Problems

Ted Eisenberg, Section Editor

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This section of the Journal offers readers an opportunity to exchange interesting mathematical problems and solutions. Please send them to Ted Eisenberg, Department of Mathematics, Ben-Gurion University, Beer-Sheva, Israel or fax to: 972-86-477-648. Questions concerning proposals and/or solutions can be sent e-mail to <eisenbt@013.net>. Solutions to previously stated problems can be seen at <http://ssmj.tamu.edu>.

Solutions to the problems stated in this issue should be posted before December 15, 2010

- **5122:** *Proposed by Kenneth Korbin, New York, NY*

  Partition the first 32 non-negative integers from 0 to 31 into two sets A and B so that the sum of any two distinct integers from set A is equal to the sum of two distinct integers from set B and vice versa.

- **5123:** *Proposed by Kenneth Korbin, New York, NY*

  Given isosceles triangle ABC with $\overline{AB} = \overline{BC} = 2011$ and with cevian $\overline{BD}$. Each of the line segments $\overline{AD}$, $\overline{BD}$, and $\overline{CD}$ have positive integer length with $\overline{AD} < \overline{CD}$.

  Find the lengths of those three segments when the area of the triangle is minimum.

- **5124:** *Proposed by Michael Brozinsky, Central Islip, NY*

  If $n > 2$ show that $\displaystyle\sum_{i=1}^{n} \sin^2\left(\frac{2\pi i}{n}\right) = \frac{n}{2}$.

- **5125:** *Proposed by José Luis Díaz-Barrero, Barcelona, Spain*

  Let a, b, c be positive real numbers such that $a^2 + b^2 + c^2 = 3$. Prove that

  $$\frac{ab}{2(c+a)+5b} + \frac{bc}{2(a+b)+5c} + \frac{ca}{2(b+c)+5a} < \frac{11}{32}.$$

- **5126:** *Proposed by Pantelimon George Popescu, Bucharest, Romania and José Luis Díaz-Barrero, Barcelona, Spain*

  Let a, b, c, d be positive real numbers and $f : [a, b] \to [c, d]$ be a function such that $|f(x) - f(y)| \ge |g(x) - g(y)|$, for all $x, y \in [a, b]$, where $g : R \to R$ is a given injective function, with $g(a),\ g(b) \in \{c, d\}$.

  Prove

  (**i**) $f(a) = c$ and $f(b) = d$, or $f(a) = d$ and $f(b) = c$.

  (**ii**) If $f(a) = g(a)$ and $f(b) = g(b)$, then $f(x) = g(x)$ for $a \le x \le b$.

- **5127:** *Proposed by Ovidiu Furdui, Cluj, Romania*

  Let $n \geq 1$ be an integer and let $T_n(x) = \sum_{k=1}^{n} (-1)^{k-1} \dfrac{x^{2k-1}}{(2k-1)!}$, denote the $(2n-1)$th Taylor polynomial of the sine function at 0. Calculate

$$\int_0^\infty \frac{T_n(x) - \sin x}{x^{2n+1}} dx.$$

Solutions

- **5104:** *Proposed by Kenneth Korbin, New York, NY*

  There are infinitely many primitive Pythagorean triangles with hypotenuse of the form $4x^4 + 1$ where x is a positive integer. Find the dimensions of all such triangles in which at least one of the sides has prime integer length.

Solution by Brian D. Beasley, Clinton, SC

It is well-known that a primitive Pythagorean triangle (a, b, c) satisfies $a = 2st$, $b = s^2 - t^2$, and $c = s^2 + t^2$ for integers $s > t > 0$ of opposite parity with $\gcd(s, t) = 1$. Then a is never prime. Letting x be a positive integer and taking

$$c = 4x^4 + 1 = (2x^2 + 2x + 1)(2x^2 - 2x + 1),$$

we see that c can only be prime if $2x^2 - 2x + 1 = 1$, meaning $x = 1$. Thus $s = 2$ and $t = 1$, which produces the triangle $(4, 3, 5)$. Similarly, $b = (s + t)(s - t)$ can only be prime if $s - t = 1$, which would yield

$$4x^4 + 1 = 2t^2 + 2t + 1 \quad \text{and hence} \quad 2x^4 = t(t + 1).$$

But this would force one of the consecutive positive integers t or $t + 1$ to be a fourth power and the other to be twice a fourth power, meaning $t = 1$. Once again, our only solution is the triangle $(4, 3, 5)$.

