Problems   Ted Eisenberg, Section Editor

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This section of the Journal offers readers an opportunity to exchange interesting mathematical problems and solutions. Please send them to Ted Eisenberg, Department of Mathematics, Ben-Gurion University, Beer-Sheva, Israel or fax to: 972-86-477-648. Questions concerning proposals and/or solutions can be sent e-mail to <eisenbt@013.net>. Solutions to previously stated problems can be seen at <http://www.ssma.org/publications>.

Solutions to the problems stated in this issue should be posted before
January 15, 2019

5511: *Proposed by Kenneth Korbin, New York, NY*

A trapezoid with perimeter $58+14\sqrt{11}$ is inscribed in a circle with diameter $17+7\sqrt{11}$. Find its dimensions if each of its sides is of the form $a+b\sqrt{11}$ where a and b are positive integers.

5512: *Proposed by Ángel Plaza, Universidad de Las Palmas de Gran Canaria. Spain*

If $a_k > 0$, $(k = 1, 2, \ldots, n)$ then $\dfrac{n}{\displaystyle\sum_{k=1}^{n} \frac{1}{\frac{1}{k}+a_k}} - \dfrac{n}{\displaystyle\sum_{k=1}^{n} \frac{1}{a_k}} \geq \dfrac{2}{n+1}$.

5513: *Proposed by Michael Brozinsky, Central Islip, NY*

In an $n \times n \times n$ cube partitioned into n^3 congruent cubes by $n-1$ equally spaced planes parallel to each pair of parallel faces, there are 20 times as many non-cubic rectangular parallelepipeds that could be formed as were cubic parallelepipeds. What is n?

5514: *Proposed by D. M. B*a*tinetu-Giurgiu, "Matei Basarab" National College, Bucharest, Romania and Neculai Stanciu, "George Emil Palade" School, Buzău, Romania*

If $a \in \left(0, \dfrac{\pi}{2}\right)$ and $b = \arcsin a$, then calculate $\displaystyle\lim_{n\to\infty} \sqrt[n]{n!}\left(\sin\left(\frac{b \cdot \sqrt[n+1]{(2n+1)!!}}{\sqrt[n]{(2n-1)!!}}\right) - a\right)$.

5515: *Proposed by José Luis Díaz-Barrero, Barcelona Tech, Barcelona, Spain*

Let n be a positive integer. Prove that

$$\frac{1}{2^n}\left(\sum_{k=1}^{n}\sqrt{\frac{1}{n^2}+\binom{n-1}{k-1}^2}\right)^2 \geq 1.$$

5516: *Proposed by Ovidiu Furdui and Alina Sîntămărian both at the Technical University of Cluj-Napoca, Cluj-Napoca, Romania*

Calculate $\displaystyle\sum_{n=1}^{\infty} n\left(\zeta(3)-1-\frac{1}{2^3}-\cdots-\frac{1}{n^3}-\frac{1}{2n^2}\right)$.

Solutions

5493: *Proposed by Kenneth Korbin, New York, NY*

Convex quadrilateral $ABCD$ is inscribed in a circle with diameter $\overline{AC}=729$. Sides $\overline{AB}$ and $\overline{CD}$ each have positive integer length. Find the perimeter if $\overline{BD}$=715.

Solution by Bruno Salgueiro Fanego, Viveiro, Spain

Let $a=AB, b=BC, c=CD$ and $d=DA$. Since AC is a diameter of the circumcircle of $ABCD$, $\angle CBA=\dfrac{\pi}{2}=\angle ADC$ and hence the Pythagorean theorem can be applied on $\triangle ABC$ and $\triangle ACD$: $a^2+b^2=7292=c^2+d^2$.

Since $ABCD$ is cyclic, by Ptolemy's theorem $ac+bd=729\times 715$. Thus,
$(729^2-a^2)(729^2-c^2)=b^2d^2=(729\cdot 715-ac)^2$, that is, the point with positive integer coordinates (a,c) lies on the ellipse whose equation is
$729x^2-1430xy+729y^2-14737464=0$.

From this it follows that

$(a,c)\in\{(279,405),(405,279),(715,729),(729,715)\}$ and since $a<729$ and $c<729$,

$(a,c)\in\{(279,405),(405,279)\}$, so the lengths of the sides of $ABCD$ are

$(a,b,c,d)\in\{(279,180\sqrt{14},\ 405,162\sqrt{14}),\ (279,162\sqrt{14},405,180\sqrt{14})\}$

and hence, the perimeter of $ABCD$ is $a+b+c+d=342(2+\sqrt{14})$.

Editor′s comment : **Ioannis D. Sfikas'** solution to this problem started off with comments about its related history. "In Euclidean geometry, Ptolemy's inequality relates the six distances determined by four points in the plane or in a higher-dimensional space. It states for any four points A,B,C,D the following inequality holds:

$$\overline{AB}\cdot\overline{CD}+\overline{BC}\cdot\overline{DA}\geq\overline{AC}\cdot\overline{BD}.$$

As a special case, Ptolemy's theorem states that the inequality becomes an equality exactly when the four points lie in cyclic order on a circle. The inequality does not generalize from Euclidean spaces to arbitrary metric spaces. The spaces where it remains valid are called the Ptolemaic spaces; they include the inner product spaces, Hadamard spaces, and shortest path distances on Ptolemaic graphs.

In other words, Ptolemy's theorem is a relation between the four sides and two diagonals of a cyclic quadrilateral (a quadrilateral whose vertices lie on a common circle). The theorem is named after the Greek astronomer and mathematician Ptolemy. Ptolemy used the theorem as an aid to creating his table of chords, a trigonometric table that he applied to astronomy." Ioannis then went on to solve the problem in the above manner.

