Problems

Ted Eisenberg, Section Editor

This section of the Journal offers readers an opportunity to exchange interesting mathematical problems and solutions. Please send them to Ted Eisenberg, Department of Mathematics, Ben-Gurion University, Beer-Sheva, Israel or fax to: 972-86-477-648. Questions concerning proposals and/or solutions can be sent e-mail to <eisenbt@013.net>. Solutions to previously stated problems can be seen at <http://www.ssma.org/publications>.

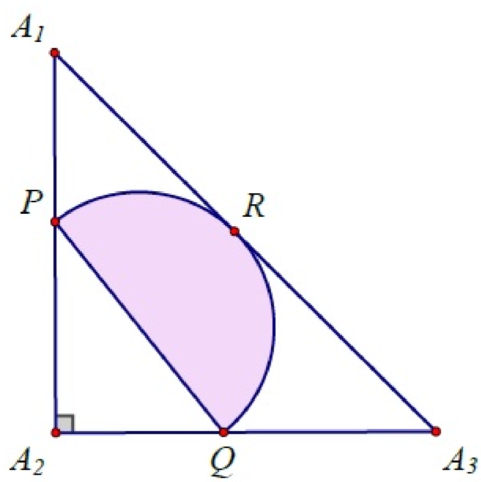
Solutions to the problems stated in this issue should be posted before January 15, 2021

5607: *Proposed by Kenneth Korbin, New York, NY*

Given $\triangle ABC$ with integer area and with altitude $\overline{CD}$. Find the sides if $\overline{BC} = \overline{AC} + 1 = \overline{CD} + 100$.

5608: *Proposed by Arsalan Wares, Valdosta State University, Valdosta, Georgia*

Triangle $A_1A_2A_3$ is a right isosceles triangle with $\angle A_1A_2A_3 = 90°$. Point P is on side A_1A_2 such that $\dfrac{A_1P}{PA_2} = \dfrac{4}{5}$. Point Q is on side A_2A_3 and arc PQR, touching side A_1A_3 at point R, is semicircular. If $A_1A_2 = 3$, find the exact length A_1R.



5609: *Proposed by Ángel Plaza, Universidad de Las Palmas de Gran Canaria, Spain*

Evaluate the following limit without using derivatives:

$$\lim_{x\to 0}\frac{(1+a\ln(1+bx^3))^{1/x}-1}{x^2}.$$

5610: *Proposed by Albert Natian, Los Angeles Valley College, Valley Glen, CA*

Find the smallest positive number x such that the following three quantities a, b and c are all integers;

$$a=a(x)=\sqrt[4]{1+\sqrt{19+\sqrt{\frac{x}{15}}}}\;+\;\sqrt[4]{1+\sqrt[3]{\frac{x}{100}}},$$

$$b=b(x)=\sqrt{\frac{5x}{48}}+\sqrt[3]{26+\frac{x}{20000}},$$

$$c=c(x)=\sqrt[3]{\frac{2x}{25}}.$$

5611: *Proposed by José Luis Díaz-Barrero, Barcelona Tech, Barcelona, Spain*

Let $n\geqslant 1$ be an integer number. Prove that the following inequality holds:

$$\sum_{k=1}^{n}\sqrt{\frac{k\tan^{-1}(k)}{\tan^{-1}(k)+\tan^{-1}(n-k)}}\leqslant\frac{(n+1)\sqrt{n}}{2}.$$

(Here, $\tan^{-1}(x)$ represents $\arctan(x)$ for all real x.)

5612: *Proposed by Ovidiu Furdui and Alina Sîntămărian, Technical University of Cluj-Napoca, Cluj-Napoca, Romania*

Let $a,b\in\mathbb{R}$ with $a-b=\dfrac{\gamma}{2}$. Calculate

$$\lim_{n\to\infty}\left(2\mathrm{e}^{1+\frac{1}{2}+\cdots+\frac{1}{n}-a}-\sqrt{n}\mathrm{e}^{1+\frac{1}{3}+\cdots+\frac{1}{2n-1}-b}\right).$$

(Note: γ is the Euler-Mascheroni constant, which is defined as the $\lim_{n\to\infty}(H_n-\ln(n))$. Here, H_n denotes the nth harmonic partial sum $1+\dfrac{1}{2}+\dfrac{1}{3}+\ldots+\dfrac{1}{n}$.)

Solutions

- **5589:** *Proposed by Kenneth Korbin, New York, NY*

  Find the dimensions of a triangle with integer length sides if it can be inscribed in a circle with diameter $16\sqrt{7}$.

