Problems

Ted Eisenberg, Section Editor

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This section of the Journal offers readers an opportunity to exchange interesting mathematical problems and solutions. Please send them to Ted Eisenberg, Department of Mathematics, Ben-Gurion University, Beer-Sheva, Israel or fax to: 972-86-477-648. Questions concerning proposals and/or solutions can be sent e-mail to <eisenbt@013.net>. Solutions to previously stated problems can be seen at <http://www.ssma.org/publications>.

Solutions to the problems stated in this issue should be posted before October15, 2016

- **5403:** *Proposed by Kenneth Korbin, New York, NY*

  Let $\phi = \dfrac{1+\sqrt{5}}{2}$. Solve the equation $\sqrt[3]{x+\phi} = \sqrt[3]{\phi} + \sqrt[3]{x-\phi}$ with $x > \phi$.

- **5404:** *Proposed Arkady Alt, San Jose, CA*

  For any given positive integer $n \geq 3$, find the smallest value of the product of $x_1 x_2 \ldots x_n$, where $x_1, x_2, x_3, \ldots x_n > 0$ and $\dfrac{1}{1+x_1} + \dfrac{1}{1+x_2} + \ldots + \dfrac{1}{1+x_n} = 1$.

- **5405:** *Proposed by D. M. Bătinetu-Giurgiu, Bucharest, Romania and Neculai Stanciu,"George Emil Palade" School, Buzău, Romania*

  If $a, b \in \Re$ such that $a+b=1$, $e_n = \left(1+\dfrac{1}{n}\right)^n$ and $c_n = -\ln n + \displaystyle\sum_{k=1}^{n}\frac{1}{k}$, then compute

$$\lim_{n\to\infty}\left((n+1)^a \sqrt[n+1]{((n+1)!c_n)^b} - n^a \sqrt[n]{(n!e_n)^b}\right).$$

- **5406:** *Proposed by Cornel Ioan Vălean, Timis, Romania*

  Calculate:

$$\sum_{n=1}^{\infty}\frac{H_n}{n}\left(\zeta(3) - 1 - \frac{1}{2^3} - \cdots - \frac{1}{n^3}\right),$$

  where $H_n = \displaystyle\sum_{k=1}^{n}\frac{1}{k}$ denotes the harmonic number.

- **5407:** *Proposed by José Luis Díaz-Barrero, Barcelona Tech, Barcelona, Spain*

Find all triples (a,b,c) of positive reals such that

$$\begin{aligned} a+b+c &= 1, \\ \frac{1}{(a+bc)^2}+\frac{1}{(b+ca)^2}+\frac{1}{(c+ab)^2} &= \frac{243}{16}. \end{aligned}$$

- **5408:** *Proposed by Ovidiu Furdui, Technical University of Cluj-Napoca, Cluj-Napoca, Romania*

  Calculate:

$$\int_0^1 \frac{\ln x \ln(1-x)}{x(1-x)}dx.$$

Solutions

- **5385:** *Proposed by Kenneth Korbin, New York, NY*

  A triangle with integer length sides and integer area has perimeter $P = 6^6$. Find the sides of the triangle when the area is minimum.

Solution by Toshihiro Shimizu, Kawasaki, Japan

Let $s = P/2 = 23328$. Let the sides of the triangle be a, b, c. The square of area of the triangle can be written as $s(s-a)(s-b)(s-c)$. Thus, $(s-a)(s-b)(s-c)$ must be minimized and this value must be twice the square of an integer. Let $\alpha = s-a$, $\beta = s-b$, $\gamma = s-c$ and $T = (s-a)(s-b)(s-c) = \alpha\beta\gamma$. Then, $\alpha+\beta+\gamma = s$ and without loss of generality, we assume $\alpha \geq \beta \geq \gamma > 0$. When $(\alpha, \beta, \gamma) = (23276, 44, 8)$, we have $T = 2 \cdot 2024^2$. We show that this case is the unique smallest case. In this case it follows that $(a, b, c) = (52, 23284, 23320)$ and $Area = 437,184$.

First, we assume that if $\beta\gamma = t$ for some positive integer t. Then, it follows that $\alpha = s - \beta - \gamma \geq s - t - 1$ and

$$T = \alpha\beta\gamma \geq (s-t-1)\cdot t$$

Thus, we need to find the case that $(s-t-1)\cdot t < 2\cdot 2024^2$ or $t^2 - 23327t + 2\cdot 2024^2 > 0$ or $t < 356.6$.
Therefore, we only need to consider the case that $\beta\gamma \leq 356$ and the range of γ is $\gamma \leq \lfloor\sqrt{356}\rfloor = 18$.
We consider the case $\gamma = 1$. The range of β is $1 \leq \beta \leq 356$.For case $(\beta,\gamma) = (1,1)$, $\alpha = s-\beta-\gamma = 23326$ and $T = 23326, T/2 = 11663$, It's not a square of an integer.
For case $(\beta,\gamma) = (2,1)$, $\alpha = s-\beta-\gamma = 23325$ and $T = 46650, T/2 = 23325$, It's not a square of an integer.
For case $(\beta,\gamma) = (3,1)$, $\alpha = s-\beta-\gamma = 23324$ and $T = 69972, T/2 = 34986$, It's not a square of an integer.

$$\vdots$$

Editor's interlude : The solution continues on in the above manner, and after 49 pages, with each line similar to the output listed above, the proof by exhaustion ends with the final entries listed as:

$$\vdots$$

We consider the case $\gamma = 17$. The range of β is $17 \le \beta \le 20$. For case $(\beta, \gamma) = (17, 17)$, $\alpha = s - \beta - \gamma = 23294$ and $T = 6731966, T/2 = 3365983$, It's not a square of an integer.
For case $(\beta, \gamma) = (18, 17)$, $\alpha = s - \beta - \gamma = 23293$ and $T = 7127658, T/2 = 3563829$, It's not a square of an integer.
For case $(\beta, \gamma) = (19, 17)$, $\alpha = s - \beta - \gamma = 23292$ and $T = 7523316, T/2 = 3761658$, It's not a square of an integer.
For case $(\beta, \gamma) = (20, 17)$, $\alpha = s - \beta - \gamma = 23291$ and $T = 7918940, T/2 = 3959470$, It's not a square of an integer.
We consider the case $\gamma = 18$. The range of β is $18 \le \beta \le 19$.For case $(\beta, \gamma) = (18, 18)$, $\alpha = s - \beta - \gamma = 23292$ and $T = 7546608, T/2 = 3773304$, It's not a square of an integer.
For case $(\beta, \gamma) = (19, 18)$, $\alpha = s - \beta - \gamma = 23291$ and $T = 7965522, T/2 = 3982761$, It's not a square of an integer.

Editor again: Each of the complete solutions submitted used Hero's formula on an expression connecting the perimeter of the triangle with its area, and then used a computer in proving that they had the minimal area. But sometimes computers get it wrong. **David Stone and John Hawkins of Southern Georgia University in Statesboro, GA** stated that the area of the triangle with integer length sides of $(1, 23327, 23328)$ is *essentially* zero, which of course they quickly dismissed. They then listed the areas of the following three Heronian triangles each having perimeter $6^6 = 46,656$.

$$\begin{pmatrix} a & b & c & s & s-a & s-b & s-c & area \\ 52 & 23284 & 23320 & 23328 & 23276 & 44 & 8 & 437184 \\ 72 & 23290 & 23294 & 23328 & 23256 & 38 & 343 & 837218 \\ 153 & 23225 & 23278 & 23328 & 23175 & 103 & 50 & 1,668,60 \end{pmatrix}$$

Ed Gray of Highland Beach, FL showed that the Heronian Triangle with side lengths of $\{1928, 21402, 23326\}$ has an area of 1386720, and **Kenneth Korbin,** proposer of the problem, showed that a triangle with side lengths $\{2600, 2073, 23319\}$ has an area of 3,357,936. **Kee-Wai Lau of Hong Kong, China** also showed that the triangle with integer side lengths of $\{52, 23284, 23320\}$ has a perimeter of 6^6 and produces the triangle with the minimal integral area.

