata" University, Rome, Italy

The AGM yields

$$\frac{3}{(n+1)(n+2)}\sum_{k=1}^{n}|z_x|^2 + \frac{3n^2+6n+1}{20}\sum_{k=1}^{n}|w_x|^2 \geq 2\sqrt{\frac{3}{20}\frac{3n^2+6n+1}{n^2+3n+2}}\sqrt{\sum_{k=1}^{n}|z_x|^2 \cdot \cdot \sum_{r=1}^{n}|w_r|^2}.$$

Then we use Cauchy–Schwarz

$$\sqrt{\sum_{k=1}^{n}|z_x|^2 \cdot \sum_{r=1}^{n}|w_r|^2} \geq \sum_{k=1}^{n}|z_x| \cdot |w_k|$$

Moreover

$$Re\left(\sum_{k=1}^{n} z_k w_k\right) \leq \left|\sum_{k=1}^{n} z_k w_k\right| \leq \sum_{k=1}^{n} |z_k w_k|,$$

and the inequality amounts to show that

$$2\sqrt{\frac{3}{20}\frac{3n^2+6n+1}{n^2+3n+2}} \geq 1 \iff n \leq -\frac{7}{4},\ n \geq 1.$$

This completes the proof.

Solution 4 by Nicusor Zlota, "Traian Vuia" Technical College, Focsani, Romania

Let $z_k = x_k + iy_k$ and $w_k = a_k + ib_k$, for $0 \leq k \leq n$. We can assume that $x_k, y_k, a_k, b_k \geq 0$, because we can increase the left hand side of the statement of the problem by using absolute values.

We wish to prove the inequality:

$$\sum_{k=1}^{n}(a_k x_k - b_k y_k) \leq \frac{3}{(n+1)(n+2)}\sum_{k=1}^{n}\left(x_k^2 + y_k^2\right) + \frac{3n^2+6n+1}{20}\sum_{k=1}^{n}\left(a_k^2 + b_k^2\right).$$

Because of symmetry, we need only show that:

$$a_k x_k \leq \frac{3}{(n+1)(n+2)}x_k^2 + \frac{3n^2+6n+1}{20}a_k^2.$$

Considering this as a quadratic inequality for the variable x_k, we see that the discriminant is negative.

$$\Delta = a_k^2 - 4\frac{3}{(n+1)(n+2)}\frac{3n^2+6n+1}{20}a_k^2 = a_k^2\left(\frac{-4n^2+3n+7}{5(n+1)(n+2)}\right) < 0.$$

Hence, the problem is solved.

Also solved by Bruno Salgueiro Fanego,Viveiro, Spain; Ed Gray, Highland Beach, FL, and the proposer.

- **5342:** *Proposed by Ovidiu Furdui, Technical University of Cluj-Napoca, Cluj-Napoca, Romania*

Let $a, b, c, \alpha > 0$, be real numbers. Study the convergence of the integral

$$I(a,b,c,\alpha) = \int_1^\infty \left(a^{1/x} - \frac{b^{1/x}+c^{1/x}}{2}\right)^\alpha dx.$$

The problem is about studying the conditions which the four parameters, a, b, c, and α, should verify such that the improper integral would converge.

Solution 1 by Arkady Alt, San Jose, CA

Case 1. If $a = b = c$, then for any nonzero x, $a^{\frac{1}{x}} - \dfrac{b^{\frac{1}{x}} + c^{\frac{1}{x}}}{2} = 0$, so $I(a, b, c, \alpha) = 0$ for any real $\alpha > 0$.

Case 2. Suppose α isn't an integer. Then $a^{\frac{1}{x}} - \dfrac{b^{\frac{1}{x}} + c^{\frac{1}{x}}}{2}$ must be nonnegative for any x and in particular, it must be positive for $x = 1$, that is $a \geq \dfrac{b + c}{2}$.

Since $\begin{cases} 2a = b + c \\ b = c \end{cases} \iff a = b = c$ then, to avoid the trivial case 1, we will consider a, b, c such that

$a > \dfrac{b + c}{2}$ or $\begin{cases} 2a = b + c \\ b \neq c. \end{cases}$

Then, by the AM-PM inequality, for $x > 1$ we have

$$\frac{b + c}{2} > \left(\frac{b^{\frac{1}{x}} + c^{\frac{1}{x}}}{2}\right)^x \iff \left(\frac{b + c}{2}\right)^{\frac{1}{x}} > \frac{b^{\frac{1}{x}} + c^{\frac{1}{x}}}{2},$$

and we obtain $a^{\frac{1}{x}} > \dfrac{b^{\frac{1}{x}} + c^{\frac{1}{x}}}{2}$ for any $x > 1$ and that the integral is defined.

