Problems Ted Eisenberg, Section Editor

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This section of the Journal offers readers an opportunity to exchange interesting mathematical problems and solutions. Please send them to Ted Eisenberg, Department of Mathematics, Ben-Gurion University, Beer-Sheva, Israel or fax to: 972-86-477-648. Questions concerning proposals and/or solutions can be sent e-mail to <eisenbt@013.net>. Solutions to previously stated problems can be seen at <http://www.ssma.org/publications>.

Solutions to the problems stated in this issue should be posted before
May 15, 2020

- **5583:** *Proposed by Kenneth Korbin, New York, NY*

  a) Given positive angles A and B with $A+B=180^\circ$. A circle with radius 16 and a circle with radius 49 are each tangent to both sides of $\angle A$. The circles are also tangent to each other. Find sine A.

  b) A circle with radius x and a circle with radius y are each tangent to both sides of $\angle B$. These circles are also tangent to each other. Find positive integers x and y with $(x,y)=1$.

- **5584:** *Proposed by Michael Brozinsky, Central Islip, NY*

  Let a and $n \geq 2$ be positive integers where $0 \leq a \leq n-1$.

  Find the number of points of intersection of the curve C_1 whose parametric equations are:

$$\begin{aligned} x &= (n-1)\cdot\cos\left(\frac{t}{n-1}\right)+\cos(t), \\ y &= (n-1)\cdot\sin\left(\frac{t}{n-1}\right)-\sin(t), \text{ where} \\ &\frac{a\cdot(n-1)\cdot 2\pi}{n} \leq t \leq \frac{(a+1)\cdot(n-1)\cdot 2\pi}{n} \end{aligned}$$

  and the curve C_2 whose parametric equations are:

$$\begin{aligned} x &= (n+1)\cdot\cos\left(\frac{t}{n+1}\right)-\cos(t), \\ y &= (n+1)\cdot\sin\left(\frac{t}{n+1}\right)-\sin(t), \text{ where} \\ &\frac{a\cdot(n+1)\cdot 2\pi}{n} \leq t \leq \frac{(a+1)\cdot(n+1)\cdot 2\pi}{n} \end{aligned}$$

- **5585:** *Proposed by Daniel Sitaru, National Economic College "Theodor Costescu," Mehedinti, Romania*

In ΔABC the following relationship holds:

$$\sin^4 A + \sin^4 B + \sin^4 C + \sin^4\left(\frac{\pi}{3} + A\right) + \sin^4\left(\frac{\pi}{3} + B\right) + \sin^4\left(\frac{\pi}{3} + C\right) \le \frac{27}{8}$$

- **5586:** *Proposed by Michel Bataille, Rouen, France*

For $n \in N$ let

$$u_n = \frac{1}{n}\sum_{k=1}^{n} ke^{k/n^2}.$$

Find real numbers α, β such that $\lim\limits_{n\to\infty}(u_n - \alpha n) = \beta$.

- **5587:** *Proposed by José Luis Díaz-Barrero, Barcelona Tech, Barcelona, Spain*

Let $f, g : \Re \to \Re$ be two real functions defined by $f(x) = x^4 + 1$ and $g(x) = a_0 + a_1x^3 + a_2x^5 + a_3x^7 - x^9$ where $a_1 < 0, a_2 < 0, a_3 < 0$ and a_0 is a real number. Find the number of real solutions to the equation

$$(g \circ f)(x) = (f \circ g)(x).$$

- **5588:** *Proposed by Ovidiu Furdui and Alina Sîntămărian, Technical University of Cluj-Napoca, Cluj-Napoca, Romania*

Let $a > 1$. Calculate

$$\lim_{x\to\infty} x\int_0^1 a^{t(t-1)x}\mathrm{d}t.$$

Solutions

- **5565:** *Proposed by Kenneth Korbin, New York, NY*

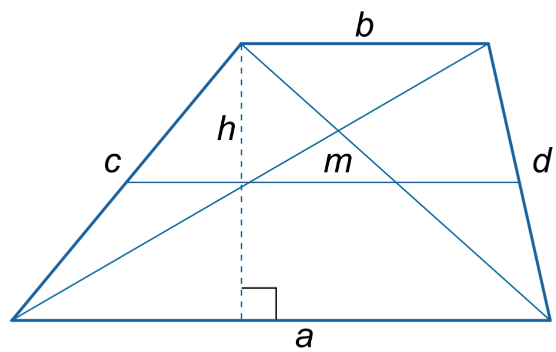
A trapezoid with integer length sides is inscribed in a circle with diameter $7^3 = 343$. Find the minimum and the maximum possible values of the perimeter.

Solution by Ioannis D. Sfikas, National and Kapodistrian University of Athens, Greece

A trapezoid (or a trapezium) is a quadrilateral in which (at least) one pair of opposite sides is parallel. Sometimes a trapezoid is defined as a quadrilateral having exactly one pair of parallel sides. The parallel sides are called the bases, and two other sides are called the legs.

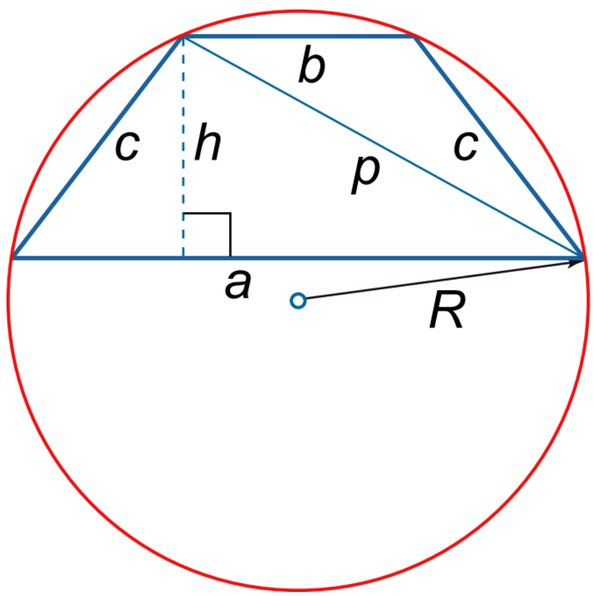
Bases of a trapezoid: a, b
Legs of a trapezoid: c, d
Midline of a trapezoid: m
Altitude of a trapezoid: h

Perimeter: P
Diagonals of a trapezoid: p, q
Angle between the diagonals: φ
Radius of the circumscribed circle: R
Radius of the inscribed circle: r
Area: S



In Euclidean geometry, an isosceles trapezoid (isosceles trapezium in British English) is a convex quadrilateral with a line of symmetry bisecting one pair of opposite sides. It is a special case of a trapezoid. Alternatively, it can be defined as a trapezoid in which both legs and both base angles are of the same measure. Note that a non-rectangular parallelogram is not an isosceles trapezoid because of the second condition, or because it has no line of symmetry. In any isosceles trapezoid, two opposite sides (the bases) are parallel, and the two other sides (the legs) are of equal length (properties shared with the parallelogram). The diagonals are also of equal length. The base angles of an isosceles trapezoid are equal in measure (there are in fact two pairs of equal base angles, where one base angle is the supplementary angle of a base angle at the other base). All four vertices of an isosceles trapezoid lie on a circumscribed circle. The radius in the circumscribed circle is given by:

$$R = c\sqrt{\frac{ab+c^2}{4c^2-(a-b)^2}} = c\sqrt{\frac{ab+c^2}{(2c+a-b)(2c-a+b)}}.$$



So, since $2R = 7^3 = 343$, then $7^3, b, c > 0$ and we have:

$$(2c^2 + ab - 7^6)^2 = (a^2 - 7^6)(b^2 - 7^6). \tag{1}$$

The number of integer solutions is 34.

- If $a = b = c$, then by (1) we have: $2a^2 = 7^6$, and we haven't any positive integer solutions.