- **5386:** *Proposed by Michael Brozinsky, Central Islip, NY.*

Determine whether or not there exit nonzero constants a and b such that the conic whose polar equation is

$$r = \sqrt{\frac{a}{\sin(2\theta) - b\cos(2\theta)}}$$

has a rational eccentricity.

Solution by Elsie M. Campbell, Dionne T. Bailey, and Charles Diminnie, Angelo State University, San Angelo, TX

To begin, the given polar equation can be written in x and y as follows:

$$by^2 + 2xy - bx^2 \quad = \quad a. \tag{1}$$

Noting that (1) has the form $Dx^2 + Exy + Fy^2 = a$, the angle of rotation is found to be

$$\tan(2\theta) \quad = \quad \frac{E}{D-F} = -\frac{1}{b}. \tag{2}$$

With some perseverance and the standard rotation formulas with $x = u\,cos(\theta) - v\,\sin(\theta)$ and $y = u\,\sin(\theta) + v\,\cos(\theta)$, (1) can be written as

$$\big(\sin(2\theta) - b\,\cos(2\theta)\big)u^2 + \big(b\,\cos(2\theta) - \sin(2\theta)\big)v^2 \quad = \quad a. \tag{3}$$

Thus, using (2), $\sin(2\theta) = \dfrac{1}{\sqrt{b^2+1}}$ and $\cos(2\theta) = -\dfrac{b}{\sqrt{b^2+1}}$. (3) can now be simplified and displayed in standard form of a conic as

$$\sqrt{b^2+1}\,u^2 - \sqrt{b^2+1}\,v^2 = a$$
$$\frac{u^2}{\dfrac{a}{\sqrt{b^2+1}}} - \frac{v^2}{\dfrac{a}{\sqrt{b^2+1}}} = 1. \tag{4}$$

If we consider A to be the distance from the center of the hyperbola to a vertex, B to be the distance from the center to an end of the conjugate axis, and C to be the distance from the center to a focus, then from (4), $A^2 = \dfrac{a}{\sqrt{b^2+1}}$, $B^2 = \dfrac{a}{\sqrt{b^2+1}}$, and

$$C^2 \quad = \quad A^2 + B^2 = \frac{2a}{\sqrt{b^2+1}}. \tag{5}$$

Using (5), eccentricity is defined to be $e = \dfrac{C}{A} = \sqrt{2}$. Thus, there do not exist nonzero constants a and b to yield a rational eccentricity.

Editor′s comment: This problem appeared before in this column as problem 5304; mea culpa, once again.

Also solved by Arkady Alt, San Jose, CA; Hatef I. Arshagi, Guilford Technical Community College, Jamestown, NC; Bruno Salgueiro Fanego, Viveiro, Spain; Ed Gray; Highland Beach, FL; Kee-Wai Lau, Hong Kong, China; Toshihiro Shimizu, Kawasaki, Japan, and the proposer.

- **5387:** *Proposed by Arkady Alt, San Jose, CA*

Let $D := \{(x,y) \mid x, y \in R_+,\ x \neq y \text{ and } x^y = y^x\}$.(Obviously $x \neq 1$ and $y \neq 1$).
Find $\sup\limits_{(x,y)\in D} \left(\dfrac{x^{-1}+y^{-1}}{2}\right)^{-1}$

Solution 1 by Henry Ricardo, New York Math Circle, NY

The power mean inequality gives us

$$M_{-1}(x,y) \;=\; \left(\frac{x^{-1}+y^{-1}}{2}\right)^{-1} \;\le\; M_0(x,y) \;=\; \sqrt{xy},$$

so that

$$\sup_{(x,y)\in D}\left(\frac{x^{-1}+y^{-1}}{2}\right)^{-1} \;\le\; \sup_{(x,y)\in D}\sqrt{xy}.$$

Now it is well known that the general solution of the equation $x^y = y^x$ in the first quadrant is given parametrically by

$$x \;=\; \left(1+\frac{1}{u}\right)^{u}, \quad y \;=\; \left(1+\frac{1}{u}\right)^{u+1}, \; u>0,$$

a form attributed to Christian Goldbach. This gives us

$$x\cdot y \;=\; \left(1+\frac{1}{u}\right)^{u}\cdot\left(1+\frac{1}{u}\right)^{u+1},$$

implying that

$$\sup_{(x,y)\in D}\left(\frac{x^{-1}+y^{-1}}{2}\right)^{-1} \;=\; \lim_{u\to\infty}\sqrt{xy} \;=\; \sqrt{e\cdot e} \;=\; e.$$

Solution 2 by Toshihiro Shimizu, Kawasaki, Japan

It is well-known that for any positive integer n,

$$(x,y) = \left(\left(1+\frac{1}{n}\right)^{n}, \left(1+\frac{1}{n}\right)^{n+1}\right)$$

satisfies the equation $x^y = y^x$ and $x \neq y$. Letting $n\to\infty$, both x and y converges to e. Thus, the value $((x^{-1}+y^{-1})/2)^{-1}$ also converges to e.
Next, we show that for any real number satisfying $x^y = y^x, x\neq y$, the equation $((x^{-1}+y^{-1})/2)^{-1} \le e$ holds. $x^y = y^x$ is equivalent to $\log x/x = \log y/y$. Since $\log x/x$ is negative and monotone decreasing for $x<1$, and it's positive and monotone increasing for $1\le x\le e$ and also it's positive and monotone decreasing on $e\le x$, it is obvious that $1<x,y$ and without loss of generality , we assume $y<e<x$. We write $x=1/s$, $y=1/t$. Then, $s<1/e<t$ and $s\log s = t\log t$. The inequality $((x^{-1}+y^{-1})/2)^{-1}\le e$ is equivalent to $1/e \le (s+t)/2$.
Let $f(x) = x\log x$. Then, $f'(x) = 1+\log x, f''(x) = 1/x, f'''(x) = -x^{-2} < 0$ for $x>0$. Thus, $f'(x)$ is concave and it follows that

$$\frac{f'(z)+f'(\frac{2}{e}-z)}{2} \le f'(\frac{z+\frac{2}{e}-z}{2}) = f'(\frac{1}{e}) = 0$$

for any $z > 0$. Integrating from $z = s$ to $z = 1/e$, we get

$$\frac{f(1/e) - f(s) + f(\frac{2}{e} - s) - f(1/e)}{2} \leq 0,$$

or $f(2/e - s) \leq f(s) = f(t)$. Since, $f(z)$ is monotone increasing on $1/e \leq z$, it follows that $2/e - s \leq t$ or $1/e \leq (s+t)/2$. Therefore we have shown that $((x^{-1} + y^{-1})/2)^{-1} \leq e$ for any $(x, y) \in D$.

Finally we conclude that the supremum value is e.

Solution 3 by Bruno Salgueiro Fanego, Viveiro, Spain

In is known that $D \cap \left\{ (x,y) \middle| x \neq 1, y \neq 1 \right\}$ can be parametrized by

$(0,1) \cup (1, +\infty) \ni t \to \left(x(t), y(t)\right) = \left(t^{\frac{1}{t-1}}, t^{\frac{t}{t-1}}\right)$.

