Problems

Ted Eisenberg, Section Editor

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This section of the Journal offers readers an opportunity to exchange interesting mathematical problems and solutions. Please send them to Ted Eisenberg, Department of Mathematics, Ben-Gurion University, Beer-Sheva, Israel or fax to: 972-86-477-648. Questions concerning proposals and/or solutions can be sent e-mail to <eisenbt@013.net>. Solutions to previously stated problems can be seen at <http://ssmj.tamu.edu>.

Solutions to the problems stated in this issue should be posted before
January 15, 2011

- **5128:** *Proposed by Kenneth Korbin, New York, NY*

  Find all positive integers less than 1000 such that the sum of the divisors of each integer is a power of two.

  For example, the sum of the divisors of 3 is 2^2, and the sum of the divisors of 7 is 2^3.

- **5129:** *Proposed by Kenneth Korbin, New York, NY*

  Given prime number c and positive integers a and b such that $a^2 + b^2 = c^2$, express in terms of a and b the lengths of the legs of the primitive Pythagorean Triangles with hypotenuses c^3 and c^5, respectively.

- **5130:** *Proposed by Michael Brozinsky, Central Islip, NY*

  In Cartesianland, where immortal ants live, calculus has not been discovered. A bride and groom start out from $A(-a, 0)$ and $B(b, 0)$ respectively where $a \neq b$ and $a > 0$ and $b > 0$ and walk at the rate of one unit per second to an altar located at the point P on line $L : y = mx$ such that the time that the first to arrive at P has to wait for the other to arrive is a maximum. Find, without calculus, the locus of P as m varies through all nonzero real numbers.

- **5131:** *Proposed by José Luis Díaz-Barrero, Barcelona, Spain*

  Let a, b, c be positive real numbers. Prove that

  $$\frac{a+b+3c}{3a+3b+2c} + \frac{a+3b+c}{3a+2b+3c} + \frac{3a+b+c}{2a+3b+3c} \geq \frac{15}{8}.$$

- **5132:** *Proposed by José Luis Díaz-Barrero, Barcelona, Spain*

  Find all all functions $f : C \to C$ such that $f(f(z)) = z^2$ for all $z \in C$.

- **5133:** *Proposed by Ovidiu Furdui, Cluj, Romania*

  Let $n \geq 1$ be a natural number. Calculate

  $$I_n = \int_0^1 \int_0^1 (x-y)^n dxdy.$$

- **5110:** *Proposed by Kenneth Korbin, New York, NY.*

Given triangle ABC with an interior point P and with coordinates $A(0,0), B(6,8)$, and $C(21,0)$. The distance from point P to side $\overline{AB}$ is a, to side $\overline{BC}$ is b, and to side $\overline{CA}$ is c where $a:b:c = \overline{AB}:\overline{BC}:\overline{CA}$.

Find the coordinates of point P

Solution 1 by Boris Rays, Brooklyn, NY

From the given triangle we have $\overline{AB} = 10, \overline{BC} = 17$ and $\overline{CA} = 21$. Also $a:b:c = 10:17:21$.

Let $a = 10t,\ b = 17t$, and $c = 21t$, where t is real number, $t > 0$. (1)

$$\text{Area }\triangle\text{ABC} = \text{Area }\triangle\text{APB} + \text{Area }\triangle\text{BPC} + \text{Area }\triangle\text{CPA}. \qquad (2)$$

Express all of the terms in (2) by using formulas in (1).

$$\begin{aligned} \frac{1}{2}\cdot 21\cdot 8 &= \frac{1}{2}\cdot 10\cdot 10t + \frac{1}{2}\cdot 17\cdot 17t + \frac{1}{2}\cdot 21\cdot 21t \\ &= \frac{1}{2}t\left(10^2+17^2+21^2\right) = \frac{1}{2}830t \end{aligned}$$

From the above we find that $t = \dfrac{84}{415} = \dfrac{2^2\cdot 3\cdot 7}{5\cdot 83}$.

