Problems

Ted Eisenberg, Section Editor

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This section of the Journal offers readers an opportunity to exchange interesting mathematical problems and solutions. Please send them to Ted Eisenberg, Department of Mathematics, Ben-Gurion University, Beer-Sheva, Israel or fax to: 972-86-477-648. Questions concerning proposals and/or solutions can be sent e-mail to <eisenbt@013.net>. Solutions to previously stated problems can be seen at <http://www.ssma.org/publications>.

Solutions to the problems stated in this issue should be posted before December 15, 2020

- **5601:** *Proposed by Kenneth Korbin, New York, NY*

  Solve:
  $$\frac{\sqrt{x(x-1)^2}}{(x+1)^2} = \frac{\sqrt{77}}{36}.$$

- **5602:** *Proposed by Pedro Henrique Oliveira Pantoja. University of Campina Grande, Brazil*

  Prove that:
  $$\det\begin{vmatrix} 1 & \cos\dfrac{\pi}{7} & \sin\dfrac{3\pi}{7} \\ \sin\dfrac{3\pi}{7} & \sin\dfrac{2\pi}{7} & \sin^2\dfrac{\pi}{7} \\ 0 & \tan\dfrac{\pi}{7} & 2\sin^2\dfrac{\pi}{7} \end{vmatrix} = \frac{\sqrt{7}}{8}.$$

- **5603:** *Proposed by Michael Brozinsky, Central Islip, NY*

  In an election 50 votes were cast for candidate A and 50 for candidate B. The candidates decide to end the tie as follows; by tallying the votes at random and if A is ever in the lead by 3 votes, then Candidate A will be declared the winner. Otherwise Candidate B wins. What is the probability that A wins?

- **5604:** *Proposed by Albert Natian, Los Angeles Vallet College, Valley Glen, CA*

  Prove:
  $$\binom{N}{r} = \lim_{n\to\infty}\frac{1}{n}\sum_{\mu=0}^{n-1} e^{-ir\mu\frac{2\pi}{n}}\left(1+e^{i\mu\frac{2\pi}{n}}\right)^N$$

  where $N, r \in \mathbb{N}$ and $i^2 = -1$.

- **5605:** *Proposed by José Luis Díaz-Barrero, Barcelona Tech, Barcelona, Spain*

Let b and c be distinct coprime numbers. Find the smallest positive integer a for which

$$\gcd(a^b - 1, a^c - 1) = 100.$$

- **5606:** *Proposed by Ovidiu Furdui and Alina Sîntămărian, Technical University of Cluj-Napoca, Cluj-Napoca, Romania*

Let $a, b > 0$, $c \geqslant 0$ and $4ab - c^2 > 0$. Calculate

$$\int_{-\infty}^{\infty} \frac{x}{a\mathrm{e}^x + b\mathrm{e}^{-x} + c} \mathrm{d}x.$$

Solutions

- **5583:** *Proposed by Kenneth Korbin, New York, NY*

(a) Given positive angles A and B with $A + B = 180^\circ$. A circle with radius 16 and a circle with radius 49 are each tangent to both sides of $\angle A$. The circles are also tangent to each other. Find sine A.

(b) A circle with radius x and a circle with radius y are each tangent to both sides of $\angle B$. These circles are also tangent to each other. Find positive integers x and y with $(x, y) = 1$.

Solution 1 by David E. Manes, Oneonta, NY

In their solution to problem **5457**: (this journal), Professors Stone and Hawkins of Georgia Southern University proved the following: Let A be an angle, $0 < A < \pi$. If two circles, C_1 of radius r and C_2 of radius R ($r < R$), are inscribed in A, with C_2 tangent to C_1, then

$$R = \left(\frac{1 + \sin\alpha}{1 - \sin\alpha}\right) r, \;\; \alpha = \frac{1}{2}A.$$

Using this result in part (a), let $R = 49$ and $r = 16$. Then $49 = \left(\dfrac{1 + \sin\alpha}{1 - \sin\alpha}\right) 16$, where $\alpha = A/2$ and $A + B = 180^\circ$. Solving for $\sin\alpha$, one obtains $\sin\alpha = \sin(A/2) = 33/65$. Then

$$\cos A = 1 - 2\sin^2\left(\frac{A}{2}\right) = 1 - 2\left(\frac{33}{65}\right)^2 = \frac{2047}{4225}.$$

Hence,

$$\sin A = \sqrt{1 - \cos^2 A} = \sqrt{1 - (2047/4225)^2} \approx 0.874\,792\,899\,408$$

and $\arcsin(\sqrt{1-(2047/4225)^2}) \approx 61.02°$ so that $B = 180° - A \approx 118.98°$ and

$$\beta = \frac{B}{2} = \frac{1}{2}\left(180° - \arcsin\left(\sqrt{1-\left(\frac{2047}{4225}\right)^2}\right)\right) \approx 59.49°.$$

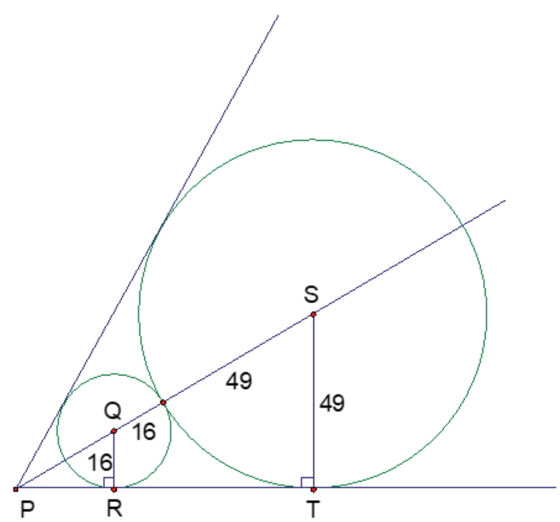
For part (b), assume $x < y$ and use the above result in the form

$$y = \left(\frac{1+\sin\beta}{1-\sin\beta}\right)x,\ \beta = \frac{1}{2}B.$$

Substituting integer values for x, one finds that if $x = 9$, then $y = 121$. Furthermore, one can show inductively that if x is any positive integer, then $y = (121/9)x$. Therefore, if $k \geqslant 1$ is a positive integer and $x = 9k$, then $y = 121k$. Since x and y are relatively prime it follows that the solution $x = 9, y = 121$ is unique.

Solution 2 by David A. Huckaby, Angelo State University, San Angelo, TX

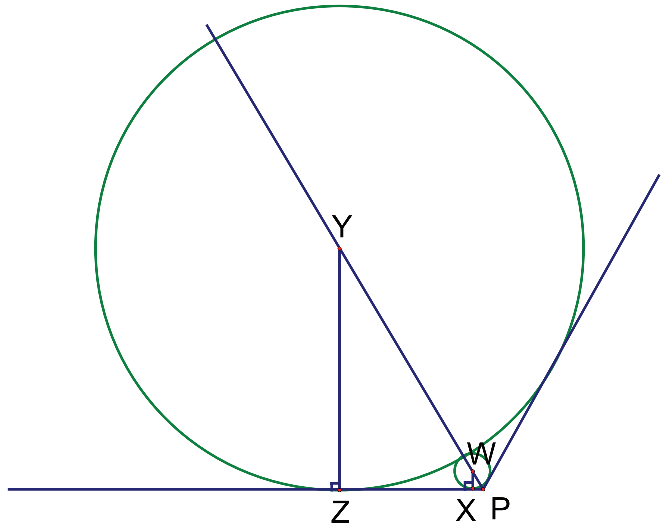
For part (a), see the figure below, in which angle QPR is $\frac{A}{2}$.



We have $\frac{QR}{PQ} = \frac{ST}{PS}$, that is, $\frac{16}{PQ} = \frac{49}{PQ+16+49}$, whence $PQ = \frac{1040}{33}$. So $\sin\left(\frac{A}{2}\right) = \frac{16}{\frac{1040}{33}} = \frac{33}{65}$ and $\cos\left(\frac{A}{2}\right) = \sqrt{1-\left(\frac{33}{65}\right)^2} = \frac{56}{65}$.

So $\sin A = 2\sin\left(\frac{A}{2}\right)\cos\left(\frac{A}{2}\right) = 2\left(\frac{33}{65}\right)\left(\frac{56}{65}\right) = \frac{3696}{4225}$.

For part (b), first note that angle A is indeed an actute angle as in the figure above: $\cos A = 1 - 2\sin^2(\frac{A}{2}) = 1 - 2(\frac{33}{65})^2 = \frac{2047}{4225} > 0$. So B is an obtuse angle as in the figure below, in which angle WPX is $\frac{B}{2}$.



Since $A + B = 180°$, we have $\dfrac{B}{2} = 90° - \dfrac{A}{2}$, so that $\sin\left(\dfrac{B}{2}\right) = \cos\left(\dfrac{A}{2}\right) = \dfrac{56}{65}$.

So $\dfrac{WX}{WP} = \sin\left(\dfrac{B}{2}\right) = \dfrac{56}{65}$, so that $56WP = 65WX$. Now $\dfrac{WX}{WP} = \dfrac{YZ}{YP}$, that is, $\dfrac{56}{65} = \dfrac{YZ}{YZ + WX + WP}$, whence $65YZ = 56(YZ + WX + WP)$, so that $9YZ = 56WX + 56WP = 56WX + 65WX = 121WX$. In other words, $3^2YZ = 11^2WX$.

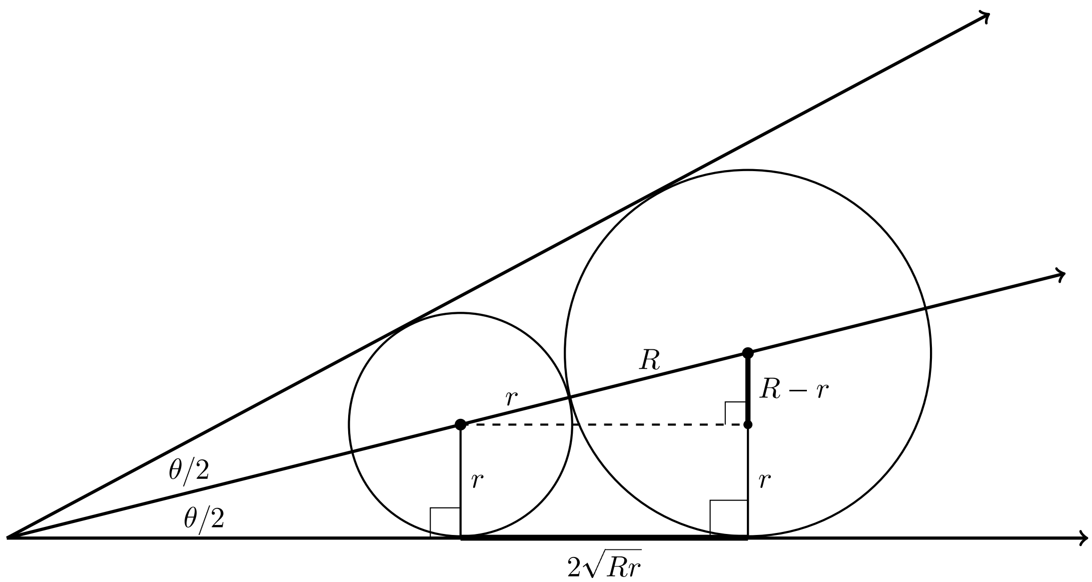
This equation holds, as does the condition $(x, y) = 1$, if $x = WX = 3^2 = 9$ and $y = YZ = 11^2 = 121$.