Also solved by the Brian D. Beasley, Presbyterian College, Clinton, SC; Cartesian Gains Student Problem Solving Group, Mountain Lakes High School, Mountain Lakes, NJ; Ed Gray, Highland Beach, FL; Paul M. Harms, North Newton, KS; Kee-Wai Lau, Hong Kong, China; Ioannis D. Sfikas, National and Kapodistrian University of Athens, Greece; Albert Stadler, Herrliberg, Switzerland; David Stone and John Hawkins, Georgia Southern University, Statesboro, GA, and the proposer.

5494: *Proposed by Moshe Stupel, "Shaanan" Academic College of Education and Gordon Academic College of Education, and Avi Sigler, "Shaanan" Academic College of Education, Haifa, Israel*

If $a \geq b \geq c \geq d$ are the lengths of four segments from which an infinite number of convex quadrilaterals can be constructed, calculate the maximal product of the lengths of the diagonals in such quadrilaterals.

Solution 1 by Ioannis D. Sfikas, National and Kapodistrian University of Athens, Greece

A classical result of Claudius Ptolemy of Alexandria (circa 85AD−165AD), known as Ptolemy's theorem, states that for a cyclic quadrilateral with side lengths a, b, c, d (in that order) and diagonals of lengths p and q, the product of the lengths of the diagonals equals the sum of the products of the lengths of the opposite sides, $pq = ac + bd$. For a general convex quadrilateral, we have *Ptolemy's inequality:*

Theorem 1. For a convex quadrilateral with sides of length a, b, c, d (in that order) and diagonals of length p and q, we have $pq \leq ac + bd$.

For the above problem, we have to order α, β, and γ, where $\alpha = ab + cd$, $\beta = ac + bd$ and $\gamma = ad + bc$. Then, we have:

(i) If we have $\alpha \geq \beta$, that means $ab + cd \geq ac + bd$, or $a(b - c) + d(c - b) \geq 0$, which holds.

(ii) If we have $\alpha \geq \gamma$, that means $ab + cd \geq ad + bc$, or $b(a - c) + d(c - a) \geq 0$, or $(b - d)(a - c) \geq 0$, which holds.

So, if $a \geq b \geq c \geq d$ are the lengths of four segments, from which an infinite number of convex quadrilaterals can be constructed, then the maximal product of the lengths of the diagonals in such quadrilaterals is $ab + cd$.

[1] Alsina, Claudi and Nelsen, Roger B. (2009). *When less is more: visualizing basic inequalities, p. 82. Mathematical Association of America.*

Solution 2 by Kee-Wai Lau, Hong Kong, China

We show that the maximal product of the length of the diagonals in such quadrilaterals equals $ab+cd$.

Let $WXYZ$ be a convex quadrilateral such that $\overline{WX}=w,\ \overline{XY}=x,\ \overline{YZ}=y,\ \overline{ZW}=z$. By a result of C.A. Bretschneider, the product of the lengths of the diagonals equals $\sqrt{w^2y^2+x^2z^2-2wxyz\cos(\angle XWZ+\angle XYZ)}$, which does not exceed $\sqrt{w^2y^2+x^2z^2+2wxyz}=wy+xz$. Note that

$$ab+cd=ac+bd+(b-c)(a-d)\geq ac+bd$$

and

$$ab+cd=ad+bc+(a-c)(b-d)\geq ad+bc.$$

Hence in order to obtain the maximum product of the diagonals, we put $w=a,\ x=c,\ y=b$, and $z=d$. It is easy to check that when $WXYZ$ is a cyclic quadrilateral, we have

$$\overline{WY}=\sqrt{\frac{(ab+cd)(bc+ad)}{(ac+bd)}}\text{ and }\overline{XZ}=\sqrt{\frac{(ab+ca)(ac+bd)}{(bc+ad)}},$$

so that the maximum product $ab+cd$ is attained. Hence our claimed maximum.

Solution 3 by Albert Stadler, Herrliberg, Switzerland

The German mathematician Carl Anton Bretschneider derived in 1842 the following generalization of Ptolemy's theorem, regarding the product pq of the diagonals in a convex quadrilateral

$$p^2q^2=a^2c^2+b^2d^2-2abcd\cos(A+C)\qquad(1)$$

This relation can be considered to be a law of cosines for a quadrilateral. Since $4\cos(A+C)\geq-1$, it also gives a proof of Ptolemy's inequality.

We note that the product p^2q^2 in (1) is maximal if $\cos(A+C)=-1$, i.e., if $A+C=180°$ which implies that the product pq is maximal if the quadrilateral is a cyclic quadrilateral. In that case we get $pq=ac+bd$.

It remains to determine for which permutation of the sides the term $ac+bd$ is maximal. There are three possibilities, namely

$ab+cd, ac+bd, ad+bc$. Of these three expressions $ab+cd$ is maximal, since

$ab+cd-(ac+bd)=(a-d)(b-c)\geq 0$, and $ab+cd-(ad+bc)=(a-c)(b-d)\geq 0$.

References
[1] Titu Andreescu & Dorian Andrica, Complex Numbers from A to . . . Z, Birkhäuser, 2006, pp. 207-209.

Solution 4 by David Stone and John Hawkins, Georgia Southern University, Statesboro, GA

The maximal product of the lengths of the diagonals is $ab+cd$. This maximum is achieved when $(AB,BC,CD,DA)=(a,c,b,d)$ is a cyclic quadrilateral.

By considering the vertices as hinges, Thomas proves [1] that any convex quadrilateral can be deformed into a cyclic quadrilateral (having the same side lengths).

In any convex quadrilateral, Ptolemy's Inequality tells us that the product of the diagonals is less than or equal to the sum of the products of the lengths of opposite sides. In a cyclic quadrilateral, Ptolemy's Theorem, tells us that the product of the diagonals equals the sum of the products of the lengths of opposite sides. Given our four appropriate segments, a, b, c, d, there are six ways to arrange them in a convex quadrilateral. By symmetry, only three of these are distinct.