The y-coordinate of point P is c, the distance to side $\overline{CA}$.

$$y_P = c = 21t = 21\cdot\frac{84}{415} = \frac{1764}{415}.$$

Let points E and F lie on side $\overline{CA}$, where $\overline{PE}\perp\overline{CA}$ and $\overline{BF}\perp\overline{CA}$.

Hence we have $\overline{PE} = C = \dfrac{42^2}{415}$, $\overline{BF} = 8$, and $\overline{AF} = 6$.

$$\text{Area }\triangle\text{APB} + \text{Area }\triangle\text{APE} + \text{Area BPEF} = \text{Area }\triangle\text{ABF}.$$

Letting $\overline{AE} = x$ we have $\overline{EF} = 6 - x$. Therefore,

$$\frac{1}{2}\cdot 10\cdot a + \frac{1}{2}\cdot x\cdot c + \frac{1}{2}\left(\overline{PE}+\overline{BF}\right)\cdot\overline{EF} = \frac{1}{2}\overline{AF}\cdot\overline{BF}$$

$$\frac{1}{2}\cdot 100\cdot\frac{84}{415} + \frac{1}{2}\cdot x\cdot\frac{42^2}{415} + \frac{1}{2}\left(\frac{42^2}{415}+8\right)(6-x) = \frac{1}{2}6\cdot 8.$$

From the above equation we find x.

$$x = \frac{1}{8}\left(\frac{8400+6(42)^2}{415}\right) = \frac{2373}{415}.$$

Hence, the coordinates of point P are $\left(\dfrac{2373}{415}, \dfrac{1764}{415}\right)$.

Solution 2 by Charles McCracken, Dayton, OH

$$\overline{AB} = 10 \qquad \overline{BC} = 17 \qquad \overline{CA} = 21$$

The equations of $\overline{AB}, \overline{BC}$ and $\overline{CA}$ are respectively,

$$4x - 3y = 0 \qquad 8x + 15y - 168 = 0 \qquad y = c$$

Then,

$$a = \frac{4x - 3y}{5} \qquad b = \frac{8x + 15y - 168}{17} \qquad c = y$$

$$\frac{\left(\dfrac{4x - 3y}{5}\right)}{y} = \frac{10}{21} \qquad \frac{\left(\dfrac{8x + 15y - 168}{-17}\right)}{y} = \frac{17}{21}$$

$$21\,(4x - 3y) = 50y \qquad 21\,(8x + 15y - 168) = -289y$$

$$84x - 113y = 0 \qquad 168x + 604y = 3528$$

These last two equations give:

$$(x, y) = \left(\frac{2373}{415}, \frac{1764}{415}\right)$$

Note that P is the Lemoine point of $\triangle ABC$, that is, the intersection of the symmedians. (*Editor:* A symmedian is the reflection of a median about its corresponding angle bisector.)

Also solved by Brian D. Beasley, Clinton, SC; Michael Brozinsky, Central Islip, NY; Elsie M. Campbell, Dionne T. Bailey, and Charles Diminnie (jointly), San Angelo, TX; Paul M. Harms, North Newton, KS; Kee-Wai Lau, Hong Kong, China; John Nord, Spokane, WA; Raúl A. Simón, Santiago, Chile; Danielle Urbanowicz, Jennie Clinton, and Bill Solyst (jointly; students at Taylor University), Upland, IN; David Stone and John Hawkins (jointly), Satetesboro, GA, and the proposer.

- **5111:** *Proposed by Michael Brozinsky, Central Islip, NY.*

In Cartesianland where immortal ants live, it is mandated that any anthill must be surrounded by a triangular fence circumscribed in a circle of unit radius. Furthermore, if the vertices of any such triangle are denoted by A, B, and C, in counterclockwise order, the anthill's center must be located at the interior point P such that $\angle PAB = \angle PBC = \angle PCA$.