For any real $p > 0$ we have $\lim\limits_{t \to 0} \dfrac{p^t - 1}{t} = \ln p$. So, $\lim\limits_{x \to \infty} x \left(a^{\frac{1}{x}} - \dfrac{b^{\frac{1}{x}} + c^{\frac{1}{x}}}{2}\right) =$

$$\lim_{x \to \infty} x \left(a^{\frac{1}{x}} - 1\right) - \frac{1}{2} \left(\lim_{x \to \infty} x \left(b^{\frac{1}{x}} - 1\right) + \lim_{x \to \infty} x \left(c^{\frac{1}{x}} - 1\right)\right) = \ln a - \frac{\ln b + \ln c}{2} = \ln \frac{a}{\sqrt{bc}} > 0,$$

because $a > \sqrt{bc}$ if $b \neq c$ or if $a > \dfrac{b + c}{2}$.

Therefore, $\lim\limits_{x \to \infty} \dfrac{\left(a^{\frac{1}{x}} - \dfrac{b^{\frac{1}{x}} + c^{\frac{1}{x}}}{2}\right)^\alpha}{\dfrac{1}{x^\alpha}} = \ln^\alpha \dfrac{a}{\sqrt{bc}} > 0$, and by the Limit Comparison Test, $I(a, b, c, \alpha)$ converges iff $\dfrac{1}{x^\alpha}$ converges; that is, $I(a, b, c, \alpha)$ converges if $\alpha > 1$ and diverges if $\alpha \in (0, 1]$.

Case 3. Let α be a positive integer. Then the expression $\left(a^{\frac{1}{x}} - \dfrac{b^{\frac{1}{x}} + c^{\frac{1}{x}}}{2}\right)^\alpha$ is defined for any positive a, b, c and since

$$\lim_{x \to \infty} \left(a^{\frac{1}{x}} - \frac{b^{\frac{1}{x}} + c^{\frac{1}{x}}}{2}\right)^\alpha = \ln^\alpha \frac{a}{\sqrt{bc}} > 0$$

is the limit of $I(a, b, c, \alpha)$ for $a > \sqrt{bc}$ and when $\alpha > 1$. So the situation of $a = \sqrt{bc}$ must be analyzed.

Then $\left(a^{\frac{1}{x}}-\dfrac{b^{\frac{1}{x}}+c^{\frac{1}{x}}}{2}\right)^{\alpha}=\dfrac{(-1)^{\alpha}\left(b^{\frac{1}{2x}}-c^{\frac{1}{2x}}\right)^{2\alpha}}{2^{\alpha}}.$

Assume, without loss of generality, $b>c$. Since $\lim\limits_{x\to\infty} x\left(b^{\frac{1}{2x}}-a^{\frac{1}{2x}}\right)=\dfrac{1}{2}\ln\dfrac{b}{c}>0$,

then $\lim\limits_{x\to\infty}\dfrac{\left(b^{\frac{1}{2x}}-a^{\frac{1}{2x}}\right)^{2\alpha}}{\frac{1}{x^{2\alpha}}}=\left(\dfrac{1}{2}\ln\dfrac{b}{c}\right)^{2\alpha}>0,$ and by the Limit Comparison Test

$I(a,b,c,\alpha)$ is convergent iff $\dfrac{1}{x^{2\alpha}}$convergent, that is $I(a,b,c,\alpha)$ convergent if $\alpha>1/2$ and divergent if $\alpha\in(0,1/2]$.

In summary,

- If $a=b=c$ then $I(a,b,c,\alpha)=0$ is convergent for any real α;

- If $\alpha\in\Re_+/N$ and $a>\dfrac{b+c}{2}$ or $\begin{cases}2a=b+c\\ b\neq c\end{cases}$ then $I(a,b,c,\alpha)$ is convergent for $\alpha>1$ and divergent for $\alpha\in(0,1]$;

- If $\alpha\in\Re_+/N$ and $a>\sqrt{bc}$ then $I(a,b,c,\alpha)$ is convergent for $\alpha>1$ and divergent for $\alpha\in(0,1]$;

- If $\alpha\in N$ and $a=\sqrt{bc}$ then $I(a,b,c,\alpha)$ is convergent for $\alpha>1/2$ and divergent for $\alpha\in(0,1/2]$.

