Problems

Ted Eisenberg, Section Editor

****

This section of the Journal offers readers an opportunity to exchange interesting mathematical problems and solutions. Please send them to Ted Eisenberg, Department of Mathematics, Ben-Gurion University, Beer-Sheva, Israel or fax to: 972-86-477-648. Questions concerning proposals and/or solutions can be sent e-mail to <eisenbt@013.net>. Solutions to previously stated problems can be seen at <http://www.ssma.org/publications>.

Solutions to the problems stated in this issue should be posted before January 15, 2018

- **5463:** *Proposed by Kenneth Korbin, New York, NY*

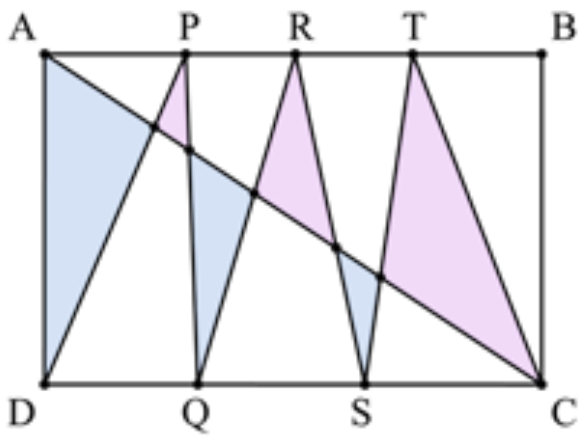
  Let N be a positive integer. Find triangular numbers x and y such that $x^2 + 14xy + y^2 = \left(72N^2 - 12N - 1\right)^2$.

- **5464:** *Proposed by Ed Gray, Highland Beach, FL*

  Let ABC be an equilateral triangle with side length s that is colored white on the front side and black on the back side. Its orientation is such that vertex A is at lower left, B is its apex, and C is at lower right. We take the paper at B and fold it straight down along the bisector of angle B, thus exposing part of the back side which is black. We continue to fold until the black part becomes $1/2$ of the existing figure, the other half being white. The problem is to determine the position of the fold, the distance defined by x (as a function of s) which is the distance from B to the fold.

- **5465:** *Proposed by Arsalan Wares, Valdosta State University, Valdosta, GA*

  Quadrilateral $ABCD$ is a rectangle with diagonal AC. Points P, R, T, Q and S are on sides AB and DC and they are connected as shown. Three of the triangles inside the rectangle are shaded pink, and three are shaded blue. Which is larger, the sum of the areas of the pink triangles or the sum of the areas of the blue triangles?



- **5466:** *Proposed by D.M. Bătinetu– Giurgiu, "Matei Basarab" National College, Bucharest, Romania and Neculai Stanciu, "Geroge Emil Palade" School, Buzău, Romania*

  Let $f : (0, +\infty) \to (0, +\infty)$ be a continuous function. Evaluate

$$\lim_{n\to\infty} \int_{\frac{n^2}{\sqrt[n]{n!}}}^{\frac{(n+1)^2}{\sqrt[n+1]{(n+1)!}}} f\left(\frac{x}{n}\right) dx.$$

- **5467:** *Proposed by José Luis Díaz-Barrero, Barcelona Tech, Barcelona, Spain*

  In an arbitrary triangle $\triangle ABC$, let a, b, c denote the lengths of the sides, R its circumradius, and let h_a, h_b, h_c respectively, denote the lengths of the corresponding altitudes. Prove the inequality

$$\frac{a^2+bc}{b+c} + \frac{b^2+ca}{c+a} + \frac{c^2+ab}{a+b} \geq \frac{3abc}{2R}\sqrt[3]{\frac{1}{h_a \cdot h_b \cdot h_c}},$$

  and give the conditions under which equality holds.

- **5468:** *Proposed by Ovidiu Furdui and Alina Sîntămărian, both at the Technical University of Cluj-Napoca, Cluj-Napoca, Romania*

  Find all differentiable functions $f : \Re \to \Re$ with $f(0) = 1$ such that $f'(x) = f^2(-x)f(x)$, for all $x \in \Re$.

Solutions

- **5445:** *Proposed by Kenneth Korbin, New York, NY*

  Find the sides of a triangle with exradii $(3, 4, 5)$.

Solution 1 by Solution by David E. Manes, Oneonta, NY

Denote the triangle by ABC with vertices A, B and C. Let $a = BC$, the side opposite the vertex $A, b = AC$ and $c = AB$. Let $r_a = 3$, the exradius of the circle tangent to side BC. Similarly, $r_b = 4$ is the exradius of the circle tangent to AC and $r_c = 5$ is the exradius of the circle tangent to AB. If r is the inradius of triangle ABC, then $\frac{1}{r_a} + \frac{1}{r_b} + \frac{1}{r_c} = \frac{1}{r}$ implies $\frac{1}{3} + \frac{1}{4} + \frac{1}{5} = \frac{1}{r}$ implies $r = \frac{60}{47}$. If Δ is the area of triangle ABC, then $\Delta^2 = r \cdot r_a \cdot r_b \cdot r_c$. Therefore,

$$\left(\frac{60}{47}\right)(3 \cdot 4 \cdot 5) = \Delta^2 \quad \text{implies} \quad \Delta = \frac{60}{\sqrt{47}} = \frac{60\sqrt{47}}{47}.$$

If $s = \dfrac{a+b+c}{2}$ is the semiperimeter of ABC, then $s = \dfrac{\Delta}{r}$. Therefore,

$$s = \frac{\left(\frac{60\sqrt{47}}{47}\right)}{\left(\frac{60}{47}\right)} = \sqrt{47}.$$