Show $\overline{PA} \cdot \overline{PB} \cdot \overline{PC} \leq 1$.

Solution by Kee-Wai Lau, Hong Kong, China

It is easy to check that $\angle APB = 180^\circ - B$, $\angle BPC = 180^\circ - C$, and $\angle CPA = 180^\circ - A$.

It is well known that the area of $\triangle ABC = 2R^2 \sin A \sin B \sin C$, where R is the circumradius of the triangle. Here we have $R = 1$. Since the area of $\triangle ABC$ equals the sum of the areas of triangles APB, BPC and CPA, we have

$$\text{Area } \triangle\text{ABC} = \text{Area } \triangle\text{APB} + \text{Area } \triangle\text{BPC} + \text{Area } \triangle\text{CPA}$$

$$2\sin A \sin B \sin C = \frac{1}{2}\left(\overline{PA}\cdot\overline{PB}\sin B + \overline{PB}\cdot\overline{PC}\sin C + \overline{PC}\cdot\overline{PA}\sin A\right).$$

By the arithmetic mean-geometric mean inequality, we have

$$\overline{PA}\cdot\overline{PB}\sin B+\overline{PB}\cdot\overline{PC}\sin C+\overline{PC}\cdot\overline{PA}\sin A \geq 3\left(\overline{PA}\cdot\overline{PB}\cdot\overline{PC}\right)^{2/3}(\sin A\sin B\sin C)^{1/3}.$$

It follows that

$$\left(\overline{PA}\cdot\overline{PB}\cdot\overline{PC}\right)^{2/3} \leq \frac{4}{3}(\sin A \sin B \sin C)^{2/3}. \qquad (1)$$

By the concavity of the function $\ln(\sin x)$ for $0 < x < \pi$, we obtain

$$\ln(\sin A) + \ln(\sin B) + \ln(\sin C) \leq 3\left(\sin\left(\frac{A+B+C}{3}\right)\right) = 3\ln\left(\frac{\sqrt{3}}{2}\right).$$

Therefore,

$$\sin A \sin B \sin C \leq \frac{3\sqrt{3}}{8}. \qquad (2)$$

The result $\overline{PA}\cdot\overline{PB}\cdot\overline{PC} \leq 1$ now follows easily from (1) and (2) immediately above.

Comments: The proposer, **Michael Brozinsky,** mentioned in his solution that point P is precisely the Brocard point of the triangle, and **David Stone and John Hawkins** noted in their solution that given an inscribed triangle and letting
$\theta = \angle PAB = \angle PBC = \angle PCA$, then the identity

$$\sin\theta = \frac{abc}{2\sqrt{a^2b^2+a^2c^2+b^2c^2}}$$

allows one to find the unique angle θ and thus sides $\overline{PA}, \overline{PB}$, and $\overline{PC}$.

Also solved by David Stone and John Hawkins (jointly), Satetesboro, GA, and the proposer.

- **5112:** *Proposed by Juan-Bosco Romero Márquez, Madrid, Spain*

Let $0 < a < b$ be real numbers with a fixed and b variable. Prove that

$$\lim_{b\to a}\int_a^b \frac{dx}{\ln\dfrac{b+x}{a+x}} = \lim_{b\to a}\int_a^b \frac{dx}{\ln\dfrac{b(a+x)}{a(b+x)}}.$$