- If $a = b$, then by (1) we have: $a^2 + c^2 = 7^6$, and we can easily check that we haven't any positive integer solutions.
- If $a = c$, then by (1) we have: $4a^3 + 7^6 b = 3 \cdot 7^6 a$, and we have the solutions: $(a, b, c) = (49, 143, 49), (98, 262, 98)$.
- If $b = c$, then by (1) we have: $4\mathrm{b}^3 + 7^6 a = 3 \cdot 7^6$, and we have the solutions: $(a, b, c) = (143, 49, 49), (262, 98, 98)$.
- If $a = 7^3$, then by (1) we have: $2c^2 + 7^3 b = 7^6$, and we have the solutions: $(a, b, c) = (7^3, 119, 196), (7^3, 217, 147), (7^3, 287, 98), (7^3, 329, 49)$.
- If $b = 7^3$, then by (1) we have: $2c^2 + 7^3 a = 7^6$, and we have the solutions: $(a, b, c) = (119, 7^3, 196), (217, 7^3, 147), (287, 7^3, 98), (329, 7^3, 49)$.
- If we assume that $2c^2 + ab = a^2$ and $2c^2 + ab = b^2$, then we have no integer solutions.
- Furthermore, with the help of Mathematica we can obtain the following solutions:

$$
\begin{array}{rcllll}
(a,b,c) & = & (18,294,161) & (28,294,294) & (49,143,329) & (98,262,287) \\
 & & (119,343,196) & (143,49,329) & (147,333,147) & (147,333,217) \\
 & & (196,332,119) & (196,332.196) & (217,343,147) & (235,245,7) \\
 & & (235,245,245) & (245,235,245) & (245,235,245) & (262,98,287) \\
 & & (287,343,98) & (294,18,161) & (294,18,294) & (329,343,49) \\
 & & (332,196,119) & (332,196,196) & (333,147,217) & (333,147,147) \\
 & & (343,119,196) & (343,217,147) & (343,287,98) & (3243,329,49)
\end{array}
$$

So, for the perimeter $P = a + b + 2c$, we have:

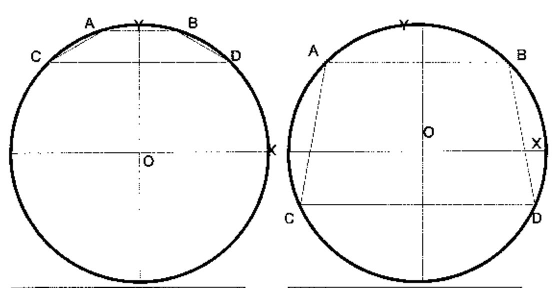
$$
\begin{aligned}
P_{max} &= 245 + 235 + 2 \cdot 245 = 970, \\
P_{min} &= 143 + 49 + 2 \cdot 49 = 290.
\end{aligned}
$$

Editor′s comments: **Albert Stadler of Herrliberg, Switzerland** noted in his solution to this problem that the formula for finding the radius of the circumscribing circle of a cyclic quadrilateral with successive sides a, b, c, d and semiperimeter s is

$$R = \frac{1}{4}\sqrt{\frac{(ab + cd)(ac + bd)(ad + bc)}{(s - a)(s - b)(s - c)(s - d)}};$$

and that this formula is attributed to the Indian mathematician Vatasseri Parameshvara who lived the 15^{th} century (http://en.wikipedia.org/wiki/Cyclicquadrilateral).

David Stone and John Hawkins of Georgia Southern University in Statesboro, GA, observed that the trapezoid had to be isosceles, and that meant that a-priori, they had an intuitive sketch of the solution.



Case I: Fits in semicircle

Case II: Larger than semicircle

They carried out calculations similar to the above and found that the trapezoid with the smallest perimeter has bases 49, 143, with a slant height of 49. Its perimeter is 290 and it is a Case I trapezoid. The largest perimeter has bases 235, 245, with a slant height of 245 and a perimeter of 970 making it a Case II trapezoid. They stated: "The largest possible quadrilateral which can be inscribed in our circle is a square of side length $\frac{343}{\sqrt{2}} \approx 242.5$, which has perimeter 970.15. Even with the restrictions on our figure, our maximum comes within an eyelash of that bound. (Of course, our maximal integer-sided trapezoid is nearly square.)"

Also solved by Kee-Wai Lau, Hong Kong, China; Albert Stadler, Herrliberg, Switzerland; David Stone and John Hawkins, Georgia Southern University, Statesboro, GA, and the proposer.

- **5566:** *Proposed by Michael Brozinsky, Central Islip, NY*

Square ABCD (in clockwise order) with all sides equal to x has point E on AB at a distance $\alpha \cdot x$ from B where $0 < \alpha < 1$. The right triangle EBC is folded along segment EC so that what was previously corner B is now at point B'. A perpendicular from B' to AD intersects AD at H. If the ratio of the areas of trapezoids $AEB'H$ to $DCB'H$ is $\frac{7}{18}$ what is α?

Solution 1 by Brian Bradie, Christopher Newport University, Newport News, VA

Let $\theta = \angle BCE = \angle ECB'$, and let F be the foot of the perpendicular drawn from point B' to side BC, as shown in the figure below. Then

$$\cos\theta = \frac{1}{\sqrt{1+\alpha^2}}, \quad \text{and} \quad \sin\theta = \frac{\alpha}{\sqrt{1+\alpha^2}}.$$

Moreover,

$$\begin{aligned} B'F &= x\sin 2\theta = 2x\sin\theta\cos\theta = \frac{2\alpha}{1+\alpha^2}x, \text{ and} \\ CF &= x\cos 2\theta = x(\cos^2\theta - \sin^2\theta) = \frac{1-\alpha^2}{1+\alpha^2}x, \end{aligned}$$

so

$$\begin{aligned} B'H &= \left(1 - \frac{2\alpha x}{1+\alpha^2}\right)x = \frac{(\alpha-1)^2}{1+\alpha^2}x, \text{ and} \\ BF &= \left(1 - \frac{1-\alpha^2}{1+\alpha^2}\right)x = \frac{2\alpha^2}{1+\alpha^2}x. \end{aligned}$$

The area of trapezoid $AEB'H$ is then

$$\frac{1}{2}\cdot\frac{2\alpha^2}{1+\alpha^2}x\cdot\left(\frac{(\alpha-1)^2}{1+\alpha^2} + 1 - \alpha\right)x = \frac{\alpha^2(1-\alpha)(\alpha^2-\alpha+2)}{(1+\alpha^2)^2}x^2,$$

and the area of trapezoid $DCB'H$ is

$$\frac{1}{2}\cdot\frac{1-\alpha^2}{1+\alpha^2}x\cdot\left(\frac{(\alpha-1)^2}{1+\alpha^2} + 1\right)x = \frac{(1-\alpha^2)(\alpha^2-\alpha+1)}{(1+\alpha^2)^2}x^2.$$

Therefore, the ratio of the area of trapezoid $AEB'H$ to the area of trapezoid $DCB'H$ is

$$\frac{\alpha^2(\alpha^2-\alpha+2)}{(1+\alpha)(\alpha^2-\alpha+1)} = \frac{\alpha^2(\alpha^2-\alpha+2)}{\alpha^3+1}.$$

Equating this expression to 7/18 and rearranging yields

$$(2\alpha-1)(9\alpha^3-8\alpha^2+14\alpha+7)=0. \qquad (1)$$

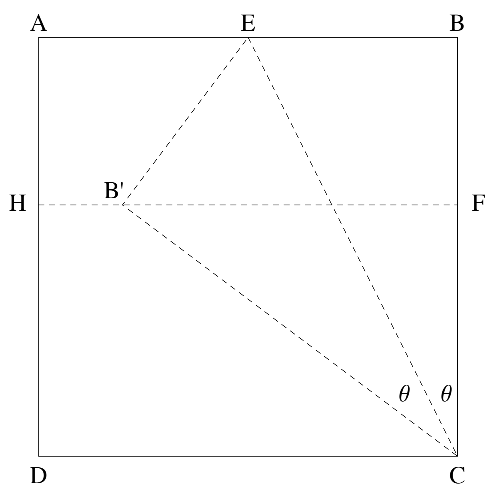
Clearly, $\alpha = 1/2$ is a solution of equation (1). Now, let

$$f(\alpha) = 9\alpha^3 - 8\alpha^2 + 14\alpha + 7.$$

Then $f(0) = 7$, and

$$f'(\alpha) = 27\alpha^2 - 16\alpha + 14 = 27\left(\alpha - \frac{8}{27}\right)^2 + \frac{314}{27} > 0.$$

It follows $f(\alpha) \geq 7$ for all $\alpha \geq 0$. Hence, $\alpha = 1/2$ is the only solution of equation (1) with $0 < \alpha < 1$. Thus, $\alpha = 1/2$.



Solution 2 by David E. Manes, Oneonta, NY

Introduce the following coordinates: $A(0,0)$, $B(0,x)$, $C(x,x)$, $D(x,0)$ and $E(0,x-\alpha x)$. Then $ABCD$ is a square in clockwise order with side length x and E is a point on AB at a distance $\alpha \cdot x$ from B where $0 < \alpha < 1$. Denote the coordinates of point B' by $B'(a,b)$ and let $[X]$ represent the area of region X in the xy-plane. If the ratio $[AEB'H]/[DCB'H] = 7/18$, then the value of α is 1/2.