We show these three possibilities with corresponding bound on the product of the diagonals, $AC \cdot BD$:

$(AB, BC, CD, DA) = (a, b, c, d); AC \cdot BD \leq ac + bd$
$(AB, BC, CD, DA) = (a, b, d, c); AC \cdot BD \leq ad + bc$
$(AB, BC, CD, DA) = (a, c, b, d); AC \cdot BD \leq ab + cd.$

The third case gives the largest possible value (because we've placed the two largest sides opposite one another).

Algebraically,

$ac + bd \leq ab + cd \iff 0 \leq (a-d)(b-c)$

which is true by the given ordering, and

$ad + bc \leq ab + cd \iff 0 \leq (a-c)(b-d)$

which is also true by the given ordering.

Summarizing: when the four segments are arranged in a quadrilateral $ABCD$, the product of the diagonals is $\leq AB \cdot CD + BC \cdot AD$; the largest possible value for $AB \cdot CD + BC \cdot AD$ is $ab + cd$, which is achieved when a, b and c, d are opposite sides of a cyclic quadrilateral.

Reference:

1. Peter, Thomas, Maximizing the Area of a Quadrilateral, The College Mathematics Journal, Vol. 34, No. 4 (September 2003), pp. 315-316.

Also solved by Kenneth Korbin, New York, NY; David E. Manes, Oneonta, NY, and the proposers.

5495: *Proposed by D.M. Bătinetu-Giurgiu, "Matei Basarab" National College, Bucharest and Neculai Stanciu, "George Emil Palade" School Buzău, Romania*

Let $\{x_n\}_{n\geq 1},\ x_1 = 1,\ x_n = 1 \cdot \sqrt{3!!} \cdot \sqrt[3]{5!!} \cdot \ldots \sqrt[n]{(2n-1)!!}$.

Find:

$$\lim_{n\to\infty} \left(\frac{(n+1)^2}{\sqrt[n+1]{x_{n+1}}} - \frac{n^2}{\sqrt[n]{x_n}} \right).$$

Solution 1 by Moti Levy, Rehovot, Israel

$$(2n-1)!! = \frac{(2n)!}{2^n n!}. \qquad (1)$$

Using Stirling's asymptotic formula, we have

$$n! \sim \frac{n^n}{e^n}. \tag{2}$$

Applying (2) to (1) yields

$$\sqrt[n]{(2n-1)!!} \sim \left(\frac{(2n)^{2n}}{e^{2n}2^n n!}\right)^{\frac{1}{n}} = \frac{2n}{e} \tag{3}$$

Now we use (3) to approximate x_n,

$$x_n \sim \prod_{k=1}^{n} 2ke = \frac{2^n n!}{e^n} \sim \frac{2^n \frac{n^n}{e^n}}{e^n} = \frac{2^n n^n}{e^{2n}},$$

or

$$\sqrt[n]{x_n} \sim \frac{2n}{e^2}.$$

Hence,

$$\frac{(n+1)^2}{\sqrt[n+1]{x_{n+1}}} - \frac{n^2}{\sqrt[n]{x_n}} \sim \frac{e^2}{2}(n+1) - \frac{e^2}{2}n = \frac{e^2}{2},$$

and we conclude that $\lim\limits_{n\to\infty}\left(\frac{(n+1)^2}{\sqrt[n+1]{x_{n+1}}} - \frac{n^2}{\sqrt[n]{x_n}}\right) = \frac{e^2}{2} \cong 3.694\,5.$

Solution 2 by Kee-Wai Lau, Hong Kong, China

We show that the limit of the problem equals $\frac{e^2}{2}$.

We need the following knows results for positive integers n.

$$\begin{aligned} \ln(n!) &= \left(n+\frac{1}{2}\right)\ln n - n + A + O\left(\frac{1}{n}\right), && (1)\\ \sum_{k=1}^{n}\frac{1}{k} &= \ln n + B + O\left(\frac{1}{n}\right), && (2)\\ \ln\left(1+\frac{1}{n}\right) &= \frac{1}{n} + O\left(\frac{1}{n^2}\right). && (3) \end{aligned}$$

where A and B are constants.

By (1) we have

$$\ln((2k)!) - \ln(k!) = \ln(k!) + (2\ln 2 - 1)k + \frac{\ln 2}{2} + O\left(\frac{1}{k}\right). \tag{4}$$

Next we show that

$$_n = n\ln n + (\ln 2 - 2)n + \frac{(1+\ln 2)\ln n}{2} + O(1) \tag{5}$$

In fact by (4) we have

$$\ln x_n = \sum_{k=1}^{n} \frac{\ln((2k-1)!!)}{k}. \quad = \quad \sum_{k=1}^{n} \frac{\ln((2k)!) - \ln(k!) - (\ln 2)k}{k}$$

$$= \sum_{k=1}^{n} \left(\ln k + \ln 2 - 1 + \frac{\ln 2}{2k} + O\left(\frac{1}{k^2}\right)\right)$$

$$= \ln(n!) + (\ln 2 - 1)n + \frac{\ln 2}{2} \sum_{k=1}^{n} \frac{1}{n} + O(1),$$

and (5) follows from (1) and (2).