Solution by Shai Covo, Kiryat-Ono, Israel

We begin with the left-hand side limit. Writing $\ln\dfrac{b+x}{a+x}$ as $\ln(b+x) - \ln(a+x)$, we have by the mean value theorem that this expression is equal to $\dfrac{1}{\xi}(b-a)$ where $\xi = \xi(a,b,x)$ is some point between $(a+x)$ and $(b+x)$. Since x varies from a to b, it thus follows that

$$\frac{b-a}{2b} \leq \ln\frac{b+x}{a+x} \leq \frac{b-a}{2a}.$$

Hence,

$$2a = \int_a^b \frac{2a}{b-a}dx \le \int_a^b \frac{dx}{\ln \dfrac{b+x}{a+x}} \le \int_a^b \frac{2b}{b-a}dx = 2b,$$

and so

$$\lim_{b \to a} \int_a^b \frac{dx}{\ln \dfrac{b+x}{a+x}} = 2a.$$

Applying this technique to the computation of the right-hand side limit gives

$$\frac{a(b-a)}{ab+b^2} \le \ln \frac{b(a+x)}{a(b+x)} \le \frac{b(b-a)}{ab+a^2},$$

from which it follows immediately that also

$$\lim_{b \to a} \int_a^b \frac{dx}{\ln \dfrac{b(a+x)}{a(b+x)}} = 2a.$$

Also solved by Michael Brozinsky, Central Islip, NY; Kee-Wai Lau, Hong Kong, China; Paolo Perfetti, Department of Mathematics, University of Rome, Italy; David Stone and John Hawkins (jointly), Statesboro, GA, and the proposer.

- **5113:** *Proposed by Paolo Perfetti, Mathematics Department, Tor Vergata University, Rome, Italy*

Let x, y be positive real numbers. Prove that

$$\frac{2xy}{x+y} + \sqrt{\frac{x^2+y^2}{2}} \le \sqrt{xy} + \frac{x+y}{2} + \frac{\left(\dfrac{x+y}{6} - \dfrac{\sqrt{xy}}{3}\right)^2}{\dfrac{2xy}{x+y}}.$$

Solution 1 by Kee-Wai Lau, Hong Kong, China

By homogeneity, we may assume without loss of generality that $xy = 1$. Let $t = x + y \ge 2\sqrt{xy} = 2$. Then the inequality of the problem is equivalent to

$$\begin{aligned}
& \frac{2}{t} + \sqrt{\frac{t^2-2}{2}} \le 1 + \frac{t}{2} + \frac{t(t-2)^2}{72} \\
\Leftrightarrow \quad & 36t\sqrt{2\left(t^2-2\right)} \le t^4 - 4t^3 + 40t^2 + 72t - 144 \\
\Leftrightarrow \quad & \left(t^4 - 4t^3 + 40t^2 + 72t - 144\right) - 2592t^2\left(t^2-2\right) \ge 0 \\
\Leftrightarrow \quad & t^8 - 8t^7 + 96t^6 - 176t^5 - 1856t^4 + 6912t^3 - 1152t^2 - 20376t + 20376 \ge 0 \\
\Leftrightarrow \quad & (t-2)^2\left(t^6 - 4t^5 + 76t^4 + 144t^3 - 1584t^2 + 5184\right) \ge 0.
\end{aligned}$$

Since

$$
\begin{aligned}
& t^6 - 4t^5 + 76t^4 + 144t^3 - 1584t^2 + 5184 \\
= \;& t^4(t-2)^2 + 72(t-2)^4 + \frac{16(3t-8)^2(15t+11) + 832}{3} > 0,
\end{aligned}
$$

the inequality of the problem holds.

Solution 2 by Paul M. Harms, North Newton, KS

Le $w = \dfrac{x+y}{2\sqrt{xy}}$ and $z = \sqrt{xy}$. For x and y positive

$$(\sqrt{x} - \sqrt{y})^2 = x + y - 2\sqrt{xy} \geq 0 \Longrightarrow w = \frac{x+y}{2\sqrt{xy}} \geq 1. \text{ Also } z > 0.$$

From the substitutions we have the following expressions :

$$
\begin{aligned}
2xy &= 2z^2 \\
x + y &= 2zw \\
x^2 + y^2 = (x+y)^2 - 2xy &= 4z^2w^2 - 2z^2 = 2z^2(2w^2 - 1)
\end{aligned}
$$

