Problems

Ted Eisenberg, Section Editor

****

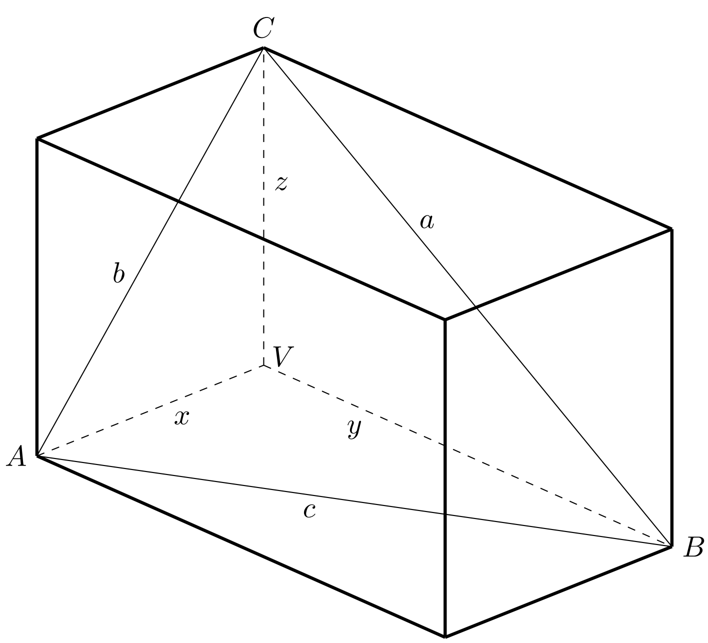
This section of the Journal offers readers an opportunity to exchange interesting mathematical problems and solutions. Please send them to Ted Eisenberg, Department of Mathematics, Ben-Gurion University, Beer-Sheva, Israel or fax to: 972-86-477-648. Questions concerning proposals and/or solutions can be sent e-mail to <eisenbt@013.net>. Solutions to previously stated problems can be seen at <http://www.ssma.org/publications>.

Solutions to the problems stated in this issue should be posted before September 15, 2020

- **5595:** *Proposed by Kenneth Korbin, New York, NY*

  Trapezoid $ABCD$ with integer length sides is inscribed in a circle with diameter 23^3. Side $\overline{AB} = 4439$. Find the other three sides.

- **5596:** *Proposed by Albert Natian, Los Angeles Valley College, Valley Glen, CA*



Let V be a vertex of a rectangular box. Let $\overline{VA}$, $\overline{VB}$ and $\overline{VC}$ be the three edges meeting at vertex V. Suppose the area of the triangle ABC is $6\sqrt{26}$. The volume of the box is 144. And the sum of the edges of the box is 76. Find the total surface area of the box.

- **5597:** *Proposed by Daniel Sitaru, National Economic College "Theodor Costescu," Mehedinti, Romania*

  If $x, y, z > 0; xyz = 1$ then:

  $$\left(x+y-\frac{1}{\sqrt{z}}\right)^2+\left(y+z-\frac{1}{\sqrt{x}}\right)^2+\left(z+x-\frac{1}{\sqrt{y}}\right)^2 \geqslant 3$$

- **5598:** *Proposed by José Luis Díaz-Barrero, Barcelona Tech, Barcelona, Spain*

  Let $A(x)$ be a polynomial of degree n such that $A(i) = 3^i$ for $0 \leqslant i \leqslant n$. Find the value of $A(n+1)$.

- **5599:** *Proposed by Ovidiu Furdui and Alina Sîntămărian, Technical University of Cluj-Napoca, Cluj-Napoca, Romania*

  Let $n \geqslant 2$ be an integer. Calculate

  $$\int_0^{\frac{\pi}{2}} \frac{\sin x + \cos x}{\sin^{2n-1} x + \cos^{2n-1} x} \mathrm{d}x.$$

- **5600:** *Proposed by Seán M. Stewart, Bomaderry, NSW, Australia*

  Evaluate:

  $$\int_0^{\pi} \log\left(1+2a\cos x+a^2\right)\log\left(1+2b\cos x+b^2\right)dx,$$

  if $a, b \in \Re$ are such that the product ab with $|a|, |b| < 1$ satisfies the equation $a^2b^2 + ab = 1$.

Solutions

5577: *Proposed by Kenneth Korbin, New York, NY*

Convex quadrilateral $ABCD$ with integer length sides is inscribed in a circle with diameter $\overline{AD} = 625$. Find the perimeter if $\left(\overline{AB}, \overline{BC}, \overline{CD}\right) = 1$.

Solution 1 by Albert Stadler, Herrliberg, Switzerland

Let M be the centre of the circle. The triangles AMB, BMC, CMD are isosceles and their vertex angles add up to π. Therefore

$$\arcsin\frac{\overline{AB}}{625} + \arcsin\frac{\overline{BC}}{625} + \arcsin\frac{\overline{CD}}{625} = \frac{\pi}{2}.$$

We take the sines of both sides of the equation and use the identity

$$\sin(a+b+c) = \cos a \cos b \sin c + \cos b \cos c \sin a + \cos c \cos a \sin b - \sin a \sin b \sin c$$

to get

$$\sqrt{1-\left(\frac{\overline{AB}}{625}\right)^2}\sqrt{1-\left(\frac{\overline{BC}}{625}\right)^2}\frac{\overline{CD}}{625}+\sqrt{1-\left(\frac{\overline{CD}}{625}\right)^2}\sqrt{1-\left(\frac{\overline{AB}}{625}\right)^2}\frac{\overline{BC}}{625}$$

$$+\sqrt{1-\left(\frac{\overline{BC}}{625}\right)^2}\sqrt{1-\left(\frac{\overline{CD}}{625}\right)^2}\frac{\overline{AB}}{625}-\frac{\overline{AB}}{625}\frac{\overline{BC}}{625}\frac{\overline{CD}}{625}=1.$$

An exhaustive computer search gives the following 60 feasible triples $\left(\overline{AB},\overline{BC},\overline{CD}\right)$.

$(0,336,527)$	$(0,527,336)$	$(25,25,623)$	$(25,623,25)$	$(47,425,425)$	$(50,50,617)$
$(50,617,50)$	$(75,75,607)$	$(75,607,75)$	$(100,100,593)$	$(100,593,100)$	$(113,400,400)$
$(125,355,433)$	$(125,433,355)$	$(150,150,553)$	$(150,553,150)$	$(175,175,527)$	$(175,527,175)$
$(200,200,497)$	$(200,497,200)$	$(220,336,375)$	$(220,375,336)$	$(225,225,463)$	$(225,463,225)$
$(233,350,350)$	$(275,275,383)$	$(275,383,275)$	$(287,325,325)$	$(300,300,337)$	$(300,337,300)$
$(325,287,325)$	$(325,325,287)$	$(336,0,527)$	$(336,220,375)$	$(336,375,220)$	$(336,527,0)$
$(337,300,300)$	$(350,233,350)$	$(350,350,233)$	$(355,125,433)$	$(355,433,125)$	$(375,220,336)$
$(375,336,220)$	$(383,275,275)$	$(400,113,400)$	$(400,400,113)$	$(425,47,425)$	$(425,425,47)$
$(433,125,355)$	$(433,355,125)$	$(463,225,225)$	$(497,200,200)$	$(527,0,336)$	$(527,175,175)$
$(527,336,0)$	$(553,150,150)$	$(593,100,100)$	$(607,75,75)$	$(617,50,50)$	$(623,25,25)$.

The perimeter is the sum $\overline{AB}+\overline{BC}+\overline{CD}+\overline{AD}$. The only values the perimeter can assume are

1298, 1342, 1382, 1418, 1478, 1488, 1502, 1522, 1538, 1556 1558, 1562.

