Problems

Ted Eisenberg, Section Editor

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This section of the Journal offers readers an opportunity to exchange interesting mathematical problems and solutions. Please send them to Ted Eisenberg, Department of Mathematics, Ben-Gurion University, Beer-Sheva, Israel or fax to: 972-86-477-648. Questions concerning proposals and/or solutions can be sent e-mail to <eisenbt@013.net>. Solutions to previously stated problems can be seen at <http://www.ssma.org/publications>.

Solutions to the problems stated in this issue should be posted before
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5499: *Proposed by Kenneth Korbin, New York, NY*

Given a triangle with sides $(21, 23, 40)$. The sum of these digits is $2+1+2+3+4+0 = 12$. Find primitive pythagorean triples in which the sum of the digits is 12 or less.

5500: *Proposed by Moshe Stupel, "Shaanan" Academic College of Education and Gordon Academic College of Education, and Avi Sigler, "Shaanan" Academic College of Education, Haifa, Israel*

Without the use of a calculator, show that: $8\sin 20^\circ \cdot \sin 40^\circ \cdot \sin 60^\circ \cdot \sin 80^\circ = \dfrac{3}{2}$.

5501: *Proposed by D.M. Bătinetu-Giurgiu, Bucharest, Romania, Neculai Stanciu, "George Emil Palade" School Buzău, Romania and Titu Zvonaru, Comănesti, Romania*

Determine all real numbers a, b, x, y that simultaneously satisfy the following relations:

$$\begin{cases} (1) \qquad ax + by = 5 \\ \\ (2) \qquad ax^2 + by^2 = 9 \\ \\ (3) \qquad ax^3 + by^3 = 17 \\ \\ (4) \qquad ax^4 + by^4 = 33. \end{cases}$$

5502: *Proposed by Daniel Sitaru, "Theodor Costescu" National Economic College, Drobeta Turnu-Severin, Mehedinti, Romania*

Prove that if $a, b, c > 0$ and $a + b + c = e$ then

$$e^{ac^e} \cdot e^{ba^e} \cdot e^{cb^e} > e^e \cdot a^{be^2} \cdot b^{ce^2} \cdot c^{ae^2}.$$

Here, $e = \lim_{n\to\infty}\left(1 + \dfrac{1}{n}\right)^n$

5503: *Proposed by José Luis Díaz-Barrero, Barcelona Tech, Barcelona, Spain*

Let $a_1, a_2, \ldots, a_n$ be positive real numbers with $n \geq 2$. Prove that

$$\frac{(a_1^m a_2 + a_2^m a_3 + \cdots + a_n^m a_1)^m}{(a_1^m + a_2^m + \cdots a_n^m)^{m+1}} \leq \frac{1}{n},$$

where m is a positive integer.

5504: *Proposed by Ovidiu Furdui and Alina Sîntămărian both at the Technical University of Cluj-Napoca, Cluj-Napoca, Romania*

Let $n \geq 0$ be an integer. Calculate

$$\int_0^1 \frac{x^n}{\left\lfloor \frac{1}{x} \right\rfloor} dx,$$

where $\lfloor x \rfloor$ denotes the integer part of x.

Solutions

5481: *Proposed by Kenneth Korbin, New York, NY*

A triangle with integer area has integer length sides $(3, x, x+1)$. Find five possible values of x with $x > 4$.

Solution 1 by Dionne Bailey, Elsie Campbell, Charles Diminnie, and Trey Smith, Angelo State University, San Angelo, TX

For our approach, we will need to find positive integer solutions for the equation

$$m^2 - 8k^2 = 9. \tag{1}$$

One way to do so is to first solve the Pell Equation

$$X^2 - 8Y^2 = 1 \tag{2}$$

and then set $m = 3X$ and $k = 3Y$.

Following the usual process for solving (2), we note that the solution with the smallest X value is $X = 3$, $Y = 1$. Then, all solutions (X_n, Y_n) of (2) can be found by setting

$$X_n + Y_n\sqrt{8} = \left(3 + \sqrt{8}\right)^n$$

for all $n \geq 1$. Then, as described above, we get solutions for (1) by setting $m_n = 3X_n$ and $k_n = 3Y_n$. The first six solutions for (1) and (2) are listed in the following table:

n	X_n	Y_n	m_n	k_n
1	3	1	9	3
2	17	6	51	18
3	99	35	297	105
4	577	204	1,731	612
5	3,363	1,189	10,089	3,567
6	19,601	6,930	58,803	20,790

(3)

For the problem at hand, the semiperimeter s of our triangle is

$$s = \frac{3 + x + (x+1)}{2} = x + 2$$

and Heron's Formula for the area A yields

$$\begin{aligned} A &= \sqrt{s\,(s-3)\,(s-x)\,(s-x-1)} \\ &= \sqrt{(x+2)\,(x-1)\,(2)\,(1)} \\ &= \sqrt{2\,(x^2+x-2)}. \end{aligned}$$

For A to be a positive integer, we must find a positive integer k for which

$$A^2 = 2\left(x^2 + x - 2\right) = 4k^2$$

or

$$x^2 + x - 2 - 2k^2 = 0. \tag{4}$$

By the Quadratic Formula, the positive solution of (4) is

$$\begin{aligned} x &= \frac{-1 + \sqrt{1 + 8\,(k^2+1)}}{2} \\ &= \frac{-1 + \sqrt{8k^2+9}}{2}. \end{aligned}$$

For x to be a positive integer, we will need

$$8k^2 + 9 = m^2$$

or

$$m^2 - 8k^2 = 9$$

for some odd positive integer m. However, table (3) gives us six solutions to use. In each case,

$$x = \frac{-1 + \sqrt{8k^2+9}}{2} = \frac{m-1}{2} \quad \text{and} \quad A = 2k.$$

The solution $m_1 = 9$ and $k_1 = 3$ yields $x = 4$, which is ruled out in the statement of the problem. The other five entries in the table provide five plausible values of x for which A is a positive integer. These values are listed in our final table:

n	m_n	k_n	x	A
2	51	18	25	36
3	297	105	148	210
4	1,731	612	865	1,224
5	10,089	3,567	5,044	7,134
6	58,803	20,790	29,401	41,580

Solution 2 by Bruno Salgueiro Fanego, Viveiro, Spain

Let p and S be the semi perimeter and the area of such a triangle respectively. Then $2p = 3 + x + x + 1 = 2x + 4$ and, by Heron's formula
$S = \sqrt{p(p-3)(p-x)(p-(x+1))} = \sqrt{2x^2 + 2x - 4}$ must be an integer.

It can be easily verified that for each of the five values of $x \in \{25, 148, 865, 5044, 29401\}$ one obtains triangles that have areas of $36, 210, 1224, 7134, 41580$, respectively.

More generally, if $\begin{pmatrix} x_1 \\ S_1 \end{pmatrix} = \begin{pmatrix} 25 \\ 36 \end{pmatrix}$, then the recurrence given by
$\begin{pmatrix} x_{n+1} \\ S_{n+1} \end{pmatrix} = \begin{pmatrix} 3 & 2 \\ 4 & 3 \end{pmatrix} \begin{pmatrix} x_n \\ S_n \end{pmatrix} + \begin{pmatrix} 1 \\ 2 \end{pmatrix}$ for any integer $n \geq 1$, gives a pair $\begin{pmatrix} x_{n+1} \\ S_{n+1} \end{pmatrix}$ where x_{n+1} is the length of a triangle with integer length sides $(3, x_{n+1}, x_{n+1} + 1)$ and S_{n+1} is the integer area of that triangle.