Observe that the length of EB' equals the length of EB so that $\alpha x = \sqrt{a^2+(x-\alpha x-b)^2}$. The arithmetic and the algebra on this equation yield

$$a^2+b^2+x^2 = 2x(\alpha x + b - \alpha b).\ (\mathbf{1})$$

Also, $B'C = BC = x = \sqrt{(x-a)^2+(x-b)^2}$ yields

$$a^2+b^2+x^2 = 2x(a+b).\ (\mathbf{2})$$

Equating the results in equations (**1**) and (**2**), one obtains

$$a^2 + b^2 + x^2 = 2x(\alpha x + b - \alpha b) = 2x(a+b) \quad \Longrightarrow \quad \alpha = \frac{a}{x-b}. \ (\mathbf{3})$$

Note that right triangles EBC and $EB'C$ are congruent so that

$$[EBCB'] = 2[EB'C] = 2(1/2)EB' \cdot B'C = \alpha x \cdot x = \alpha x^2.$$

Since the square $ABCD$ is the disjoint union of the three quadrilaterals $EBCB'$, $AEB'H$ and $DCB'H$, it follows that

$$[EBCB'] + [AEB'H] + [DCB'H] = [ABCD] = x^2. \ (\mathbf{4})$$

The areas of the trapezoids $AEB'H$ and $DCB'H$ are given by

$$[AEB'H] = (1/2)(AE + HB')AH = (1/2)(x - \alpha x + b)a$$

and

$$[DCB'H] = (1/2)(HB' + DC)HD = (1/2)(b+x)(x-a).$$

Therefore, substituting these values in equation (**4**), one obtains

$$\alpha x^2 + (1/2)(x - 2a + b)a + (1/2)(b+x)(x-a) = x^2.$$

This equation reduces to

$$\alpha = \frac{x-b}{2x-a}. \ (\mathbf{5})$$

Using the ratio 7/18 of the areas of the trapezoids $AEB'H$ to $DCB'H$, we have

$$\frac{[AEB'H]}{[DCB'H]} = \frac{7}{18} \quad \Longrightarrow \quad \frac{(1/2)(ax - \alpha ax + ab}{(1/2)(bx + x^2 - ab - ax)} = \frac{7}{18}.$$

Simplifying this equation and solving for α, we get

$$\alpha = \frac{(x+b)(25a - 7x)}{18ax}. \ (\mathbf{6})$$

Finally, one checks that the values $a = \dfrac{2}{5}x$, $b = \dfrac{1}{5}x$ and $\alpha = \dfrac{1}{2}$ satisfy each of the equations (**1**) through (**6**). For example, in equation (**6**), let $a = \dfrac{2}{5}x$ and $b = \dfrac{1}{5}x$. Then

$$\frac{(x+b)(25a-7x)}{18ax} = \frac{\left(x + \frac{1}{5}x\right)\left(25\left(\frac{2}{5}x\right) - 7x\right)}{18\left(\frac{2}{5}x\right)} = \frac{\left(\frac{6}{5}x\right)(3x)}{\frac{36}{5}x^2}$$
$$= \frac{1}{2} = \alpha.$$

Hence, $\alpha = \dfrac{1}{2}$.

Solution 3 by Michel Bataille, Rouen, France

We show that $\alpha = \frac{1}{2}$.

Let $\theta = \angle BCE$. Then $\tan\theta = \frac{EB}{BC} = \frac{\alpha \cdot x}{x} = \alpha$ and $\angle B'CE = \theta$ (since B' is the reflection of B in EC). Since E, B, C, B' lie on the circle with diameter EC, we also have $\angle EBB' = \theta$.

If G is the orthogonal projection of B' onto BC, we deduce that $\angle BB'G = \theta$ (since $B'G$ is parallel to EB).

Now, let M be the midpoint of BB'. Then,

$$BB' = 2BM = 2BE\cos\theta = 2\alpha x \cdot \frac{1}{\sqrt{1+\alpha^2}}.$$

Since $BG = B'G\tan\theta = \alpha B'G$, we readily obtain

$$\frac{4\alpha^2 x^2}{1+\alpha^2} = BB'^2 = B'G^2 + BG^2 = (1+\alpha^2)B'G^2$$

so that $B'G = \frac{2\alpha x}{1+\alpha^2}$ and $B'H = x - B'G = \frac{x(1-\alpha)^2}{1+\alpha^2}$.

Now, we calculate

$$Area(AEB'H) = \frac{BG(AE+B'H)}{2} = \frac{\alpha B'G\left((1-\alpha)x + \frac{x(1-\alpha)^2}{1+\alpha^2}\right)}{2} = \frac{\alpha^2x^2(1-\alpha)(\alpha^2-\alpha+2)}{(1+\alpha^2)^2}$$

and

$$Area(DCB'H) = \frac{HD(B'H+x)}{2} = \frac{1}{2}\left(x - \frac{2\alpha^2 x}{1+\alpha^2}\right)\left(\frac{x(1-\alpha)^2}{1+\alpha^2} + x\right) = \frac{x^2(1-\alpha^2)(\alpha^2-\alpha+1)}{(1+\alpha^2)^2}.$$

It follows that $\frac{Area(AEB'H)}{Area(DCB'H)} = \frac{7}{18}$ is equivalent to $\frac{\alpha^2(\alpha^2-\alpha+2)}{\alpha^3+1} = \frac{7}{18}$, which rewrites as

$$(2\alpha-1)(9\alpha^3 - 8\alpha^2 + 14\alpha + 7) = 0. \qquad (1)$$

Let $f(t) = 9t^3 - 8t^2 + 14t + 7$. Since the derivative $f'(t) = 27t^2 - 16t + 14$ is positive for all real t, we have $f(t) \geq f(0) = 7$ for all $t \geq 0$ and so $f(t) = 0$ has no solution in $(0,1)$. Thus, the conjunction of (1) and $\alpha \in (0,1)$ is equivalent to $\alpha = \frac{1}{2}$.

Editor's comment: **David Stone and John Hawkins of Georgia Southern University in Statesboro, GA** observed that this problem is actually the converse of problem 5560. They stated that "In that problem, α was given as $1/2$, and some distances were calculated, enabling us to show that the trapezoidal area ratio is $7/18$. So the result of this problem is the converse if 5560. Taken together, they tell us that point E cuts AB exactly in half if and only if the ratio of the two trapezoidal areas is $7/18$." They also elaborated on this finding.

"These two problems involve two ratios: $\alpha = \dfrac{EB}{AB}$ and $R = \dfrac{Area(AEB'H)}{Area(DCB'H)}$. We have shown that $\alpha = \dfrac{1}{2}$ if and only if $R = \dfrac{7}{18}$. The work above actually tells us the relationship between the two ratios (where each ratio is between 0 and 1):
$R = \dfrac{\left(\alpha^2-\alpha+2\right)\alpha^2}{\alpha^3+1} = \dfrac{\alpha^4-\alpha^3+2\alpha^2}{\alpha^3+1}$. Expressed as a polynomial in α, we have

$$\alpha^4 - \alpha^3 + 2\alpha^2 = R\left(\alpha^3+1\right)$$

$$\alpha^4 - (1+R)\alpha^3 + 2\alpha^2 - R = 0.$$

This equation implicitly defines α in terms of R, which, fortunately, we were able to solve when $R = \dfrac{7}{18}$.

But $R = \dfrac{\alpha^4 - \alpha^3 + 2\alpha^2}{\alpha^3 + 1}$ provides an explicit formula for R in terms of α. The graph of this rational fraction passes through $(0,0)$, $\left(\dfrac{1}{2}, \dfrac{7}{18}\right)$, and $(1,1)$, and is increasing, hence one-to-one. That is, it provides a one-to-one correspondence between the values of R and the values of α."

Brian Beasley of Presbyterian College in Clinton, SC also developed the formula that if $0 < \alpha < 1$ then the ratio of the areas of the trapezoids is $g(\alpha) = \dfrac{\alpha^2(\alpha^2 - \alpha + 2)}{\alpha^3 + 1}$. Observing that "$g(\alpha)$ is increasing on $(0,1)$ with $\lim\limits_{\alpha \to 0^+} g(\alpha) = 0$ and $\lim\limits_{\alpha \to 1^-} g(\alpha) = 1$ allows one to calculate $g(\alpha)$ for other values of α such as $g(1/4) = 29/260$ and $g(3/4) = 261/364$. Conversely, this also allows us to modify the original problem with different desired ratios of the areas; for example if $g(\alpha) = 1/2$, then $\alpha \approx 0.58421$."

Also solved by Brian Beasley, Presbyterian College, Clinton, SC; Bruno Salgueiro Fanego, Viveiro, Spain; Kee-Wai Lau, Hong Kong, China; Ioannis D. Sfikas, National and Kapodistrian University of Athens, Greece; Albert Stadler, Herrliberg, Switzerland; Seán M. Stewart, Bomaderry, NSW, Australia; David Stone and John Hawkins, Georgia Southern University, Statesboro, GA, and the proposer

- **5567:** *Proposed by D.M. Bătinetu-Giurgiu, National College "Matei Basarab" Bucharest, and Neulai Stanciu, "George Emil Palade" School, Buză, Romania*

Let $[A_1A_2A_3A_4]$ be a tetrahedron with total area S, and with the area of S_k being the area of the face opposite the vertex A_k, $k = 1, 2, 3, 4$. Prove that

$$\frac{20}{3} \le \sum_{k=1}^{4} \frac{S + S_k}{S - S_k} < 8.$$

Solution 1 by Albert Stadler, Herrliberg, Switzerland

The stated inequality is equivalent to

$$\frac{8}{3} = \frac{20}{3} - 4 \le \sum_{k=1}^{4} \left(\frac{S + S_k}{S - S_k} - 1 \right) = 2 \sum_{k=1}^{4} \frac{S_k}{S - S_k} < 8 - 4 = 4$$

or to

$$\frac{4}{3} \le \sum_{k=1}^{4} \frac{S_k}{S - S_k} < 2.$$

The function $f(x) := x/(1-x)$ is convex, since $f'(x) = \dfrac{2}{(1-x)^2} > 0$, for $0 < x < 1$. Therefore, by Jensens' inequality,

$$\sum_{k=1}^{4} \frac{S_k}{S - S_k} = \sum_{k=1}^{4} \frac{\frac{S_k}{S}}{1 - \frac{S_k}{S}} \ge \frac{4\left(\frac{1}{4}\sum_{k=1}^{4} \frac{S_k}{S}\right)}{1 - \frac{1}{4}\sum_{k=1}^{4} \frac{S_k}{S}} = \frac{4}{3}.$$