Comments by David Stone and John Hawkins, Georgia Southern University, Statesboro, GA. In their solution they coordinatized the quadrilateral by placing it into a semicircle with the longest side of the quadrilateral being the diameter of the circle. Letting the vertices of the quadrilateral be $A(-r,0)$, $B(b_1,b_2)$, $C(c_1,c_2)$, and $D(r,0)$ they obtained the lengths of the sides with $\overline{AB}=a,\overline{BC}=b,\overline{CD}=c$ and $\overline{DA}=d=625$. They restricted the size of a so that B fell into the second quadrant and they supposed that $a\leqslant c$. That is, $a\leqslant\sqrt{2}r=441.9$, with $1\leqslant a\leqslant 441$ and $a\leqslant c\leqslant\overline{BD}=\sqrt{4r^2-a^2}$. They claimed that given such a solution they could always reflect the digram across the $y-$axis to force $a\geqslant c$.

With the aide of a computer they showed that minimum perimeter was 1298 while the maximum perimeter was 1562 and concluded their solution with the following comments:

1) Perhaps the nicest quadrilateral which can be inscribed in our semicircle is to let $a = b = c = r = 312.5$. That is, the quadrilateral is one-half of a regular hexagon. We're not allowing this one, because it doesn't have integral sides. Note that its perimeter, 1562.5, would be an upper bound for all of the inscribed quadrilaterals. In fact, two of ours, $(300, 337, 300, 625)$ and $(325, 287, 325, 625)$, have perimeter 1562, the maximum possible using integer sides.

2) In this comment they showed how they obtained degenerate quadrilaterals, that is, Pythagorean triangles inscribed in the semicircle: they did this by letting $b = 0$, the length of $\overline{BC}$. This produced four degenerate quadrilaterals. (Using the lengths as defined above) they found them to be:

a	c	d	P
175	600	625	1400
220	585	625	1430
336	527	625	1488
375	500	625	1500

3) In this comment they wrote: Surprisingly, we also found values for a, b and c which form self-intersecting inscribed quadrilaterals with integral side lengths. Of course, they don't satisfy the requirements of our problem (and aren't usually even considered when discussing quadrilaterals). For instance, with $AB = 355, CD = 575$, we see that BD crosses AB and $CB = 125$ (vertex C actually lies between A and B on the semicircle).

Observations by Ken Korbin, proposer of the problem on Stadler's solution:

- It is interesting to see that 3 different quadrilaterals have perimeter 1538.
- Each of the quadrilaterals has just one side not a multiple of 5.
- Each of the trapezoids has 3 sides that are multiples of 25.

Also solved by Kee-Wai Lau, Hong Kong, China; David E. Manes, Oneonta, NY; Ioannis D. Sfikas, National and Kapodistrian University of Athens, Greece; David Stone and John Hawkins, Georgia Southern University, Statesboro, GA, and the proposer.

5578: *Proposed by Roger Izard, Dallas, TX*

In triangle ABC points F, E, and D lie on lines segments AB, BC, and AC respectively, such that $\dfrac{\overline{AF}}{\overline{BA}} = \dfrac{\overline{BE}}{\overline{BC}} = \dfrac{\overline{DC}}{\overline{AC}}$ and $\angle BAE = \angle CBD = \angle ACF$. Prove or disprove: Triangle ABC must be an equilateral triangle.

Solution 1 by Kee-Wai Lau, Hong Kong China

We prove that triangle ABC must be an equilateral triangle.

As usual, let $\overline{AB} = c, \overline{BC} = a, \overline{CA} = b$, and let R be the circumradius.

Suppose that $\dfrac{\overline{AF}}{\overline{BA}} = \dfrac{\overline{BE}}{\overline{BC}} = \dfrac{\overline{DC}}{\overline{AC}} = k$ and $\angle BAE = \angle CBD = \angle ACF = \theta$,

where $0 < k < 1$ and $0 < \theta < \dfrac{\pi}{3}$. Applying the sine formula to triangle BAE, we have

$$\frac{\sin\theta}{ka} = \frac{\sin(B+\theta)}{c} = \frac{\sin B\cos\theta + \cos B\sin\theta}{c},$$

so that $\cot\theta = \dfrac{c - ka\cos B}{ka\sin B} = \dfrac{\left((2-k)c^2 - ka^2 + kb^2\right)R}{kabc}$, by the sine and cosine formulas.

Similarly, by considering triangles CBD and ACF, we obtain respectively

$\cot\theta = \dfrac{\left((2-k)a^2 - kb^2 + kc^2\right)R}{kabc}$ and $\cot\theta = \dfrac{\left((2-k)b^2 - kc^2 + ka^2\right)R}{kabc}$. Hence we have

$$(2-k)c^2 - ka^2 + kb^2 = (2-k)a^2 - kb^2 + kc^2, \tag{1}$$

and

$$(2-k)c^2 - ka^2 + kb^2 = (2-k)b^2 - kc^2 + ka^2. \tag{2}$$

Simplifying (1), we have

$$(k-1)c^2 = kb^2 - a^2, \tag{3}$$

and simplifying (2), we have

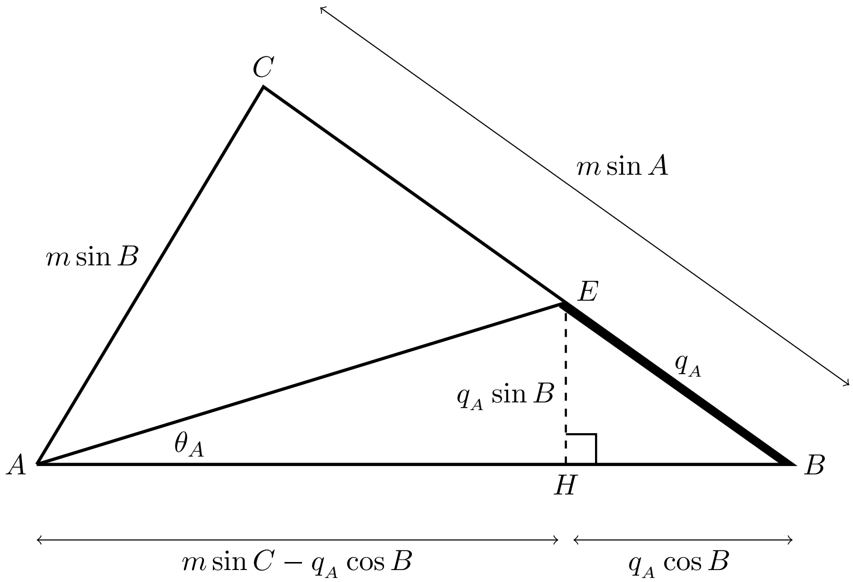
$$c^2 = ka^2 + (1-k)b^2. \tag{4}$$

By substituting c^2 of (4) into (3), we obtain $(k^2-k+1)(a+b)(a-b) = 0$.

Since $k^2 - k + 1 > 0$, so $a = b$. By putting $a = b$ into (3), we obtain $a = c$. Thus ABC is indeed an equilateral triangle.

Solution 2 Albert Natian, Los Angeles Valley College, Valley Glen, CA

Claim: Yes, triangle ABC must be an equilateral triangle.



Proof.