(Note that $t = \dfrac{y(t)}{x(t)}$ is the slope of the line from $(0,0)$ to $\left(x(t), y(t)\right)$; moreover,

$$y(t)^{x(t)} = \left(t^{\frac{t}{t-1}}\right)^{t^{\frac{1}{t-1}}} = t^{\frac{t}{t-1} \cdot t^{\frac{1}{t-1}}} = t^{\frac{t \cdot t^{\frac{1}{t-1}}}{t-1}} = t^{\frac{t^{1+\frac{1}{t-1}}}{t-1}} = t^{\frac{t^{\frac{t}{t-1}}}{t-1}} = t^{\frac{1}{t-1} \cdot t^{\frac{t}{t-1}}} = \left(t^{\frac{1}{t-1}}\right)^{t^{\frac{t}{t-1}}} = x(t)^{y(t)}.$$

Hence,

$$\left(\frac{x(t)^{-1} + y(t)^{-1}}{2}\right)^{-1} = \frac{2x(t)y(t)}{x(t) + y(t)} = \frac{2t^{\frac{1}{t-1}} \cdot t^{\frac{t}{t-1}}}{t^{\frac{1}{t-1}} + t^{\frac{t}{t-1}}} = \frac{2t^{\frac{1+t}{t-1}}}{t^{\frac{1}{t-1}} \cdot (1+t)} = \frac{2t^{\frac{t}{t-1}}}{t+1}.$$

Let us define $(0,1) \cup (1, \infty) \ni \mu \to f(u) = \dfrac{2u^{\frac{u}{u-1}}}{u+1.}$

Then $f'(u) = \dfrac{2u^{\frac{u}{u-1}} \left(2u - 2 - (u+1)\ln u\right)}{\left(u^2 - 1\right)^2}$ so $f'(u) > 0$ for $u \in (0,1)$ and $f'(u) < 0$ for $u \in (1, +\infty)$, with implies that f is strictly increasing in $(0,1)$ and strictly decreasing in $(1, +\infty)$, which implies that

$$\begin{aligned}
\sup_{u \in (0,1) \cup (1,+\infty)} f(u) &= \lim_{u \to 1} f(u) = \lim_{n \to 1} \frac{2}{u+1} \cdot \lim_{u \to 1} u^{\frac{u}{u-1}} = \lim_{n \to 1} u^{\frac{u}{u-1}} = e^{\ln \lim_{u \to 1} u^{\frac{u}{u-1}}} \\
&= e^{\lim_{u \to 1} \ln u^{\frac{u}{u-1}}} = e^{\lim_{u \to 1} \frac{u}{u-1} \ln u} = e^{\lim_{u \to 1} \frac{u}{u-1} \left(-\sum_{n=1}^{\infty} \frac{1-u^n}{n}\right)} = e^{\lim_{u \to 1} u \sum_{n=1}^{\infty} \frac{(1-u)^{n-1}}{n}} \\
&= e^{\lim_{u \to 1} u + \sum_{n=2}^{\infty} \frac{u(1-u)^{n-1}}{n}} = e^{1+0} = e.
\end{aligned}$$

Thus, $\displaystyle\sup_{(x,y) \in D} \left(\frac{x^{-1} + y^{-1}}{2}\right)^{-1} = \sup_{t \in (0,1) \cup (1,+\infty)} \left(\frac{x(t)^{-1} + y(t)^{-1}}{2}\right)^{-1} = \sup_{t \in (0,1) \cup (1,+\infty)} f(t) = e.$

Solutions 4 and 5 by Michael Brozinsky, Central Islip, NY

For simplicity, we shall use $\dfrac{2xy}{x+y}$, which equals the given expression.
We shall also use the Lambert function $W(x)$ which is the inverse of $f(x) = x \cdot e^x$ (with the domain of $f(x)$ being $\{-1, \infty\,)$ so that $W(x)$ has domain $\left[-\dfrac{1}{e}, \infty\right)$ and

$$\begin{aligned} W\left(x \cdot e^x\right) &= x \; if \; x \geq -1, \text{and} \\ x &= W(x) \cdot e^{W(x)}, \; if \; x \geq \frac{1}{e} \qquad (*) \end{aligned}$$

From $y^x = x^y$ we have $\dfrac{\ln(x)}{x} = \dfrac{\ln(y)}{y}$ (Δ), and since $F(t) = \dfrac{\ln(t)}{t}$ is one to one and negative on $(0,1)$, one to one and positive on $(1,e)$ and one to one and positive on (e,∞) and since $x \neq y$, we can assume that $1 < y < e$ and $x > e$ so that in particular $\ln(y) > -1$ and from $(*)$, $W\left(-\ln(y) \cdot e^{-\ln(y)}\right) = -\ln(y)$ which we will encounter later when we obtain $(**)$ below.

From $y^x = x^y$ we have by raising both sides to the $\dfrac{1}{xy}$ power that $y^{\frac{1}{y}} = x^{\frac{1}{x}}$. The left hand side can be written as $\left(e^{\ln(y)}\right)^{\frac{1}{y}} = \left(e^{\ln(y)}\right)^{e^{-\ln(y)}} = e^{\ln(y)\cdot e^{-\ln(y)}}$ and so we have $e^{\ln(y)\cdot e^{-\ln(y)}} = x^{\frac{1}{x}}$. If we take natural logs of both sides of this equation and multiply both sides by -1 we have

$$-\ln(y) \cdot e^{-\ln(y)} = \frac{-\ln(x)}{x} \qquad (1).$$

Now $\dfrac{-\ln(x)}{x} > -\dfrac{1}{e}$ (since $\dfrac{\ln(x)}{x}$ has it s maximum of $\dfrac{1}{e}$) when $x = e$ and thus $W\left(-\dfrac{\ln(x)}{x}\right) > -1$ and so $1 + W\left(-\dfrac{\ln(x)}{x}\right) > 0$. (Note $W(u) \geq -1$ with equality only if $u = -\dfrac{1}{e}$).

Taking W of both sides of (1) and using $(*)$ we have from (1) that
$-\ln(y) = W\left(-\dfrac{\ln(x)}{x}\right)$ $(**)$ and so

$$y = \frac{1}{e^{-\ln(y)}} = \frac{1}{e^{W\left(-\frac{\ln(x)}{x}\right)}} = \; using(*) \; \frac{W\left(-\frac{\ln(x)}{x}\right)}{-\frac{\ln(x)}{x}} = -\frac{x}{\ln(x)} \cdot W\left(-\frac{\ln(x)}{x}\right)$$

The expression whose supremum we wish to find is thus

$$\frac{2xy}{x+y} = \frac{2x\left(-\frac{x}{\ln(x)} \cdot W\left(-\frac{\ln(x)}{x}\right)\right)}{x + \left(-\frac{x}{\ln(x)} \cdot W\left(-\frac{\ln(x)}{x}\right)\right)} - \frac{2x^2 W\left(-\frac{\ln(x)}{x}\right)}{\ln(x) \cdot \left(x - \frac{xW\left(-\frac{\ln(x)}{x}\right)}{\ln(x)}\right)} \qquad (***)$$

Now differentiating the second equation in $(*)$ shows $W'(x) = \dfrac{1}{e^{W(x)} \cdot (W(x)+1)}$ and so differentiating $(***)$ gives, after simplification

$$\frac{2W\left(-\frac{\ln(x)}{x}\right)^2\left(\ln(x) - W\left(-\frac{\ln(x)}{x}\right) - 2\right)}{\left(\ln(x) - W\left(-\frac{\ln(x)}{x}\right)\right)^2\left(1 + W\left(-\frac{\ln(x)}{x}\right)\right)} = -\frac{2\ln(y)^2\left(\ln(x)+\ln(y)-2\right)}{\left(\ln(x)+\ln(y)\right)^2\left(1-\ln(y)\right)} \; using\ (**)\ \ (1).$$

Recall $1 - \ln(y) = 1 + W\left(\frac{\ln(x)}{x}\right) > 0$. The expression in (1) thus is positive when $\ln(x) + \ln(y) - 2 < 0$ and negative when $\ln(x) + \ln(y) - 2 > 0$. This last expression in $(***)$ increases if $xy < e^2$ and decreases when $xy > e^2$ and thus has maximum of e when $xy = e^2$ and so e is the desired supremum.