Solution 3 by Julio Cesar Mohnsam and Luiz Lemos Junior, both at IFSUL Campus Pelotas-RS, Brazil

Let p be the semi-perimeter $p = \dfrac{3 + x + x + 1}{2} = x + 2$
The area by Heron is given by:
$A = \sqrt{p(p-3)(p-x)(p-x-1)} = \sqrt{(x+2)(x-1)(2)(1)}$
Then $(x+2)(x-1)(2)$ must be a square, that is, $2x^2 + 2x - 4 = y^2$, follow that:

$$2x^2 + 2x - y^2 - 4 = 0 \tag{1}$$

Multiplying (1) by 8 we have:

$$16x^2 + 16x - 8y^2 - 32 = 0 \tag{2}$$

Adding 4 on both sides of (2) we have:

$$(4x+2)^2 - 8y^2 - 36 = 0 \tag{3}$$

Now make $X = 4x + 2$ and $Y = y$, we have:

$$X^2 - 8Y^2 - 36 = 0 \tag{4}$$

(4) is a diophantine equation of the form $ax^2 - by^2 + c = 0$ in the case of $c = -1$ we have the particular case of the equation of Pell $x^2 - Dy^2 = 1$. If $(a, b)|c$, the equation has a solution. Let's solve (4) using the method Florentin Smarandache [1].
We consider the equation

$$aX^2 - bY^2 + c = 0 \tag{5}$$

and

$$a\alpha^2 - b\beta^2 = a \tag{6}$$

We set the matrix A from (6) as follows:

$$A = \begin{bmatrix} \alpha_0 & \frac{b}{a}\beta_0 \\ \beta_0 & \alpha_0 \end{bmatrix}$$

where (α_0, β_0) are initial solutions of (6).

Now let (X_0, Y_0) are initial solutions of (5), then the general solutions of (5) are given by the following recurrence relation:

$$\begin{bmatrix} X_n \\ Y_n \end{bmatrix} = A^n \begin{bmatrix} X_0 \\ Y_0 \end{bmatrix}$$

Thus, by solving (4) we have to first assemble matrix A from $X^2 - 8Y^2 = 1$, note that this Pell equation has initial solution $(3, 1)$, so we have:

$$A = \begin{bmatrix} 3 & 8 \\ 1 & 3 \end{bmatrix}$$

But (4) has initial solution $(6, 0)$. So we have to:

$$\begin{bmatrix} X_1 \\ Y_1 \end{bmatrix} = \begin{bmatrix} 3 & 8 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} 6 \\ 0 \end{bmatrix} = \begin{bmatrix} 18 \\ 6 \end{bmatrix}$$

Like $18 = X_1 = 4x_1 + 2 \to x_1 = 4$ but we have to find $x > 4$.
Thus we calculate A^2, such that:

$$\begin{bmatrix} X_2 \\ Y_2 \end{bmatrix} = \begin{bmatrix} 3 & 8 \\ 1 & 3 \end{bmatrix}^2 \begin{bmatrix} 6 \\ 0 \end{bmatrix} = \begin{bmatrix} 17 & 48 \\ 6 & 17 \end{bmatrix} \begin{bmatrix} 6 \\ 0 \end{bmatrix} = \begin{bmatrix} 102 \\ 36 \end{bmatrix}$$

As $102 = X_2 = 4x_2 + 2 \to x_2 = 25$ and the lengths of the first triangle are $(3, 25, 26)$. To find the other values of x we will diagonalize the matrix A. We know that
$A^n = PD^nP^-1$.

$$A = PDP^{-1} = P \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} P^{-1}$$

Eigenvalues $\lambda_1 = 3 + 2\sqrt{3}$ and $\lambda_2 = 3 - 2\sqrt{3}$
Eigenvectors $\vec{v}_1 = \begin{pmatrix} 2\sqrt{3} \\ 1 \end{pmatrix}$ and $\vec{v}_2 = \begin{pmatrix} -2\sqrt{3} \\ 1 \end{pmatrix}$
Therefore $x = \{25, 148, 565, 5044, 29401\}$

[1] Smarandache F. **“Un metodo de resolucion de la ecuacion diofantica**. Gazeta Matematica, Serie 2, Vol. 1, Nr. 2, 1988. Madrid. p. 151-157.

Solution 4 by Brian D. Beasley, Presbyterian College, Clinton, SC

Given such a triangle, its semiperimeter is $s = (3 + x + x + 1)/2 = x + 2$. Then by Heron's formula, its area is

$$\Delta = \sqrt{((x+2)(x-1)(2)(1)} = \sqrt{2(x^2 + x - 2)},$$

so we seek integers Δ and x with $x > 4$ such that $\Delta^2 = 2x^2 + 2x - 4$. This equation in turn is equivalent to

$$2\Delta^2 + 9 = (2x+1)^2, \quad \text{or} \quad 2\left(\frac{\Delta}{3}\right)^2 + 1 = \left(\frac{2x+1}{3}\right)^2.$$

We let $a = (2x + 1)/3$ and $b = \Delta/3$ in order to solve the Pellian equation $2b^2 + 1 = a^2$. This equation has infinitely many integer solutions for a and b, which we may describe with the sequences

$$a_0 = 1, a_1 = 3, \text{ and } a_{n+2} = 6a_{n+1} - a_n \quad \text{for } n \geq 0;$$
$$b_0 = 0, b_1 = 2, \text{ and } b_{n+2} = 6b_{n+1} - b_n \quad \text{for } n \geq 0.$$

Thus there are infinitely many integers x which satisfy the requirements of the problem, given by the terms x_n (with $n \geq 2$) of the sequence

$$x_0 = 1, x_1 = 4, \text{ and } x_{n+2} = 6x_{n+1} - x_n + 2 \quad \text{for } n \geq 0.$$

In particular, the next five values of x after 4 are 25, 148, 865, 5044, and 29401.

Addenda. (i) We may also describe the above sequences by letting $\gamma = 3 + 2\sqrt{2}$ and $\delta = 3 - 2\sqrt{2}$. Then $a_n = (\gamma^n + \delta^n)/2$ for each $n \geq 0$, which implies that $x_n = (3\gamma^n + 3\delta^n - 2)/4$.

(ii) We further note that the ratios a_n/b_n for $n \geq 1$ occur as every other term in the sequence of converging to the continued fraction representation of $\sqrt{2}$.

Comments by Editor : **Ioannis D. Sfikas of Athens Greece** started his solution off with some nomenclature and bit of history about the problem.

"A triangle whose sides and area are rational numbers is called a *rational triangle.* If the rational triangle is right-angled, it is called a *right-angled rational triangle* or a *rational Pythagorean triangle* or a *numerical right triangle.* If the sides of a rational triangle is of integer length, it is called an *integer triangle.* If further these sides have no common factor greater than unity, the triangle is called a *primitive integer triangle.* If the integer triangle is right-angled, it is called a *Pythagorean triangle. A Heronian triangle* (named after Heron of Alexandria) is an integer triangle with the additional property that its area is also an integer. A Heronian triangle is called *primitive Heronian triangle* if sides have no common factor greater than unity. In the 7th century, the Indian mathematician Brahmagupta studied the special case of triangles with consecutive integer sides."

Kenneth Korbin, the proposer of this problem stated that triangles sides with lengths $(3, x, x+1)$ with $x \geq 4$ have an area of $\sqrt{2x^2 + 2x - 4}$, and are associated with the sequences of $(25, 148, 865, 5044, 29401, \ldots, x_N, \ldots)$ that satisfies the recursion of $x_{N+1} = 6x_N - x_{N-1} + 2$.