Solution 3 by Albert Natian, Los Angeles Valley College, Valley Glen, CA

Answer: $\sin A = \dfrac{3696}{4225}$, $x = 121, y = 9$.

Computation: Given mutually tangent circles with radii r and R $(r < R)$ that are also tangent to the sides of an angle of measure θ, as shown, we glean the facts that

$$\sin(\theta/2) = \frac{R - r}{R + r} \quad \text{and} \quad \cos(\theta/2) = \frac{2\sqrt{Rr}}{R + r}.$$



From the above equations we get

$$\sin\theta = 2\sin(\theta/2)\cos(\theta/2) = \frac{4(R-r)\sqrt{Rr}}{(R+r)^2}.$$

a) With $r = 16$, $R = 49$ we have $\sin A = \sin\theta = \dfrac{2^4 \cdot 3 \cdot 7 \cdot 11}{5^2 \cdot 13^2} = \dfrac{3696}{4225}$.

b) Because $A + B = 180^\circ$, then $B/2 = 90^\circ - A/2$ and so

$$\sin(B/2) = \sin(90^\circ - A/2) = \cos(A/2) = \frac{2\sqrt{49 \cdot 16}}{49+16},$$

$$\cos(B/2) = \cos(90^\circ - A/2) = \sin(A/2) = \frac{49-16}{49+16}.$$

Letting x take the place of R and y the place of r in the above figure, we see that

$$\frac{x-y}{x+y} = \sin(B/2) = \frac{2\sqrt{49 \cdot 16}}{49+16} = \frac{2^3 \cdot 7}{5 \cdot 13} = \frac{56}{65},$$

$$\frac{2\sqrt{xy}}{x+y} = \cos(B/2) = \frac{49-16}{49+16} = \frac{3 \cdot 11}{5 \cdot 13} = \frac{33}{65}.$$

Solving either of the latter equations for the ratio x/y, we get $x/y = 121/9$. Since $(121, 9) = 1$, then we have $x = 121$ and $y = 9$.

Solution 4 by Kee-Wai Lau of Hong Kong, China

We show that *a)* $\sin A = \dfrac{3696}{4225}$ and *b)* $x = 9, y = 121$ or $x = 121, y = 9$.

Clearly, we can only deal with the case $y > x$. Let the distance between the center of the smaller circle and the point of intersection of the common tangents be d. By similar triangles, we have $\dfrac{d}{d+x+y} = \dfrac{x}{y}$, so that $d = \dfrac{x(x+y)}{y-x}$ and $\sin\left(\dfrac{B}{2}\right) = \dfrac{y-x}{y+x}$.

Hence $\cos\left(\frac{B}{2}\right) = \frac{2\sqrt{xy}}{y+x}$ and $\sin B = 2\sin\left(\frac{B}{2}\right)\cos\left(\frac{B}{2}\right) = \frac{4\sqrt{xy}(y-x)}{(y+x)^2}$. In a similar way, we obtain $\sin A = \frac{4\sqrt{(16)(49)}(49-16)}{(49+16)^2} = \frac{3696}{4225}$. Hence $\sin B = \sin A = \frac{3696}{4225}$ and $924(x+y)^2 = 4225\sqrt{xy}(y-x)$. Squaring both sides, we obtain

$$924^2(x+y)^4 - 4225^2(xy)(y-x)^2 = 0,$$

or

$$(16x-49y)(49x-16y)(9x-121y)(121x-9y) = 0.$$

Since $y > x$, so $(49x-16y)(121x-9y) = 0$. From $A \neq \frac{\pi}{2}$, we see that $B \neq A$ and $49x - 16y \neq 0$. Hence our result for x and y satisfying $(x, y) = 1$.

Editor′s comment : **David Stone and John Hawkins of Georgia Southern University** mentioned in their solution that the question is really "a two-dimensional version of the ice cream cone problem: if we drop a (spherical) scoop of ice cream, with known radius, into a cone with a known vertex angle, where does it lodge?"

They approached their solution by using two general lemmas, and then applying the data of the problem the lemmas. Their lemmas were:

- Lemma 1: Suppose that a circle of radius r is inscribed in an angle θ, with $0 < \theta < \pi$. Let w be the distance from the vertex of the angle to the circle (measured along the angle bisector). Then $\sin\frac{\theta}{2} = \frac{r}{w+r}$.
  That is, if two of the quantities θ, r and w are known, the third one is determined; in particular $\theta = 2\sin^{-1}\left(\frac{r}{w+r}\right)$ and $w = r\left(\csc\frac{\theta}{2} - 1\right)$.
  They stated and proved their second lemma.
- Lemma 2: If two mutually tangent circles of radii r and R, thought $R > r$, are inscribed in an angle θ, then
  (a) $\sin\frac{\theta}{2} = \frac{R-r}{R+r}$, $\cos\frac{\theta}{2} = \frac{2\sqrt{rR}}{R+r}$, $\sin\theta = \frac{4(R-r)\sqrt{rR}}{R+r}$.
  $b)$ $\frac{R}{r} = \frac{1+\sin\frac{\theta}{2}}{1-\sin\frac{\theta}{2}}$.

Following their solution to the two-scoop situation, they made two comments; 1) Nothing in Lemma 1 or Lemma 2 assumed that r and R are integers, so there are uncountably many pairs of mutually tangent inscribed circles: $x = 9z, y = 121z$, for z any positive real.

2) Where do the scoops of ice cream lodge? The distance from the vertex of angle B to the scoop of radius 9 is $1\frac{25}{56}$, while the distance from the vertex out to the (large, chocolate) scoop of radius 121 is $19\frac{25}{56}$.

A note from Kenneth Korbin, proposer of the problem.

We are given a pair of radii with lengths 16 and 49.

$$\frac{x}{y} = \left(\frac{1-\sqrt{\frac{16}{49}}}{1+\sqrt{\frac{16}{49}}}\right)^2 = \left(\frac{1-\frac{4}{7}}{1+\frac{4}{7}}\right)^2 = \frac{9}{121}.$$

Similarly,

$$\frac{16}{49} = \left(\frac{1-\sqrt{\frac{x}{y}}}{1+\sqrt{\frac{x}{y}}}\right)^2.$$

So,

$$\begin{aligned}
\frac{49+16}{49-16} &= \frac{65}{33}\\
\frac{121+9}{121-9} &= \frac{130}{112} = \frac{65}{56}\\
\sin A &= \frac{2\cdot 33\cdot 56}{33^2+56^2} = \frac{3696}{65^2}\\
\frac{49}{16} &< \frac{121}{9}, \text{ therefore,}\\
A &< B
\end{aligned}$$

Also solved by Michel Bataille, Rouen, France; Brian Bradie, Christopher Newport University, Newport News, VA; Paul M. Harms, North Newton, KS; Ioannis D. Sfikas, National Technical University of Athens, Greece; Albert Stadler, Herrliberg, Switzerland, and the proposer.

- **5584:** *Proposed by Michael Brozinsky, Central Islip, NY*

Let a and $n \geqslant 2$ be positive integers where $0 \leqslant a \leqslant n-1$.

Find the number of points of intersection of the curve C_1 whose parametric equations are:

$$\begin{aligned}
x &= (n-1)\cdot\cos\left(\frac{t}{n-1}\right) + \cos(t),\\
y &= (n-1)\cdot\sin\left(\frac{t}{n-1}\right) - \sin(t), \text{ where}\\
&\frac{a\cdot(n-1)\cdot 2\pi}{n} \leqslant t \leqslant \frac{(a+1)\cdot(n-1)\cdot 2\pi}{n}
\end{aligned}$$

and the curve C_2 whose parametric equations are:

$$\begin{aligned}
x &= (n+1)\cdot\cos\left(\frac{t}{n+1}\right) - \cos(t),\\
y &= (n+1)\cdot\sin\left(\frac{t}{n+1}\right) - \sin(t), \text{ where}\\
&\frac{a\cdot(n+1)\cdot 2\pi}{n} \leqslant t \leqslant \frac{(a+1)\cdot(n+1)\cdot 2\pi}{n}
\end{aligned}$$

Solution 1 by Ioannis D. Sfikas, National Technical University of Athens, Greece

Given two curves parametric equations, we can consider their *intersection points* and their *collision points.*

- An intersection point is where the two equations have the same x and y values, but possibly at different times.
- A collision point is where the two equations have the same x and y values at the same time.

Intersection points are points which are on both sets of equations (i.e., where the curves cross each other). In general, this will happen at different values of t for the two sets of parametric equations. Collision points are intersection points at which the parameter in both sets of parametric equations have the same value. If the sets of parametric equations are thought of as describing the motion of different particles and t is thought of as the time, intersection points are points passed by both particles, but possibly at different times, and collision points are intersection points reached by both particles at the same time.

To find intersection points: Create simultaneous equations for t and n by setting $x = x$ and $y = y$ for the two sets of equations, and then see if there are any solutions (for t and n in their designated intervals). If there are intersection points, the simultaneous equations will have solutions. If there are no intersection points, the simultaneous equations will have no solution or will lead to a contradiction.

To repeat: Set x in one set equal to x in the second set, and likewise with the y's. This will give two simultaneous equations, both involving n and t. Solve as simultaneous equations; for example, solve one equation for n or t and plug into the second to obtain an equation for just one variable. Sometimes it is more natural to solve for a function of the variable, such as solving for $\sin t$, rather than solving for t, if this makes the substitution into the second equation easier. After completely solving for one variable, substitute back into its parametric equations to find the points of intersection (x, y).

(1) If the resultant equation in one variable has no solutions, there are no points of intersection.

(2) Make sure the values of the parameters (n and t) are actually in the intervals specified for the parametric equations.

(3) The number of different values for t, say, and the number of points of intersection may be different, since different values of the parameter may give the same (x, y) point. You can only tell this by substituting back into the formulas for x and y.