Solution 5

For simplicity, we shall use $\dfrac{2xy}{x+y}$, which equals the given expression. From $y^x = x^y$ we have $\dfrac{\ln(x)}{x} = \dfrac{\ln(y)}{y}$ (Δ), and since $F1(t) = \dfrac{\ln(t)}{t}$ is one to one and negative on $(0,1)$, one to one and positive on $(1,e)$ and one to one and positive on (e,∞) and since $x \neq y$, we can assume that $1 < x < e$ and $y > e$

Now since $y \cdot \ln(x) = x \cdot \ln(y)$, we have that $y \cdot \ln(x) - x = x \cdot (\ln(y) - 1) > 0 \quad (*)$. Since $\dfrac{d}{dx}\left(u(x)^{v(x)}\right) = u(x)^{v(x)} \cdot \left(\dfrac{v(x)}{u(x)}u'(x) + \ln\left(u(x)\right) \cdot v'(x)\right)$ we readily have from $y^x = x^y$ by implicit differentiation that $y' = \dfrac{y \cdot \ln(y) - \frac{y^2}{x}}{y \cdot \ln(x) - x}$ and since $\dfrac{d}{dx}\left(\dfrac{2xy}{x+y}\right) = \dfrac{2\left(x^2y' + y^2\right)}{(x+y)^2}$ we have by substitution that

$$\begin{aligned}
\frac{d}{dx}\left(\frac{2xy}{x+y}\right) &= \frac{2y\left(\ln(y)x^2 + \ln(x)y^2 - 2xy\right)}{\left(y\ln(x) - x\right)(x+y)} \text{ and factoring out } xy \\
&= \frac{2xy^2\left(\frac{\ln(y)}{y}x + \frac{\ln(x)}{x}y - 2\right)}{(y\ln(x) - x)(x+y)^2}, \text{ and since } x^y = y^x, \\
&= \frac{2xy^2\left(\frac{\ln(x^y)}{y} + \frac{\ln(y^x)}{x} - 2\right)}{(y\ln(x) - x)(x+y)^2} \\
&= \frac{2xy^2\left(\ln(x) + \ln(y) - 2\right.}{\left(y\ln(x) - x\right)(x+y)^2}.
\end{aligned}$$

The expression is thus positive (recall $y\ln(x) - x > 0$) when $\ln(x) + \ln(y) - 2 < 0$ and negative when $\ln(x) + \ln(y) - 2 > 0$. Thus $\sup\limits_{(x,y)\in D} \left(\dfrac{x^{-1}+y^{-1}}{2}\right)^{-1}$ increases if $xy < e^2$ and decreases when $xy > e^2$ and so e is the desired supremum.

Editor′s comment: Michael Brozinsky also submitted two more solutions to this problem, each in the spirit of solutions the above.

Also solved by Hatef I. Arshagi, Guilford Technical Community College, Jamestown, NC; Kee-Wai Lau, Hong Kong, China; David Stone and John Hawkins, Georgia Southern University, Statesboro, GA, and the proposer.

5388: *Proposed by Jiglău Vasile, Arad, Romania*

Let $ABCD$ be a cyclic quadrilateral, R and r its exradius and inradius respectively, and a, b, c, d its side lengths (where a and c are opposite sides.) Prove that

$$\frac{R^2}{r^2} \geq \frac{a^2c^2}{b^2d^2} + \frac{b^2d^2}{a^2c^2}.$$

Solution 1 by Toshihiro Shimizu, Kawasaki, Japan

Remark: We assume that $ABCD$ is inscribable (and thus $ABCD$ is bicentric) and excircle is circumcircle.

Let the circumcircle and incircle of $ABCD$ be Γ(with center O), Γ'(with center I), respectively. We fix Γ,Γ' and move A such that $ABCD$ has circumcircle Γ and incircle Γ'. The existence of such quadrilateral is assured by the Poncelet's closure theorem (see also https://en.wikipedia.org/wiki/Poncelet%27s_closure_theorem).

If Γ and Γ' are concentric, the quadrilateral is square and we can easy to check that $R = \sqrt{2}r$ and $\frac{a^2c^2}{b^2d^2} + \frac{b^2d^2}{a^2c^2} = 2$. Thus the equality holds. We assume that Γ and Γ' are not concentric.

As A vary, we only show the case when (r.h.s), that is $\frac{a^2c^2}{b^2d^2} + \frac{b^2d^2}{a^2c^2}$, is maximum. The value is maximum when $\frac{ac}{bd}$ is maximum. We calculate the maximum value.

Let P be the intersection of AC and BD. Let W, X, Y, Z be the tangency point of Γ' with AB, BC, CD, DA, respectively.

Then, we show the following lemma. The point P is a fixed point as A varies. Let E be the intersection of AB and CD. Let F be the intersection of BC and DA. Since the quadrilateral $ABCD$ is inscribable, AC, BD, ZX, WY are all concurrent at point P. (it can be shown by Brianchon's theorem and we omit) Then, ZX is the polar line of F with respect to Γ' and WY is the polar line of E with respect to Γ'. Thus, FE is the polar line of P (intersection of ZX and WY) with respect to Γ'. Moreover, E, P is on the polar line of F with respect to Γ and F, P is on the polar line of E with respect to Γ. (This fact is well known and I saw it in my Japanese book.) Therefore, EF and P are polar line and pole with respect to both Γ and Γ'. We will show that this situation only occurs when P is one of the particular two points. More precisely, since EF is polar line of P with respect to both

Γ and Γ', both PO, PO' are perpendicular to EF. Thus, P must be on OO'. We calculate the position of P (see Figure 1) Let $x = IP$ and $d = IO$. From the point P, draw a line perpendicular to OO' and let S, S' be one of the intersection with Γ, Γ', respectively. Let Q be the intersection of tangent line of Γ at S and OO' and Q' be the intersection of tangent line of Γ' at S' and OO'. We find the condition that $Q = Q'$. This situation is equivalent to the above since $\triangle QSO$ and $\triangle SPO$ is similar right triangle,
$OQ = OS \cdot OS/OP = R^2/(x + d)$. Similarly, $IQ' = IS' \cdot IS'/IP = r^2/x$. Thus,

$$\frac{R^2}{x+d} = d + \frac{r^2}{x}$$

must be hold. Since this equation is quadratic equation, there are at most two valid value of x. As A varies continuously, P moves continuously and can't jump to another point. Thus, P must be fixed point as A varies. Therefore, lemma has been shown.

Now we have fixed point P and line EF are fixed as A varies. We show that EI and FI are perpendicular. Since $WY \perp EI$ and $ZX \perp FI$, it suffices to show that $ZX \perp WY$. Since $\angle ZAP = \angle DAC = \angle DBC = \angle PBX$ and $\angle AZP = \angle FZX = \angle FXZ = \angle BXP$, we have $\triangle ZPA \sim \triangle XPB$. Thus, $\angle ZPA = \angle XPB$. Similarly, $\angle APW = \angle DPY$, $\angle WPB = \angle YPC$, $\angle XPC = \angle ZPD$. Since $\angle APW = \angle YPC$ and $\angle XPC = \angle ZPA$,
$\angle ZPA + \angle WPA = 360°/4 = 90°$. Thus, $ZX \perp WY$.

Let $\theta = \angle IEF$, $\angle DEA = 2\alpha$, $\angle DFC = 2\beta$. The distance between I and EF be $p(> r)$, this value is constant as θ vary. Then, since $EI = p/\sin\theta$ and $FI = p/\cos\theta$,
$\sin\alpha = r/EI = (r\sin\theta)/p$ and $\sin\beta = r/FI = (r\cos\theta)/p$. Thus,
$\cos 2\alpha = 1 - 2\sin^2\alpha = 1 - 2(r^2\sin^2\theta)/p^2$ and $\cos 2\beta = 1 - 2\sin^2\beta = 1 - 2(r^2\cos^2\theta)/p^2$.