Assume the hypotheses in the statement of the problem. The sides of triangle ABC can be expressed as $m \sin A$, $m \sin B$ and $m \sin C$ for some positive number m, as shown. Let q_A denote length of line segment BE and let θ_A denote the measure of angle $\angle BAE$. Draw EH perpendicular to side AB. Since $\angle A + \angle B + \angle C = \pi$, then $\angle C = \pi - (\angle A + \angle B)$ and so

$$\sin C = \sin\left(\pi - (\angle A + \angle B)\right) = \sin\left(\angle A + \angle B\right) = \sin A \cos B + \sin B \cos A.$$

Let ρ_A denote the ratio of $BE = q_A$ to $BC = m \sin A$; i.e., $\rho_A = q_A / (m \sin A)$. So

$$q_A = m\rho_A \sin A.$$

From the above figure and results we have

$$\cot \theta_A = \frac{m \sin C - q_A \cos B}{q_A \sin B} = \frac{m\left(\sin A \cos B + \sin B \cos A\right) - m\rho_A \sin A \cos B}{m\rho_A \sin A \sin B},$$

$$\cot \theta_A = \frac{1}{\rho_A}\left[(1-\rho_A)\cot B + \cot A\right],$$

$$\rho_A \cot \theta_A = (1-\rho_A)\cot B + \cot A.$$

Similar discussion with respect to vertices B and C will produce the results

$$\rho_B \cot \theta_B = (1-\rho_B)\cot C + \cot B,$$

$$\rho_C \cot \theta_C = (1-\rho_C)\cot A + \cot C$$

where $\theta_B = \angle CBD$, $\theta_C = \angle ACF$, $\rho_B = \overline{DC}/\overline{AC}$ and $\rho_C = \overline{AF}/\overline{BA}$. Since $\overline{AF}/\overline{BA} = \overline{BE}/\overline{BC} = \overline{DC}/\overline{AC}$ and $\angle BAE = \angle CBD = \angle ACF$, then $\rho := \rho_A = \rho_B = \rho_C$ and $\theta := \theta_A = \theta_B = \theta_C$. So

$$\rho_A \cot \theta_A = \rho_B \cot \theta_B = \rho_C \cot \theta_C = \rho \cot \theta,$$

$$\rho \cot \theta = (1-\rho)\cot B + \cot A = (1-\rho)\cot C + \cot B = (1-\rho)\cot A + \cot C$$

which implies

$$\cot A = \cot B = \cot C = \frac{\rho}{2-\rho}\cot\theta$$

and since $\angle A + \angle B + \angle C = \pi$ and two or more of the angles $\angle A$, $\angle B$, $\angle C$ are acute, then

$$\cot A = \cot B = \cot C = \frac{1}{\sqrt{3}} \quad \text{and} \quad \angle BAE = \angle CBD = \angle ACF = \pi/3.$$

Thus, triangle ABC is equilateral.

Also solved by Albert Stadler, Herrliberg, Switzerland, and the proposer.

5579: *Proposed by Daniel Sitaru, National Economic College "Theodor Costescu", Drobeta Turnu-Severin, Romania*

Prove: If $a, b \in \Re, a \leqslant b$, then $\log 5 \cdot \int_a^b 5^{x^2} dx + \log 5 \cdot \int_a^b 5x^4 dx \geqslant 5^b - 5^a$.

Solution 1 by Albert Stadler, Herrliberg, Switzerland

It is enough to prove that

$$5^{x^2} + 5^{x^4} \geqslant 5^x \tag{*}$$

for all real values of$x$ because then

$$\log 5 \int_a^b 5^{x^2} dx + \log 5 \int_a^b 5^{x^4} dx \geqslant \log 5 \int_a^b 5^x dx = 5^b - 5^a.$$

$(*)$ holds true for $x \leqslant 0$, because $5^{x^2} + 5^{x^4} \geqslant 2 > 1 \geqslant 5^x$, if $x \leqslant 0$.

$(*)$ holds true for $x \geqslant 1$, because $5^{x^2} + 5^{x^4} > 5^{x^2} \geqslant 5^x$, if $x \geqslant 1$.

Let $0 < x < 1$. Then, by the AM-GM inequality,

$$5^{x^2} + 5^{x^4} = \sum_{k=0}^{\infty} \frac{1}{k!}\left(x^{2k} + x^{4k}\right) \geqslant 2\sum_{k=0}^{\infty} \frac{1}{k!} x^{3k} = 2 \cdot 5^{x^3}$$

and it remains to prove that

$$f(x) := \log 2 + x^3 \log 5 - x \log 5 \geqslant 0$$

for $0 < x < 1$, for then $2 \cdot 5^{x^3} \geqslant 5^x$. $f(x)$ assumes a local minimum at $x = \dfrac{1}{\sqrt{3}}$ and

$$f\left(\frac{1}{\sqrt{3}}\right) = \log(2) - \frac{2}{3\sqrt{3}} \log 5 > 0,$$

which concludes the proof.

Solution 2 by Brian Bradie, Christopher Newport University, Newport News, VA

Because

$$\int_a^b 5^x \, dx = \frac{5^b - 5^a}{\log 5},$$

the desired inequality is equivalent to

$$\int_a^b (5^{x^2} + 5^{x^4} - 5^x) \, dx \geqslant 0.$$

If $a = b$, then

$$\int_a^b (5^{x^2} + 5^{x^4} - 5^x) \, dx = 0.$$

Next, suppose $a < b$, and consider the function

$$f(x) = \frac{1}{2}x^4 + \frac{1}{2}x^2 - x + \log_5 2.$$

Then $f'(x) = 2x^3 + x - 1$. An examination of the graphs of $y = 2x^3$ and $y = 1 - x$ reveals there exists a unique real number, say c, for which $2c^3 = 1 - c$; that is, there exists a unique real number c for which $f'(c) = 0$. Moreover, for $x < c$, $f'(x) < 0$, and for $x > c$, $f'(x) > 0$, so f achieves an absolute minimum value at $x = c$. Now,

$$\begin{aligned} f(c) &= \frac{1}{2}c^4 + \frac{1}{2}c^2 - c + \log_5 2 = \frac{1}{4}(2c^4 + c^2 - c) + \frac{1}{4}c^2 - \frac{3}{4}c + \log_5 2 \\ &= \frac{1}{4}c(2c^3 + c - 1) + \frac{1}{4}\left(c - \frac{3}{2}\right)^2 - \frac{9}{16} + \log_5 2 \\ &= \frac{1}{4}\left(c - \frac{3}{2}\right)^2 - \frac{9}{16} + \log_5 2. \end{aligned}$$

With

$$f'\left(\frac{1}{2}\right) = -\frac{1}{4} < 0 \quad \text{and} \quad f'\left(\frac{3}{5}\right) = \frac{4}{125} > 0,$$

it follows that $c < 3/5$ and

$$f(c) > \frac{1}{4}\left(\frac{3}{5} - \frac{3}{2}\right)^2 - \frac{9}{16} + \log_5 2 = -\frac{9}{25} + \log_5 2 > 0.$$

Thus,

$$\frac{1}{2}x^4 + \frac{1}{2}x^2 - x + \log_5 2 > 0 \quad \text{or} \quad x < \frac{1}{2}x^4 + \frac{1}{2}x^2 + \log_5 2$$

for all x. Because 5^x is an increasing function, it then follows that

$$5^x < 5^{\frac{1}{2}x^4 + \frac{1}{2}x^2 + \log_5 2} = 2 \cdot 5^{\frac{1}{2}x^4 + \frac{1}{2}x^2} = 2\sqrt{5^{x^4 + x^2}} \leqslant 5^{x^4} + 5^{x^2},$$

where the final inequality arises from the arithmetic mean - geometric mean inequality. Finally,

$$5^{x^2} + 5^{x^4} - 5^x > 0$$

for all x, so

$$\int_a^b (5^{x^2} + 5^{x^4} - 5^x) \, dx > 0$$

whenever $a < b$. In summary,

$$\log 5 \cdot \int_a^b 5^{x^2}\,dx + \log 5 \cdot \int_a^b 5^{x^4}\,dx \geqslant 5^b - 5^a,$$

with equality holding if and only if $a = b$.