David Stone and John Hawkins of Southern Georgia University asked in their solution, why the values of $x \geq 4$? Why not $x > 0$? They then stated that: If

$$\begin{aligned}
&x = 1, \text{ the triangle } (3,1,2) \text{ is degenerate;}\\
&x = 2, \text{ the triangle } (3,2,3) \text{ has area } \sqrt{8} \text{ and is not Heronian;}\\
&x = 3, \text{ the triangle } (3,3,4) \text{ has area } \sqrt{20} \text{ and is not Heronian; and}\\
&x = 4, \text{ the right triangle } (3,5,5) \text{ is too easy.}
\end{aligned}$$

They then asked: What about those Heronian triangles of the form $(3, x, x+2)$, where x is an integer. Applying Heron's Formula they obtained that $16A^2 = 20x^2 + 40x - 25$ and stated that there are no integer solutions to this for $x \geq 1$ because the left-and side is even while the right hand side is odd.

They then looked at triangles of the form $(3, x, x+3)$ and stated that the only triangle of this form is degenerate. Moreover, no triangle of the form $(3, x, x+d)$ can exist for $d > 3$.

They continued on with the following:

"Thus the problem poser selected the one form that does admit solutions. Still to be investigated: finding the Heronian triangles of the form $(4, x, x+1)$, and those of the form $(4, x, x+2)$, etc."

Their solution ended with the statement: "There are no Heronian triangles of the form $(n, x, x+d)$ for positive integers n and d having opposite parity."

Also solved by Hatef I. Arshagi, Guilford Technical Community College, Jamestown, NC; Jeremiah Bartz, University of North Dakota, Grand Forks, ND; Anthony Bevelacqua University of North Dakota, Grand Forks, ND; Ed Gray, Highland Beach, FL; Paul M. Harms, North Newton, KS; Kee-Wai Lau, Hong Kong, China; Carl Libis, Columbia Southern University, Orange Beach, AL; David E. Manes, Oneonta, NY; Ioannis D. Sfikas, National and Kapodistrian University of Athens, Greece; Albert Stadler, Herrliberg, Switzerland; Neculai Stanciu,"George Emil Palade" School, Buzău and Tito Zvonaru, Comănesti, Romania; David Stone and John Hawkins of Southern Georgia University, Statesboro, GA, and the proposer.

5482: *Proposed by Daniel Sitaru, "Theodor Costescu" National Economic College, Drobeta Turnu-Severin, Mehedinti, Romania*

Prove that if n is a natural number then

$$\frac{(tan5^\circ)^n}{(tan4^\circ)^n+(tan3^\circ)^n}+\frac{(tan4^\circ)^n}{(tan3^\circ)^n+(tan2^\circ)^n}+\frac{(tan3^\circ)^n}{(tan2^\circ)^n+(tan1^\circ)^n}\geq\frac{3}{2}.$$

Solutions 1 and 2 by Henry Ricardo, Westchester Area Math Circle, NY

Solution 1.

Since, for a fixed natural number n, $(\tan x)^n$ is an increasing positive function for $x\in[0,90^\circ)$, we have

$$\begin{aligned}\frac{(\tan 5^\circ)^n}{(\tan 4^\circ)^n+(\tan 3^\circ)^n}&\geq\frac{(\tan 5^\circ)^n}{(\tan 5^\circ)^n+(\tan 5^\circ)^n}=\frac{1}{2},\\ \frac{(\tan 4^\circ)^n}{(\tan 3^\circ)^n+(\tan 2^\circ)^n}&\geq\frac{(\tan 4^\circ)^n}{(\tan 4^\circ)^n+(\tan 4^\circ)^n}=\frac{1}{2},\\ \frac{(\tan 3^\circ)^n}{(\tan 2^\circ)^n+(\tan 1^\circ)^n}&\geq\frac{(\tan 3^\circ)^n}{(\tan 3^\circ)^n+(\tan 3^\circ)^n}=\frac{1}{2},\end{aligned}$$

so that adding these inequalities gives us the desired result. Equality holds if and only if $n=0$ (assuming that 0 is considered a natural number).

Solution 2.

Since, for a fixed natural number n, $(\tan x)^n$ is an increasing positive function for $x\in[0,90^\circ)$, we have

$$\frac{(\tan 3^\circ)^n}{(\tan 2^\circ)^n+(\tan 1^\circ)^n}\geq\frac{(\tan 3^\circ)^n}{(\tan 4^\circ)^n+(\tan 5^\circ)^n},$$

and

$$\frac{(\tan 4^\circ)^n}{(\tan 3^\circ)^n+(\tan 2^\circ)^n}\geq\frac{(\tan 4^\circ)^n}{(\tan 3^\circ)^n+(\tan 5^\circ)^n},$$

so that

$$\sum_{cyclic} \frac{(\tan 3^\circ)^n}{(\tan 2^\circ)^n + (\tan 1^\circ)^n} \geq \frac{(\tan 5^\circ)^n}{(\tan 4^\circ)^n + (\tan 3^\circ)^n} + \frac{(\tan 4^\circ)^n}{(\tan 3^\circ)^n + (\tan 5^\circ)^n} + \frac{(\tan 3^\circ)^n}{(\tan 4^\circ)^n + (\tan 5^\circ)^n}.$$

Setting $a = (\tan 3^\circ)^n, b = (\tan 4^\circ)^n$, and $c = (\tan 5^\circ)^n$, we see that the right-hand side of the last inequality has the form

$$\frac{a}{b+c} + \frac{b}{c+a} + \frac{c}{a+b},$$

for $a, b, c > 0$, which is greater than or equal to $3/2$ by Nesbitt's inequality. Equality holds if and only if $n = 0$ (assuming that 0 is considered a natural number).

Solution 3 by Ed Gray, Highland Beach, FL

First we retrieve the required values:
1. $\tan 1^\circ = .017455065$
2. $\tan 2^\circ = .034920769$
3. $\tan 3^\circ = .052407779$
4. $\tan 4^\circ = .069926812$
5. $\tan 5^\circ = .087488664$

We rewrite the problem's equation as:

$$\frac{1}{\dfrac{\tan 4^\circ}{\tan 5^\circ} + \dfrac{\tan 3^\circ}{\tan 5^\circ}} + \frac{1}{\dfrac{\tan 3^\circ}{\tan 4^\circ} + \dfrac{\tan 2^\circ}{\tan 4^\circ}} + \frac{1}{\dfrac{\tan 2^\circ}{\tan 3^\circ} + \dfrac{\tan 1^\circ}{\tan 3^\circ}} \geq \frac{3}{2}$$

Substituting the values from steps 1-5 and performing the indicated divisions we define:

$$f(n) = \frac{1}{(.799267114)^n + (.599023652)^n} + \frac{1}{(.794551256)^n + (.499433116)^n} + \frac{1}{(.66632797)^n + (.333062483)^n}.$$

We note that $f(n)$ is an increasing function of n since the denominators clearly decrease as n increases.

Finally we note that $f(1) = .715158838 + 1.248899272 + 1.000609919 = 2.964668029 > \dfrac{3}{2}$.
Then the equality holds for all n since $f(n)$ is an increasing function.