  Solution 1 by David A. Huckaby, Angelo State University, San Angelo, TX

  Note that since the diameter of the circle is $16\sqrt{7} \approx 42.33$, the largest each of the triangle's sides a, b, and c can be is 42. The many possible triples (a, b, c) can be checked by computer, but we will instead narrow the possibilities down to just a few triples.

  From the well-known result that the radius of the circumcircle is

$$R = \frac{abc}{4\sqrt{s(s-a)(s-b)(s-c)}},$$

  where $s = (a + b + c)$, we have

$$\begin{aligned} 16R^2 s(s-a)(s-b)(s-c) &= a^2b^2c^2 \\ 16(8\sqrt{7})^2 s(s-a)(s-b)(s-c) &= a^2b^2c^2 \\ 7 \cdot 2^{10} \cdot s(s-a)(s-b)(s-c) &= a^2b^2c^2 \\ 7 \cdot 2^{10} \cdot \frac{a+b+c}{2}\left(\frac{a+b+c}{2} - a\right)\left(\frac{a+b+c}{2} - b\right)\left(\frac{a+b+c}{2} - c\right) &= a^2b^2c^2 \\ 7 \cdot 2^6 \cdot (a+b+c)(b+c-a)(a+c-b)(a+b-c) &= a^2b^2c^2 \end{aligned}$$

  We will use the last equation to show that a, b, and c must each be divisible by 7, and we will accomplish this by showing that the left side of the equation—and hence $a^2b^2c^2$—is divisible by 7^3. Since a, b, and c are each less than or equal to 42, and thus less than $49 = 7^2$, this will imply that each must then have exacly one 7 as a factor.

  We begin by observing that since 7 is a factor of the left side of the equation, 7 must be a factor of $a^2b^2c^2$, and hence of at least one of a, b, or c. Due to the symmetry of a, b, and c in the equation, we may assume without loss of generality that 7 divides a. Then since 7^2 divides a^2, 7^2 is a factor of the left side of the equation, and therefore 7 divides at least one of the last four factors in the left side of the equation.

  Because 7 divides a, 7 divides the factor $a + b + c$ if and only if 7 divides the factor $b + c - a$. Similarly, 7 divides the factor $a + c - b$ if and only if 7 divides the factor $a + b - c$. We have already established that 7 divides at least one of these four factors, hence 7 divides at least two of them, so that the left side of the equation is indeed divisible by 7^3.

  Having thus established that a, b, and c are each divisible by 7, we see that the left side of the equation is divisible by 7^6. Combining this with the fact that a, b, and c each being divisible by 7 implies that each of the last four factors on the left side has at least one 7 as a factor, we have that exactly one of these last four factors has 7^2 as a factor. If we take a to be the minimum of a, b, and c, then it is either the factor $a + b + c$ or the factor $b + c - a$ that is divisible by 7^2. Because a, b, and c are each less than or equal to 42, we thus have $a + b + c$ equal to either 49 or 98 or $b + c - a$ equal to 49.

We now consider the implications of both sides of the equation being divisible by 2^6. Again noting that a, b, and c are each less than or equal to 42, we have that at least two of a, b, and c are even. (Indeed, if only one were even, it would have to have a factor of 2^3, and with its additional factor of 7, it would be divisible by 56, and so be larger than 42.)

So we have narrowed down the possibilities to triangles having one side that is a multiple of 7 and is less than or equal to 42 to go along with two even sides of 14 and 14, 14 and 28, 14 and 42, 28 and 28, 28 and 42, or 42 and 42, subject to the constraint that $a+b+c$ equals 49 or 98 or $b+c-a$ equals 49. Applying the constraint and checking the few remaining possibilities yields the two solutions $a=28$, $b=28$, $c=42$ and $a=28$, $b=35$, $c=42$.

Solution 2 by Bruno Salgueiro Fanego, Viveiro, Spain

Let a, b, and c be the integer lengths of the sides of triangle ABC, Let S be its area and $2R = 16\sqrt{7}$, its circumscribing diameter.