Let $f(n) = 2\ln n - \dfrac{\ln x_n}{n}$. By (5), we obtain

$$f(n) = \ln n + 2 - 2 + O\left(\frac{\ln n}{n}\right). \qquad (6)$$

We next show that

$$f(n+1) - f(n) = \frac{1}{n} + O\left(\frac{\ln n}{n^2}\right). \qquad (7)$$

In fact

$$f(n+1) - f(n) = 2(\ln(n+1) - \ln n) - \left(\frac{\ln x_{n+1}}{n+1} - \frac{\ln x_n}{n}\right)$$

$$= 2\ln\left(1 + \frac{1}{n}\right) + \frac{\ln x_n}{n(n+1)} - \frac{\ln(2n+2)! - \ln(n+1)! - (\ln 2)(n+1)}{(n+1)^2},$$

and (7) follows readily from (3),(5) and (4). By the mean value theorem, we have

$$e^{f(n+1)} - e^{f(n)} = (f(n+1) - f(n))\, e^t, \qquad (8)$$

where t is a number lying between $f(n)$ and $f(n+1)$. By (6), both $e^{f(n+1)}$ and $e^{f(n)}$ equal $\dfrac{e^2 n}{2}\left(1 + O\left(\dfrac{\ln n}{n}\right)\right)$. Hence, by (7) and (8),

$$\frac{(n+1)^2}{\sqrt[n+1]{x_{n+1}}} - \frac{n^2}{\sqrt[n]{x_n}} = e^{f(n+1)} - e^{f(n)} = \frac{e^2}{2}\left(1 + O\left(\frac{\ln n}{n}\right)\right),$$

and our claim for the limit follows.

Solution 3 by Bruno Salgueiro Fanego, Viveiro, Spain

We will use the lemma from Solution 3 to Problem 5398 that appeared in this Column (see Nov. 2016 issue) that stated: *"If the positive sequence* (p_n) *is such that* $\lim_{n\to\infty} \dfrac{p_{n+1}}{np_n} = p > 0$, *then* $\lim_{n\to\infty}\left(\sqrt[n+1]{p_{n+1}} - \sqrt[n]{p_n}\right) = \dfrac{p}{e}$."

We let $\{p_n\}_{n\geq 1},\ p_n = \dfrac{n^{2n}}{x_n}$. Then

$$\lim_{n\to\infty}\frac{p_{n+1}}{np_n} = \lim_{n\to\infty}\frac{\dfrac{(n+1)^{2n+2}}{x_n\sqrt[n+1]{(2n+1)!!}}}{n\dfrac{n^{2n}}{x_n}}$$

$$= \lim_{n\to\infty}\frac{(n+1)^{2n+2}}{n^{2n+1}\sqrt[n+1]{(2n+1)!!}} = \lim_{n\to\infty}\frac{(n+1)^{2n}(n+1)^2}{n^{2n}n\sqrt[n+1]{(2n+1)!!}}$$

$$= \lim_{n\to\infty}\left(\left(\frac{n+1}{n}\right)^n\right)^2 \lim_{n\to\infty}\frac{\sqrt[n+1]{(n+1)^{2(n+1)}}}{\sqrt[n+1]{n^{n+1}(2n+1)!!}} = e^2\lim_{n\to\infty}\sqrt[n]{\frac{n^{2n}}{(n-1)^n(2n-1)!!}}$$

$$\underset{root\ crition}{=} e^2\lim_{n\to\infty}\frac{\dfrac{(n+1)^{2(n+1)}}{n^{n+1}(2n+1)!!}}{\dfrac{n^{2n}}{(n-1)^n(2n-1)!!}} = e^2\lim_{n\to\infty}\frac{(n+1)^{2n}(n+1)^2(n-1)^n}{n^n n^{2n}(2n+1)}$$

$$= e^2\lim_{n\to\infty}\frac{(n+1)^{2n}(n+1)^2(n-1)^n}{n^{2n}n(2n+1)n^n}$$

$$= e^2\lim_{n\to\infty}\frac{(n+1)^{2n}}{n^{2n}}\lim_{n\to\infty}\frac{(n+1)^2}{n^{2n+1}}\lim_{n\to\infty}\frac{(n-1)^n}{n^n}$$

$$= e^2\lim_{n\to\infty}\left(\left(\frac{n+1}{n}\right)^n\right)^2\frac{1}{2}\lim_{n\to\infty}\left(1-\frac{1}{n}\right)^n = e^2e^2\frac{1}{2}e^{-1}$$

$$= \frac{e^3}{2} =: p > 0,$$ which implies by the lemma mentioned above,

that the required limit is $\lim_{n\to\infty}\left(\sqrt[n+1]{p_{n+1}} - \sqrt[n]{p_n}\right) = \dfrac{p}{e} = \dfrac{e^2}{2}$.

Editor′s comment : In addition to the above solution **Bruno Salgueiro Fanego** stated that a more general form of the problem was published by the authors′ of 5495 in the journal *La Gaceta de la Real Sociedad Matemática Espanãola* vol. 17 (3), 2014, pp. 523-524. (available at http://gaceta.rsme.es/abrir.phd?id=1218.) Therein they showed: If $\{a_n\}_{n\geq 1}$ is a sequence of real positive numbers such that $\lim_{n\to\infty}(a_{n+1}-a_n) = a \neq 0$, then

$$\lim_{n\to\infty}\left(\frac{(n+1)^2}{\sqrt[n+1]{\prod_{k=1}^{n+1} f(a_k)}} - \frac{(n)^2}{\sqrt[n]{\prod_{k=1}^{n} f(a_k)}}\right) = \frac{e}{ca}.$$

Letting $a_n = n$ and $f(n) = \sqrt[n]{(2n-1)!!}$ gives the desired result.

Solution 4 by Michel Bataille, Rouen, France

Let $u_n = \frac{n^2}{\sqrt[n]{x_n}} = \left(\frac{n^{2n}}{x_n}\right)^{1/n}$. We show that $\lim\limits_{n\to\infty}(u_{n+1} - u_n) = \frac{e^2}{2}$.