Using the formula $r_a = \dfrac{\Delta}{s-a}$, one obtains $a = s - \dfrac{\Delta}{r_a}$. Therefore,

$$a = \sqrt{47} - \frac{\left(\frac{60}{\sqrt{47}}\right)}{3} = \frac{47-20}{\sqrt{47}} = \frac{27\sqrt{47}}{47}.$$

Similarly,

$$b = s - \frac{\Delta}{r_b} = \sqrt{47} - \frac{\left(\frac{60}{\sqrt{47}}\right)}{4} = \frac{32\sqrt{47}}{47},$$

$$c = s - \frac{\Delta}{r_c} = \sqrt{47} - \frac{\left(\frac{60}{\sqrt{47}}\right)}{5} = \frac{35\sqrt{47}}{47}.$$

This completes the solution.

Solution 2 by Bruno Salgueiro Fanego, Viveiro, Spain

In the published soltion by Howard Eves to problem 786 in the Journal Crux Mathematicorum (1984,10(20)), a more general result of the above problem is proved. For arbitrarily chosen positive real numbers r_1, r_2, r_3 there is one and only one triangle whose exradii are r_1, r_2, r_3, and that is the one whose sides are:

$$a = \frac{r_1(r_2+r_3)}{\sqrt{r_1r_2+r_2r_3+r_3r_1}},\ b = \frac{r_2(r_3+r_1)}{\sqrt{r_1r_2+r_2r_3+r_3r_1}}\ c = \frac{r_3(r_1+r_2)}{\sqrt{r_1r_2+r_2r_3+r_3r_1}}.$$

For the exradii values of $r_1 = 3,\ r_2 = 4$ and $r_3 = 5$ we find that

$$a = \frac{3(4+5)}{\sqrt{3\cdot 4+4\cdot 5+5\cdot 3}} = \frac{27}{\sqrt{47}},\ b = \frac{4(5+3)}{\sqrt{3\cdot 4+4\cdot 5+5\cdot 3}} = \frac{32}{\sqrt{47}}, c = \frac{35}{\sqrt{47}}.$$

Solution 3 by Ed Gray, Highland Beach, FL

Letting r be the in-radius of the given triangle, r_1, r_2, r_3 the ex-radii, s its semi-perimeter, K its area and a, b, c its side lengths, then following relationships, that were developed by Feuerbach, hold:

(1) $\dfrac{1}{r} = \dfrac{1}{r_1} + \dfrac{1}{r_2} + \dfrac{1}{r_3}$.

(2) $K^2 = r \cdot r_1 \cdot r_2 \cdot r_3$

(3) $s \cdot K = r_1 \cdot r_2 \cdot r_3$

(4) $a = s - \dfrac{K}{r_1},\ b = s - \dfrac{K}{r_2}, c = s - \dfrac{K}{r_3.}$

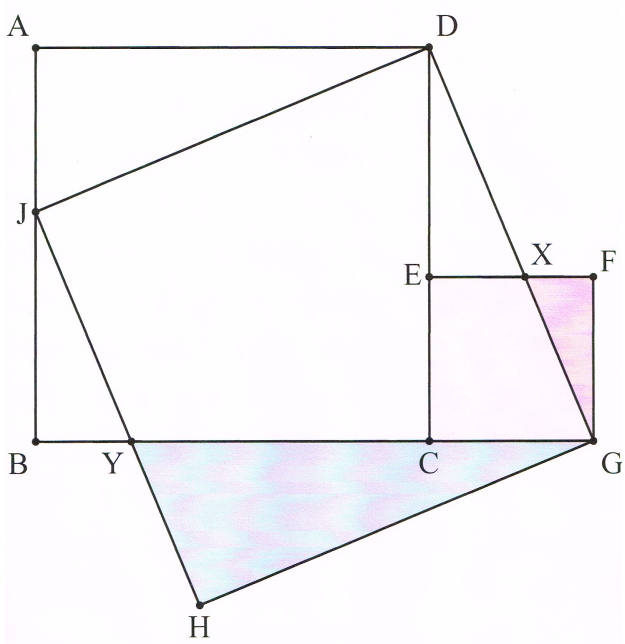
Making the substitutions we find that $a = \dfrac{27\sqrt{47}}{47},\ b = \dfrac{32\sqrt{47}}{47}, c = \dfrac{35\sqrt{47}}{47}$.

Comment by David Stone and John Hawkins of Georgia Southern University: An interesting connection to this problem (from Wolfram Mathworld) is that the curvature of the incircle equals the sum of the curvatures of the excircles: $\dfrac{1}{r} = \dfrac{1}{r_a} + \dfrac{1}{r_b} + \dfrac{1}{r_c}$ which equals $\left(\dfrac{r_br_c + r_ar_c + r_ar_b}{r_ar_br_c}\right)$. Thus the area can be written as $\Delta = \sqrt{r_ar_br_c}$.

Also solved by Arkady Alt, San Jose, CA; Kee-Wai Lau, Hong Kong, China; Charles McCracken, Dayton, OH; Daniel Sitaru, Mathematics Department, National Economic College "Theodor Costescu," Drobeta Turnu - Severin, Mehedinti, Romania; Albert Stadler, Herrliberg, Switzerland; David Stone and John Hawkins, Georgia Southern University, Statesboro, GA, and the proposer.