Also solved by Paul M. Harms, North Newton, KS; Boris Rays, Brooklyn, NY; David Stone and John Hawkins (jointly), Statesboro, GA, and the proposer

- **5105:** *Proposed by Kenneth Korbin, New York, NY*

  Solve the equation

$$x + y - \sqrt{x^2 + xy + y^2} = 2 + \sqrt{5}$$

if x and y are of the form $a + b\sqrt{5}$ where a and b are positive integers.

Solution by Shai Covo, Kiryat-Ono, Israel

We let $x = a + b\sqrt{5}$ and $y = c + d\sqrt{5}$, with $a, b, c, d \in N$. Since the solution of

$$x + y - \sqrt{x^2 + xy + y^2} = 2 + \sqrt{5} \qquad (1)$$

is symmetric in x and y, it suffices to consider the case $x \leq y$. Hence, we let $y = x\alpha$ with $\alpha \geq 1$. Substituting into (1) gives

$$(a + b\sqrt{5}) \left[(1 + \alpha) - \sqrt{1 + \alpha + \alpha^2}\right] = 2 + \sqrt{5} \qquad (2)$$

It is immediately verified by taking the derivative that the function $\varphi(\alpha) = (1+\alpha) - \sqrt{1+\alpha+\alpha^2}$ is increasing . From $\varphi(\alpha)\left[(1+\alpha)+\sqrt{1+\alpha+\alpha^2}\right] = \alpha$ it is readily seen that $\varphi(\alpha) \to \frac{1}{2}$ as $\alpha \to \infty$. On the other hand, $\varphi(1) = 2-\sqrt{3}$. We thus conclude from (2) that

$$4+2\sqrt{5} < a+b\sqrt{5} \leq \frac{2+\sqrt{5}}{2-\sqrt{3}}.$$

We verify numerically that this leaves us with the following set of pairs (a,b):

$$\begin{array}{ccccccc}\{(1,4), & (1,5), & (1,6), & (2,3), & (2,4), & (2,5), & (2,6), \\ (3,3), & (3,4), & (3,5), & (4,3), & (4,4), & (4,5), & (5,2), \\ (5,3), & (5,4), & (6,2), & (6,3), & (6,4), & (7,1), & (7,2), \\ (7,3), & (8,1), & (8,2), & (8,3), & (9,1), & (9,2), & (9,3), \\ (10,1), & (10,2), & (11,1), & (11,2), & (12,1), & (13,1)\}. & \end{array}$$

It follows straightforwardly from (1) that

$$y = \frac{\left(4+2\sqrt{5}\right)x - 9 - 4\sqrt{5}}{x-4-2\sqrt{5}}.$$

Substituting $x = a+b\sqrt{5}$ and multiplying the numerator and denominator on the right hand side by $(a-4)-(b-2)\sqrt{5}$ gives, after some algebra,

$$y = \frac{4a^2-5a+20b-20b^2-4}{(a-4)^2-5(b-2)^2} + \frac{2a^2-4a+13b-10b^2-2}{(a-4)^2-5(b-2)^2}\sqrt{5}. \qquad (3)$$

This determines the constants c and d forming y in an obvious manner, since $a,b,c,d \in N$. In particular, we see that

$$c-2d = \frac{3a-6b}{(a-4)^2-5(b-2)^2}. \qquad (4)$$

From this, noting that $c-2d$ is an integer, it follows readily that a and b cannot be both odd; furthermore if a and b are both even, then a must be divisible by 4. This restricts the set of all possible pairs (a,b)given above to

$$\begin{array}{ccccccc}\{(1,4), & (1,6), & (2,3), & (2,5), & (3,4), & (4,3), & (4,4), \\ (4,5), & (5,2), & (5,4), & (6,3), & (7,2), & (8,1), & (8,2), \\ (8,3), & (9,2), & (10,1), & (11,2), & (12,1)\}. & & \end{array}$$