Solution 3 by Seán M. Stewart, Bomaderry, NSW, Australia

Consider the function $g(x) = 5^{x^2-x} + 5^{x^4-x} - 1$. Differentiating with respect to x we have

$$g'(x) = \log 5 \cdot 5^{-x}\left(5^{x^2}(2x-1) + 5^{x^4}(4x^3-1)\right).$$

Stationary points occur when $g'(x) = 0$. Since $5^{-x} > 0$ for all x we have

$$5^{x^2}(2x-1) + 5^{x^4}(4x^3-1) = 0.$$

Solving this equation numerically, we find a single stationary point occurs when $x = x^* = 0.578\,632\,089\ldots$. At this stationary point we see that $g(x^*) = 0.147\,392\,262\ldots > 0$. Differentiating again we find

$$\begin{aligned} g''(x) = {} & 2\log 5 \cdot 5^{x^2-x} + 12\log 5 \cdot 5^{x^4-x}x^2 + \log^2 5 \cdot 5^{x^2-x}(2x-1)^2 \\ & + \log^2 5 \cdot 5^{x^4-x}(4x^3-1)^2, \end{aligned}$$

and is clearly positive for all $x \in \mathbb{R}$. Since $g''(x) > 0$, g is concave up with $x = x^*$ being a global minimum point. Since $g(x^*) > 0$, then $g(x) > 0$ for all $x \in \mathbb{R}$, or

$$5^{x^2} + 5^{x^4} > 5^x, \tag{1}$$

for all $x \in \mathbb{R}$ since $5^{-x} > 0$.
Now

$$\begin{aligned} \log 5 \cdot \int_a^b 5^{x^2}\,dx + \log 5 \cdot \int_a^b 5^{x^4}\,dx &= \log 5 \cdot \int_a^b \left(5^{x^2} + 5^{x^4}\right)\,dx \\ &> \int_a^b \log 5 \cdot 5^x\,dx \\ &= 5^x\Big|_a^b = 5^b - 5^a, \end{aligned}$$

were we have made use of the inequaility given in2 (1). Noting that equality can only occur when $a = b$, we can write

$$\log 5 \cdot \int_a^b 5^{x^2}\,dx + \log 5 \cdot \int_a^b 5^{x^4}\,dx \geqslant 5^b - 5^a,$$

for $a, b \in \mathbb{R}, a \leqslant b$, as required to prove.

Solution 4 by David Stone and John Hawkins, Georgia Southern University, Statesboro, GA

Because $\displaystyle\int_a^b 5^x dx = \frac{5^b - 5^a}{\log 5}$, the given inequality is equivalent to

$\log 5 \cdot \displaystyle\int_a^b 5^{x^2} dx + \log 5 \cdot \int_a^b 5^{x^4} dx \geqslant \log 5 \cdot \int_a^b 5^x dx$

which is equivalent to $\displaystyle\int_a^b \left(5^{x^4} + 5^{x^2} - 5^x\right) dx \geqslant 0.$

This is true because

Claim: $5^{x^4} + 5^{x^2} - 5^x \geqslant 0$ for all real x.

In this modern electronic age, we could graph $f(x) = 5^{x^4} + 5^{x^2} - 5^x$ and readily accept the truth of this claim. In fact, the TI-83 graphing calculator shows that $f(x)$ achieves an absolute minimum of .360 at $x = .5307$.

We also present an analytic proof of the claim.

If $x \geqslant 1$, then we can consider a piece of $f(x) : f(x) \geqslant 5^{x^2} - 5^x \geqslant 0$, so $f(x) \geqslant 0$.

Consider the derivative:

$f'(x) = 5^{x^4} \log 5 \cdot 4x^3 + 5^{x^2} \log 5 \cdot 2x - 5^x \log 5 = \log 5 \cdot 5^x \left\{4x^3 \cdot 5^{x^4 - x} + 2x \cdot 5^{x^2 - x} - 1\right\}.$

If $x \leqslant 0$, then each term inside the brackets is negative, so $f'(x) \leqslant 0$.

Thus $f(x)$ is decreasing to $f(0) = 1$, so $f(x) > 0$.

Now we complete our argument by showing $5^{x^4} + 5^{x^2} \geqslant 5^x$ on $[0, 1]$ by looking carefully at the behavior on small subintervals.

For convenience, let $g(x) = 5^{x^4} + 5^{x^2}$ and $h(x) = 5^x$.

We have $g'(x) = 5^{x^4} \log 5 \cdot 4x^3 + 5^{x^2} \log 5 \cdot 2x = 2x \log 5 \left\{2x^2 \cdot 5^{x^4} + 2x \cdot 5^{x^2}\right\}.$

Because $g'(x) < 0$ for $x < 0$ and $g'(x) > 0$ for $x > 0$, we see that $g(0) = 2$ is an absolute minimum.

We know that $g(x)$ and $h(x)$ are both increasing on $[0, 1]$ and $h(0) = 1 < g(0) = 2$.

We show that $h(x)$ lies below $g(x)$ on the entire interval by showing it true on six (exhaustive) subintervals. The intervals are chosen so that, on each one, $h(x)$ increases from its left-hand height to (almost) the left-hand height of $g(x)$, thus lying below $g(x)$ throughout the subinterval. One can imagine stair-steps, with the graph of $g(x)$ lying (on or) above each step and the graph of $h(x)$ lying below the step.

On the subinterval $[0, .43], h(x) \leqslant h(.43) = 5.43 = 1.998 < 2 = g(0) \leqslant g(x)$.

On the subinterval $[.43, .54], h(x) \leqslant h(.54) = 5.54 = 2.385 < 2.403 = g(.43) \leqslant g(x)$.

On the subinterval $[.54, .627], h(x) \leqslant h(.627) = 5.627 = 2.743 < 2.7455 = g(.54) \leqslant g(x)$.

On the subinterval $[.627, .715], h(x) \leqslant h(.715) = 5.715 = 3.1606 < 3.165 = g(.627) \leqslant g(x)$.

On the subinterval $[.715, .826], h(x) \leqslant h(.826) = 5.826 = 3.779 < 3.8 = g(.715) \leqslant g(x)$.

On the subinterval $[.826, 1], h(x)(1) = 51 = 5 < 5.114 = g(.826) \leqslant g(x)$.

Also solved by Michel Bataille, Rouen, France; Kee-Wai Lau, Hong Kong, China; Moti Levy, Rehovot, Israel, and the proposers.

5580: *Proposed by D.M. Bătinetu-Giurgiu "Matei Basarab" National College, Bucharest, Romania and Neculai Stanciu, "George Emil Palade" School, Buzău, Romania*

Compute: $\lim_{n\to\infty} \dfrac{1}{\left(\sqrt[n]{(2n-1)!!}\right)^2} \sum_{k=1}^{n} \left[\left(\sqrt[2k]{k!} + \sqrt[2(k+1)]{(k+1)!}\right)^2\right]$ where $[x]$ denotes the integer part of x.

Solution 1 by Brian Bradie, Christopher Newport University, Newport News, VA

By the Stolz-Cesaro theorem

$$\begin{aligned}
&\lim_{n\to\infty} \frac{1}{\left(\sqrt[n]{(2n-1)!!}\right)^2} \sum_{k=1}^{n} \left[\left(\sqrt[2k]{k!} + \sqrt[2(k+1)]{(k+1)!}\right)^2\right] \\
&= \lim_{n\to\infty} \frac{\sum_{k=1}^{n+1} \left[\left(\sqrt[2k]{k!} + \sqrt[2(k+1)]{(k+1)!}\right)^2\right] - \sum_{k=1}^{n} \left[\left(\sqrt[2k]{k!} + \sqrt[2(k+1)]{(k+1)!}\right)^2\right]}{\left(\sqrt[n+1]{(2n+1)!!}\right)^2 - \left(\sqrt[n]{(2n-1)!!}\right)^2} \\
&= \lim_{n\to\infty} \frac{\left[\left(\sqrt[2(n+1)]{(n+1)!} + \sqrt[2(n+2)]{(n+2)!}\right)^2\right]}{\left(\sqrt[n+1]{(2n+1)!!}\right)^2 - \left(\sqrt[n]{(2n-1)!!}\right)^2}.
\end{aligned}$$

Next, by Stirling's approximation, $n! \sim n^n/e^n$, so

$$\sqrt[2(n+1)]{(n+1)!} \sim \sqrt{\frac{n+1}{e}}, \qquad \sqrt[2(n+2)]{(n+2)!} \sim \sqrt{\frac{n+2}{e}},$$

and

$$\left(\sqrt[2(n+1)]{(n+1)!} + \sqrt[2(n+2)]{(n+2)!}\right)^2 \sim \frac{4n}{e}.$$

Moreover,

$$\sqrt[n]{(2n-1)!!} = \sqrt[n]{\frac{(2n)!}{2^n n!}} \sim \frac{2n}{e}, \qquad \sqrt[n+1]{(2n+1)!!} \sim \frac{2(n+1)}{e},$$

and

$$\left(\sqrt[n+1]{(2n+1)!!}\right)^2 - \left(\sqrt[n]{(2n-1)!!}\right)^2 \sim \left(\frac{2(n+1)}{e}\right)^2 - \left(\frac{2n}{e}\right)^2 \sim \frac{8n}{e^2}.$$