- **5446:** *Proposed by Arsalan Wares, Valdosta State University, Valdosta, GA*

  Polygons $ABCD, CEFG$, and $DGHJ$ are squares. Moreover, point E is on side $DC, X = DG \cap EF$, and $Y = BC \cap JH$. If GX splits square $CEFG$ in regions whose areas are in the ratio 5:19. What part of square $DGHJ$ is shaded? (Shaded region in $DGHJ$ is composed of the areas of triangle YHG and trapezoid $EXGC$.)



Solution 1 by "Get Stoked" Problem Solving Group, Mountain Lakes High School, Mountain Lakes, NJ

Since $\angle B = \angle H,\ \angle JYB = \angle GYH$,

$$\triangle JBY \sim \triangle GHY,$$

and because $\triangle JBY$ and $\triangle DAJ$ have a shared angle, $\angle B = \angle A$

$$\triangle JBY \sim \triangle DAJ,$$

and because $AD = DC, JD = GD, \angle A = \angle DCG$,

$$\triangle DAJ \cong \triangle DCG$$

and since $CG || EX$,

$$\triangle DCG \sim \triangle DEX$$

and because $\angle DEX = \angle F,\ \angle DXE = \angle GXF$,

$$\triangle DEX \sim \triangle GFX.$$

Therefore,

$$\triangle GYH \sim \triangle GXF$$

Without loss of generality, set the area of $\triangle YHG = 5$ and trapezoid $EXGC = 19$. Adding the areas of $\triangle YHG$ and trapezoid $EXGC$ and finding each side obtains:

$$\sqrt{5 + 19} = 2\sqrt{6}.$$

Drawing a perpendicular line from point X to CG creates line IX.
Because the area of $XFGI$ is double of that of $\triangle GXF$ and $FG = EC = XI$,

$$XF : CG = 5 : 12.$$

Since $FG = CG$, it can be concluded hat $\triangle GXF$ is a 5 -12 -13 triangle. Because $\triangle DGC \sim \triangle GXF$,

$$CG : DG = 5 : 13.$$

Now that we know the ratio between the two squares and that the ratio of the area between two similar polygons is the square of the ratio of the sides, it is apparent that

$$\frac{area(EXGC)}{area(JDGH)} = \frac{19}{24} \cdot \left(\frac{5}{13}\right)^2 = \frac{475}{4096}.$$

Adding the two pieces results in the part of square $DCHJ$ that is shaded

$$\frac{5}{24} + \frac{475}{4096} = \frac{55}{169}.$$

Solution 2 by Kenneth Korbin, New York, NY

Answer: $\frac{55}{169}$.

Let $\overline{XF} = 25$ and $\overline{EX} = 35$. Then each segment in the diagram will have positive integer length.

$$AD = 144,\ AJ = 60,\ JB = 89,\ BY = 35,\ YH = 65,\ HG = 156,\ GF = 60,\ FX = 25,$$

$$XE = 35,\ ED = 84,\ DJ = 156,\ JY = 91,\ YG = 169,\ CE = 60,\ CG = 60,\ XG = 65.$$

$$DX = 91.$$

Every triangle in this diagram is similar to the Pythagorean Triangle with sides $(5, 12, 13)$.

Area of square $DGHJ = (156)^2 = 24336$.

Area of triangle $YHG = \frac{1}{2}(65)(156) = 5070$.

Area of trapezoid $EXGC = (60)^2 - \frac{1}{2}(25)(60) = 2850$.

So the desired ratio is $\frac{5070 + 2850}{24336} = \frac{55}{169}$.

Also solved by Jeremiah Bartz and Nicholas Newman, University of North Dakota and Troy University respectively, Grand Forks, ND and Troy, AL; Bruno Salgueiro Fanego, Viveiro, Spain; Michael N. Fried, Ben-Gurion University of the Negev, Beer-Sheva, Israel; Ed Gray, Highland Beach, FL; Kee-Wai Lau, Hong Kong, China; David E. Manes, Oneonta, NY; Daniel Sitaru, Mathematics Department, National Economic College "Theodor Costescu," Drobeta Turnu - Severin, Mehedinti, Romania; Sachit Misra, Nelhi, India; Boris Rays, Brooklyn, NY; David Stone and John Hawkins, Georgia Southern University, Statesboro, GA, and the proposer.

- **5447:** *Proposed by Iuliana Trască, Scornicesti, Romanai*

Show that if x, y, and z is each a positive real number, then

$$\frac{x^6 \cdot z^3 + y^6 \cdot x^3 + z^6 \cdot y^3}{x^2 \cdot y^2 \cdot z^2} \geq \frac{x^3 + y^3 + z^3 + 3x \cdot y \cdot z}{2}.$$

Solution 1 by Albert Stadler, Herrliberg, Switzerland

The stated inequality is equivalent to

$$2x^6z^3 + 2y^6x^3 + 2z^6y^3 \geq x^5y^2z^2 + x^2y^5z^2 + x^2y^2z^5 + 3x^3y^3z^3. \qquad (1)$$

By the AM-GM inequality,

$$\sum_{cycl} x^6z^3 = \sum_{cycl}\left(\frac{2}{3}x^6z^3 + \frac{1}{3}y^6x^3\right) \geq \sum_{cycl}\left(x^{\frac{2}{3}\cdot 6}z^{\frac{2}{3}\cdot 6}y^{\frac{1}{3}\cdot 6}x^{\frac{1}{3}\cdot 3}\right) = \sum_{cycl} x^5y^2z^2,$$

$$\sum_{cycl} x^6z^3 \geq 3x^3y^3z^3$$

Statement (1) follows by adding these two inequalities.