The requirement that the right-handside of (4) be an integer further restricts the set to

$$\{(2,3),\ (5,2),\ (6,3),\ (7,2)\}.$$

With these values of a and b, calculating c and d according to (3) give the following x, y pairs:

$$x = 2+3\sqrt{5}, \qquad y = 118+53\sqrt{5}$$

$$x = 5+2\sqrt{5}, \qquad y = 31+14\sqrt{5}$$

$$x = 6 + 3\sqrt{5}, \qquad y = 10 + 5\sqrt{5}$$

$$x = 7 + 2\sqrt{5}, \qquad y = 13 + 6\sqrt{5}.$$

Substituting into (1) show that these x, y pairs constitute the solution of (1) for $x \leq y$. The complete solution then follows by symmetry in x and y.

Also solved by Paul M. Harms, North Newton, KS; David Stone and John Hawkins (jointly), Statesboro,GA, and the proposer.

- **5106:** *Proposed by Michael Brozinsky, Central Islip, NY*

Let a, b, and c be the sides of an acute-angled triangle ABC. Let H be the orthocenter and let d_a, d_b and d_c be the distances from H to the sides BC, CA, and AB respectively.

Show that

$$d_a + d_b + d_c \leq \frac{3}{4}D$$

where D is the diameter of the circumcircle.

Solution 1 by Bruno Salgueiro Fanego, Viveiro, Spain

From the published solution of SSM problem 5066 and Gerretsen and Euler's inequality, we have that

$$d_a + d_b + d_c = \frac{r^2 + s^2 - 4R^2}{2R} \leq \frac{r^2 + 4Rr + 3r^2}{2R} = \frac{2}{R}(r + R) \leq 1\left(\frac{R}{2} + R\right) = \frac{3}{4}D,$$

with equality if and only if $\triangle ABC$ is equilateral.

Solution 2 by Ercole Suppa, Teramo, Italy

Let H_a, H_b, H_c be the feet of A, B, C onto the sides BC, CA, AB respectively and let R be the circumradius of $\triangle ABC$. We have

$$d_a = BH_a \cdot \tan(90^\circ - C) = c \cos B \cot C.$$

Hence, taking into account the extended sine law, we get

$$d_a = 2R \sin C \cos B \cot C = 2R \cos B \cos C. \qquad (1)$$

Now, by using (1) and its cyclic permutations, the given inequality rewrites as

$$\begin{aligned} 2R\cos B\cos C + 2R\cos C\cos A + 2R\cos A\cos B &\leq \frac{3}{4} \cdot 2R \\ \cos B\cos C + \cos C\cos A + \cos A\cos B &\leq \frac{3}{4} \qquad (2) \end{aligned}$$

which is true. In fact, from the well known formulas

$$\sum \cos^2 A = 1 - 2\cos A \cos B \cos C$$

and

$$0 \leq \cos A \cos B \cos C \leq \frac{1}{8},$$

each of which is valid for an acute-angled triangle, we immediately obtain

$$\sum \cos^2 A \geq \frac{3}{4}. \qquad (3)$$

Hence, by applying the known inequality

$$1 < \cos A + \cos B + \cos C \leq \frac{3}{2},$$

we obtain

$$\begin{aligned} (\cos A + \cos B + \cos C)^2 &\leq \frac{9}{4} \Rightarrow \\ \sum \cos^2 A + 2\sum \cos B \cos C &\leq \frac{9}{4} \Rightarrow \\ 2\sum \cos B \cos C \leq \frac{9}{4} - \sum \cos^2 A &\leq \frac{9}{4} - \frac{3}{4} = \frac{3}{2} \Rightarrow \\ \sum \cos B \cos C &\leq \frac{3}{4}, \end{aligned}$$

and the conclusion follows. Equality holds for $a = b = c$.