Finally,

$$\begin{aligned}
\lim_{n\to\infty} \frac{1}{\left(\sqrt[n]{(2n-1)!!}\right)^2} \sum_{k=1}^{n} \left[\left(\sqrt[2k]{k!} + \sqrt[2(k+1)]{(k+1)!}\right)^2\right] &= \frac{e^2}{8} \lim_{n\to\infty} \frac{[4n/e]}{n} \\
&= \frac{e^2}{8} \cdot \frac{4}{e} = \frac{e}{2}.
\end{aligned}$$

Solution 2 by Moti Levy, Rehovot, Israel

Applying the Stirling's approximation $n! \sim \sqrt{2\pi n}\left(\dfrac{n}{e}\right)^n$, we obtain (1) and (2) :

$$(2n-1)!! = \frac{(2n)!}{2^n n!} \sim \frac{\sqrt{4\pi n}\left(\frac{2n}{e}\right)^{2n}}{2^n\sqrt{2\pi n}\left(\frac{n}{e}\right)^n} = 2^{n+\frac{1}{2}} n^n e^{-n},$$

$$\sqrt[n]{(2n-1)!!} \sim \frac{2n}{e}, \tag{1}$$

$$\frac{1}{\left(\sqrt[n]{(2n-1)!!}\right)^2} \sim \frac{e^2}{4n^2}. \tag{2}$$

Firstly, we get rid of the bracket by showing that

$$\lim_{n\to\infty} \frac{1}{\left(\sqrt[n]{(2n-1)!!}\right)^2} \sum_{k=1}^{n} \left[\left(\sqrt[2k]{k!} + \sqrt[2(k+1)]{(k+1)!}\right)^2\right]$$
$$= \lim_{n\to\infty} \frac{1}{\left(\sqrt[n]{(2n-1)!!}\right)^2} \sum_{k=1}^{n} \left(\sqrt[2k]{k!} + \sqrt[2(k+1)]{(k+1)!}\right)^2.$$

This follows from

$$\frac{1}{\left(\sqrt[n]{(2n-1)!!}\right)^2} \left| \sum_{k=1}^{n} \left[\left(\sqrt[2k]{k!} + \sqrt[2(k+1)]{(k+1)!}\right)^2\right] - \sum_{k=1}^{n} \left(\sqrt[2k]{k!} + \sqrt[2(k+1)]{(k+1)!}\right)^2 \right|$$

$$\leqslant \frac{1}{\left(\sqrt[n]{(2n-1)!!}\right)^2} n \sim \frac{e^2}{4n} \to 0 \quad as \quad n \to \infty.$$

Again, applying the Stirling's approximation, we get

$$\sqrt[2k]{k!} \sim \sqrt{\frac{k}{e}}. \tag{3}$$

Let $b_k := \frac{1}{k}\left(\sqrt[2k]{k!} + \sqrt[2(k+1)]{(k+1)!}\right)^2$ then by (4) we have

$$\lim_{k\to\infty} b_k = \frac{4}{e}. \tag{4}$$

Lemma: Let (b_k) be a sequence of positive numbers, such that $\lim_{k\to\infty} b_k = B$, then $\lim_{n\to\infty} \frac{1}{n^2} \sum_{k=1}^{n} k b_k = \frac{B}{2}$.

Proof: $\lim_{k\to\infty} b_k = B$ implies that for every $\varepsilon > 0$ there exists N such that $|b_k - B| < \varepsilon$ for $k > N$.

$$\frac{1}{n^2}\sum_{k=1}^{n} k b_k = \frac{1}{n^2}\sum_{k=1}^{N} k b_k + \frac{1}{n^2}\sum_{k=N+1}^{n} k b_k \geqslant \frac{1}{n^2}\sum_{k=1}^{N} k b_k + \frac{1}{n^2}\sum_{k=N+1}^{n} k\left(B+\varepsilon\right)$$

$$\frac{1}{n^2}\sum_{k=1}^{n} k b_k = \frac{1}{n^2}\sum_{k=1}^{N} k b_k + \frac{1}{n^2}\sum_{k=N+1}^{n} k b_k \leqslant \frac{1}{n^2}\sum_{k=1}^{N} k b_k + \frac{1}{n^2}\sum_{k=N+1}^{n} k\left(B-\varepsilon\right)$$

$$\frac{1}{n^2}\sum_{k=N+1}^{n} k\,(B-\varepsilon) \leqslant \lim_{n\to\infty}\frac{1}{n^2}\sum_{k=1}^{n} kb_k \leqslant \lim_{n\to\infty}\frac{1}{n^2}\sum_{k=N+1}^{n} k\,(B+\varepsilon)$$

$$\lim_{n\to\infty}\frac{1}{n^2}\sum_{k=N+1}^{n} k\,(B+\varepsilon) = \lim_{n\to\infty}(B+\varepsilon)\frac{(N+1+n)(n-N)}{2n^2} = \frac{B+\varepsilon}{2},$$
$$\lim_{n\to\infty}\frac{1}{n^2}\sum_{k=N+1}^{n} k\,(B-\varepsilon) = \lim_{n\to\infty}(B-\varepsilon)\frac{(N+1+n)(n-N)}{2n^2} = \frac{B-\varepsilon}{2}$$

$$\frac{B-\varepsilon}{2} \leqslant \lim_{n\to\infty}\frac{1}{n^2}\sum_{k=1}^{n} kb_k \leqslant \frac{B+\varepsilon}{2}$$

Since ε can be arbitrarily small then $\lim_{n\to\infty}\frac{1}{n^2}\sum_{k=1}^{n} kb_k = \frac{B}{2}$.

Now we apply the Lemma, (2) and Equation (4) to complete the evaluation of the limit.

$$\lim_{n\to\infty}\frac{1}{\left(\sqrt[n]{(2n-1)!!}\right)^2}\sum_{k=1}^{n}\left(\sqrt[2k]{k!}+\sqrt[2(k+1)]{(k+1)!}\right)^2$$
$$= \lim_{n\to\infty}\frac{e^2}{4n^2}\sum_{k=1}^{n}\left(\sqrt[2k]{k!}+\sqrt[2(k+1)]{(k+1)!}\right)^2$$
$$= \lim_{n\to\infty}\frac{e^2}{4n^2}\sum_{k=1}^{n} kb_k = \frac{e^2}{4}\frac{\frac{4}{e}}{2} = \frac{e}{2}.$$

Solution 3 by Michel Bataille, Rouen, France

The required limit is $\frac{e}{2}$.

For $n\in\mathbb{N}$, let $a_n = \sqrt[2n]{n!}$ and $S_n = \sum_{k=1}^{n}\left[(a_k+a_{k+1})^2\right]$, $\quad T_n = \sum_{k=1}^{n}(a_k+a_{k+1})^2$.