Solution 4 by David Stone and John Hawkins, Georgia Southern University, Statesboro, GA

Lemma: For fixed positive reals a, b, c with $a < c,\ b < c$ let $f(x) = \dfrac{c^x}{b^x + a^x}$ for $x \geq 0$.
Then $f(x) \geq \dfrac{1}{2}$, for $x \geq 0$, with equality holding only for $x = 0$.
Proof: We calculate the derivative:

$$f'(x) \quad = \quad \frac{(b^x + a^x)\, c^x \ln c - c^x\, (a^x \ln a + b^x \ln b)}{(b^x + a^x)^2}$$

$$
\begin{aligned}
&= c^x \frac{(b^x + a^x)\ln c - (a^x \ln a + b^x \ln b)}{(b^x + a^x)^2} \\
&= c^x \frac{b^x(\ln c - \ln b) + a^x(\ln c - \ln a)}{(b^x + a^x)^2}.
\end{aligned}
$$

The ln function is increasing, so $\ln c > \ln b$ and $\ln c > \ln a$; thus we see that the derivative is positive. Hence the function f is increasing, so $\frac{1}{2} = \text{f}(0) \le f(x)$ for $x \ge 0$. Because the derivative is strictly positive, the function f actually grows: so $f(x) > \frac{1}{2}$ for $x > 0$.

To verify the inequality of the problem, we note that the tangent function is increasing, so in each summand the tangent term in the numerator is larger that each tangent term in the denominator. Hence we can apply the lemma to each of the three summands, forcing the sum $\ge \frac{3}{2}$. Note that equality holds if and only if $n = 0$.

Comment: We can apply the lemma to obtain some ugly inequalities which are clearly true:

$$
\begin{aligned}
\frac{3^n}{1^n + 2^n} + \frac{4^n}{2^n + 3^n} + \frac{5^n}{3^n + 4^n} + \cdots + \frac{(n+2)^n}{n^n + (n+1)^n} &\ge \frac{n}{2}, \text{ and} \\
\frac{[(n+2)!]^n}{[n!]^n + [(n+1)!]^n} &\ge \frac{1}{2}.
\end{aligned}
$$

Also solved by Arkady Alt, San Jose, CA (two solutions); Paul M. Harms, North Newton, KS; Kee-Wai Lau, Hong Kong, China; Paolo Perfetti, Department of Mathematics, University of Tor Vergata, Rome, Italy; Angel Plaza, University of Las Palmas de Gran Canaria, Spain; Ioannis D. Sfikas, National and Kapodistrian University of Athens, Greece; Albert Stadler, Herrliberg, Switzerland, and the proposers

5483: *Proposed by D.M. Bătinetu-Giurgiu, "Matei Basarab" National College, Bucharest and Neculai Stanciu, "George Emil Palade" School Buzău, Romania*

If $a, b > 0$, and $x \in \left(0, \frac{\pi}{2}\right)$ then show that

$$
\begin{aligned}
&(i) \qquad (a+b) \cdot \frac{\sin x}{x} + \frac{2ab}{a+b} \cdot \frac{\tan x}{x} \ge \frac{6ab}{a+b}. \\
&(ii) \qquad a \cdot \tan x + \ b \cdot \sin x > 2x\sqrt{ab}.
\end{aligned}
$$

Solution 1 by Paolo Perfetti, Department of Mathematics, Tor Vergata University, Rome, Italy

Proof of (i).

The AHM yields

$$a+b \geq \frac{4}{\frac{1}{a}+\frac{1}{b}} \quad \Longleftrightarrow \quad (a+b)^2 \geq 4ab$$

and then

$$(a+b)\cdot\frac{\sin x}{x}+\frac{2ab}{a+b}\cdot\frac{\tan x}{x} \geq \frac{4ab}{a+b}\cdot\frac{\sin x}{x}+\frac{2ab}{a+b}\cdot\frac{\tan x}{x}$$

Thus we prove

$$\frac{4ab}{a+b}\cdot\sin x+\frac{2ab}{a+b}\cdot\tan x-\frac{6ab}{a+b}x \geq 0$$

This is equivalent to

$$f(x) \doteq 4\sin x+2\tan x-6x \geq 0$$

$$f'(x)=4\cos x+\frac{2}{\cos^2 x}-6$$

$$f''(x)=-4\sin x+\frac{4\sin x}{\cos^3 x}=4\sin x\left(\frac{4}{\cos^3 x}-1\right)>0$$

via $\cos x \in (0,1)$ for $0<x<\pi/2$. Since $f'(0)=f(0)=0$ we get $f(x) \geq 0$.

Proof of (ii).
Let

$$f(x)=a\cdot\tan x+b\cdot\sin x-2x\sqrt{ab}, \quad f(0)=0$$

$$f'(x)=\frac{a}{\cos^2 x}+b\cos x-2\sqrt{ab} \geq \frac{a}{\cos x}+b\cos x-2\sqrt{ab} \geq 2\sqrt{\frac{ab\cos x}{\cos x}}-2\sqrt{ab}=0$$

and this concludes the proof.

Solution 2 by Arkady Alt, San Jose, CA

(i) First we will prove inequality
$\tan x+2\sin x>3x \iff \dfrac{\tan x}{x}+\dfrac{2\sin x}{x}>3,\ x\in(0,\pi/2)$.
Let $h(x):=\tan x+2\sin x-3x,\ x\in(0,\pi/2)$. Since $h'(x)=\dfrac{1}{\cos^2 x}+2\cos x-3=$
$\dfrac{(2\cos x+1)(1-\cos x)^2}{\cos^2 x}>0,\ x\in(0,\pi/2)$ then $h(x)>h(0)=0$.
Hence, $(a+b)\dfrac{\sin x}{x}+\dfrac{2ab}{a+b}\cdot\dfrac{\tan x}{x}>(a+b)\sin x+\dfrac{2ab}{a+b}\cdot\left(3-\dfrac{2\sin x}{x}\right)=$
$\dfrac{\sin x}{x}\left(a+b-\dfrac{4ab}{a+b}\right)+\dfrac{6ab}{a+b}=\dfrac{\sin x}{x}\cdot\dfrac{(a-b)^2}{a+b}+\dfrac{6ab}{a+b}\geq\dfrac{6ab}{a+b}$.
(ii) Let $h(x):=a\tan x+b\sin x-2x\sqrt{ab}$. Since $h'(x)=\dfrac{a}{\cos^2 x}+b\cos x-2\sqrt{ab}\geq$
$2\sqrt{\dfrac{a}{\cos^2 x}\cdot b\cos x}-2\sqrt{ab}=2\sqrt{ab}\cdot\dfrac{1-\sqrt{\cos x}}{\sqrt{\cos x}}>0$ then $h(x)>h(0)=0 \iff$
$a\tan x+b\sin x>2x\sqrt{ab}$.

Solution 3 by Kee-Wai Lau, Hong Kong, China

It is well known that for $x \in \left(0, \dfrac{\pi}{2}\right)$, we have $\sin x - \dfrac{x^3}{6}$ and $\tan x \geq x + \dfrac{x^3}{3}$. Since $a + b = \dfrac{4ab + (a-b)^2}{a+b} \geq \dfrac{4ab}{a+b}$, so the left side of (i) is greater than or equal to

$$\frac{2ab}{a+b}\left(\frac{2\sin x + \tan x}{x}\right) \geq \frac{2ab}{a+b}\left(2\left(1 - \frac{x^2}{6}\right) + \left(1 + \frac{x^2}{3}\right)\right) = \frac{6ab}{a+b},$$

as required.

It is also well known that for $x \in \left(0, \dfrac{\pi}{2}\right)$, $\cos x \leq 1 - \dfrac{x^2}{2} + \dfrac{x^4}{24}$, so that

$$\sin^2 x = x^2 \cos x \geq \left(x - \frac{x^3}{6}\right)^2 - x^2\left(1 - \frac{x^2}{2} + \frac{x^4}{24}\right) = \frac{x^4(12 - x^2)}{72} \geq 0.$$

Hence,

$$a \cdot \tan x + b \cdot \sin x \geq 2\sqrt{(a\tan x)(b\sin x)} = 2\sqrt{ab}x\sqrt{\frac{\sin^2 x}{x^2\cos x}} \geq 2\sqrt{ab}x$$

and (ii) holds.