Then, from the Law of Sines, it follows that

$$\begin{aligned}
a &= AB \\
&= FB \cdot \frac{\sin\angle BFA}{\sin\angle FAB} \\
&= EF \cdot \frac{\sin\angle FEB}{\sin\angle EBF} \cdot \frac{\sin\angle BFA}{\sin\angle FAB} \\
&= EF \cdot \frac{\sin(\theta - \alpha)\sin 2\beta}{\cos(\beta - \alpha)\cos(\alpha + \beta)} \\
c &= CD \\
&= CF \cdot \frac{\sin\angle DFC}{\sin\angle CDF} \\
&= EF \cdot \frac{\sin\angle CEF}{\sin\angle ECF} \cdot \frac{\sin\angle DFC}{\sin\angle CDF} \\
&= EF \cdot \frac{\sin(\theta + \alpha)\sin 2\beta}{\cos(\alpha + \beta)\cos(\beta - \alpha)}
\end{aligned}$$

b, d are calculated by replacing θ by $\pi/2 - \theta$ and swapping α and β from a, c respectively.

Then, since both denominators are unchanged under these replacement, we get

$$\begin{aligned}\frac{ac}{bd} &= \frac{\sin(\theta-\alpha)\sin(\theta+\alpha)}{\sin(\pi/2-\theta-\beta)\sin(\pi/2-\theta+\beta)}\cdot\frac{\sin^2 2\beta}{\sin^2 2\alpha}\\ &= \frac{\cos 2\theta-\cos 2\alpha}{\cos(\pi-2\theta)-\cos 2\beta}\cdot\frac{1-\cos^2 2\beta}{1-\cos^2 2\alpha}\\ &= \frac{\cos 2\theta-\cos 2\alpha}{\cos(\pi-2\theta)-\cos 2\beta}\cdot\frac{1-\cos 2\beta}{1-\cos 2\alpha}\cdot\frac{1+\cos 2\beta}{1+\cos 2\alpha}\\ &= \frac{1-2\sin^2\theta-\left(1-2(r^2\sin^2\theta)/p^2\right)}{1-2\cos^2\theta-(1-2(r^2\cos^2\theta)/p^2)}\cdot\frac{\cos^2\theta}{\sin^2\theta}\cdot\frac{2-2(r^2\cos^2\theta)/p^2}{2-2(r^2\sin^2\theta)/p^2}\\ &= \frac{p^2-r^2\cos^2\theta}{p^2-r^2\sin^2\theta}.\end{aligned}$$

Since $\sin^2\theta+\cos^2\theta=1$, this value takes maximum when $\sin\theta=0$ and the maximum value is $\frac{p^2-r^2}{p^2}$. Similarly, the minimal value is $\frac{p^2}{p^2-r^2}$ when $\sin\theta=1$. Therefore, the maximal value of $\frac{a^2c^2}{b^2d^2}+\frac{b^2d^2}{a^2c^2}$ is $\left(\frac{p^2-r^2}{p^2}\right)^2+\left(\frac{p^2}{p^2-r^2}\right)^2$. Thus we only need to show that

$$\frac{R^2}{r^2}\ge\left(\frac{p^2-r^2}{p^2}\right)^2+\left(\frac{p^2}{p^2-r^2}\right)^2$$

Now we derive relation between p, r, R. Let K, L be the intersection of line QI and Γ', where K is closer to Q than L. Let the tangent line at K meet Γ at N and the tangent line at L meet Γ at N'. We can see that Q, N, N' are collinear and this line is a tangent line of Γ (see figure 2). Let the tangency point be M. Then, since $\triangle QKN$ and $\triangle QMI$ are similar, $KN = MI\cdot QK/QM = r(p-r)/\sqrt{p^2-r^2}$. Thus, $KO=\sqrt{R^2-\left(\frac{r(p-r)}{\sqrt{p^2-r^2}}\right)^2}$. Similarly, since $\triangle QLN'$ and $\triangle QMI$ are similar, $LN' = MI\cdot QL/QM = r(p+r)/\sqrt{p^2-r^2}$. Thus, $LO=\sqrt{R^2-\left(\frac{r(p+r)}{\sqrt{p^2-r^2}}\right)^2}$. Therefore, since $2r = KO+LO$, it follows that

$$\begin{aligned}2r &= \sqrt{R^2-\left(\frac{r(p-r)}{\sqrt{p^2-r^2}}\right)^2}+\sqrt{R^2-\left(\frac{r(p+r)}{\sqrt{p^2-r^2}}\right)^2}\\ 2r &= \sqrt{R^2-\frac{p+r}{p-r}\cdot r^2}+\sqrt{R^2-\frac{p-r}{p+r}\cdot r^2}\end{aligned}$$

Squaring, we get

$$4r^2 = 2R^2-\left(\frac{p+r}{p-r}+\frac{p-r}{p+r}\right)r^2+2\sqrt{R^2-\frac{p+r}{p-r}\cdot r^2}\sqrt{R^2-\frac{p-r}{p+r}\cdot r^2}$$

$$\left(4+\frac{p+r}{p-r}+\frac{p-r}{p+r}\right)r^2-2R^2 = 2\sqrt{R^2-\frac{p+r}{p-r}\cdot r^2}\sqrt{R^2-\frac{p-r}{p+r}\cdot r^2}$$

Squaring again, we get

$$\left(4+\frac{p+r}{p-r}+\frac{p-r}{p+r}\right)^2 r^4-4R^2r^2\left(4+\frac{p+r}{p-r}+\frac{p-r}{p+r}\right)+4R^4=4R^4-4R^2r^2\left(\frac{p+r}{p-r}+\frac{p-r}{p+r}\right)+4r^4$$

$$\left(\left(4+\frac{p+r}{p-r}+\frac{p-r}{p+r}\right)^2-4\right)r^2=16R^2$$

Therefore,

$$\begin{aligned}\frac{R^2}{r^2}&=\frac{1}{16}\left(\left(4+\frac{p+r}{p-r}+\frac{p-r}{p+r}\right)^2-4\right)\\&=\frac{1}{16}\left(14+\left(\frac{p-r}{p+r}\right)^2+\left(\frac{p+r}{p-r}\right)^2+8\left(\frac{p-r}{p+r}+\frac{p+r}{p-r}\right)\right)\\&=\frac{14\left(p^2-r^2\right)^2+(p-r)^4+(p+r)^4+8\left(p-r\right)\left(p+r\right)\left((p-r)^2+(p+r)^2\right)}{16\left(p-r\right)^2\left(p+r\right)^2}\\&=\frac{2p^4-p^2r^2}{\left(p^2-r^2\right)^2}\end{aligned}$$

Therefore we need to show that

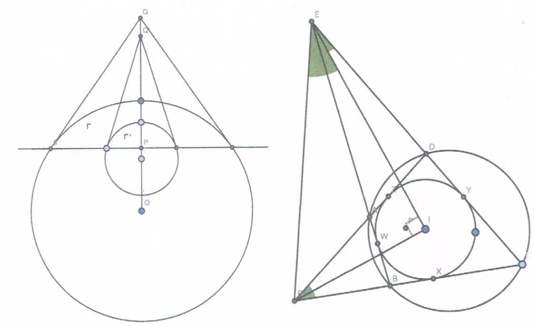
$$\frac{2p^4-p^2r^2}{\left(p^2-r^2\right)^2}\geq\left(\frac{p^2-r^2}{p^2}\right)^2+\left(\frac{p^2}{p^2-r^2}\right)^2$$

It is equivalent to

$$\begin{aligned}\frac{p^4-p^2r^2}{\left(p^2-r^2\right)^2}&\geq\left(\frac{p^2-r^2}{p^2}\right)^2\\\frac{p^2}{p^2-r^2}&\geq\left(\frac{p^2-r^2}{p^2}\right)^2\\p^6&\geq\left(p^2-r^2\right)^3.\end{aligned}$$

The last inequality is obvious.