To prove the other inequality we use that

$$\frac{S_k}{S - S_k} \le \frac{2S_k}{S},$$

which is equivalent to $2S_k \le S$. This inequality is clearly true, since the area of each face is not greater than the sum of the areas of the other three faces. Equality holds if and only if all four vertices of the tetrahedron lie in a plane and A_k lies in the convex hull of $\{A_1, A_2, A_3, A_k\}/\{A_k\}$. Therefore,

$$\sum_{k=1}^{4} \frac{S_k}{S - S_k} < \sum_{k=1}^{4} \frac{2S_k}{S} = 2.$$

Solution 2 by Michel Bataille, Rouen, France

(1) (*Left inequality.*) Consider the function f defined by $f(x) = \frac{S+x}{S-x}$. This function is convex on the interval $(0, S)$ (since $f''(x) = 4S(S-x)^{-3} > 0$ for $x \in (0, S)$). From Jensen's inequality, we deduce that

$$f(S_1) + f(S_2) + f(S_3) + f(S_4) \ge 4f\left(\frac{S_1 + S_2 + S_3 + S_4}{4}\right)$$

or, since $S = S_1 + S_2 + S_3 + S_4$ and $f(\frac{S}{4}) = \frac{5/4}{3/4} = \frac{5}{3}$,

$$\sum_{k=1}^{4} \frac{S + S_k}{S - S_k} \ge 4 \cdot \frac{5}{3} = \frac{20}{3}.$$

(2) (*Right inequality.*) We introduce the positive numbers $a_1 = S_2 + S_3 + S_4 - S_1 = S - 2S_1, a_2 = S - 2S_2, a_3 = S - 2S_3, a_4 = S - 2S_4$ (if r_k is the radius of the escribed sphere tangent to the face opposite the vertex A_k, then $r_k a_k$ is three times the volume of the tetrahedron). Since $S_k = \frac{S - a_k}{2}$, the right inequality is equivalent to

$$\sum_{k=1}^{4} \left(3 - \frac{4a_k}{S + a_k}\right) < 8,$$

that is,

$$\sum_{k=1}^{4} \frac{a_k}{S + a_k} > 1. \tag{1}$$

Now, $a_k < S$ hence $S + a_k < 2S$ and so

$$\sum_{k=1}^{4} \frac{a_k}{S + a_k} > \sum_{k=1}^{4} \frac{a_k}{2S} = 1$$

(since $a_1 + a_2 + a_3 + a_4 = 2S$) and (1) holds.

Solution 3 by Daniel Văcaru, Pitesti, Romania

$$\sum_{k=1}^{4} \left(\frac{S + S_k}{S - S_k}\right) + 4 = \sum_{k=1}^{4} \left(\frac{S + S_k}{S - S_k} + 1\right) \quad = \quad 2S \sum_{k=1}^{4} \left(\frac{1}{S - S_k}\right)$$

$$\overset{Bergstrom}{\geq} \quad 2S\left(\frac{(1+1+1+1)^2}{\sum_1^4 (S-S_k)}\right)$$

$$= \quad \frac{32S}{3S} = \frac{32}{3}$$

$$\Rightarrow \quad \sum_{k=1}^{4}\left(\frac{S+S_k}{S-S_k}\right) \geq \frac{32}{3} - 4 = \frac{20}{3},$$

with equality for equifacial tetrahedrons.

Using $\dfrac{x}{y} < \dfrac{x+z}{y+z}, (\forall)\, z > 0, (\forall)\ \ y > x > 0$, we obtain

$$\sum_{k=1}^{4}\left(\frac{S+S_k}{S-S_k}\right) = \sum_{k=1}^{4}\left(\frac{S-S_k+2S_k}{S-S_k}\right) = 4+2\sum_{k=1}^{4}\left(\frac{S_k}{S-S_k}\right) < 4+2\cdot\sum_{k=1}^{4}\left(\frac{2S_k}{S}\right) 4+2\cdot 2 = 8.$$

Solution 4 by Kee-Wai Lau, Hong Kong, China

Comment and Solution: The inequalities of the problem are equivalent to
$\dfrac{4}{3} \leq \displaystyle\sum_{k=1}^{4} \frac{S_k}{S-S_k} < 2$. In fact,it is known [**1**. p.34, inequalities (3.6)] that for $\lambda \geq 1$,

we have $\dfrac{4}{3^\lambda} \leq \displaystyle\sum_{k=1}^{4}\left(\frac{S_k}{S-S_k}\right)^\lambda < 2$. Here we record the proof therein. Since

$$\sum_{k=1}^{4}\frac{S_k}{S-S_k} = S\left(\sum_{k=1}^{4}\frac{1}{S-S_k}\right) - 4 = \frac{1}{3}\left(\sum_{k=1}^{4}(S-S_k)\right)\left(\sum_{k=1}^{4}\frac{1}{S-S_k}\right) - 4 \geq \frac{16}{3} - 4 = \frac{4}{3},$$

and $\lambda \geq 1$, so by the power-mean inequality

$$\sum_{k=1}^{4}\left(\frac{S_k}{S-S_k}\right)^\lambda \geq 4^{1-\lambda}\left(\sum_{k=1}^{4}\frac{S_k}{S-S_k}\right)^\lambda \geq 4^{1-\lambda}\left(\frac{4}{3}\right)^\lambda = \frac{4}{3^\lambda}.$$

Since $S_k < S - S_k$, or $2S_k < S$, so

$$\left(\frac{S_k}{S-S_k}\right)^\lambda = \left(\frac{2S_k}{2S-2S_k}\right)^\lambda < \left(\frac{2S_k}{S}\right)^\lambda = \frac{2S_k}{S}\left(\frac{2S_k}{S}\right)^{\lambda-1} \leq \frac{2S_k}{S},$$

and hence, $\displaystyle\sum_{k=1}^{4}\left(\frac{S_k}{S-S_k}\right)^\lambda < \sum_{k=1}^{4}\frac{2S_k}{S} = 2.$

Reference: **1** Y. W. Fan: *Inequality in the Tetrahedron* (in Chinese), Harbin Institute of Technology Press, Harbin (2017).

Solution 5 by Moti Levy, Rehovot, Israel

It is well known that the sum of the areas of any three faces is greater than the area of the fourth face. See Wikipedia entry on Tetrahedron.

It follows that

$$S_1 < S_2 + S_3 + S_4$$

which implies

$$2S_1 < S.$$

It follows that

$$S_k < \frac{S}{2}, k = 1, 2, 3, 4.$$

Thus the set of points $\mathcal{S} := \left\{ (S_1, S_2, S_3, S_4) | \quad S_k < \frac{S}{2} \text{ and } \sum_{k=1}^{4} \; S_k = S \right\}$ is a convex set (actually a convex polytope).

Let

$$f(S_1, S_2, S_3, S_4) := \sum_{k=1}^{4} \frac{S + S_k}{S - S_k}.$$

The Hessian $f(S_1, S_2, S_3, S_4)$ is $\begin{bmatrix} \frac{4S}{(S-S_1)^3} & 0 & 0 \\ 0 & \frac{4S}{(S-S_1)^3} & 0 \\ 0 & 0 & \frac{4S}{(S-S_1)^3} \end{bmatrix}$, which is clearly positive definite on the set of points $\mathcal{S}$.

A twice continuously differentiable function of several variables is convex on a convex set if and only if its Hessian matrix of second partial derivatives is positive semidefinite on the interior of the convex set.

It follows that $f(S_1, S_2, S_3, S_4)$ is convex on $\mathcal{S}$. It is known that the maximum of convex function on a convex polytope occurs at one of the extreme points of the polytope.

The extreme points of the polytope $\mathcal{S}$ are:

$(0, 0, 0, 0)$, $\left(\frac{S}{2}, \frac{S}{2}, 0, 0\right)$, $\left(\frac{S}{2}, 0, \frac{S}{2}, 0\right)$, $\left(\frac{S}{2}, 0, 0, \frac{S}{2}\right)$, $\left(0, \frac{S}{2}, 0, \frac{S}{2}\right)$, $\left(0, \frac{S}{2}, \frac{S}{2}, 0\right)$, $\left(0, 0, \frac{S}{2}, \frac{S}{2}\right)$.