Also solved by Ioannis D. Sfikas, National and Kapodistrian University of Athens, Greece; Albert Stadler, Herrliberg, Switzerland; David Stone and John Hawkins, Southern Georgia University, Statesboro, GA, and the proposers

5484: *Proposed by Mohsen Soltanifar, Dalla Lana School of Public Health, University of Toronto, Canada*

Let X_1, X_2 be two continuous positive valued random variables on the real line with corresponding mean, median, and mode $\overline{x_1}$, $\widetilde{x}_1$, $\widehat{x}_1$ and $\overline{x_2}$, $\widetilde{x}_2$, $\widehat{x}_2$ respectively. Assume for their associated CDFs, (Cumulative Distribution Functions) we have

$$F_{X_1}(t) \leq F_{X_2}(t) \qquad (t > 0).$$

Prove or give a counter example:

$$\text{(i) } \overline{x_2} \leq \overline{x_1}, \quad \text{(ii) } \widetilde{x_2} \leq \widetilde{x_1}, \quad \text{(iii) } \widehat{x}_2 \leq \widehat{x}_1.$$

Solution 1 by Albert Stadler, Herrliberg, Switzerland

(i) We have $\overline{x_2} = E(X_2) = \displaystyle\int_0^\infty (1 - F_{X2}(t))dt \leq \int_0^\infty (1 - F_{X_1}(t))dt = E\left(X_1\right) = \overline{x_1}$.

(ii) By definition, $F_{X_1}\left(\widetilde{X_1}\right) = F_{X_2}\left(\widetilde{X_2}\right) = \dfrac{1}{2}$. The functions $t \to F_{X_1}(t)$ and $t \to F_{X_2}(t)$ are monotonically increasing. Therefore $F_{X_2}\left(\widetilde{X_2}\right) \leq F_{X_1}(\widetilde{X_1})$ implies $\widetilde{x_2} \leq \widetilde{x_1}$.

(iii) We construct a counter example as a follows:
Let $p_1(x) = 0$, if $x \leq 1/3$ or $x \geq 1$, $p_1(x) = 9x - 3$, if $1/3 \leq x \leq 2/3$, and $p_1(x) = 9 - 9x$, if $2/3 \leq x \leq 1$.

Let $p_2(x) = 36x$, if $0 \le x \le 1/6$, $p_2(x) = 12 - 36x$, if $1/6 \le x \le 1/3$, $p_2(x) = 0$, if $x \le 0$ or $x \ge 1/3$.

$p_1(x)$ and $p_2(x)$ are probability density functions since $\int_0^1 p_1(x)dx = \int_0^1 p_2(x)dx = 1$.

Obviously, $F_{X_1}(x) = \int_0^x p_1(t)dt \le \int_0^x p_2(t)dt = F_{X_2}(x)$, however $\widetilde{x_1} = 3$, $\widetilde{x_2} = 6$.

Solution 2 by Kee-Wai Lau, Hong Kong, China

We answer (i) and (ii) in the affirmative and (iii) in the negative.

Let X be a continuous random variable on the real line with CDF $F_X(t)$ and mean $\overline{x}$. It is known that $\overline{x} = \int_0^\infty (1 - F_X(t))dt - \int_{-\infty}^0 F_X(t)dt$, provided at least one of the two integrals is finite. Since X_1, X_2 are positive, so $\int_{-\infty}^0 F_{X_1}(t)dt = \int_{-\infty}^0 F_{X_2}(t)dt = 0$ and (i) follows from the fact that

$$\overline{x_2} - \overline{x_1} = \int_0^\infty \left((1 - F_{X_2}(t)) - (1 - F_{X_1}(t))\right) dt = \int_0^\infty \left(F_{X_1}(t) - F_{X_2}(t)\right) dt \le 0.$$

Next we consider the medians, assuming that $\widetilde{x}$ is the least number a satisfying $F_X(a) = \frac{1}{2}$. Suppose, on the contrary, that $\widetilde{x_2} > \widetilde{x_1}$. Since $F_X(t)$ is a non-decreasing function and $x_1 < \frac{\widetilde{x_1} + \widetilde{x_2}}{2} < \widetilde{x_2}$, we have

$$F_{X_2}\left(\widetilde{x_2}\right) = \frac{1}{2} > F_{X_2}\left(\frac{\widetilde{x_1} + \widetilde{x_2}}{2}\right) \ge F_{X_1}\left(\frac{\widetilde{x_1} + \widetilde{x_2}}{2}\right) \ge F_{X_1}\left(\widetilde{x_1}\right) = \frac{1}{2},$$

which is false. This proves (ii).

We now show that (iii) does not necessarily hold. Define the probability density functions $f_{X_1}(t)$ and $f_{X_2}(t)$ of X_1 and X_2 as follows:

$$f_{X_1}(t) = \begin{cases} 0 & x \le 0, \\ \frac{x}{16} & 0 \le x \le 4, \\ \frac{8-x}{16} & 4 \le x \le 8, \\ 0 & x \ge 8 \end{cases} \quad \text{and} \quad f_{X_2}(t) = \begin{cases} 0 & x \le 0, \\ \frac{2x}{25} & 0 \le x \le 5, \\ 0 & x \ge 5. \end{cases}$$

Then

$$F_{X_1}(t) = \begin{cases} 0 & x \le 0, \\ \dfrac{x^2}{32} & 0 \le x \le 4, \\ \dfrac{-x^2+16x-32}{32} & 4 \le x \le 8, \\ 1 & x \ge 8 \end{cases} \quad \text{and} \quad F_{X_2}(t) = \begin{cases} 0 & x \le 0, \\ \dfrac{x^2}{25} & 0 \le x \le 5, \\ 1 & x \ge 5. \end{cases}$$

It is easy to check that $F_{X_1}(t) \le F_{X2}(t)$ for $t > 0$, but $\widetilde{x_1} = 4 < 5 = \widetilde{x_2}$.
This completes the solution.

Also solved by the proposer.

5485: *Proposed by José Luis Díaz-Barrero, Barcelona Tech, Barcelona, Spain*

Let x, y, z be three positive real numbers. Show that

$$\prod_{cyclic} (2x+3y+z+1) \sum_{cyclic} (4x+2y+1)^{-3} \ge 3.$$

Solution 1 by Neculai Stanciu, "George Emil Palade" School, Bazău Romania and Tito Zvonaru, Comănesti, Romania

We denote $4x+2y+1 = a,\ 4y+2z+1 = b$, and $4z+2x+1 = c$. We must prove that

$$\frac{(a+b)(b+c)(c+a)}{8}\left(\frac{1}{a^3}+\frac{1}{b^3}+\frac{1}{c^3}\right) \ge 3 \qquad (*)$$

By the AM-GM inequality we have that

$$\frac{(a+b)(b+c)(c+a)}{8} \ \ge\ \frac{2\sqrt{ab}\cdot 2\sqrt{bc}\cdot 2\sqrt{ca}}{8} = \frac{8abc}{8} = abc, \qquad (1)$$

$$\frac{1}{a^3}+\frac{1}{b^3}+\frac{1}{c^3} \ \ge\ 3\cdot\sqrt[3]{\frac{1}{a^3}\cdot\frac{1}{b^3}\cdot\frac{1}{c^3}} = \frac{3}{abc}, \qquad (2)$$

By (1) and (2) we obtain

$$\frac{(a+b)(b+c)(c+a)}{8}\left(\frac{1}{a^3}+\frac{1}{b^3}+\frac{1}{c^3}\right) \ge abc\cdot\frac{3}{abc} \ =\ 3. \quad \text{I.e. } (*)$$

Solution 2 by Nikos Kalapodis, Patras, Greece

By the AM-GM inequality we have

$$\prod_{cyclic}(2x+3y+z+1)\sum_{cyclic}(4x+2y+1)^{-3} \ge \prod_{cyclic}(2x+3y+z+1)\frac{3}{\prod\limits_{cyclic}(4x+2y+1)}.$$

So, it suffices to prove that $\prod\limits_{cyclic}(2x+3y+z+1) \ge \prod\limits_{cyclic}(4x+2y+1)$.