Since $(a_k+a_{k+1})^2 - 1 \leqslant \left[(a_k+a_{k+1})^2\right] \leqslant (a_k+a_{k+1})^2$, we have

$$T_n - n \leqslant S_n \leqslant T_n. \tag{1}$$

We will use the known results $\sqrt[n]{n!} \sim \frac{n}{e}$, $\quad \sqrt[n]{(2n-1)!!} \sim \frac{2n}{e}$ (see my solution to problem 5536). We deduce that

$$\frac{1}{\left(\sqrt[n]{(2n-1)!!}\right)^2} \sim \frac{e^2}{4n^2} \tag{2}$$

and

$$\lim_{n\to\infty}\frac{(a_n+a_{n+1})^2}{n/e} = \lim_{n\to\infty}\left(\frac{\sqrt[n]{n!}}{n/e} + \frac{\sqrt[n+1]{(n+1)!}}{(n+1)/e}\cdot\frac{n+1}{n} + \frac{2\,\sqrt[2n]{n!}\,\sqrt[2(n+1)]{(n+1)!}}{n/e}\right) = 4,$$

hence $(a_n + a_{n+1})^2 \sim \dfrac{4n}{e}$ as $n \to \infty$. From the Stolz-Cesaro theorem, we deduce

$$T_n \sim \frac{4}{e} \sum_{k=1}^{n} k = \frac{4n(n+1)}{2e} \sim \frac{2n^2}{e}. \tag{3}$$

Now, from (2) and (3), we obtain

$$\lim_{n\to\infty} \frac{T_n}{\left(\sqrt[n]{(2n-1)!!}\right)^2} = \frac{e}{2}, \quad \lim_{n\to\infty} \frac{T_n - n}{\left(\sqrt[n]{(2n-1)!!}\right)^2} = \frac{e}{2} - 0 = \frac{e}{2}.$$

Finally, (1) and the squeeze theorem yields $\displaystyle\lim_{n\to\infty} \frac{S_n}{\left(\sqrt[n]{(2n-1)!!}\right)^2} = \frac{e}{2}.$

Also solved by Kee-Wai Lau, Hong Kong, China; Ioannis D. Sfikas, National and Kapodistrian University of Athens, Greece; Albert Stadler, Herrliberg, Switzerland, and the proposer.

5581: *Proposed by José Luis Díaz-Barrero, Barcelona Tech, Barcelona, Spain*

Let a, b, c be the lengths of the sides of an acute triangle ABC. Prove that

$$\sqrt{\frac{a^2+2bc}{b^2+c^2-a^2}} + \sqrt{\frac{b^2+2ca}{c^2+a^2-b^2}} + \sqrt{\frac{c^2+2ab}{a^2+b^2-c^2}} \geqslant 3\sqrt{3}.$$

Solution 1 by Dionne Bailey, Elsie Campbell, and Charles Diminnie, Angelo State University, San Angelo, TX

Since the Law of Cosines states that

$$c^2 = a^2 + b^2 - 2ab\cos C,$$

we obtain

$$a^2 + b^2 - c^2 = 2ab\cos C$$

and

$$c^2 + 2ab = (a+b)^2 - 2ab\cos C.$$

As a result,

$$\begin{aligned} \frac{c^2+2ab}{a^2+b^2-c^2} &= \frac{(a+b)^2 - 2ab\cos C}{2ab\cos C} \\ &= \frac{(a+b)^2}{2ab\cos C} - 1. \end{aligned}$$

Because a, b, and $\cos C > 0$, the Arithmetic - Geometric Mean Inequality yields $(a+b)^2 \geqslant \left(2\sqrt{ab}\right)^2 = 4ab$ and we get

$$\begin{aligned} \frac{c^2+2ab}{a^2+b^2-c^2} &\geqslant \frac{4ab}{2ab\cos C} - 1 \\ &= \frac{2}{\cos C} - 1 \\ &= 2\sec C - 1. \end{aligned} \tag{1}$$

with equality if and only if $a = b$.

Similar steps lead to

$$\frac{a^2 + 2bc}{b^2 + c^2 - a^2} \geqslant 2\sec A - 1 \tag{2}$$

with equality if and only if $b = c$ and

$$\frac{b^2 + 2ca}{c^2 + a^2 - b^2} \geqslant 2\sec B - 1 \tag{3}$$

with equality if and only if $c = a$.

Note that since $\triangle ABC$ is acute, we have $\sec A, \sec B, \sec C > 1$. Then, using the fact that $f(x) = \sqrt{x}$ is strictly increasing on $(0, \infty)$, conditions (1), (2), and (3) imply that

$$\begin{aligned} &\sqrt{\frac{a^2 + 2bc}{b^2 + c^2 - a^2}} + \sqrt{\frac{b^2 + 2ca}{c^2 + a^2 - b^2}} + \sqrt{\frac{c^2 + 2ab}{a^2 + b^2 - c^2}} \\ &\geqslant \sqrt{2\sec A - 1} + \sqrt{2\sec B - 1} + \sqrt{2\sec C - 1} \end{aligned} \tag{4}$$

with equality if and only if $a = b = c$.

If $g(x) = \sqrt{2\sec x - 1}$ for $0 < x < \dfrac{\pi}{2}$, then after some reorganization of terms, we obtain

$$\begin{aligned} g''(x) &= \frac{\sec^3 x\,(2\sec x - 1) + \sec x \tan^2 x\,(\sec x - 1)}{(2\sec x - 1)^{\frac{3}{2}}} \\ &> 0, \end{aligned}$$

and hence, $g(x)$ is strictly convex on $\left(0, \dfrac{\pi}{2}\right)$. By Jensen's Theorem,

$$\begin{aligned} \sqrt{2\sec A - 1} + \sqrt{2\sec B - 1} + \sqrt{2\sec C - 1} &\geqslant 3\sqrt{2\sec\left(\frac{A + B + C}{3}\right) - 1} \\ &= 3\sqrt{2\sec\left(\frac{\pi}{3}\right) - 1} \\ &= 3\sqrt{4 - 1} \\ &= 3\sqrt{3}, \end{aligned} \tag{5}$$

with equality if and only if $A = B = C$.

The desired inequality follows from conditions (4) and (5) and equality is attained if and only if $a = b = c$, i.e., if and only if $\triangle ABC$ is equilateral.

Solution 2 by Bruno Salgueiro Fanego, Viveiro, Spain

$$\sqrt{\frac{a^2 + 2bc}{b^2 + c^2 - a^2}} + \sqrt{\frac{b^2 + 2ca}{c^2 + a^2 - b^2}} + \sqrt{\frac{c^2 + 2ab}{a^2 + b^2 - c^2}} =$$

$$= \sqrt{\frac{a^2 + bc + bc}{2bc\cos A}} + \sqrt{\frac{b^2 + ca + ca}{2ca\cos B}} + \sqrt{\frac{c^2 + ab + ab}{2ab\cos C}}$$

$$\geqslant \sqrt{\frac{3\sqrt[3]{a^2bcbc}}{2bc\cos A}} + \sqrt{\frac{3\sqrt[3]{b^2caca}}{2ca\cos B}} + \sqrt{\frac{3\sqrt[3]{c^2abab}}{2ab\cos C}}$$

$$
\begin{aligned}
&= \sqrt{\frac{3\sqrt[3]{a^2b^2c^2}}{2bc\cos A}} + \sqrt{\frac{3\sqrt[3]{a^2b^2c^2}}{2ca\cos B}} + \sqrt{\frac{3\sqrt[3]{a^2b^2c^2}}{2ab\cos C}} \\
&\geqslant 3\sqrt[3]{\sqrt{\frac{3\sqrt[3]{a^2b^2c^2}}{2bc\cos A}}\sqrt{\frac{3\sqrt[3]{a^2b^2c^2}}{2ca\cos B}}\sqrt{\frac{3\sqrt[3]{a^2b^2c^2}}{2ab\cos C}}} \\
&\geqslant 3\sqrt[3]{\sqrt{\frac{\left(3\sqrt[3]{a^2b^2c^2}\right)^3}{8a^2b^2c^2\cos A\cos B\cos C}}} \\
&= 3\sqrt[6]{\frac{27}{8\cos A\cos B\cos C}} \\
&\geqslant 3\sqrt[6]{\frac{27}{8\left(\frac{\cos A+\cos B+\cos C}{3}\right)}} \\
&= 3\sqrt{\frac{9}{2(\cos A+\cos B+\cos C)}} \\
&= \frac{9}{\sqrt{2\left(1+4\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}\right)}} \\
&= \frac{9}{\sqrt{2\left(1+\frac{r}{R}\right)}} \\
&\geqslant \frac{9}{\sqrt{2\left(1+\frac{1}{2}\right)}} = 3\sqrt{3}.
\end{aligned}
$$

Equality occurs if and only if the sides of the given acute triangle are equal to one another; that is, when triangle ABC is equilateral.

