Problems

Ted Eisenberg, Section Editor

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This section of the Journal offers readers an opportunity to exchange interesting mathematical problems and solutions. Please send them to Ted Eisenberg, Department of Mathematics, Ben-Gurion University, Beer-Sheva, Israel or fax to: 972-86-477-648. Questions concerning proposals and/or solutions can be sent e-mail to <eisenbt@013.net>. Solutions to previously stated problems can be seen at <http://www.ssma.org/publications>.

Solutions to the problems stated in this issue should be posted before December 15, 2011

- **5170:** *Proposed by Kenneth Korbin, New York, NY*

  Convex quadrilateral $DEFG$ has coordinates $D(-6,-3)$ and $E(2,12)$. The midpoints of the diagonals are on line l.

  Find the area of the quadrilateral if line l intersects line FG at point $P\left(\frac{672}{33}, \frac{-9}{11}\right)$.

- **5171:** *Proposed by Kenneth Korbin, New York, NY*

  A triangle has integer length sides $x, x+y$, and $x+2y$.

  Part I: Find x and y if the inradius $r = 2011$.

  Part II: Find x and y if $r = \sqrt{2011}$.

- **5172:** *Proposed by Neculai Stanciu, Buzău, Romania*

  If a, b and c are positive real numbers, then prove that,

$$\frac{a\,(b-c)}{c\,(a+b)} + \frac{b\,(c-a)}{a\,(b+c)} + \frac{c\,(a-b)}{b\,(c+a)} \geq 0.$$

- **5173:** *Proposed by Pedro H. O. Pantoja, UFRN, Brazil*

  Find all triples x, y, z of non-negative real numbers that satisfy the system of equations,

$$\begin{cases} x^2(2x^2+x+2) = xy(3x+3y-z) \\ y^2(2y^2+y+2) = yz(3y+3z-x) \\ z^2(2z^2+z+2) = xz(3z+3x-y) \end{cases}$$

- **5174:** *Proposed by José Luis Díaz-Barrero, Barcelona, Spain*

  Let n be a positive integer. Compute:

$$\lim_{n\to\infty} \frac{n^2}{2^n} \sum_{k=0}^{n} \frac{k+4}{(k+1)(k+2)(k+3)} \binom{n}{k}.$$

- **5175:** *Proposed by Ovidiu Furdui, Cluj, Romania*

Find the value of,

$$\lim_{n\to\infty}\frac{1}{n}\sum_{i,j=1}^{n}\frac{i+j}{i^2+j^2}.$$

Solutions

- **5152:** *Proposed by Kenneth Korbin, New York, NY*

Given prime numbers x and y with $x > y$. Find the dimensions of a primitive Pythagorean Triangle which has hypotenuse equal to $x^4 + y^4 - x^2y^2$.

Solution 1 by Enkel Hysnelaj, University of Technology, Sydney, Australia and Elton Bojaxhiu, Kriftel, Germany

We will use the Euclid's formula for generating Pythagorean triples given an arbitrary pair of positive integers m and n with $m > n$, which states that the integers $(a, b, c) = (m^2 - n^2, 2mn, m^2 + n^2)$ form a Pythagorean triple. The triple generated by Euclid's formula is primitive if and only if m and n are coprime and one of them is even. Obviously c will be the hypotenuse, since it is the largest side.

Now using the fact that the hypotenuse is equal to $x^4 + y^4 - x^2y^2$ we have

$$c = x^4 + y^4 - x^2y^2 = \left(x^2 - y^2\right)^2 + x^2y^2$$

and this implies that $(m, n) = (x^2 - y^2, xy)$ since $x > y$. Using the fact that the numbers x and y are primes with $x > y$, we have $\gcd(m, n) = \gcd(x^2 - y^2, xy) = 1$, so the numbers m and n are coprime.

Now if one of the numbers x or y is even, then xy is even, and if both are odd then $x^2 - y^2$ is even; the case when both x and y are even is not possible since $\gcd(x^2 - y^2, xy) = 1$. So at least one of the numbers m or n is even.

By Euclid's formula we produce primitive Pythagorean Triangles which will have side lengths:

$$\begin{aligned}
a &= m^2 - n^2 = \left(x^2 - y^2\right)^2 - x^2y^2 = x^4 + y^4 - 3x^2y^2\\
b &= 2mn = 2\left(x^2 - y^2\right)xy = 2x^3y - 2xy^3\\
c &= m^2 + n^2 = \left(x^2 - y^2\right)^2 + x^2y^2 = x^4 + y^4 - x^2y^2
\end{aligned}$$

Solution 2 by Dionne Bailey, Elsie Campbell, and Charles Diminnie, San Angelo, TX

Recall that (a, b, c) is a primitive Pythagorean triple (with a even) if and only if there are positive integers m, n with $m > n$, $\gcd(m, n) = 1$, and $m \not\equiv n \pmod 2$ such that $a = 2mn$, $b = m^2 - n^2$, and $c = m^2 + n^2$.