Also solved by Scott H. Brown, Montgomery, AL; Paul M. Harms, North Newton, KS; Kee-Wai Lau, Hong Kong, China;

- **5107:** *Proposed by Tuan Le (student, Fairmont, H.S.), Anaheim, CA*

Let a, b, c be positive real numbers. Prove that

$$\frac{\sqrt{a^3+b^3}}{a^2+b^2} + \frac{\sqrt{b^3+c^3}}{b^2+c^2} + \frac{\sqrt{c^3+a^3}}{c^2+a^2} \geq \frac{6(ab+bc+ac)}{(a+b+c)\sqrt{(a+b)(b+c)(c+a)}}$$

Solution by Kee-Wai Lau, Hong Kong, China

By the Cauchy-Schwarz inequality, we have

$$a^2+b^2 \leq \sqrt{(a+b)\,(a^3+b^3)},\ b^2+c^2 \leq \sqrt{(b+c)\,(b^3+c^3)},\ c^2+a^2 \leq \sqrt{(c+a)\,(c^3+a^3)}.$$

Hence it suffices to show that

$$\frac{1}{\sqrt{a+b}} + \frac{1}{\sqrt{b+c}} + \frac{1}{\sqrt{c+a}} \geq \frac{6(ab+bc+ac)}{(a+b+c)\sqrt{(a+b)(b+c)(c+a)}} \quad \text{or}$$

$$\sqrt{(a+b)(b+c)} + \sqrt{(b+c)(c+a)} + \sqrt{(c+a)(a+b)} \geq \frac{6(ab+bc+ac)}{(a+b+c)}.$$

By the arithmetic mean-geometric mean-harmonic inequalities, we have

$$\begin{aligned} &\sqrt{(a+b)(b+c)} + \sqrt{(b+c)(c+a)} + \sqrt{(c+a)(a+b)} \\ \geq\ & 3\sqrt[3]{(a+b)(b+c)(c+a)} \end{aligned}$$

$$\geq \frac{9}{\dfrac{1}{a+b}+\dfrac{1}{b+c}+\dfrac{1}{c+a}}$$

$$= \frac{9(a+b)((b+c)(c+a)}{a^2+b^2+c^2+3(ab+bc+ca)}.$$

It remains to show that

$$3(a+b+c)(a+b)(b+c)(c+a) \geq 2(ab+bc+ca)\left(a^2+b^2+c^2+3\left(ab+bc+ca\right)\right).$$

But this follows from the fact that

$$\begin{aligned} & 3(a+b+c)(a+b)(b+c)(c+a) - 2(ab+bc+ca)\left(a^2+b^2+c^2+3(ab+bc+ca)\right) \\ = \;& a^3b+ab^3+a^3c+ac^3+b^3c+bc^3-2a^2bc-2ab^2c-2abc^2 \\ = \;& a(b+c)(b-c)^2+b(c+a)(c-a)^2+c(a+b)(a-b)^2 \\ \geq \;& 0, \end{aligned}$$

and this completes the solution.

Also solved by Pedro H.O. Pantoja (student, UFRN), Natal, Brazil; Paolo Perfetti, Department of Mathematics, University of Rome, Italy, and the proposer.

- **5108:** *Proposed by José Luis Díaz-Barrero, Barcelona, Spain*

Compute

$$\lim_{n\to\infty}\frac{1}{n}\tan\left[\sum_{k=1}^{4n+1}\arctan\left(1+\frac{2}{k(k+1)}\right)\right].$$

Solution 1 by Ovidiu Furdui, Cluj, Romania

The limit equals 4. A calculation shows that

$$\begin{aligned} \arctan\left(1+\frac{2}{k(k+1)}\right) &= \arctan(1)+\arctan\frac{1}{k^2+k+1}, \\ &= \frac{\pi}{4}+\arctan\frac{1}{k}-\arctan\frac{1}{k+1}. \end{aligned}$$

And it follows that

$$\begin{aligned} \sum_{k=1}^{4n+1}\arctan\left(1+\frac{2}{k(k+1)}\right) &= (4n+1)\frac{\pi}{4}+\arctan 1-\arctan\frac{1}{4n+2} \\ &= (4n+1)\frac{\pi}{4}+\arctan\frac{4n+1}{4n+3}. \end{aligned}$$