Solution 3 by Kevin Soto Palacios, Huarmey, Perú

We use the law of cosines, that in any triangle ABC, with side lengths a, b, c respectively opposite the angles A, B, C, the following identities hold:

$$b^2 + c^2 - a^2 = 2bc\cos A, \quad c^2 + a^2 - b^2 = 2ca\cos B, \quad a^2 + b^2 - c^2 = 2ab\cos C.$$

We will also use the inequality that if x, y, z are positive real numbers then:

$$(x+y)(y+z)(z+x) \geqslant 8xyz.$$

Letting $x = \tan A,\ y = \tan B,\ z = \tan C$ implies that

$(x+y)(y+z)(z+x) = (\tan A \tan B \tan C)(\sec A \sec B \sec C) \geqslant \tan A \tan B \tan C$, and this implies

$$\sec A \sec B \sec C \geqslant 8.$$

Using the AM$\geqslant$GM inequality implies that:

$$\sqrt{\frac{a^2 \sec A}{2bc} + \sec A} + \sqrt{\frac{b^2 \sec B}{2ca} + \sec B} + \sqrt{\frac{c^2 \sec C}{2ab} + \sec C}$$

$$\geqslant 3\sqrt[6]{\left(\frac{a^2 \sec A}{2bc} + \sec A\right)\left(\frac{b^2 \sec B}{2ca} + \sec B\right)\left(\frac{c^2 \sec C}{2ab} + \sec C\right)}$$

$$\geqslant 3\sqrt[6]{\left(\frac{a^2 \sec A}{2bc}\frac{b^2 \sec B}{2ca}\frac{c^2 \sec C}{2ab} + \sqrt[3]{\sec A \sec B \sec C}\right)^3} = 3\sqrt{\sqrt[3]{\frac{\sec A \sec B \sec C}{8}} + \sqrt[3]{\sec A \sec B \sec C}}$$

$$\geqslant 3\sqrt{1+2} = 3\sqrt{3}$$

Solution 4 by Michel Bataille, Rouen, France

We observe that $a^2 + 2bc = a^2 + (b-c)^2 - b^2 - c^2 + 4bc \geqslant a^2 - b^2 - c^2 + 4bc$ and, since $b^2 + c^2 - a^2 = 2bc \cos A$, we deduce that

$$\frac{a^2 + 2bc}{b^2 + c^2 - a^2} \geqslant \frac{2}{\cos A} - 1.$$

If L denotes the left-hand side of the inequality, it follows that

$$L \geqslant f(\cos A) + f(\cos B) + f(\cos C) \qquad (1)$$

where $f(x) = (2x^{-1} - 1)^{1/2}$. [as usual, A, B, C denote the angles of the triangle opposite the sides a, b, c, respectively.]
Note that $\cos A, \cos B, \cos C \in (0,1)$ (the triangle ABC being acute).
We calculate

$$f'(x) = -x^{-2}(2x^{-1} - 1)^{-1/2}, \qquad f''(x) = x^{-4}(2x^{-1} - 1)^{-3/2}(3 - 2x)$$

and deduce that f is decreasing (since $f'(x) < 0$) and convex (since $f''(x) > 0$) on the interval $(0,1)$. Jensen's inequality then provides

$$f(\cos A) + f(\cos B) + f(\cos C) \geqslant 3f\left(\frac{\cos A + \cos B + \cos C}{3}\right). \qquad (2)$$

Now, the function cos being concave on $(0, \pi/2)$, we have
$\cos A + \cos B + \cos C \leqslant 3\cos\left(\frac{A+B+C}{3}\right) = \frac{3}{2}$ and so, recalling that f is decreasing,

$$f\left(\frac{\cos A + \cos B + \cos C}{3}\right) \geqslant f(1/2) = \sqrt{3}. \qquad (3)$$

From the results (1), (2), (3), we readily obtain $L \geqslant 3\sqrt{3}$, as desired.

Also solved by Hatef I. Arshagi, Guilford Technical Community College, Jamestown, NC; Arkady Alt, San Jose, CA; Brian Bradie, Christopher Newport University, Newport News, VA; Kee-Wai Lau, Hong Kong, China; Moti Levy, Rehovot, Israel; Albert Stadler, Herrliberg, Switzerland, Ioannis D. Sfikas, National and Kapodistrian University of Athens, Greece, and the proposer.

5582: *Proposed by Ovidiu Furdui and Alina Sîntămărian, Technical University of Cluj-Napoca, Cluj-Napoca, Romania*

Calculate

$$\lim_{n\to\infty}\sqrt[n]{\int_0^1\!\!\int_0^1\left(\frac{x+y^2+x^3+\cdots+x^{2n-1}+y^{2n}}{n}\right)^n \mathrm{d}x\mathrm{d}y}.$$

Solution 1 by Bin Pan, San Mateo, CA

First note, $\forall x, y, \in [0,1]$, we have

$$x+y^2+x^3+\cdots+x^{2n-1}+y^{2n} \leqslant 2n.$$

Therefore, the integral limit

$$\lim_{n\to+\infty}\sqrt[n]{\int_0^1\int_0^1\left(\frac{x+y^2+x^3+\cdots+x^{2n-1}+y^{2n}}{n}\right)^n dxdy} \leqslant 2 \qquad (1)$$

On the other hand, for any $\epsilon > 0$, $\exists c$ and d, $0 \leqslant c \leqslant d \leqslant 1$, such that $\forall x, y, \in [c,d]$, we have

$$x+y^2+x^3+\cdots+x^{2n-1}+y^{2n} \geqslant (2-\epsilon)n.$$

Therefore,

$$\begin{aligned}\sqrt[n]{\int_0^1\int_0^1\left(\frac{x+y^2+x^3+\cdots+x^{2n-1}+y^{2n}}{n}\right)^n dxdy} &\geqslant \sqrt[n]{\int_c^d\int_c^d(2-\epsilon)^n dxdy}\\ &\geqslant (2-\epsilon)\sqrt[n]{(d-c)^2}.\end{aligned}$$

It follows,

$$\lim_{n\to\infty}\sqrt[n]{\int_0^1\int_0^1\left(\frac{x+y^2+x^3+\cdots+x^{2n-1}+y^{2n}}{n}\right)^n dxdy} \geqslant 2-\epsilon. \qquad (2)$$

Because ϵ is arbitrary in (2), together with (1), we see

$$\lim_{n\to+\infty}\sqrt[n]{\int_0^1\int_0^1\left(\frac{x+y^2+_x^3+\cdots+x^{2n-1}+y^{2n}}{n}\right)^n dxdy} = 2.$$

Solution 2 by Albert Stadler, Herrliberg, Switzerland

Clearly $z^{2n} \leqslant z^k \leqslant 1$ for $z \in [0,1]$ and $0 \leqslant k \leqslant 2n$. Therefore,

$$\begin{aligned}\int_0^1\int_0^1 \left(x^{2n}+y^{2n}\right)^n dxdy &\leqslant \int_0^1\int_0^1 \left(\frac{x+y^2+x^3+\cdots+x^{2n-1}+y^{2n}}{n}\right)^n dxdy \\ &\leqslant \int_0^1\int_0^1 2^n dxdy = 2^n\end{aligned}$$

By the AM−GM inequality, $x^{2n}+y^{2n} \geqslant 2x^ny^n$. So,

$$\int_0^1\int_0^1 \left(x^{2n}+y^{2n}\right)^n dxdy \geqslant 2^n \int_0^1\int_0^1 \left(x^ny^n\right)^n dxdy = \frac{2^n}{(n^2+1)^2}.$$

Finally,

$$\frac{2}{\sqrt[n]{(n^2+1)^2}} \leqslant \sqrt[n]{\int_0^1\int_0^1 \left(\frac{x+y^2+x^3+\cdots+x^{2n-1}+y^{2n}}{n}\right)^n dxdy} \leqslant 2,$$

and

$$\lim_{n\to\infty} \sqrt[n]{\int_0^1\int_0^1 \left(\frac{x+y^2+x^3+\cdots+x^{2n-1}+y^{2n}}{n}\right)^n dxdy} = 2.$$