(a) A *hypocycloid* is a special plane curve generated by the trace of a fixed point on a small circle that rolls within a larger circle. It is comparable to the cycloid but instead of the circle rolling along a line, it rolls within a circle. If the smaller circle has radius r, and the larger circle has radius $R = kr$, then the parametric equations for the curve can be given by either:

$$x = (R-r)\cos\theta + r\cos\left(\frac{R-r}{r}\theta\right), \quad y = (R-r)\sin\theta - r\sin\left(\frac{R-r}{r}\theta\right),$$

or:

$$x = r(k-1)\cos\theta + r\cos[(k-1)\theta], \quad y = r(k-1)\sin\theta - t\sin[(k-1)\theta].$$

If k is an integer, then the curve is closed, and has k cusps (i.e., sharp corners, where the curve is not differentiable). Specially for $k = 2$ the curve is a straight line and the circles are called *Cardano circles.* Girolamo Cardano was the first to describe these hypocycloids and their applications to high-speed printing. If k is a rational number, say $k = \frac{p}{q}$ expressed in simplest terms, then the curve has p cusps.

$$x = (n-1)\cos\left(\frac{t}{n-1}\right) + \cos t, \quad y = (n-1)\sin\left(\frac{t}{n-1}\right) - \sin t.$$

Then, we have: $r = n-1$, $R = r+1 = n$ and $k = \dfrac{n}{n-1}$. So, the hypocycloid has $n \geqslant 2$ cusps.

(b) If the circle rolls on the outside of the fixed circle, the curve traced out by is called an *epicycloid.* If the smaller circle has radius r, and the larger circle has radius $R = kr$, then the parametric equations for the curve can be given by either:

$$x = (R+r)\cos\theta - r\cos\left(\frac{R+r}{r}\theta\right), \quad y = (R+r)\sin\theta - r\sin\left(\frac{R+r}{r}\theta\right),$$

or:

$$x = r(k+1)\cos\theta - r\cos\left[(k+1)\theta, \quad y = r(k+1)\sin\theta - r\sin\left[(k+1)\theta\right].\right.$$

If k is a positive integer, then the curve is closed, and has k cusps (i.e., sharp corners, where the curve is not differentiable). If k is a rational number, say $k = \frac{p}{q}$ expressed as an irreducible fraction, then the curve has p cusps.

$$x = (n+1)\cos\left(\frac{t}{n+1}\right) - cos(t), \quad y = (n+1)\sin\left(\frac{t}{n+1}\right) - \sin t.$$

Then, we have: $r = n+1$, $R = 1-r = 2-n$ and $k = \dfrac{2-n}{n+1}$. So, the epicycloid has $2-n$ cusps. Since $n \geqslant 2$, then the epicycloid has 0 cusps.

The intersection points of the hypocycloid and the epicycloid are the common cusps-points, so they don't have intersection points, except at the end points of the curves.

Solution 2 by Albert Stadler, Herrliberg, Switzerland

We identify a point (x, y) in the Euclidean plane with the complex number $x + iy$ in the complex plane. The parametric equations for C_1 and C_2 then read as

$$C_1: \; x + iy = (n-1)e^{\frac{it}{n-1}} + e^{-it} \qquad C_2: \; x + iy = (n+1)e^{\frac{it}{n+1}} + e^{-it}.$$

We need to find all pairs (u, v) with

$$\frac{2\pi a(n-1)}{n} \leqslant u \leqslant \frac{2\pi(a+1)(n-1)}{n}, \quad \frac{2\pi a(n+1)}{n} \leqslant v \leqslant \frac{2\pi(a+1)(n+1)}{n}, \qquad (*)$$

such that

$$(n-1)e^{\frac{iu}{n-1}} + e^{-iu} = n(n+1)e^{\frac{iv}{n-1}} + e^{-iv}.$$

We take the squared modulus on both sides and find

$$(n-1)^2 + 1 + 2(n-1)\cos\left(\frac{un}{n-1}\right) = (n+1)^2 + 1 + 2(n+1)\cos\left(\frac{vn}{n+1}\right)$$

or equivalently

$$2(n-1)\cos\left(\frac{un}{n-1}\right)+2(n+1)\cos\left(\frac{vn}{n+1}\right)=4n.$$

This equality holds if and only if $\cos\left(\frac{un}{n-1}\right)=\cos\left(\frac{vn}{n+1}\right)=1$ which is equivalent to say that $\frac{un}{n-1}\equiv 0\,(mod\ 2\pi)$ and $\frac{v.Tn}{n-1}\equiv 0\,(mod\ 2\pi)$. Therefore there are integers j and k such that $u=2\pi j\frac{n-1}{n}$ and $v=2\pi k\frac{n+1}{n}$.

If $t=u$ the $(n-1)e^{\frac{it}{n-1}}+e^{-it}=ne^{\frac{2\pi ij}{n}}$.

If $t=v$ the $(n+1)e^{\frac{it}{n+1}}-e^{it}=ne^{\frac{2\pi ik}{n}}$.

From $ne^{\frac{2\pi ij}{n}}=ne^{\frac{2\pi ik}{n}}$ we deduce that $j\equiv k(mod\ n)$.

To sum up: C_1 and C_2 if and only if $t=u=2\pi j\frac{n-1}{n}$ in the parametric definition C_1 and $t=v=2\pi k\frac{n+1}{n}$ in the parametric definition C_2, where j and k are integers that satisfy $j\equiv k\ (mod\ n)$. In light of $(*)$ this means that there are exactly two points of intersection, namely the endpoints of the two curves.

Solution 3 by Kee-Wai Lau, Hong Kong, China

We show that there are two points of intersection of the curves C_1 and C_2. At a point of intersection, we have

$$(n-1)\cdot\cos\left(\frac{t}{n-1}\right)+\cos t=(n+1)\cdot\cos\left(\frac{s}{n+1}\right)-\cos s, \tag{1}$$

and

$$(n-1)\cdot\sin\left(\frac{t}{n-1}\right)-\sin t=(n+1)\cdot\sin\left(\frac{t}{n+1}\right)-\sin s, \tag{2}$$

where $\frac{a\cdot(n-1)\cdot 2\pi}{n}\leqslant t\leqslant\frac{(a+1)\cdot(n-1)\cdot\pi}{n}$ and $\frac{a\cdot(n+1)\cdot 2\pi}{n}\leqslant s\leqslant\frac{(a+1)\cdot(n+1)\cdot\pi}{n}$.

We square both sides of (1) and (2) and then add up the resulting equations to obtain

$$(n-1)^2+1+2(n-1)\left(\cos\left(\frac{t}{n-1}\right)\cos t-\sin\left(\frac{t}{n-1}\right)\sin t\right)$$

$$(n+1)^2+1-2(n+1)\left(\cos\left(\frac{s}{n+1}\right)\cos s+\sin\left(\frac{s}{n+1}\right)\sin s\right).$$

Simplifying by using the well-known compound angle formulas for $\cos(a+b)$ and $\cos(a-b)$, we obtain

$$2n \quad = \quad (n-1)\cos\left(\frac{nt}{n-1}\right)+(n+1)\cos\left(\frac{ns}{n+1}\right).$$

or $$(n-1)\sin^2\left(\frac{nt}{2(n-1)}\right)+(n+1)\sin^2\left(\frac{ns}{2(n+1)}\right)=0.$$

Hence $\sin\left(\frac{nt}{2(n-1)}\right)=\sin\left(\frac{ns}{2(n-1)}\right)=0$, yielding $t=\frac{2\pi a(n-1)}{n},\frac{2\pi(a+1)(n-1)}{n}$ and $s=\frac{2\pi a(n+1)}{n},\frac{2\pi(n+1)(a+1)}{n}$. It is easy to check that the entries $(t,s)=\left(\frac{2\pi a(n-1)}{n},\frac{2\pi(a+1)(n+1)}{n}\right)$ and $\left(\frac{2\pi(a+1)(n-1)}{n},\frac{2\pi a(n+1)}{n}\right)$ are not solutions to the simultaneous equations (1) and (2).

If $(t,s)=\left(\frac{2\pi a(n-1)}{n},\frac{2\pi a(n+1)}{n}\right),\left(\frac{2\pi(a+1)(n-1)}{n},\frac{2\pi(a+1)(n+1)}{n}\right)$, we have respectively the points of intersection $\left(n\cos\frac{2\pi a}{n},n\sin\frac{2\pi a}{n}\right)$ and $\left(n\cos\frac{2\pi(a+1)}{n},n\sin\frac{2\pi(a+1)}{n}\right)$.

These two points are distinct. For if $\cos\frac{2\pi a}{n}=\cos\frac{2\pi(a+1)}{n}$ and $\sin\frac{2\pi(a+1)}{n}$ then we have $\sin\frac{2\pi(a+1)}{n}=\cos\frac{\pi(2a+1)}{n}=0$, which is impossible. Hence our claim.

Also solved by the proposer.

- **5585:** *Proposed by Daniel Sitaru, National Economic College "Theodor Costescu," Mehedinti, Romania*

In ΔABC the following relationship holds:

$$\sin^4 A+\sin^4 B+\sin^4 C+\sin^4\Big(\frac{\pi}{3}+A\Big)+\sin^4\Big(\frac{\pi}{3}+B\Big)+\sin^4\Big(\frac{\pi}{3}+C\Big)\leqslant\frac{27}{8}$$

Solution 1 by Brian Bradie, Christopher Newport University, Newport News, VA

Note:

$$\begin{aligned}\sin^4\left(x-\frac{\pi}{6}\right) &= \left(\frac{\sqrt{3}}{2}\sin x-\frac{1}{2}\cos x\right)^4\\ &= \frac{1}{16}(9\sin^4 x-12\sqrt{3}\sin^3 x\cos x+18\sin^2 x\cos^2 x-4\sqrt{3}\sin x\cos^3 x+\cos^4 x),\\ \sin^4\left(x+\frac{\pi}{6}\right) &= \frac{1}{16}(9\sin^4 x+12\sqrt{3}\sin^3 x\cos x+18\sin^2 x\cos^2 x+4\sqrt{3}\sin x\cos^3 x+\cos^4 x),\end{aligned}$$

and

$$\begin{aligned}\sin^4\left(x-\frac{\pi}{6}\right)+\sin^4\left(x+\frac{\pi}{6}\right) &= \frac{1}{8}(9\sin^4 x+18\sin^2 x\cos^2 x+\cos^4 x)\\ &= \frac{1}{8}(1+8\sin^2 x(\sin^2 x+2\cos^2 x))\\ &= \frac{9}{8}-\cos^4 x.\end{aligned}$$

Thus,

$$\sin^4\left(x-\frac{\pi}{6}\right)+\sin^4\left(x+\frac{\pi}{6}\right)\leqslant\frac{9}{8}$$

for all x, with equality when $x = \frac{\pi}{2} + n\pi$ for any integer n. Because

$$\sin^4 x + \sin^4\left(x + \frac{\pi}{3}\right) \quad \text{is just a translation of} \quad \sin^4\left(x - \frac{\pi}{6}\right) + \sin^4\left(x + \frac{\pi}{6}\right)$$

by $\pi/6$, it follows that

$$\sin^4 x + \sin^4\left(x + \frac{\pi}{3}\right) \leqslant \frac{9}{8}$$

for all x, with equality when $x = \frac{\pi}{3} + n\pi$ for any integer n. Therefore, for any angles A, B, and C,

$$\sin^4 A + \sin^4 B + \sin^4 C + \sin^4\left(\frac{\pi}{3} + A\right) + \sin^4\left(\frac{\pi}{3} + B\right) + \sin^4\left(\frac{\pi}{3} + C\right) \leqslant \frac{27}{8},$$

with equality when $A = \frac{\pi}{3} + n_1\pi$, $B = \frac{\pi}{3} + n_2\pi$, and $C = \frac{\pi}{3} + n_3\pi$ for any integers n_1, n_2, and n_3. For the special case when A, B, and C are the angles in a triangle, this becomes

$$\sin^4 A + \sin^4 B + \sin^4 C + \sin^4\left(\frac{\pi}{3} + A\right) + \sin^4\left(\frac{\pi}{3} + B\right) + \sin^4\left(\frac{\pi}{3} + C\right) \leqslant \frac{27}{8},$$

with equality when $A = B = C = \frac{\pi}{3}$.