To this end, we first recall that $(2n-1)!! = \dfrac{(2n)!}{2^n(n!)}$ and the following asymptotic expansion as $n \to \infty$:

$$\ln(n!) = n\ln(n) - n + \frac{\ln(n)}{2} + \ln(\sqrt{2\pi}) + \frac{1}{12n} + o(1/n),$$

from which we readily deduce

$$\ln\left(\frac{(2n)!}{n!}\right) = \ln((2n)!) - \ln(n!) = n\ln(n) + n(2\ln(2) - 1) + \frac{\ln(2)}{2} - \frac{1}{24n} + o(1/n).$$

Now, we have

$$\ln(u_n) = 2\ln(n) - \frac{\ln(x_n)}{n} = 2\ln(n) - \frac{1}{n}\ln\left(\prod_{k=1}^{n}\frac{1}{2}\cdot\left(\frac{(2k)!}{k!}\right)^{1/k}\right) = 2\ln(n) + \ln(2) - \frac{s_n}{n} \quad (1)$$

where $s_n = \sum\limits_{k=1}^{n}\frac{1}{k}\cdot\ln\left(\frac{(2k)!}{k!}\right)$.
Consider the sequence $\{y_n\}_{n\geq 2}$ defined by

$$y_n = s_n - n\ln(n) - (2\ln(2) - 2)n - \frac{1+\ln(2)}{2}\cdot\ln(n).$$

For $n \to \infty$, we calculate

$$\begin{aligned}
y_n - y_{n-1} &= s_n - s_{n-1} - n\ln(n) + (n-1)\ln(n-1) - (2\ln(2) - 2) + \frac{1+\ln(2)}{2}\cdot\ln\left(1 - \frac{1}{n}\right) \\
&= \frac{1}{n}\left(\ln\left(\frac{(2n)!}{n!}\right)\right) + n\ln\left(1 - \frac{1}{n}\right) - \ln(n) - \frac{1-\ln(2)}{2}\ln\left(1 - \frac{1}{n}\right) + (2 - \ln(2)) \\
&= 2\ln(2) - 1 + \frac{\ln(2)}{2n} - \frac{1}{24n^2} + o(1/n^2) + n\left(-\frac{1}{n} - \frac{1}{2n^2} - \frac{1}{3n^3} + o(1/n^3)\right) \\
&\qquad - \frac{1-\ln(2)}{2}\left(-\frac{1}{n} - \frac{1}{2n^2} + o(1/n^2)\right) + 2 - 2\ln(2) \\
&= \frac{a}{n^2} + o(1/n^2)
\end{aligned}$$

where we set $a = -\frac{1+2\ln(2)}{8}$. Thus, the series $\sum_{n=2}^{\infty}(y_n - y_{n-1})$ is convergent. Let S denotes its sum. Then, we may write $\sum_{k=2}^{n}(y_k - y_{k-1}) = S + o(1)$ and so $y_n = b + o(1)$ as $n \to \infty$ (where $b = S + y_1$).
It follows that

$$s_n = n\ln(n) + (2\ln(2) - 2)n + \frac{1+\ln(2)}{2}\cdot\ln(n) + b + o(1)$$

as $n \to \infty$.
From (1), we now obtain

$$\ln(u_n) = \ln(n) + 2 - \ln(2) - \frac{1+\ln(2)}{2}\cdot\frac{\ln(n)}{n} - \frac{b}{n} + o(1/n).$$

First, we deduce that $\ln(u_n) = \ln(n) + 2 - \ln(2) + o(1)$, hence $u_n = e^{\ln(n)+2-\ln(2)} \cdot e^{o(1)}$ and so $u_n \sim \ n \cdot \frac{e^2}{2}$. Second, the calculation of $\ln(u_{n+1}) - \ln(u_n)$ easily leads to

$$\ln(u_{n+1}) - \ln(u_n) = \ln\left(1 + \frac{1}{n}\right) + o(1/n) = \frac{1}{n} + o(1/n).$$

(Note that
$\frac{\ln(n)}{n} - \frac{\ln(n+1)}{n+1} = \frac{1}{n}\left((\ln n)\left(1 - (1+1/n)^{-1}\right) + o(1)\right) = \frac{1}{n}\left(-\frac{\ln(n)}{n} + o(1)\right) = o(1/n)$ as $n \to \infty$.)
Since

$$u_{n+1} - u_n = u_n\left(\frac{u_{n+1}}{u_n} - 1\right) = u_n\left(e^{\ln(u_{n+1})-\ln(u_n)} - 1\right)$$

we finally arrive at

$$u_{n+1} - u_n \ \sim \ u_n(\ln(u_{n+1}) - \ln(u_n)) \ \sim \ n \cdot \frac{e^2}{2} \cdot \left(\frac{1}{n}\right) \ \sim \ \frac{e^2}{2}$$

and the result follows.

Also solved by Paolo Perfetti, Department of Mathematics, Tor Vergata University, Rome, Italy; Ioannis D. Sfikas, National and Kapodistrian University of Athens, Greece; Albert Stadler, Herrliberg, Switzerland; Marian Ursarescu - Romania, and the proposers.

5496: *Daniel Sitaru, "Theodor Costescu" National Economic College, Drobeta Turnu-Severin, Mehedinti, Romania*

Let a, b, c be real numbers such that $0 < a < b < c$. Prove that:

$$\sum_{cyclic}\left(e^{a-b} + e^{b-a}\right) \geq 2a - 2c + 3 + \sum_{cyclic}\left(\frac{b}{a}\right)^{\sqrt{ab}}.$$

Solution 1 by Henry Ricardo, Westchester Area Math Circle, NY

For $x > 0$ we apply the known inequality $e^x > x + 1$ to $x = a - b, b - c$, and $a - c$ to get

$$e^{a-b} > a - b + 1, \quad e^{b-c} > b - c + 1, \quad e^{a-c} > a - c + 1,$$

respectively. Adding these inequalities yields

$$e^{a-b} \ + \ e^{b-c} \ + \ e^{a-c} \ > \ 2a \ - \ 2c \ + \ 3. \tag{1}$$

For $x > y$, we see that
$e^{x-y} > (x/y)^{\sqrt{xy}} \iff x - y > \sqrt{xy}\ln(x/y) \iff \sqrt{xy} < (x-y)/(\ln x - \ln y)$, which is the left-hand member of the *logarithmic mean inequality.* Thus we have, since $0 < a < b < c$,

$$e^{b-a} > \left(\frac{b}{a}\right)^{\sqrt{ab}}, \quad e^{c-b} > \left(\frac{c}{b}\right)^{\sqrt{bc}}, \quad e^{c-a} > \left(\frac{c}{a}\right)^{\sqrt{ac}} > \left(\frac{a}{c}\right)^{\sqrt{ac}}. \tag{2}$$