Solution 3 by Moti Levy, Rehovot, Israel

Let

$$\begin{aligned}J_n &:= \int_0^1\int_0^1 \left(\frac{x+y^2+x^3+\cdots+x^{2n-1}+y^{2n}}{n}\right)^n dxdy \\ &= 2^n \int_0^1\int_0^1 \left(\frac{x+y^2+x^3+\cdots+x^{2n-1}+y^{2n}}{2n}\right)^n dxdy. \qquad (1)\end{aligned}$$

By the power mean inequality we have

$$\left(\frac{\sum_{i=1}^n a_i}{n}\right)^n \leqslant \frac{1}{n}\sum_{i=1}^n a_i^n, \qquad a_i \geqslant 0 \qquad for \quad i=1,2,\ldots \qquad (2)$$

Applying (2) on (1) we get

$$\begin{aligned}J_n &\leqslant \frac{2^n}{n}\int_0^1\int_0^1 \left(x^n+y^{2n}+x^{3n}+\cdots+x^{(2n-1)n}+y^{2n^2}\right)dxdy \\ &= 2^n\left(\frac{1}{n}\sum_{k=1}^n \frac{1}{(2k-1)\,n+1}+\frac{1}{(2k)\,n+1}\right) \\ &\leqslant 2^n\left(\frac{1}{n}\sum_{k=1}^n \frac{1}{(2k-1)\,n+1}+\frac{1}{(2k-1)\,n+1}\right) \\ &\leqslant 2^n\left(\frac{1}{n^2}\sum_{k=1}^n \frac{2}{(2k-1)}\right) \leqslant 2^n\left(\frac{1}{n^2}\sum_{k=1}^n \frac{2}{(2k-1)}\right) \\ &\leqslant 2^n\left(\frac{1}{n^2}2n\right) \leqslant 2^n\left(\frac{2}{n}\right)\end{aligned}$$

By AM-GM inequality,

$$\left(\frac{\sum_{i=1}^{n} a_i}{n}\right)^n \geqslant \prod_{i=1}^{n} a_i \qquad a_i \geqslant 0 \qquad for \quad i = 1, 2, \ldots \tag{3}$$

Applying (3) on (1) we get

$$\begin{aligned} J_n &\geqslant 2^n \int_0^1 \int_0^1 \left(\prod_{i=1}^{n} x^{2i-1}\right)\left(\prod_{i=1}^{n} y^{2i}\right) dxdy \qquad (4) \\ &= 2^n \int_0^1 x^{n^2} dx \int_0^1 y^{n(n+1)} dy = 2^n \frac{1}{1+n^2} \frac{1}{1+n(n+1)} \end{aligned}$$

$$\lim_{n\to\infty} \sqrt[n]{2^n \left(\frac{2}{n}\right)} \geqslant \lim_{n\to\infty} \sqrt[n]{J_n} \geqslant \lim_{n\to\infty} \sqrt[n]{2^n \frac{1}{1+n^2} \frac{1}{1+n(n+1)}}$$

Now $\lim_{n\to\infty} \sqrt[n]{2^n \left(\frac{2}{n}\right)} = \lim_{n\to\infty} \sqrt[n]{2^n \frac{1}{1+n^2} \frac{1}{1+n(n+1)}} = 2$, hence (by the "sandwich rule"),

$$\lim_{n\to\infty} \sqrt[n]{J_n} = 2.$$

Solution 4 by Kee-Wai Lau, Hong Kong, Chiina

We show that the limit equals 2.

Since $x + y^2 + x^3 + \cdots + x^{2n-1} + y^{2n} \leqslant 2n$, so

$$\int_0^1 \int_0^1 \left(\frac{x + y^2 + x^3 + \cdots + x^{2n-1} + y^{2n}}{n}\right) dxdy \leqslant 2^n.$$

By the AM-GM inequality, we have

$$\frac{x + y^2 + x^3 + \cdots + x^{2n-1} + y^{2n}}{2n} \geqslant \sqrt[2n]{x^{(n^2)} y^{n(n+1)}} = x^{\frac{n}{2}} y^{\frac{n+1}{2}}.$$

Hence,

$$\begin{aligned} \int_0^1 \int_0^1 \left(\frac{x + y^2 + x^3 + \cdots + x^{2n-1} + y^{2n}}{n}\right)^n dxdy &\geqslant 2^n \int_0^1 \int_0^1 x^{\left(\frac{n^2}{2}\right)} y^{n(n+1)} dxdy \\ &= \frac{2^{n+1}}{(n^2+2)(n^2+n+1)}. \end{aligned}$$

It follows that

$$2\left(\sqrt[n]{\frac{2}{(n^2+2)(n^2+n+1)}}\right) \leqslant \sqrt[n]{\int_0^1 \int_0^1 \left(\frac{x + y^2 + x^3 + \cdots + x^{2n-1} + y^{2n}}{n}\right)^n dxdy} \leqslant 2.$$

Since $\lim_{n\to\infty} \sqrt[n]{\frac{2}{(n^2+2)(n^2+n+1)}} = 1$, our result follows.

Solution 5 by Albert Natian, Los Angeles Valley College,Valley Glen, CA

Answer. 2.

Set

$$Q_n := \sqrt[n]{\int_0^1 \int_0^1 \left(\frac{x + y^2 + x^3 + y^4 + \cdots + x^{2n-1} + y^{2n}}{n} \right)^n dx\ dy}.$$

Then

$$Q_n^n = \int_0^1 \int_0^1 \left(\frac{1}{n} \sum_{k=1}^{n} \left(x^{2k-1} + y^{2k} \right) \right)^n \ dx\ dy \leqslant \int_0^1 \int_0^1 \left(\frac{1}{n} \sum_{k=1}^{n} \left(1^{2k-1} + 1^{2k} \right) \right)^n dx\ dy,$$

$$Q_n^n \leqslant \int_0^1 \int_0^1 \left(\frac{1}{n} \cdot 2n \right)^n dx\ dy = \int_0^1 \int_0^1 2^n\ dx\ dy = 2^n.$$

Also

$$Q_n^n = \int_0^1 \int_0^1 \left(\frac{1}{n} \sum_{k=1}^{n} \left(x^{2k-1} + y^{2k} \right) \right)^n dx\ dy \geqslant \int_0^1 \int_0^1 \left(\frac{1}{n} \sum_{k=1}^{n} \left(x^{2n-1} + y^{2n} \right) \right)^n dx\ dy,$$

$$Q_n^n \geqslant \int_0^1 \int_0^1 \left(\frac{1}{n} \cdot n \left(x^{2n-1} + y^{2n} \right) \right)^n dx\ dy = \int_0^1 \int_0^1 \left(x^{2n-1} + y^{2n} \right)^n dx\ dy,$$

$$Q_n^n \geqslant \int_0^1 \int_0^1 x^{2n-2} y^{2n-1} \left(x^{2n-1} + y^{2n} \right)^n dx\ dy = \frac{2^{n+1}-1}{n\,(n+1)\,(n+2)\,(2n-1)}.$$

Combine two of the preceding inequalities to get

$$\frac{2^{n+1}-1}{n\,(n+1)\,(n+2)\,(2n-)} \leqslant Q_n^n \leqslant 2^n,$$

$$\left(\frac{2^{n+1}-1}{n\,(n+1)\,(n+2)\,(2n-1)} \right)^{1/n} \leqslant Q_n \leqslant 2$$

which implies

$$2 = \lim_{n\to\infty} \left(\frac{2^{n+1}-1}{n\,(n+1)\,(n+2)\,(2n-1)} \right)^{1/n} \leqslant \lim_{n\to\infty} Q_n \leqslant 2,$$

$$\lim_{n\to\infty} Q_n = 2,$$

whence

$$\lim_{n\to\infty} \sqrt[n]{\int_0^1 \int_0^1 \left(\frac{x + y^2 + x^3 + y^4 + \cdots + x^{2n-1} + y^{2n}}{n} \right)^n dx\ dy} = \lim_{n\to\infty} Q_n = 2.$$

Also solved by Hartef I. Arshagi, Guilford Technical Community College, Jamestown, NC, and the proposers.

Mea Culpa

Michel Bataille of Rouen, France should have been credited for having solve 5575. His name was inadvertently omitted from the list of solvers.