After expanding the inequality reduces to
$2(x^3+y^3+z^3)+x^2+y^2+z^2+3(xy^2+yz^2+zx^2) \geq 3(x^2y+y^2z+z^2x)+xy+yz+zx+6xyz.$

Since $x^2+y^2+z^2 \geq xy+yz+zx$, it remains to prove that
$2(x^3+y^3+z^3)+3(xy^2+yz^2+zx^2) \geq 3(x^2y+y^2z+z^2x)+6xyz.$

This follows again by using the AM-GM inequality properly:
$2(x^3+y^3+z^3)+3(xy^2+yz^2+zx^2) = 2(x^3+xy^2)+2(y^3+yz^2)+2(z^3+zx^2)+(xy^2+yz^2+zx^2) \geq 4x^2y+4y^2z+4z^2x+(xy^2+yz^2+zx^2) =$
$3(x^2y+y^2z+z^2x)+(x^2y+y^2z+z^2x+xy^2+yz^2+zx^2) \geq 3(x^2y+y^2z+z^2x)+6xyz.$

Also solved by Arkady Alt, San Jose, CA; Kee-Wai Lau, Hong Kong, China; Paolo Perfetti, Department of Mathematics, Tor Vergata University, Rome, Italy; Kevin Soto Palacios, Huarmey, Perú; Ioannis D. Sfikas, National and Kapodistrian University of Athens, Greece; Albert Stadler, Herrliberg, Switzerland, and the proposer.

5486: *Proposed by Ovidiu Furdui, Technical University of Cluj-Napoca, Cluj-Napoca, Romania*

Let $(x_n)_{n\geq 0}$ be the sequence defined by $x_0=0, x_1=1, x_2=1$ and
$x_{n+3}=x_{n+2}+x_{n+1}+x_n+n,\ \forall n \geq 0$. Prove that the series $\displaystyle\sum_{n=1}^{\infty}\frac{x_n}{2^n}$ converges and find its sum.

Solution 1 by Ángel Plaza, University of Las Palmas de Gran Canaria, Spain.

The recurrence sequence may be unmasked by generating functions. Let $F(z)$ be the associated generating function. That is, $F(z)=\displaystyle\sum_{n=0}^{\infty}x_nz^n$. Multiplying by z^{n+3} the recurrence relation defining (x_n) and taking into account the initial values it is obtained that

$$F(z)-\left(z+z^2\right)=z\left(F(z)-z\right)+z^2F(z)+z^3F(z)+\frac{z^4}{(1-z)^2}$$

from where $F(z)=\dfrac{z(1-z)^2+x^4}{(z-1)^2(1-z-z^2-z^3)}$.

Since $F(z)$ converges for $|z|<\dfrac{1}{3}\left(\sqrt[3]{17+3\sqrt{33}}-\dfrac{2}{\sqrt[3]{17+3\sqrt{33}}}-1\right)\sim 0.5436...$, then $\displaystyle\sum_{n=1}^{\infty}\frac{x_n}{2^n}=F(1/2)=6.$

Solution 2 by Paolo Perfetti, Department of Mathematics, Tor Vergata University, Rome, Italy

Answer: 6.

Clearly x_n increases and $x_n \geq 1$.

$$\sum_{n=1}^{p} \frac{x_n}{2^n} = \frac{x_1}{2} + \frac{x_2}{4} + \sum_{n=3}^{p} \frac{x_n}{2^n} = \frac{3}{4} + \sum_{n=0}^{p-3} \frac{x_{n+3}}{2^{n+3}} =$$

$$= \frac{3}{4} + \sum_{n=0}^{p-3} \left(\underbrace{\frac{x_{n+2}}{2^{n+3}}}_{I_1} + \underbrace{\frac{x_{n+1}}{2^{n+3}}}_{I_2} + \underbrace{\frac{x_n}{2^{n+3}}}_{I_3} \right) + \sum_{n=0}^{p-3} \frac{n}{2^{n+3}} =$$

$$= \frac{3}{4} + \underbrace{\sum_{n=2}^{p-1} \frac{x_n}{2^{n+1}}}_{I_1} + \underbrace{\sum_{n=1}^{p-2} \frac{x_n}{2^{n+2}}}_{I_2} + \underbrace{\sum_{n=1}^{p-3} \frac{x_n}{2^{n+3}}}_{I_3} + \sum_{n=0}^{p-3} \frac{n}{2^{n+3}} =$$

$$= \frac{3}{4} + \underbrace{-\frac{1}{4} + \sum_{n=1}^{p} \frac{x_n}{2^{n+1}} - \frac{x_p}{2^{p+1}}}_{I_1} + \underbrace{\sum_{n=1}^{p} \frac{x_n}{2^{n+2}} - \frac{x_{p-1}}{2^{p+1}} - \frac{x_p}{2^{p+2}}}_{I_2} +$$

$$+ \underbrace{\sum_{n=1}^{p} \frac{x_n}{2^{n+3}} - \frac{x_{p-2}}{2^{p+1}} - \frac{x_{p-1}}{2^{p+2}} - \frac{x_p}{2^{p+3}}}_{I_3} + \underbrace{\sum_{n=0}^{p-3} \frac{n}{2^{n+3}}}_{\to 1/4 \text{ as } p \to \infty}$$

It follows

$$\frac{1}{8} \sum_{n=1}^{p} \frac{x_n}{2^n} = \frac{1}{2} - \left[\frac{x_p}{2^{p+1}} + \frac{x_{p-1}}{2^{p+1}} + \frac{x_p}{2^{p+2}} + \frac{x_{p-2}}{2^{p+1}} + \frac{x_{p-1}}{2^{p+2}} \right] + \frac{1}{4}$$

Now we prove the

Lemma $x_k/2^k \to 0$.