Solution 2 by Moti Levy, Rehovot, Israel

Applying Power Means Inequality $\left(M_{\frac{1}{4}} \leqslant M_1\right)$, we get

$$\sin^4 A + \sin^4 B + \sin^4 C \leqslant \frac{1}{27}\left(\sin A + \sin B + \sin C\right)^4 \tag{1}$$

and

$$\sin^4\left(\frac{\pi}{3} + A\right) + \sin^4\left(\frac{\pi}{3} + B\right) + \sin^4\left(\frac{\pi}{3} + C\right) \leqslant \frac{1}{27}\left(\sin\left(\frac{\pi}{3} + A\right) + \sin\left(\frac{\pi}{3} + B\right) + \sin\left(\frac{\pi}{3} + C\right)\right)^4 \tag{2}$$

It is well known that

$$\sin A + \sin B + \sin C \leqslant \frac{3\sqrt{3}}{2}. \tag{3}$$

(see, for example, the bible of geometric inequalities, Bottema et al. paragraph 2.2, page 18).

We will prove that $\sin\left(\frac{\pi}{3} + A\right) + \sin\left(\frac{\pi}{3} + B\right) + \sin\left(\frac{\pi}{3} + C\right) \leqslant \frac{3\sqrt{3}}{2}$.

$$\begin{aligned}
\sin\left(A + \frac{\pi}{3}\right) + \sin\left(B + \frac{\pi}{3}\right) &= 2\sin\left(\frac{A+B}{2} + \frac{\pi}{3}\right)\cos\left(\frac{A-B}{2}\right) \\
&\leqslant 2\sin\left(\frac{A+B}{2} + \frac{\pi}{3}\right) \\
&= 2\sin\left(\frac{A+B}{2}\right)\cos\left(\frac{\pi}{3}\right) + 2\cos\left(\frac{A+B}{2}\right)\sin\left(\frac{\pi}{3}\right) \\
&= 2\cos\left(\frac{C}{2}\right)\cos\left(\frac{\pi}{3}\right) + 2\sin\left(\frac{C}{2}\right)\sin\left(\frac{\pi}{3}\right)
\end{aligned}$$

$$
\begin{aligned}
&\sin\left(\frac{\pi}{3}+A\right)+\sin\left(\frac{\pi}{3}+B\right)+\sin\left(\frac{\pi}{3}+C\right)\\
\leqslant &2\cos\left(\frac{C}{2}\right)\cos\left(\frac{\pi}{3}\right)+2\sin\left(\frac{C}{2}\right)\sin\left(\frac{\pi}{3}\right)+\sin\left(C+\frac{\pi}{3}\right)\\
&=\cos\left(\frac{C}{2}\right)+\sqrt{3}\sin\left(\frac{C}{2}\right)+\frac{1}{2}\sin\left(C\right)+\frac{\sqrt{3}}{2}\cos\left(C\right)\\
&=\cos\left(\frac{C}{2}\right)+\sqrt{3}\sin\left(\frac{C}{2}\right)+\sin\left(\frac{C}{2}\right)\cos\left(\frac{C}{2}\right)+\frac{\sqrt{3}}{2}\left(2\cos^2\left(\frac{C}{2}\right)-1\right)\\
&=\cos\left(\frac{C}{2}\right)+\sqrt{3}\sqrt{1-\cos^2\left(\frac{C}{2}\right)}+\sqrt{1-\cos^2\left(\frac{C}{2}\right)}\cos\left(\frac{C}{2}\right)+\frac{\sqrt{3}}{2}\left(2\cos^2\left(\frac{C}{2}\right)-1\right).
\end{aligned}
$$

Now let $t:=\cos\left(\frac{C}{2}\right)$, and

$$
f\left(t\right):=t+\sqrt{3}\sqrt{1-t^2}+t\sqrt{1-t^2}+\frac{\sqrt{3}}{2}\left(2t^2-1\right),
$$

then

$$
\sin\left(\frac{\pi}{3}+A\right)+\sin\left(\frac{\pi}{3}+B\right)+\sin\left(\frac{\pi}{3}+C\right)\leqslant f\left(t\right).
$$

To find the maximum of $f\left(t\right)$,

$$
\frac{df}{dt}=\frac{1}{\sqrt{1-t^2}}\left(\sqrt{1-t^2}-\sqrt{3}t-2t^2+2\sqrt{3}t\sqrt{1-t^2}+1\right), \tag{4}
$$

$$
\frac{d^2f}{dt^2}=\frac{2\sqrt{3}\left(1-t^2\right)^{\frac{3}{2}}-3t+2t^3-\sqrt{3}}{\left(1-t^2\right)^{\frac{3}{2}}}. \tag{5}
$$

By solving (4), we find that $f\left(t\right)$ has critical point at $t=\frac{\sqrt{3}}{2}$ and at $t=-\frac{\sqrt{3}}{2}$. Since $\frac{d^2f}{dt^2}\left(\frac{\sqrt{3}}{2}\right)=-12\sqrt{3}<0$, then the critical point $t=\frac{\sqrt{3}}{2}$ is maximum. Since $\frac{d^2f}{dt^2}\left(-\frac{\sqrt{3}}{2}\right)=0$ then the critical point $t=-\frac{\sqrt{3}}{2}$ is inflection point.

We conclude that

$$
\sin\left(\frac{\pi}{3}+A\right)+\sin\left(\frac{\pi}{3}+B\right)+\sin\left(\frac{\pi}{3}+C\right)\leqslant f\left(\frac{\sqrt{3}}{2}\right)=\frac{3\sqrt{3}}{2}. \tag{6}
$$

By (3) and (6) we get

$$
\sin^4 A+\sin^4 B+\sin^4 C+\sin^4\left(\frac{\pi}{3}+A\right)+\sin^4\left(\frac{\pi}{3}+B\right)+\sin^4\left(\frac{\pi}{3}+C\right)\leqslant\frac{1}{27}\left(\frac{3\sqrt{3}}{2}\right)^4+\frac{1}{27}\left(\frac{3\sqrt{3}}{2}\right)^4=\frac{27}{8}.
$$

Solution 3 by Albert Stadler, Herrliberg, Switzerland

We prove the stronger inequality

$$\sin^4 x + \sin^4\left(x + \frac{\pi}{3}\right) \leqslant \frac{9}{8}$$

from which the claimed inequality immediately follows. It is not required that $A + B + C = \pi,\ A, B, C$ and be arbitrary.

We have

$$\frac{9}{8} - \sin^4 x - \sin^4\left(x + \frac{\pi}{3}\right) = \frac{9}{8} - \sin^4\left(x + \frac{\pi}{6} - \frac{\pi}{6}\right) - \sin^4\left(x + \frac{\pi}{6} + \frac{\pi}{6}\right) =$$

$$= \frac{9}{8} - \left(\frac{\sqrt{3}}{2}\sin\left(x + \frac{\pi}{6}\right) - \frac{1}{2}\cos\left(x + \frac{\pi}{6}\right)\right)^4 - \left(\frac{\sqrt{3}}{2}\sin\left(x + \frac{\pi}{6}\right) + \frac{1}{2}\cos\left(x + \frac{\pi}{6}\right)\right)^4 =$$

$$= \frac{9}{8} - \frac{9}{8}\sin^4\left(x + \frac{\pi}{6}\right) - \frac{9}{8}\sin^2\left(x + \frac{\pi}{6}\right)\cos^2\left(x + \frac{\pi}{6}\right) - \frac{1}{8}\cos^4\left(x + \frac{\pi}{6}\right) =$$

$$= \frac{9}{8} - \frac{9}{8}\sin^4\left(x + \frac{\pi}{6}\right) - \frac{9}{8}\sin^2\left(x + \frac{\pi}{6}\right)\left(1 - \sin^2\left(x + \frac{\pi}{6}\right)\right) - \frac{1}{8}\left(1 - \sin^2\left(x + \frac{\pi}{6}\right)\right)^2 =$$

$$1 - \frac{7}{8}\sin^2\left(x + \frac{\pi}{6}\right) - \frac{1}{8}\sin^4\left(x + \frac{\pi}{6}\right) \geqslant 0.$$

Equality holds if and only if $\sin\left(x + \frac{\pi}{6}\right) = \pm 1, if x \equiv \frac{\pi}{3}(pd\pi.$

Solution 4 by Kevin Soto Palacios, Huarmey, Perú

It is sufficient to show that

$$\sin^4 A + \sin^4\left(\frac{\pi}{3} + A\right) = \sin^4 A + \sin^4\left(\frac{2\pi}{3} - A\right) \leqslant \frac{9}{8}.$$

$$\sin^4 A + 8\sin^4\left(\frac{2\pi}{3} - A\right) = 2\left(1 - \cos 2A\right)^2 + 2\left(1 - \cos\left(\frac{4\pi}{3} - 2A\right)\right)^2 =$$

$$= 4 + 2\left(\cos^2 2A + \cos^2\left(\frac{4\pi}{3} - 2A\right)\right) - 4\left(\cos 2A + \cos\left(\frac{4\pi}{3} - 2A\right)\right).$$

$$LHS = 6 + \cos 4A + \cos\left(\frac{8\pi}{3} - 4A\right) - 8\cos\left(\frac{2\pi}{3}\right)\cos\left(2A - \frac{2\pi}{3}\right) =$$

$$= 6 + 2\cos\left(\frac{4\pi}{3}\right)\cos\left(4A - \frac{4\pi}{3}\right) + 4\cos\left(2A - \frac{2\pi}{3}\right).$$

$$LHS = 6 - 2\cos\left(4A - \frac{4\pi}{3}\right) + 4\cos\left(2A - \frac{2\pi}{3}\right) = 7 - 2\cos^2\left(2A - \frac{2\pi}{3}\right) + 4\cos\left(2A - \frac{2\pi}{3}\right) =$$

$$= 9 - 2\left(1 - \cos\left(2A - \frac{2\pi}{3}\right)\right)^2 \leqslant 9.$$

$$\Longrightarrow \sin^4 A + \sin^4\left(\frac{\pi}{3}+A\right) \leqslant \frac{9}{8}$$

$$\Longrightarrow \sin^4 A+\sin^4\left(\frac{\pi}{3}+A\right)+\sin^4 B+\sin^4\left(\frac{\pi}{3}+B\right)+\sin^4 C+\sin^4\left(\frac{\pi}{3}+C\right) \leqslant \frac{9}{8}+\frac{9}{8}+\frac{9}{8}=\frac{27}{8}.$$

Also solved by Michel Bataille, Rouen, France; Kee-Wai Lau, Hong Kong, China; Ioannis D. Sfikas, National Technical University of Athens, Greece, and the proposer.