Therefore the maximum of $f(S_1, S_2, S_3, S_4)$ on $\mathcal{S}$ is

$$f\left(\frac{S}{2}, \frac{S}{2}, 0, 0\right) = 2\frac{S + \frac{S}{2}}{S - \frac{S}{2}} + 2 = 8.$$

Now since $g(x) := \frac{S+x}{S-x}$ is convex on $(0, S)$ then by Jensen's inequality

$$\frac{1}{4}\sum_{k=1}^{4} g(S_k) \geq g\left(\frac{1}{4}\sum_{k=1}^{4} S_k\right) = g\left(\frac{S}{4}\right) = \frac{S + \frac{S}{4}}{S - \frac{S}{4}} = \frac{5}{3}.$$

We conclude that

$$\sum_{k=1}^{4} g(S_k) = \sum_{k=1}^{4} \frac{S + S_k}{S - S_k} \geq \frac{20}{3}.$$

Remark: The inequality $\frac{20}{3} \leq \sum_{k=1}^{4} \frac{S+S_k}{S-S_k}$ has nothing to do with the tetrahedron, while the inequality $\sum_{k=1}^{4} \frac{S+S_k}{S-S_k} < 8$ relies on the property $S_1 < S_2 + S_3 + S_4$ of the tetrahedron.

Also solved by Ángel Plaza, Universidad de Las Palmas de Gran Canaria, Spain; Ioannis D. Sfikas, National and Kapodistrian University of Athens, Greece, and the proposer.

- **5568:** *Proposed by Daniel Sitaru, National Economic College "Theodor Costescu," Mehedinti, Romania*

  Given: $A \in M_5(\Re), \det(A^5 + I_5) \neq 0$, and $A^{20} - I^5 = A^5(A^5 + I_5)$.
  Prove that $\sqrt[4]{\det A} \in \Re$.

  Solution 1 by Albert Stadler, Herrliberg, Switzerland

  We note that

$$0 = A^{20} - I_5 - A^5(A^5 - I_5) = (A^5 + I_5)(A^{15} - A^{10} - I^5).$$

By assumption, $\det(A^5 + I_5) \neq 0$. So $A^5 + I_5$ is non-singular and we deduce that

$$A^{15} - A^{10} - I_5 = 0.$$

Let $B = A^5$. So B satisfies the matrix equation

$$B^3 - B^2 - I_5 = 0$$

which implies that B is non-singular. For if B were singular there would be a non-zero vector x such that $Bx = 0$. But then $0 = (B^3 - B^2 - I_5)x = B(B(Bx)) - B(Bx) - I_5 x = x$ leads to a contradiction.

The equation $x^3 - x^2 - 1 = 0$ has one positive root and two complex conjugate roots. Let us denote by ρ the positive root and by σ and $\overline{\sigma}$ the two complex roots. We find that approximately $\rho \approx 1.46557, \sigma \approx -0.232786 + 0.792552i, \overline{\sigma} \approx -0.232786 - 0.792552i$. By Vieta's formula,

$$\rho\sigma\overline{\sigma} = 1.$$

Let $\lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5$ be the eigenvalues of B. They are the zeros of the characteristic polynomial $p(\lambda) = \det(\lambda I_5 - B)$. Let x_j be an eigenvector associated with the eigenvalue λ_j. So $Bx_j = \lambda_j x_j$. We claim that every eigenvalue of B is a root of $x^3 - x^2 - 1 = 0$. Indeed,

$$0 = (B^3 - B^2 - I_5)x_j = (\lambda_j^3 - -\lambda j^2 - 1)x_j$$

which implies that $\lambda_j^3 - \lambda_j^2 - 1 = 0$ and so $\lambda_j \in \{\rho, \sigma, \overline{\sigma}\}$ for $1 \leq j \leq 5$. Finally $\det(B) = \lambda_1 \cdot \lambda_2 \cdot \lambda_3 \cdot \lambda_4 \cdot \lambda_5$. So either $\det(B) = \rho^5$, or $\det(B) = \rho^3 \cdot \sigma \cdot \overline{\sigma} = \rho^2$ or $\det(B) = \rho\,(\sigma \cdot \overline{\sigma})^2$ or $\det(B) = \rho(\cdot\overline{\sigma})^2 = \frac{1}{\rho}$, since B is a real matrix and therefore the determinant is real, and

$$\det A = \sqrt[5]{\det B} \in \left\{\rho, \rho^{\frac{2}{5}}, \rho^{-\frac{1}{5}}\right\},$$

So $\sqrt[4]{\det A}$ is positive and can assume at most three (positive) values.

Solution 2 by Kee-Wai Lau, Hong Kong, China

Since $A^{20} - I_5 - A^5(A^5 + I) = 0_5$, the zero matrix, so $(A^5 + I_5)(A^{15} - A^{10} - I_5) = 0_5$. Given that $\det(A^5 + I_5) \neq 0$, so the inverse of $A^5 + I_5$ exists and we have

$$A^{15} - A^{10} - I_5 = 0_5. \tag{1}$$

We now show that $\det A > 0$, so that the statement $\sqrt[4]{\det A} \in \Re$ holds.

By (1), we have

$$A^{10}\left(A^5 - I_5\right) = I_5. \tag{2}$$

and

$$A^{15} = A^{10} + I_5 \tag{3}$$

By (2), we have $(\det A)^{10}\det(A^5 - I_5) = \det\left(A^{10}(A^5 - I_5)\right) = 1$ and so $\det A \neq 0$. Let $i = \sqrt{-1}$. By (3), we have

$$\begin{aligned}
(\det A)^{15} &= \det(A^{10} + I_5) \\
&= \det\left((A^5 + iI_5)(A^5 - iI_5)\right) \\
&= \det(A^5 + iI_5)(\det(A^5 - iI_5) \\
&= \det(A^5 + iI_5)\overline{\det(A^5 + iI_5)} \\
&= \left|\det(A^5 + iI_5)\right|^2 \\
&\geq 0.
\end{aligned}$$

Thus $\det A > 0$ and this completes the solution.

Also solved by Daniel Văcaru, Pitesti, Romania, and the proposer.

5569. *Proposed by José Luis Díaz-Barrero, Barcelona Tech, Barcelona, Spain*

Compute

$$\lim_{x\to 0} \frac{\tan(x\cdot\cos(x)) - \tan(x)\cdot\cos(\tan(x))}{x^7}.$$

Solution 1 by Ángel Plaza, Universidad de Las Palmas de Gran Canaria, Spain

By the Taylor expansions $\tan(x\cdot\cos(x)) = x - \frac{x^3}{6} - \frac{13x^5}{40} + \frac{11x^7}{1008} + \cdots$, and $\tan(x)\cdot\cos(\tan(x)) = x - \frac{x^3}{6} - \frac{13x^5}{40} - \frac{137x^7}{560} + \cdots$, it follows that

$$\lim_{x\to 0} \frac{\tan(x\cdot\cos(x)) - \tan(x)\cdot\cos(\tan(x))}{x^7} = \frac{11}{1008} + \frac{137}{560} = \frac{23}{90}.$$

Solution 2 by Albert Stadler, Herrliberg, Switzerland

We use the Taylor expansions of $\cos x$ and $\tan x$ around $x = 0$:

$$x\cos x = x - \frac{x^3}{2} + \frac{x^5}{24} - \frac{x^7}{720} + O(x^9),$$

$$\tan x = x + \frac{x^3}{3} + \frac{2x^5}{15} + \frac{17x^7}{315} + O(x^9),$$

as $x \to 0$. Then

$$\begin{aligned}
(x\cos x)^3 &= x^3 - \frac{3x^5}{2} + \frac{7x^7}{8} + O(x^9),\\
(x\cos x)^5 &= x^5 - \frac{5x^7}{2} + O(x^9),\\
(x\cos x)^7 &= x^7 + O(x^9),\\
(\tan x)^3 &= x^3 + x^5 + \frac{11x^7}{15} + O(x^9),\\
(\tan x)^5 &= x^5 + \frac{5x^7}{3} + O(x^9),\\
(\tan x)^7 &= x^7 + O(x^9),
\end{aligned}$$

as $x \to 0$, and

$$\tan(x\cos x) - (\tan x)\cos(\tan x) =$$

$$(x\cos x) + \frac{(x\cos x)^3}{3} + \frac{2(x\cos x)^5}{15} + \frac{17(x\cos x)^7}{315} - \tan x + \frac{(\tan x)^3}{2} - \frac{(\tan x)^5}{24} + \frac{(\tan x)^7}{720} + O(x^9)$$

$$= \left(x - \frac{x^3}{2} + \frac{x^5}{24} - \frac{x^7}{720}\right) + \frac{1}{3}\left(x^3 - \frac{3x^5}{2} + \frac{7x^7}{8}\right) + \frac{2}{15}\left(x^5 - \frac{5x^7}{2}\right) + \frac{17}{315}x^7$$

$$-\left(x + \frac{x^3}{3} + \frac{2x^5}{15} + \frac{17x^7}{315}\right) + \frac{1}{2}\left(x^3 + x^5 + \frac{11x^7}{15}\right) - \frac{1}{24}\left(x^5 + \frac{5x^7}{3}\right) + \frac{1}{720}x^7 + O(x^9) =$$

$$= \left(-\frac{1}{720} + \frac{7}{24} - \frac{1}{3} + \frac{17}{315} - \frac{17}{315} + \frac{11}{30} - \frac{5}{72} + \frac{1}{720}\right)x^7 + O(x^9) = \frac{23}{90}x^7 + O(x^9).$$

Therefore the limit in question equals $\frac{23}{90}$.