The inequality becomes

$$\frac{2z^2}{2zw} + \sqrt{\frac{2z^2\,(2w^2-1)}{2}} \leq z + \frac{2zw}{2} + \frac{\left(\dfrac{2zw - 2z}{6}\right)^2}{\dfrac{2z^2}{2zw}}$$

Simplifying and dividing both sides of the inequality by z yields the inequality

$$\frac{1}{w} + \sqrt{2w^2 - 1} \leq 1 + w + \frac{1}{9}\,(w-1)^2\, w.$$

After multiplying both sides by $9w$ and isolating the square root term we get

$$9w\sqrt{2w^2-1} \leq -9 + 9w + 9w^2 + (w-1)^2w^2 = w^4 - 2w^3 + 10w^2 + 9w - 9.$$

Now let $w = L + 1$. Since $w \geq 1$, we check the resulting inequality for $L \geq 0$. Replacing w by $L+1$ and squaring both sides of the inequality we obtain

$$
\begin{aligned}
81\,(L+1)^2\left[2L^2 + 4L + 1\right] &= 81\left(2L^4 + 8L^3 + 11L^2 + 6L + 1\right) \\
&\leq \left(L^4 + 2L^3 + 10L^2 + 27L + 9\right)^2 \\
&= L^8 + 4L^7 + 24L^6 + 94L^5 + 226L^4 + 576L^3 + 909L^2 + 486L + 81
\end{aligned}
$$

Moving all terms to the right side, we need to show for $L \geq 0$, that

$$0 \leq L^2\left[L^6 + 4L^5 + 24L^4 + 94L^3 + 64L^2 - 72L + 18\right].$$

Let

$$g(L) = 94L^3 + 64L^2 - 72L + 18.$$

If $g(L) \geq 0$ for $L \geq 0$, then the inequality holds since all other terms and factors of the inequality not involved with $g(L)$ are non-negative.

The derivative $g'(L) = 2\left[141L^2 + 64L - 36\right]$. The zeroes of $g'(L)$ are $L = -0.7810$ and $L = 0.3297$ with a negative derivative between these two L values. It is easy to check that $g(0.3297) > 0$ is the only relative minimum and that $g(L) > 0$ for all $L \geq 0$. Thus the inequality holds.

A comment by the editor: **David Stone and John Hawkins of Statesboro, GA** sent in a solution path that was dependent on a computer, and this bothered them. They let $y = ax$ in the statement of the problem and then showed that the original inequality was equivalent to showing that

$$\frac{2a}{1+a} + \sqrt{\frac{1+a^2}{2}} \leq \frac{(\sqrt{a}+1)^2}{2} + \frac{(a+1)(\sqrt{a}-1)^4}{72a}.$$

They then had Maple graph the left and right hand sides of the inequality respectively; they analyzed the graphs and concluded that the inequality held (with equality holding for $a = 1$.) But this approach bothered them and so they let $a = z^2$ in the above inequality and they eventually obtained the following:

$$(z-1)^4\Big(z^{12} - 4z^{11} + 82z^{10} + 124z^9 - 1265z^8$$
$$+392z^7 + 2492z^6 + 392z^5 - 1265z^4 + 124z^3 + 82z^2 - 4z + 1\Big) \leq 0.$$

Again they called on Maple to factor the above polynomial, and it did into linear and irreducible quadratic factors. They then showed that there were no positive real zeros and so the inequality must be true. They also noted that equality holds if and only if $z = 1$; that is, equality holds for the original statement if and only if $x = y$. They ended their submission with the statement:

"The bottom line: with the use of a machine's assistance, we believe the original inequality to be true."

In their letter submitting the above to me David wrote:

"Last week I mentioned that our solution to Problem 5113 was dependent upon machine help. We are still in that position, so I send this to you as a comment, not as a solution. There is a nice reduction to an inequality in a single variable, but we never found an analytic verification for the inequality."