- **5586:** *Proposed by Michel Bataille, Rouen, France*

For $n \in \mathbb{N}$ let

$$u_n = \frac{1}{n}\sum_{k=1}^{n} k e^{k/n^2}.$$

Find real numbers α, β such that $\lim_{n\to\infty}(u_n - \alpha n) = \beta$.

Solution 1 by Ulrich Abel, Technische Hochschule Mittelhessen, Germany

Starting with

$$\sum_{k=1}^{n} k e^{kt} = \frac{e^t + n e^{(n+2)t} - (n+1)\, e^{(n+1)t}}{\left(e^t - 1\right)^2}$$

we obtain

$$u_n := \frac{1}{n}\sum_{k=1}^{n} k e^{k/n^2} = e^{1/n^2}\frac{1 + n e^{1/n+1/n^2} - (n+1)\, e^{1/n}}{n\left(e^{1/n^2}-1\right)^2}.$$

Substitution $x = 1/n$ yields

$$\begin{aligned}
u_n &= e^{x^2}\frac{x + e^x\left(e^{x^2}-1-x\right)}{\left(e^{x^2}-1\right)^2} \\
&= \left(1 + x^2 + \frac{1}{2}x^4 + O\left(x^6\right)\right)\frac{x + \left(1 + x + \frac{1}{2}x^2 + \frac{1}{6}x^3 + \frac{1}{24}x^4 + O\left(x^5\right)\right)\left(-x + x^2 + \frac{1}{2}x^4 + O\left(x^6\right)\right)}{\left(x^2 + \frac{1}{2}x^4 + \frac{1}{6}x^6 + \cdots\right)^2} \\
&= \frac{1 + x^2 + \frac{1}{2}x^4 + O\left(x^6\right)}{x^4\left(1 + \frac{1}{2}x^2 + \frac{1}{6}x^4 + O\left(x^6\right)\right)^2}\left(\frac{1}{2}x^3 + \frac{5}{6}x^4 + \frac{5}{8}x^5 + O\left(x^6\right)\right) = \frac{1}{2x} + \frac{5}{6} + \frac{5}{8}x + O\left(x^2\right)
\end{aligned}$$

as $x \to \infty$. Therefore,

$$u_n = \frac{n}{2} + \frac{5}{6} + \frac{5}{8n} + O\left(n^{-2}\right) \qquad (n \to \infty).$$

This implies

$$\lim_{n\to\infty}\left(u_n - \frac{1}{2}n\right) = \frac{5}{6}.$$

Solution 2 by Albert Natian, Los Angeles Valley College,Valley Green, CA

Answer. $\alpha = \dfrac{1}{2}$, $\beta = \dfrac{5}{6}$.

Computation. It's rather immediate that

$$\sum_{k=0}^{n} e^{k\theta} = \frac{e^{(n+1)\theta} - 1}{e^{\theta} - 1},$$

and

$$\sum_{k=1}^{n} ke^{k\theta} = \frac{ne^{(n+2)\theta} - (n+1)\, e^{(n+1)\theta} + e^{\theta}}{\left(e^{\theta} - 1\right)^2}.$$

So

$$u_n = \frac{1}{n} \sum_{k=1}^{n} ke^{k/n^2} = \frac{1}{n} \cdot \frac{ne^{(n+2)/n^2} - (n+1)\, e^{(n+1)/n^2} + e^{1/n^2}}{\left(e^{1/n^2} - 1\right)^2}.$$

Set $n = 1/x$ and write

$$Q_n := u_n - \alpha n - \beta = x \cdot \frac{\frac{1}{x} e^{\left(x+2x^2\right)} - \left(\frac{1}{x} + 1\right) e^{\left(x+x^2\right)} + e^{x^2}}{\left(e^{x^2} - 1\right)^2} - \frac{\alpha}{x} - \beta,$$

$$Q_n = \frac{xe^{\left(x+2x^2\right)} - x\,(x+1)\, e^{\left(x+x^2\right)} + x^2 e^{x^2} - (\alpha + \beta x) \left(e^{x^2} - 1\right)^2}{x\left(e^{x^2} - 1\right)^2}.$$

Since the denominator of the preceding fraction can be expressed as

$$x\left(e^{x^2} - 1\right)^2 = x^5 \left(\sum_{k=1}^{\infty} \frac{x^{2(k-1)}}{k!}\right)^2$$

and since

$$\lim_{x \to 0^+} \left(\sum_{k=1}^{\infty} \frac{x^{2(k-1)}}{k!}\right)^2 = 1,$$

then

$$\lim_{n \to \infty} Q_n = \lim_{x \to 0^+} \frac{xe^{\left(x+2x^2\right)} - x\,(x+1)\, e^{\left(x+x^2\right)} + x^2 e^{x^2} - (\alpha + \beta x) \left(e^{x^2} - 1\right)^2}{x^5}.$$

In order to find α and β so that $\lim\limits_{n \to \infty} Q_n = 0$, we write the numerator N of the preceding fraction in Taylor expansion, and then simplify the fraction:

N_0 :

$$xe^{\left(x+2x^2\right)} = x\left(1 + \left(x + 2x^2\right) + \frac{1}{2}\left(x + 2x^2\right)^2 + \frac{1}{6}\left(x + 2x^2\right)^3 + \frac{1}{24}\left(x + 2x^2\right)^4 + \cdots\right),$$

$$-x\,(x+1)\, e^{\left(x+x^2\right)} = -x\,(x+1)\left(1 + \left(x + x^2\right) + \frac{1}{2}\left(x + x^2\right)^2 + \frac{1}{6}\left(x + x^2\right)^3 + \frac{1}{24}\left(x + x^2\right)^4 + \cdots\right),$$

$$x^2 e^{x^2} = x^2\left(1 + x^2 + \frac{1}{2}x^4 + \cdots\right),$$

$$-(\alpha + \beta x)\left(e^{x^2} - 1\right)^2 = -x^4 (\alpha + \beta x)\left(1 + \frac{1}{2}x^2 + \cdots\right)^2,$$

D_0: x^5.

Multiply N_0 and D_0 by $1/x$ and simplify to get:

$D_1 = x^4$,

$N_1 = \left(1 + x + \frac{5}{2}x^2 + \frac{13}{6}x^3 + \frac{73}{24}x^4 + \cdots\right)$
$- \left(1 + 2x + \frac{5}{2}x^2 + \frac{8}{3}x^3 + \frac{53}{24}x^4 + \cdots\right)$
$+ \left(x + x^3 + \frac{1}{2}x^5 + \cdots\right)$
$- \left(\alpha x^3 + \beta x^4 + \alpha x^5 + \cdots\right)$,

$N_1 = \left(\frac{1}{2} - \alpha\right) x^3 + \left(\frac{5}{6} - \beta\right) x^4 + \cdots = \left(\frac{1}{2} - \alpha\right) x^3 + \left(\frac{5}{6} - \beta\right) x^4 + \mathcal{O}\left(x^5\right).$

We thus have

$$\lim_{n\to\infty} Q_n = \lim_{x\to 0^+} \frac{N_1}{D_1} = \lim_{x\to 0^+} \frac{\left(\frac{1}{2} - \alpha\right) x^3 + \left(\frac{5}{6} - \beta\right) x^4 + \mathcal{O}\left(x^5\right)}{x^4} = 0$$

which implies

$$\alpha = \frac{1}{2} \quad \text{and} \quad \beta = \frac{5}{6}.$$

Solution 3 by Brian Bradie, Christopher Newport University,Newport News, VA

We will show that $\alpha = \frac{1}{2}$ and $\beta = \frac{5}{6}$. We start with

$$\begin{aligned}
\lim_{n\to\infty} (u_n - \alpha n) &= \lim_{n\to\infty} \frac{\sum_{k=1}^n ke^{k/n^2} - \alpha n^2}{n} \\
&= \lim_{n\to\infty} \left(\sum_{k=1}^{n+1} ke^{k/(n+1)^2} - \sum_{k=1}^{n} ke^{k/n^2} - \alpha(n+1)^2 + \alpha n^2\right) \\
&= \lim_{n\to\infty} \left((n+1)e^{1/(n+1)} - \alpha(2n+1) + \sum_{k=1}^{n} k\left(e^{k/(n+1)^2} - e^{k/n^2}\right)\right),
\end{aligned}$$

where the second line follows from the Stolz-Cesaro theorem. Now, as $n \to \infty$,

$$\begin{aligned}
(n+1)e^{1/(n+1)} &= (n+1)\left(1+\frac{1}{n+1}+O\left(\frac{1}{(n+1)^2}\right)\right) \\
&= n+2+O\left(\frac{1}{n}\right), \\
e^{k/(n+1)^2}-e^{k/n^2} &= k\left(\frac{1}{(n+1)^2}-\frac{1}{n^2}\right)+O\left(k^2n^{-5}\right) \\
&= -\frac{2n+1}{n^2(n+1)^2}k+O\left(k^2n^{-5}\right), \text{ and} \\
\sum_{k=1}^{n} k\left(e^{k/(n+1)^2}-e^{k/n^2}\right) &= -\frac{2n+1}{n^2(n+1)^2}\sum_{k=1}^{n}k^2+O\left(\frac{1}{n}\right) \\
&= -\frac{(2n+1)^2}{6n(n+1)}+O\left(\frac{1}{n}\right).
\end{aligned}$$

Hence,

$$\lim_{n\to\infty}(u_n-\alpha n)=\lim_{n\to\infty}\left((1-2\alpha)n+2-\alpha-\frac{(2n+1)^2}{6n(n+1)}+O\left(\frac{1}{n}\right)\right).$$

This limit exists if and only if $1-2\alpha=0$; that is, if and only if $\alpha=\dfrac{1}{2}$. With $\alpha=\dfrac{1}{2}$,

$$\lim_{n\to\infty}(u_n-\alpha n)=\lim_{n\to\infty}\left(\frac{3}{2}-\frac{(2n+1)^2}{6n(n+1)}+O\left(\frac{1}{n}\right)\right)=\frac{3}{2}-\frac{2}{3}=\frac{5}{6}=\beta.$$

Solution 4 by Hatef I. Arshagi, Guilford Technical Community College, Jamestown, NC

We will show that $\alpha=\dfrac{1}{2}$ and $\beta=\dfrac{5}{6}$.