Solution 3 by David E. Manes, Oneonta, NY

Let $\alpha(x) = \dfrac{\tan(x\cdot\cos(x)) - \tan(x)\cdot\cos(\tan(x))}{x^7}$. Then $\lim_{x\to 0}\alpha(x) = \dfrac{23}{90}$.
Note that

$$\lim_{x\to 0}\tan(x\cdot\cos(x)) = \lim_{x\to 0} -\tan(x)\cdot\cos(\tan(x)) = \lim_{x\to 0} x^7 = 0.$$

Since all three of the above functions are non-zero and differentiable on the open interval $(0, \pi/4)$, it follows that $L'H\hat{o}pital's$ rule can be used to find $\lim_{x\to 0}\alpha(x)$. Define functions f, g and h such that $f(x) = \tan(x\cdot\cos(x))$, $g(x) = -\tan(x)\cdot\cos(\tan(x))$ and $h(x) = x^7$. Then $\alpha(x) = \dfrac{f(x)+g(x)}{h(x)}$ and $f'(x) = \sec^2(x\cdot\cos(x))(\cos x - x\cdot\sin x)$, $g'(x) = -\sec^2(x)(\cos(\tan(x)) - \tan x\cdot\sin(\tan x))$ and $h'(x) = 7x^6$. Then

$$\lim_{x\to 0} f'(x) = f'(0) = 1,\ \lim_{x\to 0} g'(x) = g'(0) = -1,\ \text{and}\ \lim_{x\to 0} h'(x) = h'(0) = 0.$$

Therefore, $\lim_{x\to 0}\dfrac{f'(x)+g'(x)}{h'(x}$ is indeterminante. For $k = 2nd, 4th$ or $6th$ derivative,

$$\lim_{x\to 0} f^{(k)}(x) = \lim_{x\to 0} g^{(k)}(x) = \lim_{x\to 0} h^{(k)}(x) = 0.$$

For the third derivative, $f^{(3)}(x) = -1$, $g^{(3)}(x) = 1$ and $h^{(3)}(x) = 0$. For the fifth derivative, $f^{(5)}(x) = -39$, $g^{(5)}(x) = 39$ and $h^{(5)}(x) = 0$. Therefore, none of these derivatives (first through sixth) produce a value for $\lim_{x\to 0}\alpha(x)$. However, the seventh derivative for each of the three functions yields the limit. Since the seventh derivative of $f(x)$ is the sum of more than thirty terms, we give only the five terms that contribute non-zero values to $f^{(7)}(0)$. These terms are:

$$\begin{gathered}
\lim_{x\to 0} 272(\cos x - x\cdot\sin x)^7 \sec^8(x\cdot\cos(x)) = 272,\\
\lim_{x\to 0} 560(\cos x - x\cdot\sin x)^4(x\cdot\sin x - 3\cos x)\sec^6(x\cdot\cos(x)) = -3(560),\\
\lim_{x\to 0} 140(\cos x - x\cdot\sin x)(x\cdot\sin x - 3\cos x)^2\sec^4(x\cdot\cos x) = 9\cdot 140,\\
\lim_{x\to 0} 42(\cos x - x\cdot\sin x)^2(5\cos x - x\cdot\sin x)\sec^4(x\cdot\cos(x)) = 5\cdot 42,\\
\lim_{x\to 0}(x\cdot\sin x - 7\cos x)\sec^2(x\cdot\cos(x)) = -7.
\end{gathered}$$

Therefore, $\lim_{x\to 0} f^{(7)}(x) = f^{(7)}(0) = 272 - 1\,680 + 1\,260 + 210 - 7 = 55$. The seventh derivative of the function $g(x)$ is somewhat more tractable than $f^{(7)}(x)$ in that

$$\begin{gathered}
g^{(7)}(x) = \frac{1}{32}\sec^{14}(x)((43\,856\cos(2x) - 43\,297\cos(4x) - 24\,672\cos(6x)+\\
6\,702\cos(8x) - 240\cos(10x) + \cos(12x) + 57\,106\cos(\tan(x))+\\
(24\,716\cos(2x) + 35\,793\cos(4x) + 3\,238\cos(6x) - 5\,890\cos(8x)+\\
366\cos(10x) - \cos(12x) - 1\,614)\tan(x)\cdot\sin(\tan(x))).
\end{gathered}$$

Therefore,

$$\lim_{x\to 0} g^{(7)}(x) = g^{(7)}(0) = \left(\frac{1}{32}\right)(43856 - 43297 - 24672 + 6702 - 240 + 1 + 57106) = 1233.$$

Thirdly, $\lim_{x\to 0} h^{(7)}(x) = h^{(7)}(0) = 7! = 5040$. Therefore,

$$\lim_{x\to 0}\frac{f^{(7)}(x)+g^{(7)}(x)}{h^{(7)}(x)} = \frac{f^{(7)}(0)+g^{(7)}(0)}{h^{(7)}(0)} = \frac{1233+55}{5040} = \frac{23}{90}.$$

Hence, by *L'Hôpital's* rule, $\lim_{x\to 0}\alpha(x) = \lim_{x\to 0}\dfrac{\tan(x\cdot\cos(x)) - \tan(x)\cdot\cos(\tan(x))}{x^7} = \dfrac{23}{90}$.

Solution 4 by Albert Natian, Los Angeles Valley College, Valley Glen, CA

Answer. $\dfrac{23}{90}$.

Justification. We will find

$$\lim_{x\to 0}\frac{Q(x)}{x^7}$$

where

$$Q(x) = \tan(x\cdot\cos(x)) - \tan(x)\cdot\cos(\tan(x)).$$

Introduce the notation to mean the following: Given two functions $F(x)$ and $G(x)$, we write $F(x)G(x)$ if there exists a power series $\sum_{k=8}^{\infty} a_k x^k$ such that $F(x) = G(x) + \sum_{k=8}^{\infty} a_k x^k$. Thus

$$\tan(x)\, x + bx^3 + cx^5 + dx^7$$

where

$$b = \frac{1}{3}, \quad c = \frac{2}{15}, \quad d = \frac{17}{315}$$

and

$$\cos(x)\, 1 + fx^2 + gx^4 + hx^6,$$

$$x \cdot \cos(x)\, x + fx^3 + gx^5 + hx^7$$

where

$$f = -\frac{1}{2}, \quad g = \frac{1}{24}, \quad h = -\frac{1}{720}.$$

By brute computation we reach at

$$\tan(x \cdot \cos(x))\, x + (f+b)\, x^3 + (g + 3bf + c)\, x^5 + \left(h + 3bf^2 + 3bg + 5cf + d\right) x^7$$

and

$$\cos(\tan(x))\, 1 + fx^2 + (2bf + g)\, x^4 + \left(fb^2 + 2fc + 4bg + h\right) x^6,$$

$$\tan(x) \cdot \cos(\tan(x))\, x + (f+b)\, x^3 + (3bf + g + c)\, x^5 + \left(3fb^2 + 3fc + 5bg + h + d\right) x^7.$$

So

$$Q(x) = \tan(x \cdot \cos(x)) - \tan(x) \cdot \cos(\tan(x))\, \left(2cf - 2bg + 3bf^2 - 3fb^2\right) x^7.$$

Now

$$Q(x) = \left(2cf - 2bg + 3bf^2 - 3fb^2\right) x^7 + x^8 \sum_{j=0}^{\infty} a_{j+8} x^j,$$

$$\frac{Q(x)}{x^7} = \left(2cf - 2bg + 3bf^2 - 3fb^2\right) + x \sum_{j=0}^{\infty} a_{j+8} x^j,$$

$$\lim_{x \to 0} \frac{Q(x)}{x^7} = 2cf - 2bg + 3bf^2 - 3fb^2 = \frac{23}{90}.$$

Also solved by Ulrich Abel, Technische Hochschule Mittelhessen, Germany; Michel Bataille, Rouen, France; Brian Bradie, Christopher Newport University, Newport News, VA; Kee-Wai Lau, Hong Kong, China; Moti Levy, Rehovot, Israel; Ioannis D. Sfikas, National and Kapodistrian University of Athens, Greece; Seán M. Stewart, Bomaderry, NSW, Australia; David Stone and John Hawkins, Georgia Southern University, Statesboro, GA, and the proposer.