From Heron's formula, $S^2 = \dfrac{a+b+c}{2} \cdot \dfrac{-a+b+c}{2} \cdot \dfrac{a-b+c}{2} \cdot \dfrac{a+b-c}{2}$ and from the generalized theorem of sines, $S = \dfrac{1}{2}ab\sin C = \dfrac{1}{2}ab\dfrac{c}{2R} = \dfrac{abc}{4\cdot 8\sqrt{7}}$, so

$$448(a+b+c)(-a+b+c)(a-b+c)(a+b-c) = a^2b^2c^2.$$

Moreover, since ABC is inscribed in a circle with diameter $R = 8\sqrt{7}$, then $a, b.c < 16\sqrt{7}$; taking into account that a, b, and c are integers, then $a, b, c \leqslant 42$. Let us suppose without loss of generality that $a \leqslant b \leqslant c$. Hence, we can check which of those integer pairs (a, b) with $1 \leqslant a, b \leqslant 42$ satisfy the equality $448(a+b+c)(-a+b+c)(a-b+c)(a+b-c) = a^2b^2c^2$ when $c = 1, 2, \ldots, 42$.

Note also that a and b must satisfy the triangle inequalities, $a+b > c, a-b < c$ and $b-a < c$, which are equivalent, since $a \leqslant b$, to $b-a < c < b+a$.

The only case when that equality has a positive integer solution is when $c = 42$, and in that case $a = 28$ and $b \in \{28, 35\}$.

Hence, the dimensions of a triangle with integer length sides that can be inscribed in a circle with diameter $16\sqrt{7}$ are $28, 28$ and 42 (which is an isosceles triangle) or $28, 35$ and 42 (which is a scalene triangle.)

Solution 3 by Kee-Wai Lau, Hong Kong, China

It is well known that the diameter of the circle circumscribing the triangle with sides a, b and c equals $\dfrac{2abc}{\sqrt{(a+b+c)(a+b-c)(b+c-a)(c+a-b)}}$, so that

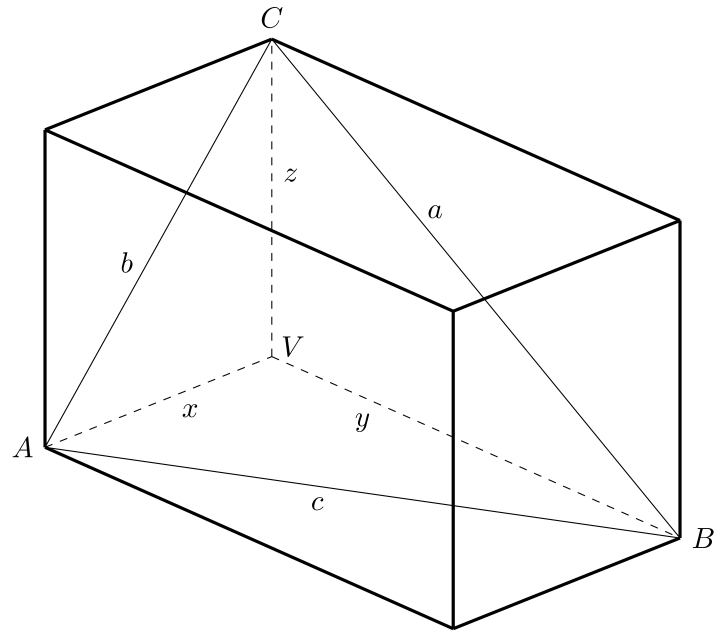
$$8\sqrt{7(a+b+c)(a+b-c)(b+c-a)(c+a-b)} - abc = 0.$$

Since $a, b, c \leqslant 16\sqrt{7}$, so with the help of a computer, we find that the sides of the triangles are 28, 28, 42 and 28, 35, 42.

Also solved by Brian D. Beasely, Presbyterian College, Clinton, SC; Pratik Donga, India; Ronald Martins, Brasília, Brazil; Albert Stadler, Herrliberg, Switzerland; David Stone and John Hawkins, Georgia Southern University, Statesboro, GA;

Daniel Văcaru, Piteşti, Romania; Titu Zvonaru, Comănesti, Romania, and the proposer.

- **5590:** *Proposed Albert Natian, Los Angeles Valley College, Valley Glen, CA*



Let V be a vertex of a rectangular box. Let $\overline{VA}$, $\overline{VB}$ and $\overline{VC}$ be the three edges meeting at vertex V. We are given that the perimeter of the triangle $\triangle ABC$ is $(28 + \sqrt{106})$, the total surface area of the box is 426, and the length of the main diagonal of the box is $5\sqrt{10}$. Find the area of the triangle $\triangle ABC$.

Editor′s Comment: This problem has a hidden subtlety buried in it that was missed by almost everyone, and for some who noticed it, they ended up assuming it anyway. Different approaches eventually yielded an equation for which the authors assumed had a unique solution; and in finding this solution they assumed that the dimensions of the box x, y, z were integers. But nowhere in the problem is this stated. In effect they solved the problem using data that was not stated, and obtained the correct answer that the area of $\triangle ABC = \dfrac{3}{2}\sqrt{1921}$.