Proof of the Lemma

First step: the sequence $x_k/2^k$ in monotonic not increasing.

$$\frac{x_{k+3}}{2^{k+3}} = \frac{x_{k+2} + x_{k+1} + x_k + k}{2^{k+3}} \leq \frac{x_{k+2}}{2^{k+2}} \iff \frac{x_{k+1} + x_k + k}{2^{k+3}} \leq \frac{x_{k+2}}{2^{k+3}}$$

that is

$$x_{(k-1)+2} + x_{(k-1)+1} + (k-1) + 1 \leq x_{(k-1)+3}$$

and this is implied by

$$x_{(k-1)+2} + x_{(k-1)+1} + (k-1) + 1 \leq x_{(k-1)+2} + x_{(k-1)+1} + x_{k-1} + (k-1) = x_{(k-1)+3}$$

via $x_{k-1} \geq 1$. The monotonicity of the sequence means that the limit L of $x_k/2^k$ does exist and moreover $0 \leq L < +\infty$. If $L = 0$ the proof is concluded yielding

$$\lim_{p \to \infty} \frac{1}{8} \sum_{n=1}^{p} \frac{x_n}{2^n} = \frac{3}{4} \iff \sum_{n=1}^{\infty} \frac{x_n}{2^n} = 6$$

$L \neq 0$ is impossible as shown by the following argument. We employ the Cesaro–Stolz theorem that states:

$$\lim_{k \to \infty} \frac{x_k}{2^k} = \lim_{k \to \infty} \frac{x_{k+1} - x_k}{2^{k+1} - 2^k}$$

provided that the second limit does exist. We write

$$\frac{x_{k+3}-x_{k+2}}{2^{k+3}-2^{k+2}}=\frac{x_{k+1}+x_k+k}{2^{k+2}}=\frac{1}{2}\frac{x_{k+1}}{2^{k+1}}+\frac{1}{4}\frac{x_k}{2^k}+\frac{k}{2^{k+2}}$$

The existence of the limit $L=\lim\limits_{k\to\infty}\frac{x_k}{2^k}$ would imply

$$L=\frac{1}{2}L+\frac{1}{4}L \implies L=0$$

Solution 3 by Arkady Alt, San Jose, CA

For any sequence $(x_n)_{n\geq 0}$ let $T(x_n):=x_{n+3}-x_{n+2}-x_{n+1}-x_n, n\in N\cup\{0\}$.
Obvious that such defined operator T (we will call it Tribonacci Operator) is linear.
Since $T\left(-\frac{n}{2}\right)=-\frac{n+3}{2}+\frac{n+2}{2}+\frac{n+1}{2}+\frac{n}{2}=n$ then denoting
$u_n:=x_n+\frac{n}{2}, n\in N\cup\{0\}$
we obtain $x_n=u_n-\frac{n}{2}, n\in N\cup\{0\}$ where $T(u_n)=0$ and,
$u_0=0, u_1=1+\frac{1}{2}=\frac{3}{2}, u_2=1+\frac{2}{2}=2.$
Let $(t_n)_{n\geq 0}$ be the sequence defined by $t_0=0, t_1=1, t_2=1$ and
$T(t_n)=0, n\in N\cup\{0\}$.
(Tribonacci Sequence). We have $t_3=2, t_4=4, t_5=7, t_6=13, t_7=24, t_8=44, \ldots$
Since $\det\begin{pmatrix}0&1&1\\1&1&2\\1&2&4\end{pmatrix}\neq 0$ then for any sequence $(x_n)_{n\geq 0}$ there is triple (c_1,c_2,c_3) of real numbers such that $x_n=c_2t_n+c_2t_{n+1}+c_3t_{n+2}$, that is sequences $(t_n)_{n\geq 0}$, $(t_{n+1})_{n\geq 0}$, $(t_{n+2})_{n\geq 0}$
form a basis of 3-dimesion space $\ker T:=\{(x_n)_{n\geq 0}\mid T(x_n)=0, n\in N\cup\{0\}\}$.
We will find representation u_n as linear combination of t_n, t_{n+1}, t_{n+2},
namely, $u_n=c_1t_n+c_2t_{n+1}+c_3t_{n+2},\ n\in N\cup\{0\}$.
We
have $u_0=c_1t_0+c_2t_1+c_3t_2 \iff c_2+c_3=0, u_1=c_1t_1+c_2t_2+c_3t_3 \iff c_1+c_2+2c_3=\frac{3}{2}$,
$u_2=c_1t_2+c_2t_3+c_3t_4 \iff c_1+2c_2+4c_3=2$. From this system of equations we obtain
$c_3=-c_2, c_1-c_2=\frac{3}{2}, c_1-2c_2=2$.Hence, $c_1=1, c_2=-\frac{1}{2}, c_3=\frac{1}{2}$ and since
$u_n=t_n-\frac{t_{n+1}}{2}+\frac{t_{n+2}}{2}$ we obtain $x_n=t_n-\frac{t_{n+1}}{2}+\frac{t_{n+2}}{2}-\frac{n}{2}=\frac{2t_n-t_{n+1}+t_{n+2}-n}{2}$.
Since radius of convergence of seies $\sum\limits_{n=1}^{\infty}nx^{n-1}$ is 1 and $\sum\limits_{n=1}^{\infty}nx^{n-1}=\frac{1}{(1-x)^2}$
then $\sum\limits_{n=1}^{\infty}\frac{n}{2^n}=\frac{1}{2}\sum\limits_{n=1}^{\infty}\frac{n}{2^{n-1}}=\frac{1}{2}\,\frac{1}{(1-1/2)^2}=2$ and, therefore, for convergency of $\sum\limits_{n=1}^{\infty}\frac{x_n}{2^n}$ suffice to prove convergency of series $\sum\limits_{n=1}^{\infty}\frac{t_n}{2^n}$.
We can prove that using another basis of $\ker T$ which form sequences $(\alpha^n)_{n\geq 0}, (\beta^n)_{n\geq 0}$, $(\gamma^n)_{n\geq 0}$
where α,β,γ are roots of characteristic equation $x^3-x^2-x-1=0$.
Substitution $x=\frac{4u+1}{3}$ in equation $x^3-x^2-x-1=0$ give us equivalent equation
$4u^3-3u=\frac{19}{8}$

which we solve using substitution $u := \frac{1}{2}\left(t+\frac{1}{t}\right)$. Then equation $4u^3-3u=\frac{19}{8}$ becomes $4\left(\frac{1}{2}\left(t+\frac{1}{t}\right)\right)^3-3\cdot\frac{1}{2}\left(t+\frac{1}{t}\right)=\frac{19}{8} \iff \frac{1}{t^3}+t^3=\frac{19}{4}$.Denoting $z:=t^3$ we obtain

$\frac{1}{z}+z=\frac{19}{4} \iff z=\frac{19-3\sqrt{33}}{8},\frac{19+3\sqrt{33}}{8} \iff t^3=\frac{19-3\sqrt{33}}{8},\frac{19+3\sqrt{33}}{8}$.

Since $\frac{19-3\sqrt{33}}{8}\cdot\frac{19+3\sqrt{33}}{8}=1$ and $u=\frac{1}{2}\left(t+\frac{1}{t}\right)$ then suffices to find $t^3=\frac{19+3\sqrt{33}}{8}$.

We have $t=r\left(\cos\varphi+i\sin\varphi\right)$, where $r=\frac{\sqrt[3]{19+3\sqrt{33}}}{2}$ and $\varphi=\frac{2k\pi}{3}, k=1,2,3$.

that is $t_k=\frac{\sqrt[3]{19+3\sqrt{33}}}{2}\omega^k, k=1,2,3$ and $\omega=\ \cos\frac{2\pi}{3}+i\sin\frac{2\pi}{3},\omega^3=1$.

Thus, denoting $\theta:=\sqrt[3]{19+3\sqrt{33}},\theta^*:=\ \sqrt[3]{19-3\sqrt{33}}$ we obtain

$\alpha=\frac{1+\theta+\theta^*}{3},\beta=\frac{1+\omega\theta+\omega^2\theta^*}{3}$,

$\gamma=\frac{1+\omega^2\theta+\omega\theta^*}{3}$, the three roots of the equation $x^3-x^2-x-1=0$.

We will prove that $\alpha=\frac{1+\theta+\theta^*}{3}<2$.

First note that by Power Mean–Arithmetic Mean inequality

$p:=\sqrt[3]{19+3\sqrt{33}}+\sqrt[3]{19-3\sqrt{33}}\ <2\sqrt[3]{\frac{19+3\sqrt{33}+19-3\sqrt{33}}{2}}=2\sqrt[3]{19}<2\sqrt[3]{27}=6$.