- **5570:** *Proposed by Ovidiu Furdui and Alina Sîntămărian, Technical University of Cluj-Napoca, Cluj-Napoca, Romania*

Calculate

$$\int_0^1 \frac{(\ln(1-x) + x)^2}{x^3} \mathrm{d}x.$$

Solution 1 by Albert Natian, Los Angeles Valley College, Valley Glen, California

First note that,

$$-\int_0^1 \frac{\ln(1-x)}{x}\ dx = \int_0^1 \frac{1}{x}\sum_{k=1}^{\infty}\frac{1}{k}\,x^k\ dx = \sum_{k=1}^{\infty}\int_0^1 \frac{1}{k}\,x^{k-1}\ dx = \sum_{k=1}^{\infty}\left(\frac{1}{k^2}\,x^k\Big|_0^1\right) = \sum_{k=1}^{\infty}\frac{1}{k^2} = \frac{\pi^2}{6},$$

$$\lim_{b\to 1^-}(b-1)\left[\ln(1-b)\right]^2 = \lim_{b\to 1^-}(b-1)\ln(1-b) = 0, \qquad \lim_{a\to 0^+}\frac{\ln(1-a)}{a} = -1,$$

$$\left[\frac{x^2-1}{2}\cdot\left(\frac{\ln(1-x)}{x}\right)^2 + (x-1)\cdot\frac{\ln(1-x)}{x}\right]_0^1 =$$

$$= \lim_{a\to 0^+}\lim_{b\to 1^-}\left[\frac{x^2-1}{2}\cdot\left(\frac{\ln(1-x)}{x}\right)^2 + (x-1)\cdot\frac{\ln(1-x)}{x}\right]_a^b = -\frac{1}{2}.$$

Now,

$$\int_0^1 \frac{(\ln(1-x)+x)^2}{x^3}\ dx = \left[\frac{x^2-1}{2}\cdot\left(\frac{\ln(1-x)}{x}\right)^2 + (x-1)\cdot\frac{\ln(1-x)}{x}\right]_0^1 - \int_0^1 \frac{\ln(1-x)}{x}\ dx.$$

So,

$$\int_0^1 \frac{(\ln(1-x)+x)^2}{x^3}\ dx = \frac{\pi^2}{6} - \frac{1}{2}.$$

Solution 2 by Albert Stadler, Herrliberg, Switzerland

We integrate by parts and get

$$\int_0^1 \frac{(\ln(1-x)+x)^2}{x^2}dx =$$

$$\frac{1}{2}\left(1-\frac{1}{x^2}\right)(\ln(1-x)+x^2)^2\Big|_{x=0}^{x=1} - \int_0^1\left(1-\frac{1}{x^2}\right)(\ln(1-x)+x)\left(1-\frac{1}{1-x}\right)dx =$$

$$-\int_0^1\left(1+\frac{1}{x}\right)(\ln(1-x)+x)dx =$$

$$-\int_0^1 (x+1)dx - \int_0^1 \ln(1-x)dx - \int_0^1 \frac{\ln(1-x)}{x}dx =$$

$$-\frac{3}{2}+1+\sum_{k=1}^{\infty}\frac{1}{k}\int_0^1 x^{k-1}dx = -\frac{1}{2}+\sum_{k=1}^{\infty}\frac{1}{k^2} = -\frac{1}{2}+\frac{\pi^2}{6}.$$

The interchange of summation and integration is permitted, since all involved terms are positive.

Solution 3 by Brian Bradie, Christopher Newport University, Newport News VA

Starting from the series representation for $\ln(1-x)$,

$$\ln(1-x) = -\sum_{n=1}^{\infty}\frac{x^n}{n},$$

we find

$$\ln(1-x)+x=-\sum_{n=2}^{\infty}\frac{x^n}{n}$$

and

$$(\ln(1-x)+x)^2=\left(\sum_{n=2}^{\infty}\frac{x^n}{n}\right)^2=\sum_{n=4}^{\infty}\left(\sum_{j=2}^{n-2}\frac{1}{j(n-j)}\right)x^n.$$

Now,

$$\sum_{j=2}^{n-2}\frac{1}{j(n-j)}=\frac{1}{n}\sum_{j=2}^{n-2}\left(\frac{1}{j}+\frac{1}{n-j}\right)=\frac{2}{n}(H_{n-2}-1),$$

where H_n denotes the nth harmonic number. Next,

$$(\ln(1-x)+x)^2=2\sum_{n=4}^{\infty}\frac{H_{n-2}-1}{n}x^n=2\sum_{n=2}^{\infty}\frac{H_n-1}{n+2}x^{n+2},$$

so

$$\frac{(\ln(1-x)+x)^2}{x^3}=2\sum_{n=2}^{\infty}\frac{H_n-1}{n+2}x^{n-1}$$

and

$$\int_0^1\frac{(\ln(1-x)+x)^2}{x^3}\,dx=2\sum_{n=2}^{\infty}\frac{H_n-1}{n(n+2)}.$$

From here, we calculate

$$\begin{aligned}\sum_{n=2}^{\infty}\frac{H_n-1}{n(n+2)} &= \sum_{n=2}^{\infty}\sum_{j=2}^{n}\frac{1}{jn(n+2)}=\sum_{j=2}^{\infty}\frac{1}{j}\sum_{n=j}^{\infty}\frac{1}{n(n+2)}\\ &= \frac{1}{2}\sum_{j=2}^{\infty}\frac{1}{j}\sum_{n=j}^{\infty}\left(\frac{1}{n}-\frac{1}{n+2}\right)\\ &= \frac{1}{2}\sum_{j=2}^{\infty}\frac{1}{j}\left(\frac{1}{j}+\frac{1}{j+1}\right)=\frac{1}{2}\sum_{j=2}^{\infty}\frac{1}{j^2}+\frac{1}{2}\sum_{j=2}^{\infty}\frac{1}{j(j+1)}\\ &= \frac{1}{2}\left(\frac{\pi^2}{6}-1\right)+\frac{1}{2}\cdot\frac{1}{2}.\end{aligned}$$

Therefore,

$$\int_0^1\frac{(\ln(1-x)+x)^2}{x^3}\,dx=2\left[\frac{1}{2}\left(\frac{\pi^2}{6}-1\right)+\frac{1}{2}\cdot\frac{1}{2}\right]=\frac{\pi^2}{6}-\frac{1}{2}.$$

Solution 4 by Seán M. Stewart, Bomaderry, NSW, Australia

The value of the integral will be shown to be equal to $\dfrac{\pi^2-3}{6}$.

We will find a power series expansion for the term $(\log(1-x)+x)^2$ by finding the Cauchy product of the two identical series for $\log(1-x)+x$. As is well known

$$\log(1-x)=-\sum_{n=1}^{\infty}\frac{x^n}{n},\quad |x|\le 1, x\ne 1$$

Thus

$$\log(1-x)+x=-\sum_{n=2}^{\infty}\frac{x^n}{n}=-\sum_{n=1}^{\infty}\frac{x^{n+1}}{n+1},$$

after a shift in the index of $n\mapsto n+1$ has been made. So

$$\begin{aligned}(\log(1-x)+x)^2&=(\log(1-x)+x)\cdot(\log(1-x)+x)\\&=x\sum_{n=1}^{\infty}\frac{x^n}{n+1}\cdot x\sum_{n=1}^{\infty}\frac{x^n}{n+1}\\&=x^2\sum_{n=1}^{\infty}\sum_{k=1}^{n}\frac{1}{k+1}\cdot\frac{1}{n+2-k}x^{n+1}\\&=\sum_{n=1}^{\infty}\frac{x^{n+3}}{n+3}\sum_{k=1}^{n}\left(\frac{1}{k+1}+\frac{1}{n+2-k}\right).\end{aligned}$$

Reindexing the two sums appearing in the inner sum. In the first, let $k\mapsto k-1$ while in the second, let $k\mapsto n+2-k$. Thus

$$\begin{aligned}(\log(1-x)+x)^2&=\sum_{n=1}^{\infty}\frac{x^{n+3}}{n+3}\left(\sum_{k=2}^{n+1}\frac{1}{k}+\sum_{k=2}^{n+1}\frac{1}{k}\right)\\&=\sum_{n=1}^{\infty}\frac{2}{n+3}\left(\sum_{k=1}^{n+1}\frac{1}{k}-1\right)\\&=\sum_{n=1}^{\infty}\frac{2(H_{n+1}-1)}{n+3}x^{n+3}.\end{aligned}$$

Here $H_n=\sum_{k=1}^{n}\frac{1}{k}$ denotes the nth harmonic number.