Let $m = \max\left\{x^2 - y^2, xy\right\}$ and $n = \min\left\{x^2 - y^2, xy\right\}$. Since x and y are distinct primes, it is easily shown that $\gcd(m, n) = 1$ and $m > n$. If $y = 2$, then since x is prime and $x > y = 2$, x must be odd and consequently, xy is even and $x^2 - y^2$ is odd. If $y > 2$, then since x and y are primes and $x > y > 2$, x and y must be odd. In this case, xy is odd and $x^2 - y^2$ is even. It follows that $m \not\equiv n \pmod 2$ in all cases.

As a result, $a = 2mn = 2xy\,(x^2 - y^2)$, $b = m^2 - n^2 = \left|(x^2 - y^2)^2 - x^2y^2\right|$, $c = m^2 + n^2 = (x^2 - y^2)^2 + x^2y^2 = x^4 + y^4 - x^2y^2$ yields a primitive Pythagorean triple (a, b, c). Some examples are listed in the following table:

x	y	a	b	c
3	2	60	11	61
5	2	420	341	541
7	2	1260	1829	2221
5	3	480	31	481
7	3	1680	1159	2041
7	5	1680	649	1801

Also solved by Brian D. Beasley, Clinton, SC; Paul M. Harms, North Newton, KS; David E. Manes, Oneonta, NY; Boris Rays, Brooklyn, NY; Raul A. Simon, Santiago, Chile; David Stone and John Hawkins (jointly), Statesboro, GA, and the proposer.

- **5153:** *Proposed by Kenneth Korbin, New York, NY*

A trapezoid with sides $(1, 1, 1, x)$ and a trapezoid with sides $(1, x, x, x)$ are both inscribed in the same circle. Find the diameter of the circle.

Solution 1 by David E. Manes, Oneonta, NY

Let D be the diameter of the circle. If $x = 1$, then $D = \sqrt{2}$. If $x = \dfrac{3+\sqrt{5}}{2}$ or $x = \dfrac{3-\sqrt{5}}{2}$, then $D = \sqrt{6 + 2\sqrt{5}}$ or $\sqrt{6 - 2\sqrt{5}}$ respectively.

Given a cyclic quadrilaterals with successive sides a, b, c, d and semiperimeter s, then the diameter D of the circumscribed circle is given by

$$D = \frac{1}{2}\sqrt{\frac{(ab+cd)(ac+bd)(ad+bc)}{(s-a)(s-b)(s-c)(s-d)}}.$$

For the trapezoid with sides $(1, 1, 1, x)$,

$$D = \frac{1}{2}\sqrt{\frac{(1+x)(1+x)(1+x)}{\left(\frac{1+x}{2}\right)\left(\frac{1+x}{2}\right)\left(\frac{1+x}{2}\right)\left(\frac{3-x}{2}\right)}} = \frac{2}{\sqrt{3x-1}}.$$

For the trapezoid with sides $(1, x, x, x)$,

$$D = \frac{1}{2}\sqrt{\frac{(x+x^2)(x+x^2)(x+x^2)}{\left(\frac{3x-1}{2}\right)\left(\frac{x+1}{2}\right)\left(\frac{x+1}{2}\right)\left(\frac{x+1}{2}\right)}} = \frac{2x\sqrt{x}}{\sqrt{3-x}}.$$

Setting the two expressions for D equal and simplifying,one obtains the quartic equation

$$x^4 - 3x^3 + 3x - 1 = \left(x^2 - 1\right)\left(x^2 - 3x + 1\right) = 0$$

whose positive roots are $x = 1$, or $x = \dfrac{3 \pm \sqrt{5}}{2}$.

If $x = 1$, then $D = \dfrac{2}{\sqrt{2}} = \sqrt{2}$.

If $x = \dfrac{3+\sqrt{5}}{2}$, then $D = \dfrac{2}{\sqrt{3 - \left(\dfrac{3+\sqrt{5}}{2}\right)}} = \sqrt{6+2\sqrt{5}}$.

If $x = \dfrac{3-\sqrt{5}}{2}$, then $D = \sqrt{6-2\sqrt{5}}$.

Finally, note that if $x = 1$, then the two trapezoids are the same unit square.