All of this reminded me of the comments in 1976 surrounding Appel and Haken's proof of the four color problem which was done with the aid of a computer. The concerns raised then, still exist today.

Also solved by Shai Covo, Kiryat-Ono, Israel; Boris Rays, Brooklyn, NY, and the proposer.

- **5114:** *Proposed by José Luis Díaz-Barrero, Barcelona, Spain*

Let M be a point in the plane of triangle ABC. Prove that

$$\frac{\overline{MA}^2 + \overline{MB}^2 + \overline{MC}^2}{\overline{AB}^2 + \overline{BC}^2 + \overline{CA}^2} \geq \frac{1}{3}.$$

When does equality hold?

Solution by Michael Brozinsky, Central Islip, NY

Without loss of generality let the vertices of the triangle be $A(0,0), B(a,0)$, and $C(b,c)$ and let M be (x,y). Now completing the square shows

$$\overline{AM}^2+\overline{BM}^2+\overline{CM}^2-\frac{1}{3}\left(\overline{AB}^2+\overline{BC}^2+\overline{AC}^2\right)$$

$$= \left(x^2+y^2+(x-a)^2+y^2+(x-b)^2+(y-c)^2-\frac{1}{3}\left(a^2+\left(b-a\right)^2+c^2+b^2+c^2\right)\right)$$

$$= 3\cdot\left(\left(x-\frac{a+b}{3}\right)+\left(y-\frac{c}{3}\right)^2\right)$$

and thus the given inequality follows at once and equality holds iff M is $\frac{2}{3}$ of the way from vertex C to side $\overline{AB}$. Relabeling thus implies that M is the centroid of the triangle.

Comments in the solutions of others: 1) From Kee-Wai Lau, Hong Kong, China. The inequality of the problem can be found at the top of p. 283, Chapter XI in *Recent Advances in Geometric Inequalities* by Mitrinovic, Pecaric, and Volenec, (Kluwer Academic Press), 1989.

The inequality was obtained using the Leibniz identity

$$\overline{MA}^2+\overline{MB}^2+\overline{MC}^2=3\overline{MG}^2+\frac{1}{3}\left(\overline{AB}^2+\overline{BC}^2+\overline{CA}^2\right)$$

where G is the centroid of triangle ABC. Equality holds if and only if $M=G$.

2) From Bruno Salgueiro Fanego, Viveiro Spain. This problem was solved for any point M in space using vectors. (See page 303 in *Problem Solving Strategies* by Arthur Engel, (Springer-Verlag), 1998.) Equality holds if, and only if, M is the centroid of ABC.

Another solution and a discussion of where the problem mostly likely originated can be found on pages 41 and 42 of

$$http://www.cpohoata.com/wp-conent/uploads/2008/10/inf081019.pdf.$$

Also, a local version of the Spanish Mathematical Olympiad of 1999 includes a version of this problem and it can be seen at
$http://platea.pntic.mec.es/\sim csanchez/local99.htm.$

3) From David Stone and John Hawkins (jointly), Statesboro, GA. Because the given problem has the sum of the squares of the triangle's sides as the denominator, one might conjecture the natural generalization

$$\frac{\sum_{i=1}^{n}\overline{MA_i}^2}{\sum_{i=1}^{n}\overline{A_iA_{i+1}}^2}\geq\frac{1}{n},$$

but this is not true. Instead, we must also allow all squares of diagonals to appear in the sum in the denominator. Of course, a triangle has no diagonals.

Also solved by Shai Covo, Kiryat-Ono, Israel; Michael N. Fried, Kibbutz Revivim, Israel; Paul M. Harms, North Newton, KS; Michael N. Fried, Kibbutz Revivim, Israel; Raúl A. Simón, Santiago, Chile, and the proposer.

- **5115:** *Proposed by Mohsen Soltanifar (student, University of Saskatchewan), Saskatoon, Canada*

Let G be a finite cyclic group. Compute the number of distinct composition series of G.