From $e^x=1+\dfrac{x}{1!}+\dfrac{x^2}{2!}+\dfrac{x^3}{3!}+\ldots$, we conclude that $e^x>1+x$ for all $x\geqslant 1$, and this implies that for $1\leqslant k\leqslant n$, $1+\dfrac{k}{n^2}<e^{\frac{k}{n^2}}$, and $\dfrac{k}{n}\left(1+\dfrac{k}{n^2}\right)=\dfrac{k}{n}+\dfrac{k^2}{n^3}<\dfrac{k}{n}\cdot e^{\frac{k}{n^2}}$, that is $\dfrac{k}{n}+\dfrac{k^2}{n^3}<\dfrac{k}{n}\cdot e^{\frac{k^2}{n^2}}$, and from this we can write

$$\sum_{k=1}^{n}\left(\frac{k}{n}+\frac{k^2}{n^3}\right)<\frac{1}{n}\sum_{k=1}^{n}ke^{\frac{k}{n^2}}=u_n, \tag{1}$$

It is known that $\displaystyle\sum_{k=1}^{n}k=\frac{n(n+1)}{2}$ and $\displaystyle\sum_{k=1}^{n}k^2=\frac{n(n+1)(2n+1)}{6}$ (2)

then

$$\sum_{k=1}^{n}\left(\frac{k}{n}+\frac{k^2}{n^2}\right)=\frac{n(n+1)}{2n}+\frac{n(n+1)(2n+1)}{6n^3}=\frac{1}{2}n+\frac{5}{6}+\frac{3n+1}{6n^2}$$

and (1) becomes

$$\frac{1}{2}n + \frac{5}{6} + \frac{3n+1}{6n^2} < u_n. \tag{3}$$

For $0 < x < 1$,

$$e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots < 1 + x + x^2 + = \frac{1}{1-x},$$

from this we conclude that for $n > 1$, and $k \leqslant n$, we have

$$e^{\frac{k}{n^2}} < \frac{1}{1 - \frac{k}{n^2}} = 1 + \frac{k}{n^2 - k} < 1 + \frac{k}{n^2 - n},$$

or $e^{\frac{k}{n^2}} < 1 + \dfrac{k}{n^2 - n}$, then

$$\frac{k}{n}\left(e^{\frac{k}{n^2}}\right) < \frac{k}{n}\left(1 + \frac{k}{n^2 - n}\right) = \frac{k}{n} + \frac{k^2}{n(n^2-n)}, \text{ and by using (2),}$$

$$\mathrm{u}_n = \frac{1}{n}\sum_{k=1}^{n} k e^{\frac{k}{n^2}} < \sum_{k=1}^{n}\left(\frac{k}{n} + \frac{k^2}{n(n^2-n)}\right) = \frac{n(n+1)}{2n} + \frac{n(n+1)(2n+1)}{6n(n^2-n)} =$$

$$= \frac{n+1}{2} + \frac{(n+1)(2n+1)}{6(n^2-n)} = \frac{1}{2}n + \frac{5}{6} + \frac{5n+1}{6n^2 - 6n},$$

that is,

$$u_n < \frac{1}{2}n + \frac{5}{6} + \frac{5n+1}{6n^2 - 6n}. \tag{4}$$

Combining (3) and (4), we write

$$\frac{1}{2}n + \frac{5}{6} + \frac{3n+1}{6n^2} - \alpha n < u_n - \alpha n < \frac{1}{2}n + \frac{5}{6} + \frac{5n+1}{6n^2 - 6n} - \alpha n,$$

or

$$\left(\frac{1}{2} - \alpha\right)n + \frac{5}{6} + \frac{3n+1}{6n^2} < u_n - \alpha n < \left(\frac{1}{2} - \alpha\right)n + \frac{5}{6} + \frac{5n+1}{6n^2 - 6n}. \tag{5}$$

Now, we argue that, we must have $\alpha = \dfrac{1}{2}$.

If $\alpha < \dfrac{1}{2}$, then $\dfrac{1}{2} - \alpha > 0$, and $\left(\dfrac{1}{2} - \alpha\right)n + \dfrac{5}{6} + \dfrac{3n+1}{6n^2} < u_n - \alpha n$, implies that $\lim\limits_{n\to\infty}(u_n - \alpha n) = \infty$, contradiction to $\lim\limits_{n\to\infty}(u_n - \alpha n) = \beta$, and if $\alpha > \dfrac{1}{2}$, then $\dfrac{1}{2} - \alpha < 0$, and $u_n - \alpha n < \left(\dfrac{1}{2} - \alpha\right)n + \dfrac{5}{6} + \dfrac{5n+1}{6n^2 - n}$, implies $\lim\limits_{n\to\infty}(u_n - \alpha n) = -\infty$, another contradiction to $\lim\limits_{n\to\infty}(u_n - \alpha n) = \beta$, therefore $\alpha = \dfrac{1}{2}$ and for any large n , we rewrite the (5) as

$$\frac{5}{6}+\frac{3n+1}{6n^2}<u_n-\alpha n<\frac{5}{6}+\frac{5n+1}{6n^2-6n} \tag{6}$$

and $\lim_{n\to\infty}\left(\frac{5}{6}+\frac{3n+1}{6n^2}\right)=\frac{5}{6}=\lim_{n\to\infty}\left(\frac{5}{6}+\frac{5n+1}{6n^2-6n}\right)$, then by the Squeeze Theorem from (6), we conclude that $\lim_{n\to\infty}(u_n-\alpha n)=\frac{5}{6}$.

Also solved by G. C. Greubel, Newport News, VA; Kee-Wai Lau, Hong Kong, China; Moti Levy, Rehovot, Israel; Albert Stadler, Herrliberg, Switzerland; Ioannis D. Sfikas, National Technical University of Athens, Greece, and the proposer

- **5587:** *Proposed by José Luis Díaz-Barrero, Barcelona Tech, Barcelona, Spain*

Let $f,g:\Re\to\Re$ be two real functions defined by $f(x)=x^4+1$ and $g(x)=a_0+a_1x^3+a_2x^5+a_3x^7-x^9$ where $a_1<0, a_2<0, a_3<0$ and a_0 is a real number. Find the number of real solutions to the equation

$$(g\circ f)(x)=(f\circ g)(x).$$

Solution 1 by David E. Manes, Oneonta, NY

With the assumptions on $a_1,a_2,a_3<0$, we will show that there are no solutions to the equation $(g\circ f)(x)=(f\circ g)(x)$. Observe that the function

$$(f\circ g)(x)=f(g(x))=(g(x))^4+1=(a_0+a_1x^3+a_2x^5+a_3x^7-x^9)^4+1$$

has a total of 69 non constant terms before combining like terms. However, the constant term is easy to evaluate; namely, $(a_0)^4+1$. The constant term for the function

$$(g\circ f)(x)=g(f(x))=a_0+a_1(x^4+1)^3+a_2(x^4+1)^5+a_3(x^4+1)^7-(x^4+1)^9$$

is $a_0+a_1+a_2+a_3-1$. Therefore, if there is a solution to the equation $f\circ g=g\circ f$, then the constant terms for the two functions have to be equal. Thus,

$$(a_0)^4+1=a_0+a_1+a_2+a_3-1$$

which can be rewritten as $(a_0)^4-a_0=a_1+a_2+a_3-2<-2$ since $a_1,a_2,a_3<0$. If $a_0\leqslant 0$ or $a_0\geqslant 1$, then $(a_0)^4-a_0\geqslant 0$. If $0<a_0<1$, then $-1<a_0<0$. Hence, for each of the possible values for a_0, the term $(a_0)^4-a_0$ is not less than -2. Accordingly then, there are no solutions to the equation $(g\circ f)(x)=(f\circ g)(x)$.

Solution 2 by Moti Levy, Rehovot, Israel

$$(g\circ f)(x)=a_0+a_1\left(x^4+1\right)^3+a_2\left(x^4+1\right)^5+a_3\left(x^4+1\right)^7-\left(x^4+1\right)^9$$
$$(f\circ g)(x)=\left(a_0+a_1x^3+a_2x^5+a_3x^7-x^9\right)^4+1$$

$$(g\circ f)(0)=a_0+a_1+a_2+a_3-1$$
$$(f\circ g)(0)=(a_0+a_1+a_2+a_3-1)^4+1$$

Since $(g \circ f)(0) = (f \circ g)(0)$, then

$$(a_0 + a_1 + a_2 + a_3 - 1)^4 + 1 = a_0 + a_1 + a_2 + a_3 - 1.$$

Let $A := a_0 + a_1 + a_2 + a_3 - 1$, then we seek real solution to $A^4 - A + 1 = 0$. But $A^4 - A + 1 = \left(A^2 - 1\right)^2 + A^2 > 0$.
Therefore the polynomial $A^4 - A + 1$ has no real root and our equation has no real solution.

Solution 3 by Albert Stadler, Herrliberg, Switzerland

We claim that the equation has no real solutions. Suppose that there is a real value of x such that

$$(g \circ f)(x) = g(x^4 + 1) = g^4(x) + 1 = (f \circ g)(x).$$

We observe that $x^4 + 1 - x = x^4 + (1 - x) > 0$ for all real x, since $x^4 + 1 - x = x^4 + (1 - x) > 0$ for $x < 1$, and $x^4 + 1 - x = (x^4 - x) + 1 > 0$ for $x \geqslant 1$.

Therefore

$$g(x^4 + 1) - g(x) = g^4(x) + 1 - g(x) > 0$$

for all real x.

By the mean-value theorem (https://.wikipedia.org/wiki/Meanvalue theorem) there is a real y, such that $x \leqslant y \leqslant x^4 + 1$ and for which $g(x^4 + 1) - g(x) = g'(y)(x^4 + 1 - x)$.
This implies that $g'(y) > 0$, since $g(x^4 + 1) - g(x) > 0$ and $x^4 + 1 - x > 0$.
However, $g'(y) = 3a_1y^2 + 5a_2y^4 + 7a_3y^6 - 9y^8 \leqslant 0$, since by assumption $a_1 < 0,\ a_2 < 0,\ a_3 < 0$.
This contradiction shows that the equation$(g \circ f)(x) =(f \circ g)(x)$ has no real solutions.