Following are the diagrams for Lemma 1.



Solution 2 by Kee-Wai Lau, Hong Kong, China

It is well known that $R = \dfrac{1}{4}\sqrt{\dfrac{(ab+cd)(ac+bd)(ad+bc)}{abcd}}$ and $r = \dfrac{\sqrt{abcd}}{a+c} = \dfrac{\sqrt{abcd}}{b+d}$ for the bicentric quadrilateral $ABCD$. Hence the inequality of the problem is equivalent to

$$(ab+cd)(ac+bd)(ad+bc)((a+c)(b+d) - 16\left(a^4c^4 + b^4d^4\right) \geq 0 \qquad (1)$$

By homogeneity, we may assume without loss of generality that

$$c = 1 - a \qquad (2)$$

and

$$d = 1 - b \qquad (3)$$

It can be checked readily, using (2) and (3), that the left hand side of (1) equals

$$(1+4a(1-a))\,a^2\,(a-1)^2\,(2a-1)^2+(1+4b(1-b))\,b^2(b-1)^2(2b-1)^2$$

$$+ab(1-a)(1-b)((2a-1)^2+(2b-1)^2).$$

Since $0<a<1$ amnd $0<b<1$, so the last expression is nonnegative. Thus (1) holds and this completes the solution.

Also solved by Ed Gray, Highland Beach, FL, and the proposer

5389: *Proposed by José Luis Díaz-Barrero, Barcelona Tech, Barcelona, Spain*

Let ABC be a scalene triangle with semi-perimeter s and area $\mathcal{A}$. Prove that

$$\frac{3a+2s}{a(a-b)(a-c)}+\frac{3b+2s}{b(b-a)(b-c)}+\frac{3c+2s}{c(c-a)(c-b)}<\frac{3\sqrt{3}}{4\mathcal{A}}.$$

Solution 1 by Neculai Stanciu, "George Emil Palade" School Buzău, Romania and Titu Zvonaru, Comaesti, Romania

$$\frac{bc^3-b^3c+a^3c-ac^3+ab^3-a^3b}{abc\,(a-b)(b-c)(c-a)}<\frac{3\sqrt{3}}{4A}\iff\frac{(a-b)(b-c)(c-a)(a+b+c)}{abc(a-b)(b-c)(c-a)}<\frac{3\sqrt{3}}{4A}$$

$$\iff\frac{2s}{4AR}<\frac{3\sqrt{3}}{4A}$$

$$\iff 2s<3R\sqrt{3},$$

which is the well-known Mitrinović's inequality (see, e.g., item 5.3 in Geometric Inequalities by O.Bottema et. al., Groningen, 1969.)

Solution 2 by Ed Gray, Highland Beach, FL

Let the statement of the problem be labeled as (1).

Outline of solution: We will show that the left hand side of (1), l.h.s. (1) =l.h.s. (9)
Statement (12) below is derived from a well known identity.
Statement (13) and onward shows that the l.h.s (9)=l.h.s. (12). So in summary,
l.h.s (1)=l.h.s. (9) =l.h.s.(12) $\le \dfrac{3\sqrt{3}}{4A}$.

Collecting the terms on the l.h.s.(1) gives us

(2) $\dfrac{bc(c-b)(4a+b+c)+ac(a-c)(a+4b+c)+ab(bb-a)(a+b+4c)}{abc(a-b)(b-c)(c-a)}$ or

(3) $\dfrac{bc(c-b)4a+(bc)(c^2-b^2)+(ac)(a-c)4b+ac(a^2-c^2)+ab(b-a)4c+ab(b^2-a^2)}{abc(a-b)(b-c)(c-a)}$

(4) $\dfrac{4abc(c-b)+4abc(a-c)+4abc)(b-a)+bc(c^2-b^2)+ab(a^2-c^2)+ab(b^2-a^2)}{abc(a-b)(b-c)(c-a)}$ or

(5) $\dfrac{4abc(c-b+a-c+b)+bc(c-b)(c+b)+ac(a-c)(a+c)+ab(b-a)(b+a)}{abc(a-b)(b-c)(c-a)} =$

(6) $\dfrac{bc(c-b)(c+b)+a-c+b)}{abc(a-b)(b-c)(c-a)} + \dfrac{ac(a-c)(a+c)+a-c+b)}{abc(a-b)(b-c)(c-a)} + \dfrac{ab(b-a)(b+a)}{abc(a-b)(b-c)(c-a)} =$

(7) $\dfrac{c+b}{a(a-b)(a-c)} + \dfrac{a+c}{b(c-b)(a-b)} + \dfrac{a+b}{c(c-b)(c-a)} =$

(8) $\dfrac{bc(c^2-b^2)+ac(a^2-c^2)+ab(b^2-a^2)}{abc(a-b)(b-c)(c-a)}$.

Slightly re-arranging (1) becomes

(9) $\dfrac{a^3(c-b)+b^3(a-c)+c^3(b-a)}{bc(a^3)(c-b)+ac(b^3)(a-c)+ab(c^3)(b-a)} < \dfrac{3\sqrt{3}}{4A}$.

A well known identity (GOOGLE) is:

(10) $\dfrac{9abc}{a+b+c} \geq 4A\sqrt{3}$, or inverting

(11) $\dfrac{a+b+c}{9abc} \leq \dfrac{1}{4A\sqrt{3}} = \dfrac{\sqrt{3}}{12A} = \dfrac{3\sqrt{3}}{36A}$

Multiplying by 9

(12) $\dfrac{a+b+c}{abc} \leq \dfrac{3\sqrt{3}}{4A}$

Hence, it is sufficient to show:

(13) $\dfrac{a^3(c-b)+b^3(a-c)+c^3(b-a)}{bc(a^3)(c-b)+ac(b^3)(a-c)+ab(c^3)(b-a)} \leq \dfrac{a+b+c}{abc}$ or

(14)

$a^3(c-b)+b^3(a-c)+c^3(b-a) \leq (a+b+c)\left(\dfrac{bc(a^3)(c-b)+ac(b^3)(a-c)+ab(c^3)(b-a)}{abc}\right)$

or

(15)

$a^3(c-b)+b^3(a-c)+c^3(b-a) \leq (c-b)\left(a^3+a^2b+a^2c\right)+(a-c)\left(ab^2+b^3+cb^2\right)+(b-a)\left(ac^2+bc^2+c^3\right))$

Transposing from left to right

(16)

$$0 \leq \ (c-b)(a^2)(b+c)+(a-c)(b^2)(a+c)+(b-a)(c^2)(b+a)$$

$$(a^2)(c^2-b^2)+(b^2)(a^2-c^2)+(c^2)(b^2-a^2) \text{ or}$$

(17)

$$0 \leq a^2c^2-a^2b^2+b^2a^2-b^2c^2+b^2c^2-a^2c^2 = 0. \text{ Q.E.D.}$$

Solution 3 by Moti Levy, Rehovot, Israel

The left hand side of the inequality can be simplified,

$$\frac{3a+2s}{a\,(a-b)\,(a-c)}+\frac{3b+2s}{b\,(b-c)\,(b-a)}+\frac{3c+2s}{c\,(c-a)\,(c-b)}=2sabc.$$

Hence the original inequality is equivalent to

$$\frac{2s}{abc}<\frac{3\sqrt{3}}{4\mathcal{A}} \quad \text{or to} \quad 4\mathcal{A}\sqrt{3}<\frac{9abc}{a+b+c}. \qquad (1)$$

It is well known that in any triangle, $\sin A+\sin B+\sin C\leq\frac{3\sqrt{3}}{2}$. Hence

$$\sin A+\sin B+\sin C \;=\; \frac{a}{2R}+\frac{b}{2R}+\frac{c}{2R}\leq\frac{3\sqrt{3}}{2} \quad \text{or}$$

$$a+b+c \;\leq\; 3R\sqrt{3}. \qquad (2)$$

It is well known that $R=\frac{abc}{4\mathcal{A}}$. Labeling this equation as (3), it follows from (2) and (3) that $a+b+c\leq 3\frac{abc}{4\mathcal{A}}\sqrt{3}$, which implies (1).