Solution 2 by Arkady Alt, San Jose, CA

Note that,

$$\frac{x^6z^3 + y^6x^3 + z^6y^3}{x^2\ y^2\ z^2} \geq \frac{x^3 + y^3 + z^3 + 3xyz}{2} \quad \iff \quad 2\left(x^6z^3 + y^6x^3 + z^6y^3\right)$$

$$\geq \quad x^5\ y^2\ z^2 + x^2\ y^5\ z^2 + x^2\ y^2\ z^5 + 3x^3\ y^3\ z^3.$$

By AM-GM Inequality,

$$x^6z^3 + y^6x^3 + z^6y^3 \geq 3\sqrt[3]{x^6z^3 \cdot y^6x^3 \cdot z^6y^3} = 3\sqrt[3]{x^9y^9z^9} = 3x^3\ y^3\ z^3.$$

And again by AM-GM Inequality.

$$2x^6z^3 + y^6x^3 \geq 3\sqrt[3]{(x^6z^3)^2\, y^6x^3} = 3\sqrt[3]{x^{15}y^6z^6} = 3x^5y^2z^2,$$

and therefore,

$$3\sum_{cyc} x^6z^3 = \sum_{cyc}\left(2x^6z^3 + y^6x^3\right) \geq \sum_{cyc} 3x^5y^2z^2 \iff \sum_{cyc} x^6z^3 \geq \sum_{cyc} x^5\ y^2\ z^2.$$

Thus, $2\sum_{cyc} x^6z^3 = \sum_{cyc} x^6z^3 + \sum_{cyc} x^6z^3 \geq \sum_{cyc} x^5\ y^2\ z^2 + 3x^3\ y^3\ z^3.$

Solution 3 by Moti Levy, Rehovot, Israel

By Muirhead inequality ($(6,3,0)$ majorizes $(5,2,2)$),

$$\sum_{sym} x^6x^3z^0 \geq \sum_{sym} x^5y^2z^2,$$

or explicitly,

$$\left(x^6z^3 + y^6x^3 + z^6y^3\right) + \left(x^6y^3 + v^6z^3 + z^6x^3\right) \geq 2\left(x^5y^2z^2 + x^2y^5z^2 + x^2y^2z^5\right). \qquad (1)$$

Again, by Muirhead inequality ($(5,2,2)$ majorizes $(3,3,3)$),

$$\sum_{sym} x^5y^2z^2 \geq \sum_{sym} x^3y^3z^3$$

or explicitly,

$$x^5y^2z^2 + x^2y^5z^2 + x^2y^2z^5 \geq 3x^3y^3z^3. \qquad (2)$$

Given three positive numbers a, b, c. We can always assign their values to x, y and z respectively, such that $x^6z^3 + y^6x^3 + z^6y^3 \geq x^6y^3 + y^6z^3 + z^6x^3$. Hence, without loss of generality, we can assume that

$$x^6z^3 + y^6x^3 + z^6y^3 \geq x^6y^3 + y^6z^3 + z^6x^3, \qquad (3)$$

then by (1), (2) and (3)

$$\begin{aligned} 2\left(x^6z^3 + y^6x^3 + z^6y^3\right) &\geq \left(x^6z^3 + y^6x^3 + z^6y^3\right) + \left(x^6y^3 + v^6z^3 + z^6x^3\right) \\ &\geq 2\left(x^5y^2z^2 + x^2y^5z^2 + x^2y^2z^5\right) \\ &\geq x^5y^2z^2 + x^2y^5z^2 + x^2y^2z^5 + 3x^3y^3z^3. \end{aligned}$$

which is equivalent to

$$\frac{x^6z^3 + y^6x^3 + z^6y^3}{x^2y^2z^2} \geq \frac{x^3 + y^3 + z^3 + 3xyz}{2}.$$

Solution 4 by Kee-Wai Lau, Hong Kong, China

We prove the stronger inequality

$$\frac{x^6z^3 + y^6x^3 + z^6y^3}{x^2y^2z^2} \geq x^3 + y^3 + z^3. \qquad (1)$$

Since $x^3 + y^3 + z^3 \geq 3xyz$ by the AM-GM inequality, the inequality of the problem follows immediately from (1).

By homogeneity, we assume without loss of generality that $xyz = 1$. By substituting $z = \dfrac{1}{xy}$ into (1), we deduce after some algebra that (1) is equivalent to

$$x^9 + x^9y^9 + 1 - x^9y^3 - x^6y^6 - x^3 \geq 0. \qquad (2)$$

Denote the left side of (2) by f. It can be checked readily by expanding both sides that

$$\begin{aligned}(1 + 2x^3 + x^3y^3)f &= x^9(1 + x^3)(1 + y^3)(1 - y^3)^2 + (1 + x^3)^2(1 - x^3)^2 + \\ &\quad x^3(1 + y^3)(1 + x^3y^3)(1 - x^3y^3)^2,\end{aligned}$$

which is nonnegative. Thus (2) holds and this completes the solution.

Solution 5 by Hatef I. Arshagi, Guilford Technical Community College, Jamestown, NC

It is well-known that for all $a, b, c \geq 0$ we have $a^3 + b^3 + c^3 \geq a^2b + b^2c + c^2a$, and $a^3 + b^3 + c^3 \geq 3abc$. Now for all positive real numbers x, y, and z we can write

$$\frac{2\left(x^6 \cdot z^3 + y^6 \cdot x^3 + z^6 \cdot y^3\right)}{x^2 \cdot y^2 \cdot z^2} \geq \frac{2\left[(x^2z)^2y^2x + (y^2x)^2z^2y + (z^2y)^2x^2z\right]}{x^2y^2z^2} =$$

$$2(x^3 + y^3 + z^3) = x^3 + y^3 + z^3 + (x^3 + y^3 + z^3) \geq x^3 + y^3 + z^3 + 3xyz.$$

Now, multiplying both sides of the inequality

$$\frac{2(x^6 \cdot z^3 + y^6 \cdot x^3 + z^6 \cdot y^3)}{x^2 \cdot y^2 \cdot z^2} \geq x^3 + y^3 + z^3 + 3xyz,$$

by $\dfrac{1}{2}$, will give us the desired result.