Albert Natian proposed this problem and he and I have exchanged many e-mails over some of the solutions submitted. In each one discussed he found an unstated assumption being used. An exception to this is the solution submitted by Titu Zvonaru, With respect to this solution Albert wrote:

"On the other hand, the Solution by Titu Zvonaru is excellent and very rigorous. As you can see, there's a lot of work that he had to do in order to reach a solution. I commend him for his hard work and success!"

Although most of the submitted solutions solved the problem by assuming along their paths that the sides lengths of the box were integers, some were very novel and instructive; one of them is included below.

Albert's solution to this problem is long, and so I have moved it to the end of this column, along with some insightful and instructive comments he made in our correspondence about problem solving in general.

Solution 1 by Titu Zvonaru, Comănesti, Romania

We have $x^2+y^2+z^2=250,\ xy+yz+zx=213$ and this yields that $x+y+z=26.$ It follows that

$$\sqrt{x^2+y^2}+\sqrt{y^2+z^2}+\sqrt{z^2+x^2}=28+\sqrt{106}$$

$$2(x^2+y^2+z^2)+2\left(\sqrt{(x^2+y^2)(y^2+z^2)}+\sqrt{(y^2+z^2)(z^2+z^2)}+\sqrt{(z^2+x^2)(x^2+y^2)}\right)=890+56\sqrt{106}$$

$$\sqrt{(x^2+y^2)(y^2+z^2)}+\sqrt{(y^2+z^2)(z^2+x^2)}+\sqrt{(z^2+x^2)(x^2+y^2)}=195+28\sqrt{106}$$

$$x^4+y^4+z^4+3\left(x^2y^2+y^2z^2+z^2x^2\right)+2\sqrt{(x^2+y^2)(y^2+z^2)(z^2+x^2)}\left(\sqrt{x^2+y^2}+\sqrt{y^2+z^2}+\sqrt{z^2+x^2}\right)=$$
$$\left(195+28\sqrt{106}\right)^2$$

$$x^2y^2+y^2z^2+z^2x^2+2(28+\sqrt{106})\sqrt{x^2+y^2)(y^2+z^2)(z^2+x^2)}=\left(195+28\sqrt{106}\right)^2-(x^2+y^2+z^2)^2$$

$$213^2-2xyz(x+y+z)+2(28+\sqrt{106})\sqrt{(x^2+y^2+z^2)(x^2y^2+y^2z^2+z^2x^2)-x^2y^2z^2}=$$
$$\left(195+28\sqrt{106}\right)^2-250^2$$

$$-52xyz+2(28+\sqrt{106})\sqrt{250(213^2-52xyz)-x^2y^2z^2}=13260+10920\sqrt{106})$$

$$-26xyz+(20+\sqrt{106})\sqrt{250(213^2-52xyz)-x^2y^2z^2}=6630+5460\sqrt{106}$$

$$(28+\sqrt{106})^2(250\cdot 213^2-250\cdot 52xyz-x^2y^2z^2)=(6630+5460\sqrt{106}+26xyz)^2 \qquad (1)$$

The quadratic equation (1),with unknown xyz, has two real roots: one is equal to 540 and the other is negative. It results that $xyz = 540$. Since $x+y+z = 26,\ xy+yz+zx = 216$ and$xyz = 540$, the real numbers x, y, z are the roots of the equation $t^3 - 26t^2 + 213t - 540 = 0$.

We obtain $(x, y, z) = (5, 9, 12)$ since $a^2 = 225,\ b^2 = 169,\ c^2 = 106$. If F is the area the triangle ABC, then:

$$16F^2 = 2a^2b^2 + 2b^2c^2 + 2c^2a^2 - a^4 - b^4 - c^4 = 69156 = 36 \cdot 1921 \implies F = \frac{3\sqrt{1921}}{2}.$$

Solution 2 by Brian D. Beasley, Presbyterian College, Clinton, SC

We let $P = a + b + c = 28 + \sqrt{106}$. Then $x^2 + y^2 + z^2 = 250$, $xy + yz + zx = 213$, and