Since $\sqrt[3]{19+3\sqrt{33}}\cdot\sqrt[3]{19-3\sqrt{33}}=\sqrt[3]{19^2-9\cdot 33}=4$ then

$p^3=38+3\sqrt[3]{19+3\sqrt{33}}\cdot\sqrt[3]{19-3\sqrt{33}}\cdot p=38+12p<38+12\cdot 6=110<125=5^3$.

Hence, $\alpha<2$. Also, we obtain $\left|\beta\right|,\left|\gamma\right|\le\frac{1+\theta+\theta^*}{3}<2$.

Since series $\sum_{n=1}^{\infty}\left(\frac{\alpha}{2}\right)^n,\sum_{n=1}^{\infty}\left(\frac{\beta}{2}\right)^n,\sum_{n=1}^{\infty}\left(\frac{\gamma}{2}\right)^n$ are convergent and t_n is linear combination of

$(\alpha^n)_{n\ge 0},(\beta^n)_{n\ge 0},(\gamma^n)_{n\ge 0}$ then series $\sum_{n=1}^{\infty}\frac{t_n}{2^n}$ convergent as well.

Now we ready to find sum of series $\sum_{n=1}^{\infty}\frac{x_n}{2^n}$.

Let $s_n:=\sum_{k=1}^{n}\frac{t_n}{2^n}$ and $s\left(x\right)=\sum_{k=0}^{n}t_{n+1}x^n$. Note also that function

$\frac{1}{1-x-x^2-x^3}$ generates

Tribonacci numbers. Indeed, let $\frac{1}{1-x-x^2-x^3}=\sum_{n=0}^{\infty}a_nx^n$. Then

$\sum_{n=0}^{\infty}a_nx^n\cdot\left(1-x-x^2-x^3\right)=1$

and since

$\sum_{n=0}^{\infty}a_nx^n\cdot\left(1-x-x^2-x^3\right)=\sum_{n=0}^{\infty}a_nx^n-\sum_{n=0}^{\infty}a_nx^{n+1}\ -_{n=0}^{\infty}a_nx^{n+2}-\sum_{n=0}^{\infty}a_nx^{n+3}=$

$a_0+\left(a_1-a_0\right)x+\left(a_2-a_1-a_0\right)x^2+\sum_{n=3}^{\infty}\left(a_{n+3}-a_{n+2}-a_{n+1}-a_n\right)x^{n+3}$ then

$a_0=1,a_1-a_0=a_2-a_1-a_0=0$ implies $a_1=1,a_2=2$ and

$a_{n+3} - a_{n+2} - a_{n+1} - a_n = 0, n \in N \cup \{0\}$. Thus, $a_n = t_{n+1}, n \in N \cup \{0\}$ and, therefore, $\sum\limits_{k=0}^{n} t_{n+1}x^n = s\left(x\right) = \dfrac{1}{1 - x - x^2 - x^3}$. In,

particular, $\sum\limits_{n=1}^{\infty} \dfrac{t_n}{2^n} = \dfrac{1}{2} \sum\limits_{n=1}^{\infty} \dfrac{t_n}{2^{n-1-}} = \dfrac{1}{2} s\left(\dfrac{1}{2}\right) =$

$\dfrac{1}{2} \cdot \dfrac{1}{1 - \dfrac{1}{2} - \left(\dfrac{1}{2}\right)^2 - \left(\dfrac{1}{2}\right)^3} = 4.$

Then, $\sum\limits_{n=1}^{\infty} \dfrac{x_n}{2^n} = \sum\limits_{n=1}^{\infty} \dfrac{t_n}{2^n} - \sum\limits_{n=1}^{\infty} \dfrac{t_{n+1}}{2^{n+1}} + 2\sum\limits_{n=1}^{\infty} \dfrac{t_{n+2}}{2^{n+2}} - \sum\limits_{n=1}^{\infty} \dfrac{n}{2^{n+1}} =$

$\sum\limits_{n=1}^{\infty} \dfrac{t_n}{2^n} - \sum\limits_{n=2}^{\infty} \dfrac{t_n}{2^n} + 2_{n=3}^{\infty} \dfrac{t_n}{2^n} - \dfrac{1}{2} \sum\limits_{n=1}^{\infty} \dfrac{n}{2^n} =$

$\dfrac{t_1}{2^1} + 2\sum\limits_{n=3}^{\infty} \dfrac{t_n}{2^n} - \dfrac{1}{2} \cdot 2 = -\dfrac{1}{2} + 2\sum\limits_{n=1}^{\infty} \dfrac{t_n}{2^n} - 2\left(\dfrac{t_1}{2^1} + \dfrac{t_2}{2^2}\right) = -\dfrac{1}{2} + 2 \cdot 4 - 2\left(\dfrac{1}{2} + \dfrac{1}{4}\right) = 6.$

Solution 4 by Brian D. Beasley, Presbyterian College, Clinton, SC

We show that the given series converges by first using induction to prove that $x_n < 1.95^n$ for each positive integer n. Note that this claim holds for $n \in \{1, 2, 3\}$. Given a positive integer k, if $x_n < 1.95^n$ for $n \in \{k, k+1, k+2\}$, then

$$x_{k+3} < 1.95^{k+2} + 1.95^{k+1} + 1.95^k + k = 1.95^k(6.7525) + k.$$

Thus it suffices to show that $1.95^k(6.7525) + k \le 1.95^{k+3}$, or equivalently $k \le 1.95^k(0.662375)$. This latter inequality holds for each positive integer k (using a separate induction argument). Hence $x_n < 1.95^n$ for $n \ge 1$, so for any positive integer m,

$$\sum_{n=1}^{m} \frac{x_n}{2^n} < \sum_{n=1}^{\infty} \frac{x_n}{2^n} < \sum_{n=1}^{\infty} \frac{1.95^n}{2^n} = \frac{0.975}{1 - 0.975} = 39.$$

Since its sequence of partial sums is increasing and bounded above, the given series converges.

Next, we let $\sum\limits_{n=1}^{\infty} \dfrac{x_n}{2^n} = L$. Then

$$L = \frac{1}{2} + \frac{1}{4} + \sum_{n=0}^{\infty} \frac{x_{n+2} + x_{n+1} + x_n + n}{2^{n+3}} = \frac{3}{4} + \frac{1}{2}\left(L - \frac{1}{2}\right) + \frac{1}{4}L + \frac{1}{8}L + \sum_{n=0}^{\infty} \frac{n}{2^{n+3}}.$$

Since $\sum\limits_{n=0}^{\infty} \dfrac{n}{2^n} = 2$, we conclude $L = \frac{7}{8}L + \frac{1}{2} + \frac{1}{8}(2)$ and hence $L = 6$.

Also solved by Kee-Wai Lau, Hong Kong, China; Ioannis D. Sfikas, National and Kapodistrian University of Athens, Greece; Albert Stadler (two solutions), Herrliberg, Switzerland; David Stone and John Hawkins, Southern Georgia University, Statesboro, GA, and the proposer.

$Mea - Culpa$

Arkady Alt of San Jose, CA should have been credited with having solved 5477, and 5478.

Dionne Bailey, Elsie Campbell, Charles Diminnie, and Trey Smith, all of Angelo State University in San Angelo, TX should have been credited with having solved 5475.

Paul M. Harms, of North Newton, KS should have been credited for having solved 5476.

Anna Valkova Tomova of Varna, Bulgaria should have been credited with having solved 5475 and 5477.

Mea Culpa.