Also solved by Bruno Salgueiro Fanego, Viveiro, Spain; Ed Gray, Highland Beach, FL; Moti Levy, Rehovot, Israel; David E. Manes, Oneonta, NY; Sachit Misra, Delhi, India; Paolo Perfetti, Department of Mathematics, Tor Vergata University of Rome, Italy; Daniel Sitaru, Mathematics Department, National Economic College "Theodor Costescu," Drobeta Turnu - Severin, Mehedinti, Romania, and the proposer.

- **5448:** *Proposed by Yubal Barrios and Ángel Plaza, University of Las Palmas de Gran Canaria, Spain*

Evaluate: $\displaystyle\lim_{n\to\infty} \sqrt[n]{\sum_{\substack{0\leq i,j\leq n \\ i+j=n}} \binom{2i}{i}\binom{2j}{j}}.$

Solution 1 by Brian Bradie, Christopher Newport University, Newport News, VA

The generating function for the central binomial coefficients is $(1-4x)^{-1/2}$; that is,

$$\sum_{i=0}^{\infty}\left(\begin{array}{c}2i\\ i\end{array}\right)x^i = \frac{1}{\sqrt{1-4x}}.$$

It follows that

$$\sum_{0\le i,j\le n, i+j=n}\left(\begin{array}{c}2i\\ i\end{array}\right)\left(\begin{array}{c}2j\\ j\end{array}\right) = \sum_{i=0}^{n}\left(\begin{array}{c}2i\\ i\end{array}\right)\left(\begin{array}{c}2(n-i)\\ n-i\end{array}\right)$$

is the coefficient of x^n in the function

$$\frac{1}{\sqrt{1-4x}}\cdot\frac{1}{\sqrt{1-4x}} = \frac{1}{1-4x} = \sum_{n=0}^{\infty}(4x)^n,$$

which is 4^n. Thus,

$$\lim_{n\to\infty}\sqrt[n]{\sum_{0\le i,j\le n, i+j=n}\left(\begin{array}{c}2i\\ i\end{array}\right)\left(\begin{array}{c}2j\\ j\end{array}\right)} = \lim_{n\to\infty}\sqrt[n]{4^n} = \lim_{n\to\infty} 4 = 4.$$

Solution 2 by Daniel Sitaru, Mathematics Department, National Economic College "Theodor Costescu," Drobeta Turnu - Severin, Mehedinti, Romania

$$(1+x)^0(1+x)^{2n} + (1+x)^2(1+x)^{2n-2} + (1+x)^4(1+x)^{2n-4} + \dots$$
$$\dots + (1+x)^{2n}(1+x)^0 = (2n+1)(1+x)^{2n}$$

The coefficient of x^n in LHS and RHS are equal:

$$\binom{2n}{n} + \binom{2}{1}\binom{2n-2}{n-1} + \binom{4}{2}\binom{2n-4}{n-2} + \dots + \binom{2n}{n} = (2n+1)\binom{2n}{n}$$

$$\sum_{\substack{0\le i,j\le n\\ i+j=n}}\binom{2i}{i}\binom{2j}{j} = (2n+1)\binom{2n}{n}$$

$$\lim_{n\to\infty}\sqrt[n]{\sum_{0\le i,j\le n}\binom{2i}{i}\binom{2j}{j}} = \lim_{n\to\infty}\sqrt[n]{(2n+1)\binom{2n}{n}}$$

$$\overbrace{=}^{CAUCHY-D'ALEMBERT} \lim_{n\to\infty}\frac{2n+3}{2n+1}\cdot\frac{\frac{(2n+2)!}{((n+1)!)^2}}{\frac{(2n)!}{((n)!)^2}}$$

$$= 1\cdot\lim_{n\to\infty}\frac{(2n+1)(2n+2)}{(n+1)^2} = 4.$$

Solution 3 by Kee-Wai Lau, Hong Kong, China

It is well known that that $|x| \le \frac{1}{4}$, $\sum_{i=0}^{\infty} \binom{2i}{i} x^i = \frac{1}{\sqrt{1-4x}}$, with the usual convention that $0! = 1$ and $\binom{0}{0} = 1$. Hence,

$$\sum_{i=0}^{\infty} (4x)^i = \frac{1}{1-4x} = \left(\sum_{i=0}^{\infty} \binom{2i}{i} x^i \right)^2 = \sum_{n=0}^{\infty} \sum_{\substack{0 \le i,j \le n \\ i+j=n}} \binom{2i}{i}\binom{2j}{j} x^n.$$

Thus for nonnegative integers n,

$$\sum_{n=0}^{\infty} \sum_{\substack{0 \le i,j \le n \\ i+j=n}} \binom{2i}{i}\binom{2j}{j} = 4^n,$$

so that the limit of the problem equals 4.

Also solved by Arkady Alt, San Jose, CA; Bruno Salgueiro Fanego, Viveiro, Spain; Ed Gray, Highland Beach, FL; Perfetti Paolo, Department of Mathematics, Tor Vergata University, Rome Italy; Albert Stadler, Herrliberg, Switzerland; Anna V. Tomova, Varna, Bulgaria, and the proposer.