Solution 2 by Kee-Wai Lau, Hong Kong, China

Call the first trapezoid $ABCD$ such that $AB = BC = CD = 1$ and $DA = x$.

Let $\angle ABC = \theta$ so that $\angle ADC = \pi - \theta$. Applying the cosine formula to triangles ABC and ADC, we obtain respectively

$$AC^2 = 2(1 - \cos\theta) \text{ and } AC^2 = x^2 + 2x\cos\theta + 1.$$

Eliminating AC from these two equations, we obtain $\cos\theta = \dfrac{1-x}{2}$ and hence $AC = \sqrt{x+1}$.

Let the diameter of the circle be d. By the sine formula, we have

$$d = \frac{AC}{\sin\theta} = \frac{2}{\sqrt{3-x}}. \qquad (1)$$

Call the second trapezoid $PQRS$ such $PQ = QR = RS = x$ and $SP = 1$.

Let $\angle$PQR=ϕ so that $\angle PSR = \pi - \phi$. By the procedure similar to that for trapezoid $ABCD$, we obtain

$$d = \frac{PR}{\sin\phi} = \frac{2x\sqrt{x}}{\sqrt{3x-1}}. \qquad (2)$$

From (1) and (2), we obtain $x^4 - 3x^3 + 3x - 1 = 0$, whose positive roots are $1, \dfrac{3-\sqrt{5}}{2}, \dfrac{3+\sqrt{5}}{2}$. The corresponding values of d are $\sqrt{2}$, $\sqrt{5}-1$, and $\sqrt{5}+1$.

Also solved by Paul M. Harms, North Newton, KS; Enkel Hysnelaj, University of Technology, Sydney, Australia (jointly with) Elton Bojaxhiu, Kriftel, Germany; Charles McCracken, Dayton, OH; John Nord, Spokane, WA; Boris Rays, Brooklyn, NY; Raul A. Simon, Santiago, Chile; Trey Smith, San Angelo, TX; Jim Wilson, Athens, GA, and the proposer.

- **5154:** *Proposed by Andrei Răzvan Băleanu (student, George Cosbuc National College) Motru, Romania*

Let a, b, c be the sides, m_a, m_b, m_c the lengths of the medians, r the in-radius, and R the circum-radius of the triangle ABC. Prove that:

$$\frac{m_a^2}{1+\cos A} + \frac{m_b^2}{1+\cos B} + \frac{m_c^2}{1+\cos C} \geq 6Rr\left(\frac{a}{b+c} + \frac{b}{c+a} + \frac{c}{a+b}\right).$$

Solution by Arkady Alt, San Jose, California, USA

Since

$$\begin{aligned}
\frac{m_a^2}{1+\cos A} &= \frac{m_a^2}{2\cos^2\frac{A}{2}} = \frac{m_a^2}{2}\left(1+\tan^2\frac{A}{2}\right) = \frac{m_a^2}{2} + \frac{m_a^2}{2}\tan^2\frac{A}{2} \\
&= \frac{m_a^2}{2} + \frac{m_a^2}{2}\cdot\frac{r^2}{(s-a)^2} = \frac{2\left(b^2+c^2\right)-a^2}{8} + \frac{(b+c)^2-a^2+(b-c)^2}{8}\cdot\frac{r^2}{(s-a)^2} \\
&\geq \frac{2\left(b^2+c^2\right)-a^2}{8} + \frac{s(s-a)r^2}{2(s-a)^2} = \frac{2\left(b^2+c^2\right)-a^2}{8} + \frac{sr^2}{2(s-a)},
\end{aligned}$$

then

$$\sum_{cyc}\frac{m_a^2}{1+\cos A} \geq \frac{3\left(a^2+b^2+c^2\right)}{8} + \frac{sr^2}{2}\sum_{cyc}\frac{1}{s-a}.$$

Noting that

$$\begin{aligned}
\frac{sr^2}{2}\sum_{cyc}\frac{1}{s-a} &= \frac{(s-a)(s-b)(s-c)}{2}\sum_{cyc}\frac{1}{s-a} = \frac{1}{2}\sum_{cyc}(s-b)(s-c) \\
&= \frac{2ab+2bc+2ca-a^2-b^2-c^2}{8},
\end{aligned}$$

we obtain

$$\frac{3\left(a^2+b^2+c^2\right)}{8} + \frac{sr^2}{2}\sum_{cyc}\frac{1}{s-a} = \frac{ab+bc+ca+a^2+b^2+c^2}{4}.$$

Hence,

$$\sum_{cyc}\frac{m_a^2}{1+\cos A} \geq A,\ \text{ where A} = \frac{ab+bc+ca+a^2+b^2+c^2}{4}.$$

Also since,

$$6Rr = \frac{12Rrs}{2s} = \frac{3abc}{2s} \quad\text{and}\quad \frac{a}{\text{b}+\text{c}} \leq \frac{a^2(b+c)}{4abc},$$

we have,

$$B \geq 6Rr\sum_{cyc}\frac{a}{b+c},\ \text{ where B} = \frac{3abc}{2s}\sum_{\text{cyc}}\frac{a^2(b+c)}{4abc} = \frac{3}{4(a+b+c)}\sum_{\text{cyc}}a^2(b+c).$$

Thus, it suffices to prove inequality $A \geq B$.