Thus,

$$\tan\left[\sum_{k=1}^{4n+1}\arctan\left(1+\frac{2}{k(k+1)}\right)\right] = \tan\left((4n+1)\frac{\pi}{4}+\arctan\frac{4n+1}{4n+3}\right)$$

$$= \frac{\tan((4n+1)\frac{\pi}{4}) + \frac{4n+1}{4n+3}}{1 - \tan((4n+1)\frac{\pi}{4})\frac{4n+1}{4n+3}}$$

$$= \frac{1 + \frac{4n+1}{4n+3}}{1 - \frac{4n+1}{4n+3}}$$

$$= 4n+2.$$

So the limit equals 4, and the problem is solved.

Solution 2 by Shai Covo, Kiryat-Ono, Israel

We will show that

$$\lim_{n\to\infty} \frac{1}{n} \tan\left[\sum_{k=1}^{4n+1} \arctan\left(1 + \frac{2}{k(k+1)}\right)\right] = 4. \qquad (1)$$

From the identity $\tan(x + m\pi) = \tan(x)$, m integer, it follows that the equality (1) will be proved if we show that

$$\sum_{k=1}^{4n+1} \arctan\left(1 + \frac{2}{k(k+1)}\right) = \arctan(4n+2) + m\pi, \qquad (2)$$

for some integer m. In fact, as we will see at the end, the m in (2) is equal to n. We first prove the following lemma.

Lemma. Define a sequence $(a_k)_{k\geq 1}$ recursively by $a_1 = 2$ and, for $k \geq 2$,

$$a_k = \frac{a_{k-1} + \left(1 + \frac{2}{k(k+1)}\right)}{1 - a_{k-1}\left(1 + \frac{2}{k(k+1)}\right)}. \qquad (3)$$

If $a_{k-1} = k$, for some $k \geq 2$, then

$$a_k = -\frac{k+2}{k},\ a_{k+1} = -\frac{1}{k+2},\ a_{k+2} = \frac{k+2}{k+4},\ a_{k+3} = k+4.$$

Hence, in particular, $a_{4n+1} = 4n+2$ for all $n \geq 0$.

Proof. Suppose that $a_{k-1} = k, k \geq 2$. Substituting this into (3) gives

$$a_k = -\frac{(k^2+1)(k+2)}{(k^2+1)k} = -\frac{k+2}{k}. \qquad (4)$$

From (3) and (4) we find

$$a_{k+1} = -\frac{k^2+2k+2}{(k^2+2k+2)(k+2)} = -\frac{1}{k+2}. \qquad (5)$$

From (3) and (5) we find

$$a_{k+2} = \frac{(k^2+4k+5)(k+2)}{(k^2+4k+5)(k+4)} = \frac{k+2}{k+4}. \qquad (6)$$

Finally, from (3) and (6) we find

$$a_{k+3} = \frac{(k^2+6k+10)(k+4)}{k^2+6k+10} = k+4.$$

The lemma is thus established.

We make use of the addition formula for arctan:

$$\arctan(x)+\arctan(y) = \begin{cases} \arctan\left(\dfrac{x+y}{1-xy}\right), & \text{if } xy<1, \\ \arctan\left(\dfrac{x+y}{1-xy}\right)+\pi\text{sign}(x), & \text{if } xy>1, \end{cases} \qquad (7)$$

the case where $xy=1$ being irrelevant here. Now, let $(a_k)_{k\geq 1}$ be the sequence defined in the above lemma. If follows readily from (7) and the lemma that, for all $k \geq 2$,

$$\arctan(a_{k-1}) + \arctan\left(1+\frac{2}{k(k+1)}\right) = \arctan(a_k)+\pi\sigma_k,$$

where $\sigma_k = 1$ or 0 accordingly, as k is or is not of the form $k=4j+2, j\geq 0$ integer. From this it follows that

$$\sum_{k=1}^{l} \arctan\left(1+\frac{2}{k(k+1)}\right) = \arctan(a_l) + \pi\sum_{k=1}^{l}\sigma_k.$$

Recalling the conclusion in the lemma, it thus follows that (2) holds with $m=n$, and so we are done.