Adding (1) and (2) , we find that

$$\sum_{cyclic}\left(e^{a-b}+e^{b-a}\right) > 2a-2c+3+\sum_{cyclic}\left(\frac{b}{a}\right)^{\sqrt{ab}}.$$

Solution 2 by Albert Stadler, Herrliberg, Switzerland

We will prove the slightly stronger inequality

$$\sum_{cyclic}\left(e^{a-b}+e^{b-a}\right) \geq a-c+3+\sum_{cyclic}\left(\frac{b}{a}\right)^{\sqrt{ab}}.$$

We will use the inequalities

$e^x \geq 1+x,\ x$ real, (1)

$1 \geq \left(\frac{y}{x}\right)^{\sqrt{xy}}, 0 \leq y \leq x,$ (2)

$e^{y-x} \geq \left(\frac{y}{x}\right)^{\sqrt{xy}},\ y \geq x,$ (3)

(1) and (2) are clear, while (3) is equivalent to each of the following lines:

$y-x \geq \sqrt{xy}\log\left(\frac{y}{x}\right),$

$\sqrt{\frac{y}{x}}-\sqrt{\frac{x}{y}} \geq \log\left(\frac{y}{x}\right),$

$x-\frac{1}{x}-\log x = \int_1^x\left(1+\frac{1}{t^2}-\frac{1}{t}\right)dt \geq 0,\ x \geq 1$ which holds true.

Thus

$$\sum_{cyclic}\left(e^{a-b}+e^{b-a}\right) \geq 1+a-b+\left(\frac{b}{a}\right)^{\sqrt{ab}}+1+b-c+\left(\frac{b}{c}\right)^{\sqrt{bc}}+1+c-a+a^{a-c}$$

$$\begin{aligned}
&= 3+\left(\frac{b}{a}\right)^{\sqrt{ab}}+\left(\frac{c}{b}\right)^{\sqrt{bc}}+e^{a-c}\\
&\geq 3+\left(\frac{b}{a}\right)^{\sqrt{ab}}+\left(\frac{c}{b}\right)^{\sqrt{bc}}+1+a-c\\
&\geq 3+\left(\frac{b}{a}\right)^{\sqrt{ab}}+\left(\frac{c}{b}\right)^{\sqrt{bc}}+\left(\frac{a}{c}\right)^{\sqrt{bc}}+a-c.
\end{aligned}$$

Also solved by Ed Gray, Highland Beach, FL; Kee-Wai Lau, Hong Kong, China, and the proposer.

5497: *Proposed by José Luis Díaz-Barrero, Barcelona Tech, Barcelona, Spain*

For all integers $n \geq 2$, show that $\prod_{k=1}^{n-1} 2 \sin\left(\frac{k\pi}{n}\right)$ is an integer and determine it.

Solution 1 by Kee-Wai Lau, Hong Kong, China

It is proved as formula 12 on p. 227 Chapter XII of [1] that

$$\sin n\theta = \sin\theta \prod_{k=1}^{n-1} 2\sin\left(\theta + \frac{k\pi}{n}\right).$$

Since $\lim_{\theta\to\infty} \frac{\sin n\theta}{\sin\theta} = n$, so $\prod_{k=1}^{n-1} 2\sin\left(\theta + \frac{k\pi}{n}\right) = n$.

Reference:
1. D.V. Durell and A. Robinson: *Advanced Trigonometry,* Dover Publication, Inc., New York 2003.

Solution 2 by David E. Manes, Oneonta, NY

Subtracting the complex equation $e^{-ix} = \cos(-x) + i\sin(-x) = \cos x - i\sin x$ from the equation $e^{ix} = \cos x + i\sin x$, one obtains the formula

$$2\sin x = \frac{1}{i}\left(e^{ix} - e^{-ix}\right).$$

Therefore,

$$\begin{aligned}\prod_{k=1}^{n-1} 2\sin\left(\frac{k\pi}{n}\right) &= \prod_{k=1}^{n-1} \frac{1}{i}\left(e^{i\pi k/n} - e^{-i\pi k/n}\right)\\ &= \left(\frac{1}{i}\right)^{n-1}\left(\prod_{k=1}^{n-1} e^{i\pi k/n}\right)\left(\prod_{k=1}^{n-1}\left(1 - e^{-2i\pi k/n}\right)\right).\end{aligned}$$

Note that

$$\begin{aligned}\prod_{k=1}^{n-1} e^{i\pi k/n} &= e^{\left(\sum_{k=1}^{n-1} i\pi k/n\right)} = e^{(i\pi/n)\sum_{k=1}^{n-1} k} = e^{(i\pi/n)((n-1)(n)/2)}\\ &= e^{(i\pi/2)(n-1)} = \left(e^{i\pi/2}\right)^{n-1} = (\cos(\pi/2) + i\sin(\pi/2))^{n-1} = i^{n-1}.\end{aligned}$$