Since $\sum_{cyc} a(a-b)(a-c) \geq 0$ (by the Schur Inequality), we have

$$4(a+b+c)(A-B) = (a+b+c)\left(ab+bc+ca+a^2+b^2+c^2\right) - 3\sum_{cyc}a^2(b+c)$$

$$
\begin{aligned}
&= && (a+b+c)\left((a+b+c)^2-ab-bc-ca\right)\\
&- && 3(a+b+c)(ab+bc+ca)+9abc\\
&\Longleftrightarrow && 9abc+(a+b+c)^3\geq 4(a+b+c)(ab+bc+ca)\\
&\Longleftrightarrow && \sum_{cyc} a(a-b)(a-c)\geq 0.
\end{aligned}
$$

Also solved by Kee-Wai Lau, Hong Kong, China, and the proposer.

- **5155:** *Proposed by José Luis Díaz-Barrero, Barcelona, Spain*

Let a, b, c, d be the roots of the equation $x^4 + 6x^3 + 7x^2 + 6x + 1 = 0$. Find the value of

$$\frac{3-2a}{1+a}+\frac{3-2b}{1+b}+\frac{3-2c}{1+c}+\frac{3-2d}{1+d}.$$

Solution 1 by Valmir Bucaj (student, Texas Lutheran University), Seguin, TX

Since a, b, c, d are the roots of the equation $x^4 + 6x^3 + 7x^2 + 6x + 1 = 0$, by Viéte's formulas we have

$$
\begin{aligned}
a+b+c+d &= -6\\
ab+ac+ad+bc+bd+cd &= 7\\
abc+abd+acd+bcd &= -6\\
abcd &= 1.
\end{aligned}
$$

For convenience we adopt the following notation:

$$
\begin{aligned}
x &= a+b+c+d\\
y &= ab+ac+ad+bc+bd+cd\\
z &= abc+abd+acd+bcd\\
w &= abcd.
\end{aligned}
$$

Finally, we have:

$$
\begin{aligned}
\frac{3-2a}{1+a}+\frac{3-2b}{1+b}+\frac{3-2c}{1+c}+\frac{3-2d}{1+d} &= \frac{-(8w+3z-2y-7x-12)}{w+z+y+x+1}\\
&= \frac{-(8\times 1+3\times(-6)-2\times 7-7\times(-6)-12)}{1-6+7-6+1}\\
&= 2.
\end{aligned}
$$

Solution 2 by Brian D. Beasley, Clinton, SC

Since $x^4 + 6x^3 + 7x^2 + 6x + 1 = (x^2 + x + 1)(x^2 + 5x + 1) = 0$, we calculate the four roots and assign the values $a = (-1 + i\sqrt{3})/2$, $b = (-1 - i\sqrt{3})/2$, $c = (-5 + \sqrt{21})/2$, and $d = (-5 - \sqrt{21})/2$. This yields:

$$\frac{3-2a}{1+a}=\frac{1-5i\sqrt{3}}{2}; \qquad \frac{3-2b}{1+b}=\frac{1+5i\sqrt{3}}{2};$$
$$\frac{3-2c}{1+c}=\frac{3+5\sqrt{21}}{6}; \qquad \frac{3-2d}{1+d}=\frac{3-5\sqrt{21}}{6}.$$

Hence the desired sum is 2.

Solution 3 by David E. Manes, Oneonta, NY

The value of the expression is 2.