Solution 4 by Kee-Wai Lau, Hong Kong, China

We show that the equation $(g \circ f)(x) = (f \circ g)(x)$ has no real solutions. We have

$$\begin{aligned}(f \circ g)(x) - (g \circ f)(x) &= (a_0 + a_1x^3 + a_2x^5 + a_3x^7 - x^9)^4 + 1 - a_0 - a_1(x^4 + 1)^3 - \\ &\quad a_2(x^4 + 1)^5 - a_3(x^4 + 1)^7 + (x_4 + 1)^9.\end{aligned}$$

We consider $(f \circ g)(x) - (g \circ f)(x)$ as a function of a_0 and denote it by $h(a_0)$.
To prove our claim, it suffices to show that $h(a_0) > 0$ for any real numbera_0.
It is clear that $\lim\limits_{a_0 \to -\infty} h(a_0) = +\infty$. Hence it remains to find its

stationary values. Since $\dfrac{dh}{da_0} = 4(a_0 + a_1x^3 + a_2x^5 + a_3x^7 - x^9)^3 - 1$, so it vanishes if and only if $a_0 = a_0^*$, where $a_0 = \dfrac{1}{2^{2/3}} - a_1x^3 - a_2x^5 - a_3x^7 + x^9$.

Now $h(a_0^*) = \dfrac{1}{2^8/3} + 1 - \dfrac{1}{2^2/3} - a_1\left((x^4 + 1)^3 - x^3\right) - a_2\left((x^4 + 1)^5 - x^5\right)$
$-a_2\left((x^4 + 1)^7 - x^7\right) + (x^4 + 1)^9 - x^9$.
For any odd positive integer n, it is clear that $(x^4 + 1)^n - x^n > 0$ if $x \leqslant 0$. If $x > 0$, then $(x^4 + 1)^n - x^n > 0 \iff x^4 + 1 > x$, which can be verified easily by

considering the cases $0 < x \leqslant 1$ and $x > 1$ separately. Given that $a_1 < 0$, $a_2 < 0, a_3 < 0$, we conclude that $h(a_0^*) > 0$, and hence our claim.

Also solved by the proposer.

- **5588:** *Proposed by Ovidiu Furdui and Alina Sîntămărian, Technical University of Cluj-Napoca, Cluj-Napoca, Romania*

Let $a > 1$. Calculate

$$\lim_{x\to\infty} x\int_0^1 a^{t(t-1)x}\mathrm{d}t.$$

Solution 1 by Bruno Salgueiro Fanego, Viveiro, Spain

$$\lim_{x\to\infty} x\int_0^1 a^{t(t-1)x}dt = \lim_{x\to\infty} x\int_0^1 a^{\left(\left(t-\frac{1}{2}\right)^2-\frac{1}{4}\right)x}dt \underbrace{=}_{u=t-\frac{1}{2}} \lim_{x\to\infty} x\int_{-\frac{1}{2}}^{\frac{1}{2}} a^{\left(u^2-\frac{1}{4}\right)x}du$$

$$= \lim_{x\to\infty} 2x\int_0^{\frac{1}{2}} a^{\left(u^2-\frac{1}{4}\right)x}du$$

$$= \lim_{x\to\infty} \frac{2x\int_0^{\frac{1}{2}} a^{u^2x}du}{a^{\frac{x}{4}}}$$

$$\underbrace{=}_{v=u\sqrt{x}} \lim_{v\to\infty} \frac{2x\int_0^{\frac{\sqrt{x}}{2}} a^{v^2}dv}{\sqrt{x}a^{\frac{x}{4}}}$$

$$\underbrace{=}_{w=\sqrt{x}} \lim_{w\to\infty} \frac{2\int_0^{\frac{w}{2}} a^{v^2}dv}{w^{-1}a^{\frac{w^2}{4}}} \left[=\frac{\infty}{\infty} = \text{Indeterminate}\right]$$

$$\underbrace{=}_{L'H\hat{o}pital} \lim_{w\to\infty} \frac{2a^{\frac{w^2}{4}}\frac{1}{2}}{-w^{-2}a^{\frac{w^2}{4}} + w^{-1}a^{\frac{w^2}{4}}\ln a\frac{2w}{4}}$$

$$= \lim_{w\to\infty} \frac{1}{-w^{-2}+\frac{\ln a}{2}} = \frac{2}{\ln a}.$$

Solution 2 by Seán M. Stewart, Bomaderry, NSW, Australia

The limit will be shown to have a value equal to $\dfrac{2}{\ln a}$, where $a > 1$.
For $x > 0$, denote the integral appearing in the limit by $I(x)$. Then

$$I(x) = \int_0^1 a^{t(t-1)x}\,dx = \int_0^1 e^{t(t-1)x\ln a}\,dt,$$

or

$$I(x) = \int_0^1 \exp\left[\left\{\left(t-\frac{1}{2}\right)^2 - \frac{1}{4}\right\} x\ln a\right] dt,$$

after completing the square. Here $a > 1$. Substituting $u = t - \frac{1}{2}$ gives

$$\begin{aligned} I(x) &= \int_{-\frac{1}{2}}^{\frac{1}{2}} \exp\left[\left(u^2 - \frac{1}{4}\right) x \ln a\right] du \\ &= 2\exp\left(-\frac{x \ln a}{4}\right) \int_0^{\frac{1}{2}} e^{x \ln a u^2}\, du, \end{aligned}$$

since the integrand is an even function between symmetric limits. Next, enforcing a substitution of $u \mapsto u/\sqrt{x \ln a}$ yields

$$\begin{aligned} I(x) &= \frac{2}{\sqrt{x \ln a}} \exp\left(-\frac{x \ln a}{4}\right) \int_0^{\frac{1}{2}\sqrt{x \ln a}} e^{u^2}\, du \\ &= \sqrt{\frac{\pi}{x \ln a}} \exp\left(-\frac{x \ln a}{4}\right) \operatorname{erfi}\left(\frac{1}{2}\sqrt{x \ln a}\right). \end{aligned}$$

Here $\operatorname{erfi}(x)$ denotes the *imaginary error function* defined by

$$\operatorname{erfi}(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{u^2}\, du.$$

Denoting the required limit to be found by ℓ, we have

$$\ell = \lim_{x\to\infty} xI(x) = \frac{\sqrt{\pi}}{\ln a} \lim_{x\to\infty} \sqrt{x \ln a} \exp\left(-\frac{x \ln a}{4}\right) \operatorname{erfi}\left(\frac{1}{2}\sqrt{x \ln a}\right),$$

or

$$\ell = \frac{\sqrt{\pi}}{\ln a} \lim_{x\to\infty} \sqrt{x} e^{-\frac{x}{4}} \operatorname{erfi}\left(\frac{1}{2}\sqrt{x}\right),$$

after a substitution of $x \mapsto x/\ln a$ in the limit has been enforced. The asymptotic expansion for the imaginary error function as $x \to \infty$ is known (see Eq. (42:6:4) on page 429 of Keith Oldham, Jan Myland, and Jerome Spanier *An Atlas of Functions* (Second Edition), Springer, New York, 2009). The result is

$$\operatorname{erfi}(x) \sim \frac{e^{x^2}}{x\sqrt{\pi}} \sum_{n=0}^{\infty} \frac{(2n)!}{4^n n! x^{2n}}.$$

Replacing x with $\sqrt{x}/2$ in the asymptotic expansion for the imaginary error function gives

$$\operatorname{erfi}\left(\frac{\sqrt{x}}{2}\right) \sim \frac{2}{\sqrt{\pi x}} e^{\frac{x}{4}} \sum_{n=0}^{\infty} \frac{(2n)!}{n! x^n} = \frac{2}{\sqrt{\pi x}} e^{\frac{x}{4}} \left(1 + \frac{2}{x} + \frac{12}{x^2} + \mathcal{O}\left(\frac{1}{x^3}\right)\right),$$

as $x \to \infty$. Thus

$$\ell = \frac{2}{\ln a} \lim_{x\to\infty} \left(1 + \frac{2}{x} + \frac{12}{x^2} + \mathcal{O}\left(\frac{1}{x^3}\right)\right) = \frac{2}{\ln a},$$

as announced.

Solution 3 by Michel Bataille, Rouen, France

The required limit is $\dfrac{2}{\ln(a)}$.

First, we treat the case $a = e$. For $x > 0$, let $I(x) = \int_0^1 e^{t(t-1)x}\,dt$. The change of variables $t = \frac{1}{2} - \frac{u}{\sqrt{x}}$ yields

$$I(x) = \frac{1}{\sqrt{x}} \int_{-\sqrt{x}/2}^{\sqrt{x}/2} e^{u^2 - \frac{x}{4}}\,du = \frac{2e^{-x/4}}{\sqrt{x}} \int_0^{\sqrt{x}/2} e^{u^2}\,du.$$

Now, let $f(u) = \frac{e^{u^2}}{2(u+1)}$. The derivative $f'(u) = \frac{e^{u^2}(2u^2+2u-1)}{2u^2+4u+2}$ satisfies $f'(u) \sim e^{u^2}$ as $u \to \infty$, hence $\int_0^\infty f'(u)\,du$ is divergent and $\int_0^X f'(u)\,du \sim \int_0^X e^{u^2}\,du$ as $X \to \infty$.
As a result, we obtain

$$\int_0^X e^{u^2}\,du \sim f(X) - f(0) = \frac{e^{X^2}}{2(X+1)} - \frac{1}{2} \sim \frac{e^{X^2}}{2X}$$

as $X \to \infty$. Returning to $I(x)$, we see that

$$I(x) \sim \frac{2e^{-x/4}}{\sqrt{x}} \cdot \frac{e^{x/4}}{\sqrt{x}} = \frac{2}{x}$$

as $x \to \infty$ and therefore $\lim\limits_{x\to\infty} xI(x) = 2$.
Finally, since $x\int_0^1 a^{t(t-1)x}\,dt = x\int_0^1 e^{t(t-1)x\ln(a)}\,dt = \frac{1}{\ln(a)}\left((x\ln(a))I(x\ln(a))\right)$, we obtain

$$\lim_{x\to\infty} x\int_0^1 a^{t(t-1)x}\,dt = \frac{2}{\ln(a)}.$$

Solution 4 by G. C. Greubel, Newport News, VA

Define the complex error functions by

$$erfi(x) = \frac{2}{\sqrt{\pi}} \int^x e^{u^2}\,du$$

which has the series form

$$erfi(x) = \frac{2}{\sqrt{\pi}} \sum_{k=0}^{\infty} \frac{x^{2n+1}}{n!\,(2n+1)}$$

and has the asymptotic form

$$erfi(x) \sim \frac{1}{i} + \frac{1}{\sqrt{\pi}\,x}\, e^{x^2}\, {}_2F_1\left(1, \frac{1}{2}; -; \frac{1}{x^2}\right) \qquad (2)$$

$$\sim \frac{1}{i} + \frac{1}{\sqrt{\pi}\,x}\, e^{x^2}\left(1 + \frac{1}{2\,x^2} + \frac{3}{4\,x^4} + \mathcal{O}\left(\frac{1}{x^6}\right)\right). \qquad (3)$$