Hence,

$$\prod_{k=1}^{n-1} 2\sin\left(\frac{k\pi}{n}\right) = \prod_{k=1}^{n-1}\left(1 - e^{-2i\pi k/n}\right) = f(1),$$

where $f(x) = \prod_{k=1}^{n-1}\left(x - e^{-2i\pi k/n}\right)$. The zeros of the polynomial $f(x)$ are the non-trivial n^{th} roots of unity so that

$$f(x) = \frac{x^n - 1}{x - 1} = 1 + x + x^2 + \cdots + x^{n-1}.$$

Therefore, $f(1) = n$. Hence, if $n \geq 0$, then

$$\prod_{k=1}^{n-1} 2\sin\left(\frac{k\pi}{n}\right) = n.$$

Solution 3 by Anthony J. Bevelacqua, University of North Dakota, Grand Forks, ND

Let $\zeta = e^{\pi i/n}$. Then $\zeta^2 = e^{2\pi i/n}$ is a primitive n-th root of unity. So $\zeta^2, \ldots, \zeta^{2(n-1)}$ are the roots of

$$\frac{x^n - 1}{x - 1} = x^{n-1} + \cdots + 1$$

and therefore

$$x^{n-1} + \cdots + 1 = \prod_{k=1}^{n-1}(x - \zeta^{2k}).$$

Let $x = 1$ to find

$$\begin{aligned} n &= \prod_{k=1}^{n-1}(1 - \zeta^{2k}) \\ &= \prod_{k=1}^{n-1} -\zeta^k(\zeta^k - \zeta^{-k}). \end{aligned}$$

Since $\zeta^k = e^{k\pi i/n} = \cos(k\pi/n) + i\sin(k\pi/n)$ we have $\zeta^k - \zeta^{-k} = 2i\sin(k\pi/n)$. Thus

$$n = \prod_{k=1}^{n-1} -2i\zeta^k \sin\left(\frac{k\pi}{n}\right).$$

Finally, since each $\sin(k\pi/n) > 0$ and each $|-2i\zeta^k| = 2$ for $k = 1, \ldots, n-1$ we have

$$n = \prod_{k=1}^{n-1} 2\sin\left(\frac{k\pi}{n}\right)$$

by taking the absolute value of the last expression.

Editor's comment : **Paul M. Harms of North Newton KS** mentioned in his solution to 5497 that Wikipedia's "List of Trigonometric Identities" includes $\prod_{k=1}^{n-1} \sin\left(\frac{k\pi}{n}\right) = \frac{n}{2^{n-1}}$, and from this the value of n immediately follows.

Also solved by Michel Bataille, Rouen, France; Bruno Salgueiro Fanego (three solutions), Viveiro, Spain; Ed Gray, Highland Beach, FL; Paul M. Harms, North Newton KS; Moti Levy, Rehovot, Israel; Paolo Perfetti, Department of Mathematics, Tor Vergata University, Rome, Italy; Ángle Plaza, University of Las Palmas de Gran Canaria, Spain; Henry Ricardo, Westchester Area Math Circle, NY; Ioannis D. Sfikas, National and Kapodistrian University of Athens, Greece; Albert Stadler, Herrliberg, Switzerland; Marian Ursărescu, Romania, and the proposer.

5498: *Proposed by Ovidiu Furdui and Alina Sîntămărian, both at the Technical University of Cluj-Napoca, Cluj-Napoca, Romania*

Prove that

$$\sum_{n=1}^{\infty} \frac{\{n!e\}}{n} = \int_0^1 \frac{e^x - 1}{x} dx$$

where $\{a\}$ denotes the fractional part of a.

Solution 1 by Pedro H. O. Pantoja, Natal/RN, Brazil

By Taylor's formula,

$$e = 1 + \frac{1}{1!} + \frac{1}{2!} + \cdots + \frac{1}{n!} + \frac{e^{\alpha}}{(n+1)!}, \quad \alpha \in (0,1),$$

and this implies that

$$n!e = n!\left(\frac{1}{1!} + \frac{1}{2!} + \cdots + \frac{1}{n!}\right) + \frac{e^{\alpha}}{n+1}, \quad \alpha \in (0,1).$$

Therefore,

$$\{n!e\} = n!e - \lfloor n!e \rfloor = n!\left(e - 1 - \frac{1}{1!} - \frac{1}{2!} - \cdots - \frac{1}{n!}\right) \Rightarrow$$

$$\{n!e\} = n!\left(\frac{1}{(n+1)!} + \frac{1}{(n+2)!} + \cdots\right) = \sum_{k=1}^{\infty} \frac{1}{(n+1)(n+2)\cdots(n+k)}.$$

We have,

$$\begin{aligned}
\sum_{n=1}^{\infty} \frac{\{n!e\}}{n} &= \sum_{n=1}^{\infty}\sum_{k=1}^{\infty} \frac{1}{n(n+1)(n+2)\cdots(n+k)} \\
&= \sum_{n=1}^{\infty}\sum_{k=1}^{\infty} \frac{1}{k!}\int_0^1 (1-x)^k x^{n-1} dx \\
&= \int_0^1 \sum_{k=1}^{\infty} \frac{(1-x)^k}{k!} \sum_{n=1}^{\infty} x^{n-1} dx \\
&= \int_0^1 (e^{1-x} - 1) \cdot \frac{1}{1-x} dx \\
&= \int_0^1 \frac{e^{1-x} - 1}{1-x} dx \\
&= \int_0^1 \frac{e^y - 1}{y} dy,
\end{aligned}$$

where in the last integral, we used the substitution $y = 1 - x$.