Note that if r is a root of the equation, then $r \neq 0$ and moreover $\frac{1}{r}$ also satisfies the equation since

$$\frac{1}{r^4}+\frac{6}{r^3}+\frac{7}{r^2}+\frac{6}{r}+1=\frac{1+6r+7r^2+6r^3+r^4}{r^4}=0.$$

Therefore, the roots of the equation can be labeled $a, b=\frac{1}{a}, c$, and $d=\frac{1}{c}$. Then

$$\frac{3-2a}{1+a}+\frac{3-2b}{1+b}=\frac{3-2a}{1+a}+\frac{3-\frac{2}{a}}{1+\frac{1}{a}}=\frac{3-2a}{1+a}+\frac{3a-2}{1+a}=1.$$

Similarly, $\frac{3-2c}{1+c}+\frac{3-2d}{1+d}=1$. Hence,

$$\frac{3-2a}{1+a}+\frac{3-2b}{1+b}+\frac{3-2c}{1+c}+\frac{3-2d}{1+d}=2.$$

Solution 4 by Michael Brozinsky, Central Islip, New York

We first note that if $f(x)=\frac{3-2x}{1+x}$, then $f^{-1}(x)=\frac{-x+3}{x+2}$.

If we denote the given polynomial by $P(x)$, then the given expression is just the sum of the roots of the equation $-3x^4+6x^3-17x^2+14x+817=0$, obtained by clearing fractions in the equation $P\left(f^{-1}(x)\right)=0$. So the answer is $-6/(-3)=2$.

Solution 5 by Pedro H. O. Pantoja, UFRN, Brazil

Let a, b, c, d be the roots of equation $x^4+6x^3+7x^2+6x+1=0$, then $1+a, 1+b, 1+c, 1+d$ will be the roots of the equation $x^4+2x^3-5x^2+6x-3=0$. So,

$$\frac{1}{1+a}, \frac{1}{1+b}, \frac{1}{1+c}, \frac{1}{1+d},$$

will be the roots of the equation $3x^4-6x^3+5x^2-2x-1=0$.

Then,

$$\frac{1}{1+a}, \frac{1}{1+b}, \frac{1}{1+c}, \frac{1}{1+d} \quad = \quad \frac{-(-6)}{3}=2, \quad \text{implies}$$

$$\frac{3-2a}{1+a}+\frac{3-2b}{1+b}+\frac{3-2c}{1+c}+\frac{3-2d}{1+d} \quad = \quad 5\cdot 2-4\cdot 2=2.$$

Comments by David Stone and John Hawkins of Statesboro, GA

(1) If we fix h and $j=k$ and define $G(x)=\dfrac{h+kx}{1+x}$, then

$$G(r)+G\left(\frac{1}{r}\right)=\frac{h+kr}{1+r}+\frac{h+k\dfrac{1}{r}}{1+\dfrac{1}{r}}=\frac{h+kr}{1+r}+\frac{rh+k}{r+1}=\frac{h+rh+k+kr}{1+r}=h+k.$$

Thus, in the setting of the posed problem,

$$G(a)+G(b)+G(c)+G(d)=G(a)+G\left(\frac{1}{a}\right)+G(b)+G\left(\frac{1}{b}\right)=2\left(h+k\right).$$

In fact, if $p(x)=\sum\limits_{i=0}^{2n}a_ix^k$ is any palindromic polynomial (i.e., $a_k=a_{2n-k}$) of even degree with distinct zeros unequal to ± 1 and $G(x)=\dfrac{h+kx}{1+x}$, then $\sum G(r)=n(h+k)$, where the sum is taken over all zeros of $p(x)$.

(2) We had to be careful with the pairing of the zeros because it is conceivable that some r is being paired with multiple reciprocals−say all the zeros were $2,2,2,$ and $\dfrac{1}{2}$. Actually, the polynomial given in the problem has a pair of real zeros (both negative and reciprocals of each other) and a pair or complex zeros (which must lie on the unit circle since the reciprocal equals the conjugate).

(3) We wonder if the expression $\dfrac{3-2x}{1+x}$ and the total sum have some deep connection to the polynomial. Perhaps there is some algebraic relationship, since algebraists carefully analyze properties of the zeros of polynomials.

Also solved by Daniel Lopez Aguayo (student, Institute of Mathematics, UNAM), Morelia, Mexico; Brian D. Beasley (two solutions), Clinton, SC; Elsie M. Campbell, Dionne T. Bailey, and Charles Diminnie (jointly), Angelo TX; Bruno Salgueiro Fanego, Viveiro, Spain; G. C. Greubel, Newport News, VA; Paul M. Harms, North Newton, KS; Enkel Hysnelaj, University of Technology, Sydney, Australia with Elton Bojaxhiu, Kriftel, Germany; Talbot Knighton, Stephen Chou and Tom Peller (jointly, students at Taylor University). Upland, IN; Bradley Luderman, San Angelo, TX; Kee-Wai Lau, Hong Kong, China; Sugie Lee, Jon Patton, and Matthew Fox (jointly, students at Taylor University), Upland, IN; Paolo Perfetti, Department of Mathematics, "Tor Vergata" University, Rome, Italy; Aaron Milauskas, Daniel Perrine, and Kari Webster (jointly, students at Taylor University), Upland, IN; John Nord, Spokane WA; Boris, Rays, Brooklyn, NY, and the proposer.