Remark: Inequality (1) was proposed by T. R. Curry in the "American Mathematical Monthly", Vol. 73 (1966) as elementary problem number 1861.
The solution by Leon Bankoff (who served as the editor of the Problem Department of PME magazine for several years) was selected.

Also solved by Arkady Alt, San Jose, CA; Hatef I. Arshagi, Guilford Technical Community College, Jamestown, NC; Elsie M. Campbell, Dionne T. Bailey, and Charles Diminnie, Angelo State University, San Angelo, TX; Bruno Salgueiro Fanego, Viveiro, Spain; Toshihiro Shimizu, Kawasaki, Japan; Kee-Wai Lau, Hong Kong, China; Nicusor Zlota "Trian Vuia" Technical College, Focsani, Romania, and the proposer.

5390: *Proposed by Ovidiu Furdui, Technical University of Cluj-Napoca, Cluj-Napoca, Romania*

Let $A\in\mathcal{M}_2\,(R)$ such that $AA^T=\begin{pmatrix} a & b \\ b & a \end{pmatrix}$, where $a>b\geq 0$. Prove that $AA^T=A^TA$ if and only if $A=\begin{pmatrix} \alpha & \beta \\ \beta & \alpha \end{pmatrix}$ or $A=\begin{pmatrix} \beta & \alpha \\ \alpha & \beta \end{pmatrix}$, where $\alpha=\frac{\pm\sqrt{a+b}\pm\sqrt{a-b}}{2}$ and $\beta=\frac{\pm\sqrt{a+b}\mp\sqrt{a-b}}{2}$. Here A^T denotes the *transpose* of A.

Solution 1 by Toshihiro Shimizu of Kawasaki, Japan

Remark: I assume that $a > b > 0$.
Let

$$A = \begin{pmatrix} x & y \\ z & w \end{pmatrix}.$$

Then, from the $AA^T = A^T A$, it follows that $x^2 + y^2 = x^2 + z^2 = y^2 + w^2 = z^2 + w^2 = a$, $xz + yw = xy + zw = b$. Thus, $y^2 = z^2, x^2 + z^2 = y^2 + w^2$ and $(x - w)(y - z) = 0$. Thus, it follows that $x = w$ or $y = z$ and $y = \pm z$.
If $y \neq z$, $y = -z \neq 0$ and $x = w$ must be satisfied. Then, we can write

$$A = \begin{pmatrix} x & y \\ -y & x \end{pmatrix}.$$

Then, $a = x^2 + y^2, b = 0$, a contradiction.
Thus $y = z$, then $x = \pm w$. Since $xz + yw = b > 0$, the plus sign must be occured. Thus, we can write

$$A = \begin{pmatrix} x & y \\ y & x \end{pmatrix},$$

and $x^2 + y^2 = a,\ 2xy = b$. Then, $(x + y)^2 = a + b$ implies $x + y = \pm\sqrt{a + b}$. Thus, x, y is a two root of the equation $t^2 \mp \sqrt{a + b}t + b/2 = 0$. Thus,
$\{x, y\} = \left\{\frac{\pm\sqrt{a+b}+\sqrt{a+b-2b}}{2}, \frac{\pm\sqrt{a+b}-\sqrt{a+b-2b}}{2}\right\}$.

Solution 2 by Kee-Wai Lau, Hong Kong, China

If $a > b = 0$, then the matrix $A_0 = \begin{pmatrix} \sqrt{a} & 0 \\ 0 & -\sqrt{a} \end{pmatrix}$ satisfying $A_0 A_0^T = \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix}$ and $A_0 A_0^T = A_0^T A_0$, is neither of the form $\begin{pmatrix} \alpha & \beta \\ \beta & \alpha \end{pmatrix}$ nor of the form $\begin{pmatrix} \beta & \alpha \\ \alpha & \beta \end{pmatrix}$.
Hence in what follows we always assume that $a > b > 0$.

Let $\begin{pmatrix} w & x \\ y & z \end{pmatrix}$ so that $AA^T = \begin{pmatrix} w^2 + x^2 & wy + xz \\ wy + xz & y^2 + z^2 \end{pmatrix}$ and $AA^T = \begin{pmatrix} w^2 + y^2 & wx + yz \\ wx + yz & x^2 + z^2 \end{pmatrix}$.
Hence if $AA^T = \begin{pmatrix} a & b \\ b & a \end{pmatrix}$, then

$$w^2 + x^2 = y^2 + z^2 = a, \qquad (1)$$

and

$$wy + xz = b. \qquad (2)$$

Suppose that $AA^T = A^T A$, then

$$x^2 = y^2, \qquad (3)$$

and

$$wy + xz = wx + yz. \qquad (4)$$

From (4) we obtain $(x - y)(z - w) = 0$. We first suppose that $x = y$. Then by (1), we have $w^2 = z^2$ and by (2) we have $x(w + z) = b$. Since $b > 0$, so $w = z$ and we have

$$\begin{array}{rcl} w^2 + x^2 &=& a \\ 2wx &=& b \qquad (5) \end{array}$$

Solving (5), we obtain $(w, x) =$

$$\left(\frac{\sqrt{a+b}+\sqrt{a-b}}{2}, \frac{\sqrt{a+b}-\sqrt{a-b}}{2}\right), \left(-\frac{\sqrt{a+b}+\sqrt{a-b}}{2}, \frac{\sqrt{a-b}-\sqrt{a+b}}{2}\right),$$
$$\left(\frac{\sqrt{a+b}-\sqrt{a-b}}{2}, \frac{\sqrt{a+b}+\sqrt{a-b}}{2}\right), \left(\frac{\sqrt{a-b}-\sqrt{a+b}}{2}, -\frac{\sqrt{a+b}+\sqrt{a-b}}{2}\right),$$

with corresponding matrices

$$A_1 = \begin{pmatrix} \frac{\sqrt{a+b}+\sqrt{a-b}}{2} & \frac{\sqrt{a+b}-\sqrt{a-b}}{2} \\ \frac{\sqrt{a+b}-\sqrt{a-b}}{2} & \frac{\sqrt{a+b}+\sqrt{a-b}}{2} \end{pmatrix},$$

$$A_2 = \begin{pmatrix} -\frac{\sqrt{a+b}+\sqrt{a-b}}{2} & \frac{\sqrt{a-b}-\sqrt{a+b}}{2} \\ \frac{\sqrt{a-b}-\sqrt{a+b}}{2}, & -\frac{\sqrt{a+b}+\sqrt{a-b}}{2} \end{pmatrix},$$

$$A_3 = \begin{pmatrix} \frac{\sqrt{a+b}-\sqrt{a-b}}{2} & \frac{\sqrt{a+b}+\sqrt{a-b}}{2} \\ \frac{\sqrt{a+b}+\sqrt{a+b}}{2}, & \frac{\sqrt{a+b}-\sqrt{a-b}}{2} \end{pmatrix}, \text{and}$$

$$A_4 = \begin{pmatrix} \frac{\sqrt{a-b}-\sqrt{a-b}}{2} & -\frac{\sqrt{a+b}+\sqrt{a-b}}{2} \\ -\frac{\sqrt{a+b}+\sqrt{a-b}}{2}, & \frac{\sqrt{a-b}-\sqrt{a+b}}{2} \end{pmatrix}.$$

It is easy to check that A_k satisfies $A_k A_k^T = \begin{pmatrix} a & b \\ b & a \end{pmatrix}$ and that $\mathrm{A}_k A_k^T = A_k^T A_k$ for $k = 1, 2, 3, 4$.

Next we suppose that $w = z$. Then by (2), we have $w(x + y) = b$. Since $b > 0$, so by (3), we have $x = y$, and we arrive at (5) again. This completes the solution.