Solution 2 by Paolo Perfetti, Department of Mathematics, Tor Vergata University, Rome, Italy

Both the sides of the equality are equal to

$$\sum_{k=1}^{\infty} \frac{1}{k \cdot k!}$$

$$\{n!e\} = n! \left\{ 2 + \frac{1}{2} + \frac{1}{3!} + \ldots + \frac{1}{m!} + \frac{1}{(m+1)!} \ldots \right\} = \left\{ \frac{1}{n+1} + \frac{1}{(n+1)(n+2)} + \ldots \right\}$$

Since

$$\sum_{k=1}^{\infty} \frac{1}{(n+1)(n+2)\ldots(n+k)} < \sum_{k=1}^{\infty} 2^{-k} = 1$$

it follows that

$$\left\{ \frac{1}{n+1} + \frac{1}{(n+1)(n+2)} + \ldots \right\} = \sum_{k=1}^{\infty} \frac{1}{(n+1)(n+2)\ldots(n+k)}$$

and

$$\sum_{n=1}^{\infty} \frac{\{n!e\}}{n} = \sum_{n=1}^{\infty} \sum_{k=1}^{\infty} \frac{1}{n(n+1)(n+2)\ldots(n+k)} =$$

$$= \sum_{k=1}^{\infty} \sum_{n=1}^{\infty} \frac{1}{n(n+1)(n+2)\ldots(n+k)} = \sum_{k=1}^{\infty} \frac{1}{k \cdot k!} \qquad (1)$$

and finally

$$\int_0^1 \frac{e^x - 1}{x} dx = \int_0^1 \sum_{k=1}^{\infty} \frac{x^{k-1}}{k!} dx = \sum_{k=1}^{\infty} \int_0^1 \frac{x^{k-1}}{k!} dx = \sum_{k=1}^{\infty} \frac{1}{k \cdot k!}$$

For proving (1) let's write $a_n = 1/(n(n+1)\cdots(n+k))$.

$$\frac{a_{n+1}}{a_n} = \frac{n}{n+k+1} \iff a_{n+1}(n+1) - na_n = -a_{n+1}k$$

Telescoping

$$\sum_{n=1}^{N} a_{n+1}(n+1) - na_n = \underbrace{a_{N+1}(N+1)}_{\to 0} - a_1 = -k \sum_{n=1}^{N} a_{n+1}$$

and

$$\sum_{k=1}^{\infty} a_k = \frac{1+k}{k(k+1)!} = \frac{1}{k \cdot k!}$$

Solution 3 by Michel Bataille, Rouen, France

From $\frac{e^x-1}{x} = \sum_{n=1}^{\infty} \frac{x^{n-1}}{n!}$ for $x \in (0,1]$ and

$$\sum_{n=1}^{\infty} \int_0^1 \left| \frac{x^{n-1}}{n!} \right| dx = \sum_{n=1}^{\infty} \frac{1}{n \cdot (n!)} < \infty,$$

we deduce that

$$\int_0^1 \frac{e^x - 1}{x}\, dx = \int_0^1 \left(\sum_{n=1}^{\infty} \frac{x^{n-1}}{n!} \right) dx = \sum_{n=1}^{\infty} \left(\int_0^1 \frac{x^{n-1}}{n!}\, dx \right) = \sum_{n=1}^{\infty} \frac{1}{n \cdot (n!)}. \qquad (1)$$

On the other hand, for $n \geq 1$ we have

$$(n!)e = (n!) \sum_{j=0}^{\infty} \frac{1}{j!} = a_n + \sum_{k=1}^{\infty} \frac{1}{(n+1)\cdots(n+k)}$$

where $a_n = n! + (n-1)!\binom{n}{1} + (n-2)!\binom{n}{2} + \cdots + 1!\binom{n}{n-1} + 1$ is a positive integer and

$$0 < \sum_{k=1}^{\infty} \frac{1}{(n+1)\cdots(n+k)} < \sum_{k=1}^{\infty} \frac{1}{(n+1)^k} = \frac{1}{n} \leq 1.$$

It follows that

$$\{n!e\} = \sum_{k=1}^{\infty} \frac{1}{(n+1)\cdots(n+k)}$$

and so

$$\begin{aligned}
\sum_{n=1}^{\infty} \frac{\{n!e\}}{n} &= \sum_{n=1}^{\infty} \sum_{k=1}^{\infty} \frac{1}{n(n+1)\cdots(n+k)} \\
&= \sum_{k=1}^{\infty} \sum_{n=1}^{\infty} \frac{1}{n(n+1)\cdots(n+k)} \\
&= \sum_{k=1}^{\infty} \left(\frac{1}{k} \sum_{n=1}^{\infty} \left(\frac{1}{n(n+1)\cdots(n+k-1)} - \frac{1}{(n+1)\cdots(n+k)} \right) \right) \\
&= \sum_{k=1}^{\infty} \frac{1}{k} \cdot \frac{1}{1 \cdot 2 \cdots k}.
\end{aligned}$$

Finally we obtain $\sum\limits_{n=1}^{\infty} \frac{\{n!e\}}{n} = \sum\limits_{k=1}^{\infty} \frac{1}{k \cdot (k!)}$, and comparing with (1) gives the required result.

Also solved by Ed Gray, Highland Beach, FL; Kee-Wai Lau, Hong Kong, China; Moti Levy, Rehovot, Israel; Ioannis D. Sfikas, National and Kapodistrian University of Athens, Greece; Albert Stadler, Herrliberg, Switzerland, and the proposers.

Mea Culpa

Brian D. Beasley of Presbyterian College in Clinton, SC should have been credited with having solved 5482, and **Albert Stadler of Herrliberg, Switzerland** should have been credited with having solved 5488.

Titu Zvonaru of Comănesti, Romania noted that proof number 2 (of the 6 shown) for problem 5492 is incomplete. The question asked us to prove that a certain inequality held that was subject to a constraint on the variables. The author of the solution found values of the variables that produced equality, and then by taking other values in small epsilon neighborhoods around this point that produced equality, showed that the resulting values of the expression were smaller than the value that gave equality. Up to here, everything is fine. But it was then concluded that the point giving equality was a local maximum. The method used was very similar to the one that is often used in obtaining saddle and extrema points vis-a-vis Lagrange Multipliers. Admittedly there is some hand-waving in using this approach, and this is what Titu noticed. The approach used in this problem can tell us when the inequality goes awry, but it cannot be used to prove with absolute certainty that the inequality holds. For that, derivative tests within the theory of Lagrange Multipliers, must be used.