- **5156:** *Proposed by Yakub N. Aliyev, Khyrdalan, Azerbaijan*

Given two concentric circles with center O and let A be a point different from O in the interior of the circles. A ray through A intersects the circles at the points B and C. The ray OA intersects the circles at the points B_1 and C_1, and the ray through A

perpendicular to line OA intersects the circles at the points B_2 and C_2. Prove that

$$B_1C_1 \leq BC \leq B_2C_2.$$

Solution 1 by Charles McCracken, Dayton, OH

$$B_1C_1 < BC$$

(because the shortest path between two concentric circle is along a ray from the center.)

Rotate $\triangle OBC$ until C is at C_2, and let D be the new position of B.

$$\begin{aligned}
\triangle ODC_2 &\quad \cong \quad \triangle OBC \\
OD = OB, \quad OC_2 &\quad = \quad OC, \quad \angle DOC = \angle BOC \\
\triangle ODC &\quad \textit{lies inside} \quad \triangle OB_2C_2 \\
DC_2 &\quad < \quad B_2C_2 \\
DC_2 &\quad = \quad BC \\
BC &\quad < \quad B_2C_2 \\
B_1C_1 < &\quad BC \quad < B_2C_2.
\end{aligned}$$

This is almost a proof without words!

Solution 2 by David Stone and John Hawkins, Statesboro, GA

We will employ the *line* through the point A, rather than a ray. This is satisfactory because the two line segments formed by the line intersecting the given annulus have the same length. So if we impose a coordinate system with origin at the circles' center and assume that the point A is on the non-negative $y-$axis, we can restrict our attention to the right half plane.

Suppose the two concentric circles have radii $r_1 < r_2$. By a rotation we can position A at $(0, b)$, where $b \geq 0$ in the case where the line is vertical, the distance BC is $\mathrm{B}_1C_1 = r_2 - r_1$.
A non-vertical line though A has equation $y = mx + b$ where m may vary from $-\infty$ to $+\infty$.

It is straight forward to determine that the line that intersects the right half circle of radius r at the point $(x, mx+b)$, where $x = \dfrac{-mb + \sqrt{(1+m^2)r^2 - b^2}}{1+m^2}$. Now we have $B = (x_1, y_1)$ the point in the right half plane where the line $y = mx + b$ intersects the inner circle of radius r_1 and $C = (x_2, y_2)$ the point in the right half pane where the line $y = mx + b$ intersects the outer circle of radius r_2.

Thus, $x_1 = \dfrac{-mb + \sqrt{(1+m^2)r_1^2 - b^2}}{1+m^2}$ and $x_2 = \dfrac{-mb + \sqrt{(1+m^2)r_2^2 - b^2}}{1+m^2}$. Note that $x_1 < x_2$.

We compute the distance BC, which depends only upon m by the distance formula:

$$\begin{aligned} d(m) &= \sqrt{(x_2-x_1)^2+(mx_2+b-mx_1-b)^2} \\ &= \sqrt{(x_2-x_1)^2+m^2(x_2-x_1)^2} \\ &= \sqrt{(1+m^2)(x_2-x_1)^2} \\ &= \sqrt{(1+m^2)}\,(x_2-x_1) \end{aligned}$$

Because $x_2-x_1=\dfrac{\sqrt{(1+m^2)\,r_2^2-b^2}-\sqrt{(1+m^2)\,r_1^2-b^2}}{1+m^2}$, we see that

$$d(m)=\frac{\sqrt{(1+m^2)\,r_2^2-b^2}-\sqrt{(1+m^2)\,r_1^2-b^2}}{\sqrt{1+m^2}}=\sqrt{r_2^2-\frac{b^2}{1+m^2}}-\sqrt{r_1^2-\frac{b^2}{1+m^2}}.$$

Our goal is to show that the length of the line segment BC is a maximum when $m=0$ and is a minimum when the line is vertical, using $BC=d(m)$ for all non-vertical lines.

Note the behavior of the function d:

(1) d is even, so its graph is symmetric about the $y-$axis.

(2) As m grows to positive or negative infinity, $d(m)$ approaches r_2-r_1.

(3) $d(0)=\sqrt{r_2^2-b^2}-\sqrt{r_1^2-b^2}=B_2C_2$.