Solution 1 by Kee-Wai Lau,Hong Kong, China

Denote the order of a group S by $|S|$. Let $E = G_0, < G_1 < G_2 < \ldots < G_m = G$ be a composition series for G, where E is the subgroup of G consisting of the identity element only. A composition series is possible if and only if the factor groups $G_1/G_0, G_2/G_1, \ldots, G_m/G_{m-1}$ are simple. For cyclic group G, where all these factor groups are also cyclic, this is equivalent to saying that

$$|G_1/G_0| = p_1,\ |G_2/G_1| = p_2,\ \ldots,\ |G_m/G_{m-1}| = p_m,$$

where $p_1, p_2, \ldots, p_m$ are primes, not necessarily distinct. By the Jordan-Hölder theorem, m is uniquely determined and the prime divisors, $p_1, p_2, \cdots, p_m$ themselves are unique. Any other composition series therefore correspond with a permutation of the primes $p_1, p_2, \ldots, p_m$. Note that

$$|G| = |G_m| = \frac{|G_m|}{|G_{m-1}|}\frac{|G_{m-1}|}{|G_{m-2}|}\cdots\frac{|G_2|}{|G_1|}\frac{|G_1|}{1} = p_m p_{m-1}\cdots p_2 p_1.$$

We rewrite $|G|$ in standard form $|G| = q_1^{a_1} q_2^{a_2} \ldots q_k^{a_k}$, where $a_1, a_2, \ldots, a_k$ are positive integers and $q_1 < q_2 < \ldots q_k$ are primes. The number of distinct composition series of G then equals

$$\frac{(a_1 + a_2 + \cdots + a_k)!}{a_1! a_2! \ldots a_k!}$$

Solution 2 by David Stone and John Hawkins (jointly), Statesboro, GA

Let G have order n, where n has prime factorization $n = \prod_{i=1}^{m} p_i^{e_i}$. Then the number of distinct composition series of G is the multinomial coefficient $\binom{e_1 + e_2 + e_3 + \ldots + e_m}{e_1, e_2, e_3, \ldots, e_m}$. Letting e=$e_1 + e_2 + e_3 + \ldots + e_m$, this can be computed as

$$\binom{e}{e_1}\binom{e-e_1}{e_2}\binom{e-e_1-e_2}{e_3}\cdots\binom{e_{m-1}+e_m}{e_{m-1}}\binom{e_m}{e_m} = \frac{e!}{(e_1!)(e_2!)(e_3!)\cdots(e_m!)}.$$

Our rationale follows.

We'll simply let G be Z_n, written additively and denote the cyclic subgroup generated by a as $< a >= \{ka| \in Z\}$.

Note that $< a >$ is a subgroup of $< b >$ if and only if $a = bc$ for some c in G. We'll denote this by $< a > \leq < b >$. That is, to enlarge the subgroup $< a >$ to $< b >$, we divide a by some group element c to obtain b. In particular, if we divide a by a prime p to obtain b, then the factor group $< b > / < a >$ is isomorphic to the simple group Z_p.

In the lattice of subgroups of G, any two subgroups have a greatest lower bound, given by intersection , and a least upper bound, given by summation.The maximal length (ascending) chains are the distinct composition series. All such chain have the same length (by the Jordan-Hölder Theorem).