Solution 2 by Paolo Perfetti, Department of Mathematics, "Tor Vergata" University, Rome, Italy

To have the integral well defined, a necessary condition is $2a\geq b+c$.

The convergence occurs in one of the following cases:

1) if $a=b=c$ we have convergence for any value of α

2) if $\alpha>1$ we have convergence regardless the values of a,b,c

3) if $1/2<\alpha\leq 1$ and $a=\sqrt{bc}$ we have convergence.

Proof

If α is irrational or it is a rational p/q reduced to the lowest terms with q even, we must impose

$$2a^{1/x}-b^{1/x}-c^{1/x}\geq 0$$

but this doesn't seem to me easy to prove. A necessary condition is $2a\geq b+c$ corresponding to $x=1$.

If $a=b=c$ the integrand is identically zero and then the integral converges regardless the value of α.

From now on, $a\neq b$ or $b\neq c$ or $a\neq c$.

We have $a^{1/x}=e^{\frac{\ln a}{x}}=1+\dfrac{\ln a}{x}+\dfrac{\ln^2 a}{2x^2}+\dfrac{\ln^3 a}{6x^3}+O(x^{-4})$ whence

$$\left[a^{1/x} - \frac{b^{1/x}+c^{1/x}}{2}\right]^\alpha = \Bigg\{1 + \frac{\ln a}{x} + \frac{\ln^2 a}{2x^2} + \frac{\ln^3 a}{6x^3} +$$

$$-\frac{1}{2}\Big(1 + \frac{\ln b}{x} + \frac{\ln^2 b}{2x^2} + \frac{\ln^3 b}{6x^3} + 1 + \frac{\ln c}{x} + \frac{\ln^2 c}{2x^2} + \frac{\ln^3 c}{6x^3} + O(x^{-4})\Big)\Bigg\}^\alpha =$$

$$= \frac{1}{x^\alpha}\Big(\ln\frac{a}{\sqrt{bc}} + \frac{\ln^2 a - \frac{\ln^2 b}{2} - \frac{\ln^2 c}{2}}{2x} + xA\Big)^\alpha$$

$$A = \frac{1}{6}\left(\frac{\ln^3 a}{x^3} - \frac{\ln^3 b}{2x^3} - \frac{\ln^3 c}{2x^3}\right) + O(x^{-4})$$

The positivity of $\ln\frac{a}{\sqrt{bc}} + \frac{\ln^2 a - \frac{\ln^2 b}{2} - \frac{\ln^2 c}{2}}{2x} + xA$ for x large enough, imposes $\ln\frac{a}{\sqrt{bc}} > 0$ that is $a^2 \geq bc$ which in turn follows by $2a \geq b + c$. Indeed

$$a^2 \geq \frac{(b+c)^2}{4} = \frac{b^2+c^2+2bc}{4} \geq \frac{4bc}{4} = bc$$

Let $\alpha > 1$. Since for any x large enough it is

$$\Big(\ln\frac{a}{\sqrt{bc}} + \frac{\ln^2 a - \frac{\ln^2 b}{2} - \frac{\ln^2 c}{2}}{2x} + xA\Big)^\alpha \leq C$$

if $\alpha > 1$ the integral $\int_1^\infty \frac{1}{x^\alpha}\Big(\ln\frac{a}{\sqrt{bc}} + \frac{\ln^2 a - \frac{\ln^2 b}{2} - \frac{\ln^2 c}{2}}{2x} + xA\Big)^\alpha dx$ converges.

Let $1/2 < \alpha \leq 1$ and $a = \sqrt{bc}$.

$$0 \leq \left(a^{1/x} - \frac{b^{1/x}+c^{1/x}}{2}\right)^\alpha = \frac{1}{x^{2\alpha}}\Big(\frac{1}{4}(\ln b - \ln c)^2 + x^2 A\Big)^\alpha \leq \frac{C_1}{x^{2\alpha}}$$

whence convergence.

Let $0 < \alpha \leq 1/2$, and $a = \sqrt{bc}$. To have convergence we need $\ln b = \ln c$ that is $b = c$, but this would yield $a = b = c$, a forbidden condition.

Also solved by the proposer.