- **5449:** *Proposed by José Luis Díaz-Barrero, Barcelona Tech, Barcelona, Spain*

Without the use of a computer, find the real roots of the equation

$$x^6 - 26x^3 + 55x^2 - 39x + 10 = (3x-2)\sqrt{3x-2}.$$

Solution 1 by Albert Stadler, Herrliberg, Switzerland

We need to consider only the values of $x \ge 2/3$, since the square root is not real for $x < 2/3$.

We see that $x = 1$ and $x = 2$ are roots of the given equation Suppose that $x \notin \{1, 2\}$. We find that

$$\begin{aligned} x^6 - 27x^3 + 55x^2 - 39x + 10 &= (x-1)(x-2)\left(x^4 + 3x^3 + 7x^2 - 12x + 5\right) \\ &= (3x-2)\sqrt{3x-2} - x^3 \\ &= \frac{(3x-2)^3 - x^6}{(3x-2)\sqrt{3x-2} + x^3} \\ &= \frac{(x-1)(x-2)\left(x^4 + 3x^3 + 7x^2 - 12x + 4\right)}{(3x-2)\sqrt{3x-2} + x^3}, \text{ implying} \end{aligned}$$

$$x^4 + 3x^3 + 7x^2 - 12x + 5 = \frac{x^4 + 3x^3 + 7x^2 - 12x + 4}{(3x-2)\sqrt{3x-2} + x^3}. \qquad (1)$$

We note that $x^4 + 3x^3 + 7x^2 - 12x + 4 = x^4 + x^2(3x-2) + (3x-2)^2 \geq 0$, since $x \geq 2/3$. So (1) has no other real solutions than $x = 1$ and $x = 2$.

Solution 2 by Ed Gray, Highland Beach, FL

Define the function:

(1) $f(x) = x^6 - 26x^3 + 55x^2 - 39x + 10 - (3x-2)^{3/2} = 0$.

Consider the term $(3x-2)^{3/2}$. Since the values of $x = 1$ and $x = 2$ both provide integer values, it is worth trying these values as a first guess. In fact,

(2) $f(1) = 1 - 26 + 55 - 39 + 10 - 1 = 0$, so in fact, $x = 1$ is a root.

(3) $f(2) = 64 - 26(8) + 55(4) - 39(2) + 10 - 8 = 64 - 208 + 220 - 78 + 10 - 8 = 0$, so, in fact $x = 2$ is also a root.

It may be fruitful to utilize the derivative which is:

(4) $f'(x) = 6x^5 - 78x^2 + 110x - 39 - (3/2) * (3)\sqrt{3x-2}$.

We note that

(5) $f''(2) = (6)(32) - (78)(4) + 110)(2) - 39 - 9 = 192 - 312 + 220 - 39 - 9 = 52$.

(6) $f'(1) = 6 - 78 + 110 - 39 - 4.5 = -5.5$, so the function is 0 at $x = 1$ and $x = 2$. At $x = 1$, it is decreasing and at $x = 2$ it is increasing. Therefore, there is a point x_0 with $1 < x_0 < 2$ and $f'(x_0) = 0$. The function is increasing at $x = 2$, where the derivative is greater than 0, so if $x > 2$, the function is greater than 0. It would be good if the function stays positive for $x > 2$.

Note the second derivative is:

(7) $f''(x) = 30x^4 - 156x + 110 - (27/4) * (3x-2)^(- 1/2)$.

At $x = 2$, $f''(2)480 - 312 + 110 - 27/8 \ = 274.625$, and clearly increases as x increases. We conclude there can be no real roots greater than 2.

Now we look at the situation where $x = 1$, which is a root. $f'(1) = -5.5$. Therefore, values of $f(x)$ for x slightly less than 1 must be positive. If $x < 2/3$, we note the radical term becomes negative and complex terms will be introduced, negating the existence of real roots. We need to consider the region $2/3 < x < 1$. The value of

$$\begin{aligned} f(2/3) &= (2/3)^6 - 26((2/3)^3 + 56(3/3)^2 - 39(2/3) + 10 \\ &= 64/729 - 26(8/27) + 55(4/9) - 39(2/3) + 10 = 604/729 > 0. \end{aligned}$$

Also, $f'(2/3) = 111/243$. This is unexpectedly positive, which means the function rises from $604/729$ at $x = 2/3$ as x increases, then there must be a point x_1 such that $2/3 < x_1 < 1$ and $f'(x_1) = 0$. After $x > x_1$, the derivative turns negative and the function descends to 0 at $x = 1$. Therefore, there can be no other real roots other than $x = 1$ and $x = 2$.

Solution 3 by Anna V. Tomova, Varna, Bulgaria

The decision area is: $3x - 2 \geq 0 \implies x \geq \dfrac{2}{3}$, $x^6 - 26x^3 + 55x^2 - 39x + 10 \geq 0$. We let $\sqrt{3x-2} = t \geq 0 \iff x = \dfrac{t^2+2}{3}$, and after substitution we obtain

$$t^{12} + 12t^{10} + 60t^8 - 542t^6 + 483t^4 - 729t^3 + 111t^2 + 604 = 0.$$

Looking for low-valued positive integer roots to the above equation so that we can use the factor theorem, we see that $t = 1$ and $t = 2$ allows us to rewrite the equation as

$$(t-1)(t-2)\left(t^{10} + 3t^9 + 19t^8 + 51t^2 + 175t^6 + 423t^5 + 377t^4 + 285t^3 + 584t^2 + 453t + 302\right) = 0.$$

Because all of the coefficients in the third factor are positive, we see that there are no other positive roots. So, $\begin{cases} x = \dfrac{t^2+2}{3} = 1 \implies x = 1 \\ x = \dfrac{t^2+2}{3} = 2 \implies x = 2. \end{cases}$

Solution 4 by Brian D. Beasley, Presbyterian College, Clinton, SC

We note that $x = 1$ and $x = 2$ satisfy the given equation, and we show that those are the only real roots.