For a specific example, let $n = 12 = 2^2 \cdot 3^1$. In Z_{12}, the distinct subgroups are:

$$\begin{aligned} 0 &= \{0\}, \\ < 2 > &= \{0, 2, 4, 6, 8, 10\}, \\ < 4 > &= \{0, 4, 8\}, \\ < 3 > &= \{0, 3, 6, 9\}, \\ < 6 > &= \{0, 6\}, \\ < 1 > &= \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11\} = Z_{12}, \end{aligned}$$

and the maximal length ascending chains (composition series) are

$$0 \leq < 4 > \ \leq \ < 2 > \ \leq \ < 1 >,$$

$$0 \leq < 6 > \ \leq \ < 2 > \ \leq \ < 1 >,$$

$$0 \leq < 6 > \ \leq \ < 3 > \ \leq \ < 1 > .$$

Note that the composition factors (the simple factor groups) of the first chain are

$$\begin{aligned} < 4 > /0 &\cong Z_3 \\ < 2 > / < 4 > &\cong Z_2, \text{ and} \end{aligned}$$

$$< 1 > / < 2 > \;\; \cong \;\; Z_2.$$

Thus, the sequence of composition factors is Z_3, Z_2, Z_2.

Similarly for the second chain, the sequence of composition factors is Z_2, Z_3, Z_2, and for the third chain the sequence of composition factors is Z_2, Z_2, Z_3. The three elements of each chain are Z_2, Z_2, and Z_3, forced by the factorization of 12. The number of possible chains is simply the number of ways to arrange these three simple groups: 3. Note that

$$\binom{2+1}{2,1} = \binom{3}{2,1} = \binom{3}{2} \cdot \binom{1}{1} = 3.$$

Method: For arbitrary $n = \prod_{i=1}^{m} p_i^{e_i}$, this example demonstrates a constructive method for generating (and counting) all such maximal chains:

(i) Start with $0 =< n >$.

(ii) Divide (in the usual sense, not mod n) by one of $n's$ prime divisors, p, to obtain $k = \frac{n}{p}$, so that $0 =< n > \leq < k >$ and the factor group $< k > / < n > \cong Z_p$.

(iii) Next, divide k by any unused prime divisor, say q of n to obtain $h = \frac{k}{q}$, so that $< k > \leq < h >$ and the factor group $< h > / < k > \cong Z_q$.

(In this process, each prime factor p will be used e_i times, so there will be $e = e_1 + e_2 + e_3 = \ldots + e_m$ steps.)

We now have the beginning of a composition series: $0 \;\leq < k > \leq < h >$. Continue with the division steps until the supply of prime divisors of n is exhausted, so the final division will produce the final element of the chain: $< 1 > = \; Z_n$. We will have thus constructed a composition series. In the procedure there will be e_1 divisions by p_1, e_2 divisions by p_2, etc.

Therefore, the number of ways to carry out this procedure is the number of ways to carry out these dvisions: choose e_1 places from e possible spots to divide by p, choose e_2 places from the remaining $e - e_1$ possible spots to divide by p_2 etc.
So we can count the total number of ways to carry out the process:

$$\binom{e}{e_1}\binom{e-e_1}{e_2}\binom{e-e_1-e_2}{e_3}\cdots\binom{e_{m-1}+e_m}{e_{m-1}}\binom{e_m}{e_m}.$$

Moreover, if we let S be the sequence of simple groups consisting of e_1 copies of Z_{p_1}, e_2 copies of Z_{p_2}, etc., then S will have $e = e_1 + e_2 + e_3 + \cdots + e_m$ elements and each of our composition series will have some rearrangement of S as its sequence of compositions factors.

Example: Let $n = 360 = 2^3 \cdot 3^2 \cdot 5^1$.
Then the sequence of divisors $3, 5, 2, 2, 3, 2$ will produce he composition series

$$0 =< 360 > \leq < 120 > \leq < 24 > \leq < 12 > \leq < 6 > \leq < 2 > \leq < 1 > = Z_{360},$$

with composition factors $Z_3,\; Z_5,\; Z_2,\; Z_2,\; Z_3,\; Z_2$.

There are $\binom{3+2+1}{3,2,1} = \binom{6}{3} \cdot \binom{3}{2} \cdot \binom{1}{1} = 60$ different ways to construct a divisors sequence from $2, 2, 2, 3, 3, 5$, so Z_{360} has 60 distinct composition series.

Also solved by the proposer.