(4) $d'(m)=-\dfrac{mb^2}{(1+m^2)^2\sqrt{r_1^2-\dfrac{b^2}{1+m^2}}\sqrt{r_2^2-\dfrac{b^2}{1+m^2}}}d(m)$.

By the expression for the derivative, we see:

$$\text{for } m<0,\ d'(m)>0, \text{ so } d \text{ is increasing;}$$
$$\text{for } m>0,\ d'(m)<0, \text{ so } d \text{ is decreasing.}$$

Therefore, d achieves its maximum, given in (3), when $m=0$; that is, in the direction perpendicular to the line along OA. The minimum value of BC is $B_1C_1=r_2-r_1$. All other values of BC lie between these extremes.

Also solved by Michael Brozinsky, Central Islip, NY; Michael N. Freid, Kibbutz Revivim, Israel; Raul A. Simon, Santiago, Chile, and the proposer.

- **5157:** *Proposed by Juan-Bosco Romero Márquez, Madrid, Spain*

Let $p\geq 2, \lambda\geq 1$ be real numbers and let $e_k(x)$ for $1\leq k\leq n$ be the symmetric elementary functions in the variables $x=(x_1,\ldots,x_n)$ and $x^p=(x_1^p,\ldots,x_n^p)$, with $n\geq 2$ and $x_i>0$ for all $i=1,2,\ldots,n$.

Prove that

$$e_n^{(pk/n)}(x)\leq\frac{e_k(x^p)+\lambda\left(e_k^p(x)-e_k(x^p)\right)}{\binom{n}{k}+\lambda\left(\binom{n}{k}^p-\binom{n}{k}\right)}\leq\left(\frac{e_1(x)}{n}\right)^{pk},\quad 1\leq k\leq n.$$

Solution by the proposer

The elementary symmetric functions in the real variables $x_1, x_2, \ldots, x_n$ for $1 \le k \le n$ are defined as follows:

$$\begin{aligned}
e_1(x) &= e_1(x_1, x_2, \ldots, x_n) = \sum_{i=1}^{n} x_i = \sum_{1\le i\le n} x_i = \sum_{1\le i\le \binom{n}{1}} x_i \\
e_2(x) &= \sum_{1\le i<j\le n} x_i x_j = \sum_{1\le i\le \binom{n}{2}} x_i^*, \\
\textit{where } x_i^* &= x_{i_1}x_{i_2}, 1 \le i_1 < i_2 \le n;\ \textit{and similarly}, \\
e_k(x) &= \sum_{1\le i\le \binom{n}{k}} x_i^* = \sum_{1\le i_1<i_2\ldots<k_k\le n} x_{i_1}x_{i_2}\ldots x_{i_k} \\
\textit{where } x_i^* &= x_{i_1}x_{i_2}\ldots x_{i_k},\ \ 1 \le i_1 < i_2 < \ldots < i_k \le n, \\
&\ldots \\
e_n(x) &= x_1 x_2 \ldots x_n.
\end{aligned}$$

We present some results, that we will need:

Theorem 1 (Mac Laurin's Inequalities)

$$If\ E_k(x) = \frac{e_k(x)}{\binom{n}{k}}, 1 \le k \le n,\ is\ the\ k^{th}\ symmetric\ function\ mean,\ then$$

$$(E_n(x))^{1/n} \le \ldots \le (E_k(x))^{1/k} \le \ldots \le (E_2(x))^{1/2} \le E_1(x).$$

Theorem 2 (Power Means Inequality)

$$If x_i > 0, i = 1, 2, \ldots, n,\ p > 1\ are\ real\ numbers\ then,$$

$$\sum_{i=1}^{n} x_i^p < \left(\sum_{i=1}^{n} x_i\right)^p \le n^{p-1} \sum_{i=1}^{n} x_i^p.$$

See, reference [1].

And by reference [2], we have:

Corollary (Shanse Wu)

$$If\ x_i > 0, i = 1, 2, \ldots, n;\ n \ge 2,\ p \ge 2,\ 1 \le k \le n\ are\ real\ numbers,\ then$$

$$(e_k(x))^p - e_k(x^p) \ge \left(\binom{n}{k}^p - \binom{n}{k}\right)(e_n(x))^{pk/n}.$$

We denote the homographic function of the real variable $\lambda \ge 1$, *with* $x \in \Re_+^n$ *and* $p \ge 2$ fixed as follows:

$$f(\lambda) = \frac{a + \lambda(b-a)}{\binom{n}{k} + \lambda\left(\binom{n}{k}^p - \binom{n}{k}\right)}$$

where $a = e_k(x^p)$ and $b = e_k^p(x),\ 1 \le k \le n.$

Properties of the function f

1) f is a positive function for $\lambda \geq 1$ since by Corollary 1, we have

$$b - a = e_k^p(x) - e_k(x^p) \geq \left(\binom{n}{k}^p - \binom{n}{k}\right) e_n^{pk/n}(x) > 0, \; for \; 1 \leq k \leq n,$$

and so by the definition of f we obtain, $f(\lambda) \geq 0$, for $\lambda \geq 1$.