Squaring both sides of the equation and factoring yields $(x-1)(x-2)f(x) = 0$, where

$$f(x) = x^{10} + 3x^9 + 7x^8 - 37x^7 - 15x^6 - 49x^5 + 579x^4 - 1025x^3 + 820x^2 - 327x + 54.$$

Since we must have $x \geq 2/3$ in the original equation, it suffices to show that $f(x) \neq 0$ for each $x \geq 2/3$. We write $f(x) = (g(x))^2 + h(x)$, where

$$g(x) = x^5 + \tfrac{3}{2}x^4 + \tfrac{19}{8}x^3 - \tfrac{353}{16}x^2 + \tfrac{2915}{128}x - \tfrac{1603}{256}$$

and

$$h(x) = \tfrac{1463}{512}x^4 + \tfrac{2463}{256}x^3 + \tfrac{410783}{16384}x^2 - \tfrac{684823}{16384}x + \tfrac{969335}{65536}.$$

Then $h'(x) = \frac{1463}{128}x^3 + \frac{7389}{256}x^2 + \frac{410783}{8192}x - \frac{684823}{16384}$. Since $h''(x) > 0$ on $(0,\infty)$, h' is increasing on $(0,\infty)$. Also, $h'(2/3) > 0$, so we have $h'(x) > 0$ for each $x \geq 2/3$. Thus h is increasing on $[2/3,\infty)$ with $h(2/3) > 0$, so $h(x) > 0$ for each $x \geq 2/3$ as needed. Hence $f(x) > 0$ for each $x \geq 2/3$.

Also solved by Bruno Salgueiro Fanego, Viveiro, Spain; Kee-Wai Lau, Hong Kong, China; Perfetti Paolo, Department of Mathematics, Tor Vergata University, Rome Italy; David Stone and John Hawkins, Georgia Southern University, Statesboro, GA, and the proposer.

- **5450:** *Proposed by Ovidiu Furdui, Technical University of Cluj-Napoca, Cluj-Napoca, Romania*

Let k be a positive integer. Calculate

$$\int_0^1 \int_0^1 \left\lfloor \frac{x}{y} \right\rfloor^k \frac{y^k}{x^k}\, dxdy,$$

where $\lfloor a \rfloor$ denotes the floor (the integer part) of a.

Solution 1 by Brian Bradie, Christopher Newport University, Newport News, VA

Reverse the order of integration, and then write

$$\begin{aligned} \int_0^1 \int_0^1 \left\lfloor \frac{x}{y} \right\rfloor^k \frac{y^k}{x^k}\, dx\, dy &= \int_0^1 \int_0^1 \left\lfloor \frac{x}{y} \right\rfloor^k \frac{y^k}{x^k}\, dy\, dx \\ &= \int_0^1 \int_x^1 \left\lfloor \frac{x}{y} \right\rfloor^k \frac{y^k}{x^k}\, dy\, dx + \sum_{n=1}^{\infty} \int_0^1 \int_{x/(n+1)}^{x/n} \left\lfloor \frac{x}{y} \right\rfloor^k \frac{y^k}{x^k}\, dy\, dx. \end{aligned}$$

For $x \le y \le 1$, $\lfloor x/y \rfloor = 0$, while for $x/(n+1) \le y \le x/n$, $\lfloor x/y \rfloor = n$, so

$$\int_0^1 \int_0^1 \left\lfloor \frac{x}{y} \right\rfloor^k \frac{y^k}{x^k}\, dx\, dy = \sum_{n=1}^{\infty} n^k \int_0^1 \int_{x/(n+1)}^{x/n} \frac{y^k}{x^k}\, dy\, dx.$$

Now,

$$\begin{aligned} \int_0^1 \int_{x/(n+1)}^{x/n} \frac{y^k}{x^k}\, dy\, dx &= \frac{1}{k+1} \int_0^1 \frac{1}{x^k} \left(\frac{x^{k+1}}{n^{k+1}} - \frac{x^{k+1}}{(n+1)^{k+1}} \right) dx \\ &= \frac{1}{k+1} \left(\frac{1}{n^{k+1}} - \frac{1}{(n+1)^{k+1}} \right) \int_0^1 x\, dx \\ &= \frac{1}{2(k+1)} \left(\frac{1}{n^{k+1}} - \frac{1}{(n+1)^{k+1}} \right), \end{aligned}$$

so

$$\begin{aligned} \int_0^1 \int_0^1 \left\lfloor \frac{x}{y} \right\rfloor^k \frac{y^k}{x^k}\, dx\, dy &= \frac{1}{2(k+1)} \sum_{n=1}^{\infty} n^k \left(\frac{1}{n^{k+1}} - \frac{1}{(n+1)^{k+1}} \right). \\ &= \frac{1}{2(k+1)} \sum_{n=1}^{\infty} \frac{n^k - (n-1)^k}{n^{k+1}}. \end{aligned}$$