Using this result for the Cauchy product in the integral, we have

$$\begin{aligned}I&=\int_0^1\frac{(\log(1-x)+x)^2}{x^3}\,dx\\&=\sum_{n=1}^{\infty}\frac{2(H_{n+1}-1)}{n+3}\int_0^1x^n\,dx\\&=\sum_{n=1}^{\infty}\frac{2(H_{n+1}-1)}{(n+1)(n+3)}.\end{aligned}$$

Note the interchange made in the order of the integration with the summation is permissible as a result of Tonelli's theorem. From the partial fraction decomposition of

$$\frac{2}{(n+1)(n+3)}=\frac{1}{n+1}-\frac{1}{n+3},$$

as the series for I converges absolutely it can be rewritten as

$$I=\sum_{n=1}^{\infty}\left(\frac{H_{n+1}}{n+1}-\frac{H_{n+1}}{n+3}\right)-\sum_{n=1}^{\infty}\left(\frac{1}{n+1}-\frac{1}{n+3}\right)=S_1-S_2.$$

For the first of the series, from the recurrence relation for the harmonic numbers, namely

$$H_{n+1} = H_n + \frac{1}{n+1},$$

we have

$$H_{n+1} = H_{n+3} - \frac{1}{n+2} - \frac{1}{n+3}.$$

Thus the summand in S_1 can be rewritten as

$$\begin{aligned}\frac{H_{n+1}}{n+1} - \frac{H_{n+1}}{n+3} &= \frac{H_{n+1}}{n+1} - \frac{H_{n+3}}{n+3} + \frac{1}{(n+2)(n+3)} + \frac{1}{(n+3)^2} \\ &= \left(\frac{H_{n+1}}{n+1} - \frac{H_{n+3}}{n+3}\right) + \left(\frac{1}{n+2} - \frac{1}{n+3}\right) + \frac{1}{(n+3)^2}.\end{aligned}$$

The series for S_1 can therefore be written as

$$S_1 = \sum_{n=1}^{\infty} \left(\frac{H_{n+1}}{n+1} - \frac{H_{n+3}}{n+3}\right) + \sum_{n=1}^{\infty} \left(\frac{1}{n+2} - \frac{1}{n+3}\right) + \sum_{n=1}^{\infty} \frac{1}{(n+3)^2}.$$

The first and second series telescope, while in the third, after reindexing $n \mapsto n-3$ we have

$$\begin{aligned}S_1 &= \frac{H_2}{2} + \frac{H_3}{3} + \frac{1}{3} + \sum_{n=4}^{\infty} \frac{1}{n^2} \\ &= \frac{1}{2}\left(1 + \frac{1}{2}\right) + \frac{1}{3}\left(1 + \frac{1}{2} + \frac{1}{3}\right) + \frac{1}{3} + \sum_{n=1}^{\infty} \frac{1}{n^2} - \left(1 + \frac{1}{2^2} + \frac{1}{3^2}\right) \\ &= \frac{49}{36} + \frac{1}{3} + \zeta(2) - \frac{49}{36} \\ &= \frac{\pi^2 + 2}{6}.\end{aligned}$$

Here the well-known value of $\frac{\pi^2}{6}$ for the Riemann zeta function $\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}$ at $s = 2$ has been used.

For the second series, as it telescopes we have

$$S_2 = \sum_{n=1}^{\infty} \left(\frac{1}{n+1} - \frac{1}{n+3}\right) = \frac{1}{2} + \frac{1}{3} = \frac{5}{6}.$$

Thus

$$I = \frac{\pi^2 + 2}{6} - \frac{5}{6} = \frac{\pi^2 - 3}{6},$$

as announced.

Solution 5 by Moti Levy, Rehovot, Israel

Path A: Using Integration by parts.

$\mathbf{Li}_2(x)$ denotes the polylogarithm function defined as

$$\mathbf{Li}_s(x) := \sum_{k=1}^{\infty} \frac{x^k}{k^s}.$$

For example, $\mathbf{Li}_2(1) = \sum_{k=1}^{\infty} \frac{1}{k^2} = \zeta(2)$.

$$\begin{aligned}
\int \frac{(\ln(1-x)+x)^2}{x^3} dx &= -\frac{(\ln(1-x)+x)^2}{2x^2} - \int \frac{x+\ln(1-x)}{x(1-x)} dx \\
&= -\frac{(\ln(1-x)+x)^2}{2x^2} - \int \frac{x+\ln(1-x)}{x} dx - \int \frac{x+\ln(1-x)}{1-x} dx \\
&= -\frac{(\ln(1-x)+x)^2}{2x^2} - x + \mathbf{Li}_2(x) + x + \ln(1-x) + \frac{1}{2}\ln^2(1-x) \\
&= -\frac{(\ln(1-x)+x)^2}{2x^2} + \ln(1-x) + \frac{1}{2}\ln^2(1-x) + \mathbf{Li}_2(x) \\
&= \ln(1-x)\left(1-\frac{1}{x}\right) + \frac{1}{2}\ln^2(1-x)\left(1-\frac{1}{x^2}\right) - \frac{1}{2} + \mathbf{Li}_2(x)
\end{aligned}$$

$$\lim_{x\to 1}\left((\ln(1-x))\left(1-\frac{1}{x}\right) + \frac{1}{2}\left(\ln^2(1-x)\right)\left(1-\frac{1}{x^2}\right)\right) = 0$$

$$\int_0^1 \frac{(\ln(1-x)+x)^2}{x^3} dx = -\frac{1}{2} + \mathbf{Li}_2(1) = -\frac{1}{2} + \zeta(2).$$

Path B: Using differentiation under the integral sign.
Define $I(a)$,

$$I(a) := \int_0^1 \frac{(\ln(1-ax)+ax)^2}{x^3} dx,$$

and note that $I(0) = 0$.

$$\begin{aligned}
\frac{\partial I(a)}{\partial a} &= \int_0^1 \frac{\partial\left(\frac{(\ln(1-ax)+ax)^2}{x^3}\right)}{\partial a} dx \\
&= -2a\int_0^1 \frac{\ln(1-ax)+ax}{x(1-ax)} dx \\
&= -2a\int_0^1 \left(\frac{\ln(1-ax)}{x} + a\right) dx - 2a^2 \int_0^1 \left(\frac{\ln(1-ax)}{1-ax} + \frac{ax}{1-ax}\right) dx \\
&= 2a\mathbf{Li}_2(a) - 2a^2 + a\ln(1-a) + 2a^2 + a\ln(1-a)(2+\ln(1-a)) \\
&= 2a\mathbf{Li}_2(a) + 2a\ln(1-a) + a\ln^2(1-a)
\end{aligned}$$

Since $I(0) = 0$, we have $I(1) = \int_0^1 \frac{\partial I(a)}{\partial a} da$.

$$I(1) = \int_0^1 \left(2a\mathbf{Li}_2(a) + 2a\ln(1-a) + a\ln^2(1-a)\right) da$$

Since

$$\int_0^1 2a\mathbf{Li}_2(a)\, da = \mathbf{Li}_2(1) + \int_0^1 a\ln(1-a)\, da,$$

we have

$$I(1) = \mathbf{Li}_2(1) + \int_0^1 \left(3a\ln(1-a) + a\ln^2(1-a)\right) da.$$

One can check that

$$\int \left(3a\ln(1-a) + a\ln^2(1-a)\right) da = a^2\ln(1-a) + \frac{1}{2}a^2\ln^2(1-a) - a\ln(1-a) - \frac{1}{2}\ln^2(1-a) - \frac{1}{2}a^2.$$

$$\lim_{a\to 1}\left(a^2\ln(1-a)+\frac{1}{2}a^2\ln^2(1-a)-a\ln(1-a)-\frac{1}{2}\ln^2(1-a)-\frac{1}{2}a^2\right)=-\frac{1}{2}.$$

It follows that

$$\int_0^1 \left(3a\ln(1-a)+a\ln^2(1-a)\right)da=-\frac{1}{2}.$$

We conclude that

$$I(1)=\mathbf{Li}_2(1)-\frac{1}{2}=\zeta(2)-\frac{1}{2}.$$

Solution 6 by Ángel Plaza, University of Las Palmas de Gran Canaria, Spain

Let $f(x)$ the function defined by $f(x)=\displaystyle\int_0^x \frac{(\ln(1-t)+t)^2}{t^3}\,dt$. Notice that $f(0)=0$ and $f'(x)=\dfrac{(\ln(1-x)+x)^2}{x^3}$. By using the Taylor expantion of $\ln(1-x)$ at $x=0$, then

$$\begin{aligned} f'(x) &= \frac{\left(\sum_{n=2}^{\infty}\frac{x^n}{n}\right)^2}{x^3} \\ &= \frac{\left(\sum_{n=2}^{\infty}\frac{x^{2n}}{n^2}\right)^2}{x^3}+\frac{2\left(\sum_{2\le i<j}^{\infty}\frac{x^{i+j}}{ij}\right)^2}{x^3}. \end{aligned}$$

Therefore, $f(x)=\displaystyle\sum_{n=2}^{\infty}\frac{x^{2n-2}}{(2n-2)n^2}+\sum_{2\le i<j}^{\infty}\frac{2x^{i+j-2}}{(i+j-2)ij}$.

Then, the proposed integral, say I, is $I=f(1)=\displaystyle\sum_{n=2}^{\infty}\frac{1}{(2n-2)n^2}+\sum_{2\le i<j}^{\infty}\frac{2}{(i+j-2)ij}=\sum_{2\le i,j}^{\infty}\frac{1}{(i+j-2)ij}=\frac{\pi^2}{6}-\frac{1}{2}$.

Also solved by Ulrich Abel, Technische Hochschule Mittelhessen, Germany; Michel Bataille, Rouen, France; Pat Costello, Eastern Kentucky University, Richmond, KY; Kee-Wai Lau, Hong Kong, China; Ioannis D. Sfikas, National and Kapodistrian University of Athens, Greece, and the proposers.

Mea Culpa

There are two mathematicians with the name of Stanley Rabinowitz, and each is an excellent problem solver and also a contributor to this column. I inadvertently attributed the solution of one to the other. Sorry.

The citation for Solution 1 to problem 5501 should have been attributed to **Stanley Rabinowitz of Brooklyn, NY,** and not as it is listed in the issue. Mea culpa.