Remark: From (2), where $m=n$, and the fact that

$$\begin{aligned}\int \arctan\left(1+\frac{2}{x(x+1)}\right)dx &= \frac{1}{2}\log(x^2+1)-\frac{1}{2}\log(x^2+2x+2) \\ &+ \arctan\left(1+\frac{2}{x(x+1)}\right)+\arctan(x+1)+C,\end{aligned}$$

it follows readily the following interesting result:

$$\int_0^{4n+1}\arctan\left(1+\frac{2}{x(x+1)}\right)dx - \sum_{k=1}^{4n+1}\arctan\left(1+\frac{2}{k(k+1)}\right) \longrightarrow \frac{1}{2}\log 2, \text{as } n\to\infty.$$

Also solved by Kee-Wai, Hong Kong, China, and the proposer.

- **5109:** *Proposed by Ovidiu Furdui, Cluj, Romania*

Let $k\geq 1$ be a natural number. Find the value of

$$\lim_{n\to\infty}\frac{(k\sqrt[n]{n}-k+1)^n}{n^k}.$$

Solution 1 by Angel Plaza and Sergio Falcon, Las Palmas de Gran Canaria, Spain

Let

$$\begin{aligned}
x_n &= \frac{(k\sqrt[n]{n}-k+1)^n}{n^k}. \quad \text{Then,}\\
\ln x_n &= \ln(k\sqrt[n]{n}-k+1)^n - \ln n^k = n\ln(k\sqrt[n]{n}-k+1) - k\ln n\\
&= n\left(\ln(k\sqrt[n]{n}-k+1) - k\ln\sqrt[n]{n}\right)\\
&= \frac{\ln\dfrac{k\sqrt[n]{n}-k+1}{(\sqrt[n]{n})^k}}{\dfrac{1}{n}} \approx \frac{\dfrac{k\sqrt[n]{n}-k+1}{(\sqrt[n]{n})^k}-1}{\dfrac{1}{n}} = \frac{k\sqrt[n]{n}-k+1-(\sqrt[n]{n})^k}{(\sqrt[n]{n})^k\,\dfrac{1}{n}}.
\end{aligned}$$

Now, taking into account that $\lim\limits_{n\to\infty}\sqrt[n]{n}=1$ and the equivalence of the infinitesimals $k(x-1)+1-x^k \approx \dfrac{k(k-1)}{2}(x-1)^2$ when $x\to 1$, we have

$$\begin{aligned}
\lim_{n\to\infty}\ln x_n &= \lim_{n\to\infty}\frac{\dfrac{k(k-1)}{2}\left(\sqrt[n]{n}-1\right)^2}{\dfrac{1}{n}} = \frac{k(k-1)}{2}\lim_{n\to\infty}\frac{(\ln n)^2}{\dfrac{1}{n}}\\
&= \frac{k(k-1)}{2}\lim_{n\to\infty}\frac{(\ln n)^2}{n} = 0. \quad \text{Therefore,}\\
\lim_{n\to\infty} x_n &= 1.
\end{aligned}$$

Solution 2 by Kee-Wai Lau of Hong Kong, China

As $n\to\infty$, we have $\sqrt[n]{n}=e^{\ln n/n}=1+\dfrac{\ln n}{n}+O\left(\dfrac{\ln^2 n}{n^2}\right)$. Since $\ln(1+x)=x+O\left(x^2\right)$ as $x\to 0$, so

$$n\ln\left(1+k\left(\sqrt[n]{n}-1\right)\right)-k\ln n = n\left(\frac{k\ln n}{n}+O\left(\frac{\ln^2 n}{n^2}\right)\right)-k\ln n = O\left(\frac{\ln^2 n}{n}\right),$$

where the constant implied by the last O depends at most on k. It follows that the limit of the problem equal 1, independent of k.

Also solved by Shai Covo, Kiryat-Ono, Israel; Paolo Perfetti, Department of Mathematics, University of Rome, Italy, and the proposer.

Late Solution

A late solution to 5099 was received from **Charles McCracken of Dayton, OH.**