Consider the integral

$$I(a) = \int_0^1 a^{x\,t(t-1)}\,dt.$$

Using

$$x\,t(t-1) = x\left(t-\frac{1}{2}\right)^2 - \frac{x}{4}$$

$$\begin{aligned}
\int a^{x\,t(t-1)}\,dt &= \int e^{\ln(a)\,x\,t(t-1)}\,dt \\
&= e^{-x\ln a/4}\int e^{\left(\sqrt{x\ln a}\,(t-\frac{1}{2})\right)^2}\,dt \\
&= \frac{1}{\sqrt{x\,\ln a}}\,e^{-x\ln a/4}\int e^{u^2}\,du \quad \text{where} \quad u=\sqrt{x\ln a}\left(t-\frac{1}{2}\right) \\
&= \frac{1}{2}\sqrt{\frac{\pi}{x\,\ln a}}\,e^{-x\ln a/4}\,erfi\left(\sqrt{x\ln a}\left(t-\frac{1}{2}\right)\right).
\end{aligned}$$

The integral in question, $I(a)$, can be seen as

$$I(a) = \sqrt{\frac{\pi}{x\,\ln a}}\,e^{-x\ln a/4}\,erfi\left(\frac{\sqrt{x\,\ln a}}{2}\right).$$

The asymptotic form of the integral follows from

$$erfi\left(\frac{\sqrt{x\,\ln a}}{2}\right) \sim -i + \frac{2}{\sqrt{\pi\,\ln a}}\,e^{x\,\ln a/4}\left(1+\frac{2}{x\,\ln a}+\frac{12}{x^2\,\ln^2 a}+\mathcal{O}\left(\frac{1}{x^3}\right)\right)$$

and is seen to be

$$I(a) \sim -i\sqrt{\frac{\pi}{x\,\ln a}}\,e^{-x\,\ln a/4} + \frac{2}{x\,\ln a}\left(1+\frac{2}{x\,\ln a}+\frac{12}{x^2\,\ln^2 a}+\mathcal{O}\left(\frac{1}{x^3}\right)\right).$$

Since

$$\lim_{x\to\infty}\sqrt{x}\,e^{-x\,\ln a/4} \to 0$$

it follows that

$$\begin{aligned}
\lim_{x\to\infty} x\int_0^1 a^{x\,t(t-1)}\,dt &= \lim_{x\to\infty} x\,I(a) \\
&= -i\sqrt{\frac{\pi}{\ln a}}\lim_{x\to\infty}\sqrt{x}\,e^{-x\,\ln a/4} \\
&\qquad + \lim_{x\to\infty}\frac{2}{\ln a}\left(1+\frac{2}{x\,\ln a}+\frac{12}{x^2\,\ln^2 a}+\mathcal{O}\left(\frac{1}{x^3}\right)\right) \\
\lim_{x\to\infty} x\int_0^1 a^{x\,t(t-1)}\,dt &= \frac{2}{\ln a}.
\end{aligned}$$

This last result is the desired result.

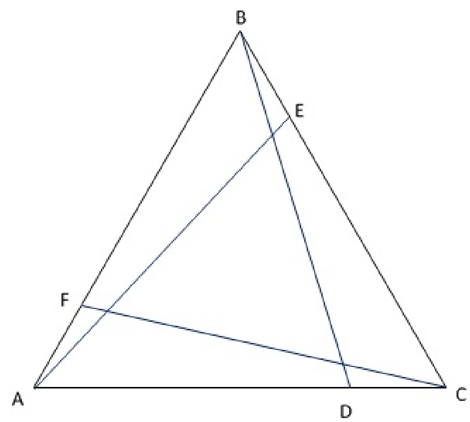
Also solved by Ulrich Abel, Technische Hochschule Mittelhessen, German; Brian Bradie, Christopher Newport University, Newport News,VA; Kee-Wai Lau, Hong Kong, China; Moti Levy, Rehovot, Israel; Albert Natian, Los Angeles Valley College, Valley Glen, CA; Ioannis D. Sfikas, National Technical University of Athens, Greece; Albert Stadler, Herrliberg, Switzerland, and the proposers.

Editor's Comment

The following solution by Michael Fried to 5578 arrived late, but it is so different in direction from the solutions paths that were featured in the May issue of the column, that I thought it would be instructive to also publish his solution.

5578: Problem poser: Roger Izard, Dallas, TX; Solution by Michael Fried, Ben-Gurion University of the Negev, Beer-Sheva, Israel

In triangle ABC points F, E, and D lie on lines segments AB, BC, and AC respectively, such that 1) $\dfrac{AF}{BA} = \dfrac{BE}{BC} = \dfrac{DC}{AC}$ and 2) $\angle BAE = \angle CBD = \angle ACF$.



Prove or disprove that triangle ABC must be an equilateral triangle**.**

Solution:
We shall show that indeed if both $\dfrac{AF}{BA} = \dfrac{BE}{BC} = \dfrac{DC}{AC}$ and $\angle BAE = \angle CBD = \angle ACF$, then triangle ABC must be equilateral.

First, observe that when ABC is equilateral, $\dfrac{AF}{BA} = \dfrac{BE}{BC} = \dfrac{DC}{AC}$ if and only if $\angle BAE = \angle CBD = \angle ACF$.

Suppose then that ABC is an arbitrary triangle for which conditions 1) and 2) hold. Since any triangle can be mapped onto any other via an affine transformation, let T be an affine transformation mapping ABC onto some equilateral triangle $A'B'C'$. We might as well choose one with the same centroid O as ABC for this will then be a fixed point for T. Since T is affine, we will have

$$\frac{A'F'}{B'A'} = \frac{B'E'}{B'C'} = \frac{D'C'}{A'C'}$$

(where $T(F) = F'$ and so on).

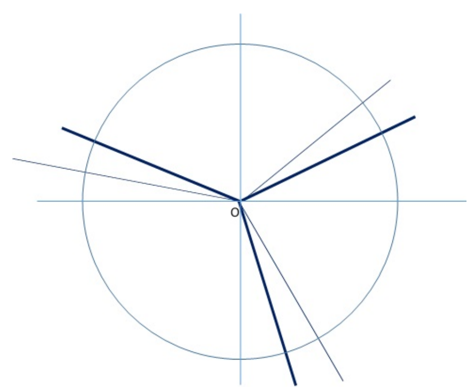
However, by the observation above we will also have, $\angle B'A'E' = \angle C'B'D' = \angle A'C'F'$. What will show is that, in fact, at least one of these angles must be unequal to the others, *unless*

T is a similarity in which case ABC was equilateral to begin with. Therefore, we will have shown that there cannot be a triangle ABC which is not equilateral for which conditions 1) and 2) hold.

To show that if T is not a similarity one of the angles $\angle B'A'E'$, $\angle C'B'D'$, $\angle A'C'F'$ must be unequal to the others, recall that it is always possible to represent an affine transformation as the composition of a similarity and a compression (see reference below [p. 124]GTrans). By the latter we mean a transformation K such that, for an appropriately chosen x and y axes, $K(P(x, y) = P'(x', y')$ with $x' = x$ and $y' = \rho y$ for some positive number ρ which can take, without loss of generality to lie in the interval $(0, 1]$ (thus the x-axis is a fixed line).

To investigate the effect of K (the compression part of T) on the equal angles, draw the angles $\angle BAE$, $\angle CBD$, $\angle ACF$ through the centroid O keeping the sides of the angles parallel to original angles. In the figure the thicker lines are parallel to the sides of the triangle: the angles between those thicker lines are naturally the exterior angles at each vertex. Since O is a fixed point the fixed axis and the compression axis for K must run through O.

The transformation K, being affine, takes parallel lines to parallel lines, so it will change these angles at O in the same way that it changes the original angles. The similarity part of T has no effect on the angles. Hence, we need only show that K will leave at least one of these angles unequal to the others.

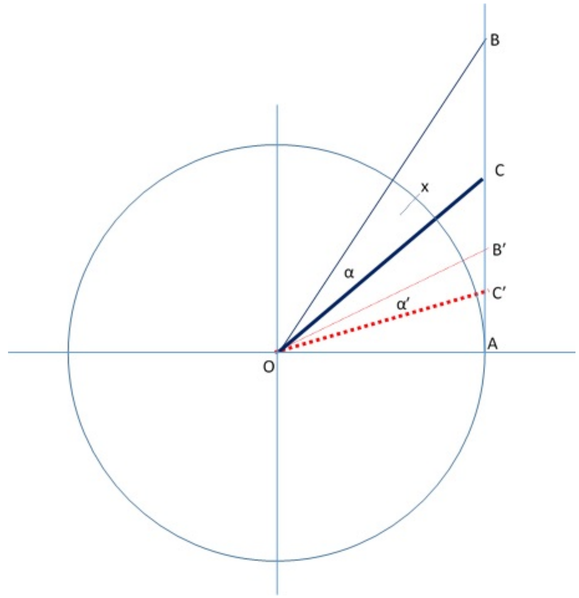


For convenience, define the *position* to be the position of its angle bisector. Thus, for example, we will say that two angles are placed symmetrically with respect to an axis when their angle bisectors form equal angles with the axis. This is actually an important case for us, for it is clear that if two of the angles are placed symmetrically with respect to the horizontal fixed axis or to the vertical compression axis, they will obviously be equal after compression. So, by means of one or two reflections, we can place all three angles in the first quadrant.

Note that having transposed the angles into the first quadrant, two angles can be in the same position, *but not all three.* For, if three angles are in the same position, two sides of the original triangle must be parallel which is impossible.

Let us, then, consider the effect of K on an angle α whose position is x radians from A. In the figure we are assuming the radius of the circle is 1 and AB is drawn tangent at A. So, let $\angle BOC = \alpha$, $\angle BOA - \dfrac{\alpha}{2} = \angle COA + \dfrac{\alpha}{2} = x$, and $\angle B'OC' = \alpha' = f(x)$. By the compression

K, then, $AC' = \rho AC$ and $AB' = \rho AB$.



The function $f(x)$ is then:

$$f(x) = \arctan\left(\rho \tan\left(x + \frac{\alpha}{2}\right)\right) - \arctan\left(\rho \tan\left(x - \frac{\alpha}{2}\right)\right)$$

The derivative of $f(x)$, after some simplifications, is,

$$f'(x) = \rho\left(\frac{1}{\rho^2 + (1-\rho^2)\cos^2\left(x + \frac{\alpha}{2}\right)} - \frac{1}{\rho^2 + (1-\rho^2)\cos^2\left(x - \frac{\alpha}{2}\right)}\right)$$

From this one can check that this function has a minimum at $x = 0$ and a maximum at $x = \dfrac{\pi}{2}$ so that it is strictly increasing in the first quadrant. This means that the only way all three angles can remain equal after compression is that they are all in the same position in the first quadrant, which, as we said, is impossible.

References

Modenov, P. S. and Parkhomenko, A. S. (1965). *Geometric Transformations, vol.1.* Translation, Michael B. P. Slater. New York: Academic Press

Mea − Culpa

Because of the Coronavirus and the chaos it caused in many countries, some solutions were mailed on time but did not arrive on time for publication and acknowledgment. Such was the case with the solution to 5571, by **Paul M. Harms of North Newton, KS.**