2) f is an infinitely differentiable continuous function for $\lambda \geq 1$.

Monotonicity of f.

We have:

$$f(1) = \frac{a + b - a}{\binom{n}{k}^p} = \frac{b}{\binom{n}{k}^p} = \frac{e_k^p(x)}{\binom{n}{k}^p} = \left(\frac{e_k(x)}{\binom{n}{k}}\right)^p = E_k^p(x) \leq (E_1^k(x)^p = [E_1(x)]^{pk}$$

by application of the Theorem 1 (MacLaurin's Inequalities).

Now by computing and evaluating the first derivative of the function f we obtain:

$$\begin{aligned} f'(\lambda) &= \frac{\left[\binom{n}{k} + \lambda\left(\binom{n}{k}^p - \binom{n}{k}\right)(b-a)\right] - [a + \lambda(b-a)]\left[\binom{n}{k}^p - \binom{n}{k}\right]}{\left[\binom{n}{k} + \lambda\left[\binom{n}{k}^p - \binom{n}{k}\right]\right]^2} = \frac{\binom{n}{k}b - a\binom{n}{k}^p}{D} \\ &= \frac{\binom{n}{k}}{D}\left[b - a\binom{n}{k}^{p-1}\right] \\ &= \frac{\binom{n}{k}}{D}\left[\Big(\sum_{1 \leq j_1 < j_2 < \ldots < j_k \leq n} x_{j_1}x_{j_2}\ldots x_{j_k}\Big)^p - \binom{n}{k}^{p-1} \sum_{1 \leq j_1 < j_2 < \ldots < j_k \leq n} (x_{j_1}x_{j_2}\ldots x_{j_k})^p\right] \\ &= \frac{\binom{n}{k}}{D}\left[\left(\sum_{1 \leq j \leq \binom{n}{k}} x_j^*\right)^p - \binom{n}{k}^{p-1} \sum_{1 \leq j \leq \binom{n}{k}} x^{p_j}\right] \leq 0. \end{aligned}$$

by application of Theorem 2 and where $D = \left[\binom{n}{k} + \lambda\left[\binom{n}{k}^p - \binom{n}{k}\right]\right]^2 > 0$, and where the products are defined as: $x_j^* = x_{j_1}x_{j_2}\ldots x_{j_k}$ where j is the number of elements in (or cardinality of) each set $J = \{j_1, j_2, \ldots, j_k\}$ with k elements. The number of all the products is also equal to all subsets of k elements of the set $\{1, 2, \ldots, n\}$ which is in total $\binom{n}{k}$.

Using Theorem 2 (the power means inequality), function f is decreasing for $\lambda \geq 1$. And so,

$$f(+\infty) = \lim_{\lambda \to +\infty} f(\lambda) \leq f(\lambda) \leq f(1), \; for \; \lambda \geq 1.$$

From the above corollary we have:

$$f(+\infty) = \lim_{\lambda \to +\infty} f(\lambda) = \frac{b - a}{\binom{n}{k}^p - \binom{n}{k}} = \frac{e_k^p(x) - e_k(x^p)}{\binom{n}{k}^p - \binom{n}{k}} \geq \left(\prod_{i=1}^{n} x_i^{p/n}\right)^k = e_n^{pk/n}(x)).$$

And so, for the conditions of the problem, we have shown that the original inequality holds.

$$e_n^{pk/n}(x) \leq \frac{e_k(x^p) + \lambda\left(e_k^p(x) - e_k(x^p)\right)}{\binom{n}{k} + \left(\binom{n}{k}^p - \binom{n}{k}\right)} \leq \left(\frac{e_1(x)}{n}\right)^{pk}.$$

References

[1] P.S. Bullen, D.S. Mitrinovic, P.M. Vasic. **The Handbook of Means and their Inequalities,** Reidel, Dordrecht, 1988.

[2] Shanse, Wu. "Generalizations and sharpness of the power means inequality and their applications," **Journal of Mathematical Analysis and Applications,** (2005; Corollary 1, pp. 642-643).

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[4] Mihaly Bencze. "New means and new refinements of Cesaro's inequality and of the AM-GM-HM inequalities," **Libertas Mathematica,** Vol 28 (2008; pp.123-124).