By the binomial theorem,

$$(n-1)^k = \sum_{j=0}^{k} (-1)^j \binom{k}{j} n^{k-j}.$$

It follows that

$$\begin{aligned} n^k - (n-1)^k &= \sum_{j=1}^{k} (-1)^{j+1} \binom{k}{j} n^{k-j}, \\ \frac{n^k - (n-1)^k}{n^{k+1}} &= \sum_{j=1}^{k} (-1)^{j+1} \binom{k}{j} \frac{1}{n^{j+1}}, \end{aligned}$$

and

$$\sum_{n=1}^{\infty} \frac{n^k - (n-1)^k}{n^{k+1}} = \sum_{j=1}^{k} (-1)^{j+1} \binom{k}{j} \sum_{n=1}^{\infty} \frac{1}{n^{j+1}} = \sum_{j=1}^{k} (-1)^{j+1} \binom{k}{j} \zeta(j+1),$$

where $\zeta(x)$ denotes the Riemann zeta function. Finally,

$$\int_0^1 \int_0^1 \left\lfloor \frac{x}{y} \right\rfloor^k \frac{y^k}{x^k}\, dx\, dy = \frac{1}{2(k+1)} \sum_{j=1}^{k} (-1)^{j+1} \binom{k}{j} \zeta(j+1).$$

Solution 2 by Ángel Plaza, University of Las Palmas de Gran Canaria, Spain.

For any positive integer m, let us assume that $m \le \dfrac{x}{y} < m+1$, which it is equivalent to

$\dfrac{x}{m+1} \le y < \dfrac{x}{m}$. The proposed integral, say I, becomes

$$\begin{aligned} I &= \sum_{m=1}^{\infty} m^k \int_0^1 \frac{1}{x^k} \int_{\frac{x}{m+1}}^{\frac{x}{m}} y^k dy\, dx \\ &= \sum_{m=1}^{\infty} m^k \int_0^1 \frac{1}{x^k} \frac{y^{k+1}}{k+1}\Big]_{\frac{x}{m+1}}^{\frac{x}{m}} dx \\ &= \sum_{m=1}^{\infty} \frac{m^k}{2(k+1)} \left(\frac{1}{m^{k+1}} - \frac{1}{(m+1)^{k+1}} \right) \\ &= \frac{1}{2(k+1)} \sum_{m=1}^{\infty} \left(\frac{1}{m} - \frac{m^k}{(m+1)^{k+1}} \right) \\ &= \frac{1}{2(k+1)} \sum_{m=1}^{\infty} \left(\frac{1}{m} - \frac{1}{m+1} - \frac{\sum_{j=1}^{k} (-1)^j \binom{k}{j}}{(m+1)^{k+1-j}} \right) \end{aligned}$$

from where,

$$I = \frac{\displaystyle\sum_{j=1}^{k} (-1)^{j+1} \binom{k}{j} \zeta(j+1)}{2(k+1)}.$$

Solution 3 by Perfetti Paolo, Department of Mathematics, Tor Vergata University, Rome Italy

We change variables $x/y = t$, $x = u$ and the integral becomes

$$\int_0^1 dt \int_0^t du \frac{1}{t^{k+2}} \lfloor t \rfloor^k u + \int_1^{\infty} dt \int_0^1 du \frac{1}{t^{k+2}} \lfloor t \rfloor^k u = \int_1^{\infty} dt \int_0^1 du \frac{1}{t^{k+2}} \lfloor t \rfloor^k u.$$

The first integral is zero so we get

$$\begin{aligned} &\frac{1}{2} \sum_{q=1}^{\infty} \int_q^{q+1} q^k \frac{dt}{t^{k+2}} = \frac{1}{2(k+1)} \lim_{n\to\infty} \sum_{q=1}^{n} q^k \left(\frac{1}{q^{k+1}} - \frac{1}{(q+1)^{k+1}} \right) \\ &= \frac{1}{2(k+1)} \sum_{q=1}^{\infty} \left(\frac{1}{q} - \frac{1}{q+1} \frac{(q+1-1)^k}{(q+1)^k} \right) \\ &= \frac{1}{2(k+1)} \sum_{q=1}^{\infty} \left(\frac{1}{q} - \frac{1}{(q+1)^{k+1}} \left((q+1)^k + \sum_{j=0}^{k-1} \binom{k}{j} (-1)^{j-k} (q+1)^j \right) \right) \\ &= \frac{1}{2(k+1)} \sum_{q=1}^{\infty} \left(\underbrace{\frac{1}{q} - \frac{1}{q+1}}_{telescope} - \sum_{j=0}^{k-1} \binom{k}{j} (-1)^{j-k} (q+1)^{j-k-1} \right) \\ &= \frac{1}{2(k+1)} \left(1 - \sum_{q=1}^{\infty} \sum_{i=1}^{k} \binom{k}{i} (-1)^i (q+1)^{-i-1} \right) \\ &= \frac{1}{2(k+1)} \left(1 - \sum_{i=1}^{k} \binom{k}{i} (-1)^i (\zeta(i+1) - 1) \right) = \frac{-1}{2(k+1)} \sum_{i=1}^{k} \binom{k}{i} (-1)^i (\zeta(i+1). \end{aligned}$$

Also solved by Bruno Salgueiro Fanego, Viveiro, Spain; Ed Gray, Highland Beach, FL; Kee-Wai Lau, Hong Kong, China; Moti Levy, Rehovot, Israel; Albert Stadler, Herrliberg, Switzerland; David Stone and John Hawkins, Georgia Southern University, Statesboro, GA, and the proposer.

End Notes

Sachit Misra of Delhi, India should have been credited with having solved 5440, and **David Stone and John Hawkins of Georgia Southern University, Statesboro, GA** should have been credited for having solved 5444. Once again, *mea culpa*.