Solution 3 by the Proposer

One implication is easy to prove. If $A = \begin{pmatrix} \alpha & \beta \\ \beta & \alpha \end{pmatrix}$ or $A = \begin{pmatrix} \beta & \alpha \\ \alpha & \beta \end{pmatrix}$, with $\alpha = \dfrac{\pm\sqrt{a+b} \pm \sqrt{a-b}}{2}$ and $\beta = \dfrac{\pm\sqrt{a+b} \mp \sqrt{a-b}}{2}$, then

$$AA^T = A^T A = \begin{pmatrix} \alpha^2 + \beta^2 & 2\alpha\beta \\ 2\alpha\beta & \alpha^2 + \beta^2 \end{pmatrix} = \begin{pmatrix} a & b \\ b & a \end{pmatrix}.$$

Now we prove the other implication. First we note, since $\det(AA^T) = \det^2 A = a^2 - b^2 > 0$, that A is invertible. The equation $AA^T = \begin{pmatrix} a & b \\ b & a \end{pmatrix}$ implies that $A^T = A^{-1}\begin{pmatrix} a & b \\ b & a \end{pmatrix} = A^{-1}(aI_2 + bJ)$, where $J = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. The equation $AA^T = A^T A$ implies that $AA^T = aI_2 + bJ = (aA^{-1} + bA^{-1}J)A = A^T A$, and this in turn implies $bA^{-1}JA = bJ$ and, since $b \neq 0$, we get that $JA = AJ$. Let $A = \begin{pmatrix} x & y \\ u & v \end{pmatrix}$. Since $JA = AJ$ we get that $u = y$ and $v = x$, so $A = \begin{pmatrix} x & y \\ y & x \end{pmatrix}$. We have

$$AA^T = \begin{pmatrix} x^2 + y^2 & 2xy \\ 2xy & x^2 + y^2 \end{pmatrix} = \begin{pmatrix} a & b \\ b & a \end{pmatrix}$$

and this implies that $x^2 + y^2 = a$ and $2xy = b$. Since we have a symmetric system it is clear that the values of x and y could be interchanged. Adding and subtracting these equations we get that $(x+y)^2 = a+b$ and $(x-y)^2 = a-b$, and we have $x + y = \pm\sqrt{a+b}$ and $x - y = \pm\sqrt{a-b}$. Thus, $x = \dfrac{\pm\sqrt{a+b} \pm \sqrt{a-b}}{2}$, $y = \dfrac{\pm\sqrt{a+b} \mp \sqrt{a-b}}{2}$ and the problem is solved.

Also solved by Boris Rays, Brooklyn, NY; Dexter Harrell (Undergraduate Student), Auburn University Montgomery, AL; David Stone and John Hawkins, Georgia Southern University, Statesboro, GA.

Problem 5375* once again

Toshihiro Shimizu of Kawasaki, Japan has the solved 5375*. (We can now remove the asterisk from its label.) Following is a restatement of Kenneth Korbin's problem and Toshihiro's solution to it.

5375 (revised): Prove or disprove the following conjecture. Let k be the product of N different prime numbers each congruent to 1(mod 4).

The total number of different rectangles and trapezoids with integer length sides and diagonals that can be inscribed in a circle with diameter k is exactly $\dfrac{5^N - 3^N}{2}$.

Solution

Let a, b be the length of longer and shorter sides of the trapezoid (or rectangle, in this case let $a = b$), c be the length of other sides and d be the length of the diagonal. Let $\alpha, \beta, \gamma, \delta$ be the central angle with respect to the circumcircle of the segment (side or diagonal) with length a, b, c, d, respectively. We can see that

$$\sin\frac{\alpha}{2} = \frac{a}{k}$$
$$\sin\frac{\beta}{2} = \frac{b}{k}$$
$$\sin\frac{\gamma}{2} = \frac{c}{k}$$
$$\sin\frac{\delta}{2} = \frac{d}{k}.$$

Moreover, $\alpha = \delta - \gamma$, $\beta = \delta + \gamma$. Thus, it follows that

$$\frac{a}{k} = \sin\frac{\alpha}{2} = \sin\left(\frac{\delta}{2} - \frac{\gamma}{2}\right) = \frac{d\sqrt{k^2 - c^2} - c\sqrt{k^2 - d^2}}{k^2}$$
$$\frac{b}{k} = \sin\frac{\beta}{2} = \sin\left(\frac{\delta}{2} + \frac{\gamma}{2}\right) = \frac{d\sqrt{k^2 - c^2} + c\sqrt{k^2 - d^2}}{k^2}.$$

Thus, $k^2 - c^2$ and $k^2 - d^2$ must be perfect square. (♡)Let these perfect squares be c'^2, d'^2, respectively. Then, $ak = dc' - cd'$, $bk = dc' + cd'$. Thus, both dc', cd' must be divisible by k. Since $dc' - cd' > 0$, it must follow that $d > c$.

Conversely, if we are given (c, d) with these condition, we can get a, b and the trapezoid (or rectangle) is determined. Thus, we calculate the number of (c, d).

It follows that

$$k^2 = c^2 + c'^2$$
$$k^2 = d^2 + d'^2$$

Let $k_1 = \gcd(c, k)$ and $k_2 = k/k_1$. Then, $\gcd(c', k) = k_1$ and d is divisible by k_2.

Let k_1 be the product of M prime numbers. We calculate the number of (c, d) with the fixed k_1. Since, the case that $c = d$ is impossible we ignore the condition $d > c$ and divide the result by 2.

The number of c with simply $k_1 \mid c$ is $3^{N-M} - 1$(see Note 2), since the condition is $(k/k_1)^2 = (c/k_1)^2 + (c'/k_1)^2$. But this value over-counts the case that $k_1 p \mid c$, where p is a prime divisor of k but not of k_1. Thus, we need to subtract $3^{N-M-1} - 1$. We also undercounted the case that cpq, where p, q is a prime divisor of k but not k_1, and so on. Thus the number of c is calculated, by Inclusion-exclusion principle, that

$$\sum_{t=0}^{N-M} \binom{N-M}{t} \cdot (-1)^t \left(3^{N-M-t} - 1\right) = (3-1)^{N-M} - (1-1)^{N-M}$$
$$= 2^{N-M}$$

The number of d can be simply calculated as $3^M - 1$. Thus, summing up about M, the total number of $\{c, d\}$ is

$$\begin{aligned} \sum_{M=0}^{N} \binom{N}{M} 2^{N-M} \cdot (3^M - 1) &= (2+3)^N - (2+1)^N \\ &= 5^N - 3^N \end{aligned}$$

Thus, the total number of (c, d) is $\dfrac{5^N - 3^N}{2}$.

Note 1: about ($\heartsuit$): precisely, I think we can show that if $a\sqrt{x} + b\sqrt{y}$ is rational, where $a, b \in Q^+$ and x, y are non-negative integer, then both x, y must be perfect square.

Note 2: From *Jacobi's two square theorem* (http://web.maths.unsw.edu.au/~mikeh/webpapers/paper21.pdf), the number of integer (x, y) with $k^2 = x^2 + y^2$ is

$$4 \sum_{2 \nmid d \mid n} (-1)^{\frac{d-1}{2}} = 4 \sum_{d \mid n} 1 = 4 \cdot 3^N.$$

Among these integer roots, there are four with at least one of them is zero $(\pm k, 0), (0, \pm k)$. Other $4 \cdot 3^N - 4$ of them are classified to $(\pm x, \pm y)$ with $x, y > 0$. Thus, the number of positive integer roots can be written as $3^N - 1$.

$$Mea - Culpa$$

Mistakes happen. **Arkady Alt of San Jose, CA** should have been credited with having solved 5381, and **G. C. Greubel of Newport News, VA** should have been listed for having solved, in two different ways, 5384. I am sorry for these mistakes.