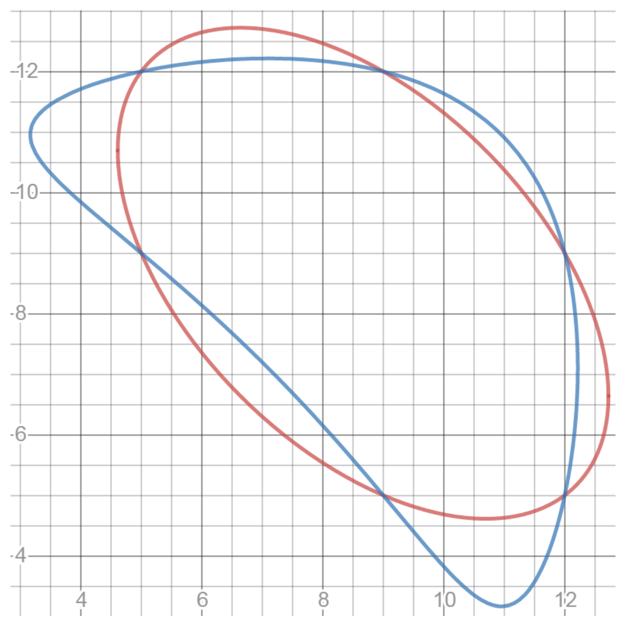
$$\sqrt{y^2+z^2} + \sqrt{z^2+x^2} + \sqrt{x^2+y^2} = P.$$

This implies $(x+y+z)^2 = 250+2(213) = 676$, so $x+y+z = 26$. Hence $x^2+y^2+(26-x-y)^2 = 250$, or equivalently $x^2 + y^2 + xy - 26x - 26y + 213 = 0$ (the equation of an ellipse). Since we also have

$$\sqrt{250-x^2} + \sqrt{250-y^2} + \sqrt{x^2+y^2} = P,$$

there are six points in the intersection of the graphs of these two equations (see below), yielding the solution $\{x, y, z\} = \{5, 9, 12\}$. Hence $\{a, b, c\} = \{\sqrt{106}, 13, 15\}$, so the area of triangle ΔABC is $3\sqrt{1921}/2$.

Addenda. (1) Here is a graph in the first quadrant of the equations $x^2+y^2+xy-26x-26y+213 = 0$ (in red) and $\sqrt{250-x^2} + \sqrt{250-y^2} + \sqrt{x^2+y^2} = P$ (in blue).



(2) The corresponding equations for a, b, and c are $a^2+b^2+ab-Pa-Pb+\dfrac{1}{2}P^2-250=0$ and

$$\sqrt{250-a^2}+\sqrt{250-b^2}+\sqrt{a^2+b^2-250}=26.$$

The six points in their intersection yield the solution $\{a,b,c\}=\{\sqrt{106},13,15\}$.

Comment by Albert Natian, proposer of the problem. This solution involves the graphs of two curves and the visual inspection of their intersections, which, by design, have integral coordinates. If these coordinates were chosen to be non-integer, graphing techniques may be unsuccessful at solving the problem.

Also solved (meaning a correct answer was submitted; names followed with an $*$ means that this particular solution was not discussed with Albert.) Bruno Salgueiro Fanego*, Viveiro, Spain; Kee-Wai Lau*, Hong Kong, China; David E. Manes, Oneonta, NY; Ronald Martins, Brasília, Brazil; David Stone and John Hawkins, Georgia Southern University, Statesboro, GA; Daniel Văcaru, Piteşti, Romania, and the proposer.

5591: *Proposed by Daniel Sitaru, National Economic College "Theodor Costescu," Mehedinti, Romania*

Solve for real numbers:

$$3^{\cos x+\cos y+\cos z}=3^{\cos^2 x+\cos x}+3^{\cos^2 y+\cos y}+3^{\cos^2 z+\cos z}.$$

Solution 1 by Brian Bradie, Christopher Newport University, Newport News, VA

The given equation is equivalent to

$$3^{\cos^2 x-\cos y-\cos z}+3^{\cos^2 y-\cos x-\cos z}+3^{\cos^2 z-\cos x-\cos y}=1.$$

By Jensen's inequality,

$$\begin{aligned}
&3^{\cos^2 x-\cos y-\cos z}+3^{\cos^2 y-\cos x-\cos z}+3^{\cos^2 z-\cos x-\cos y}\\
&\quad\geqslant\ 3\cdot 3^{\frac{1}{3}(\cos^2 x-2\cos x+\cos^2 y-2\cos y+\cos^2 z-2\cos z)}\\
&\quad=\ 3^{\frac{1}{3}(\cos x-1)^2+\frac{1}{3}(\cos y-1)^2+\frac{1}{3}(\cos z-1)^2}\\
&\quad\geqslant\ 1.
\end{aligned}$$

At the first inequality, equality holds if and only if $\cos^2 x+\cos x=\cos^2 y+\cos y=\cos^2 z+\cos z$; at the second inequality, equality holds if and only if $\cos x=\cos y=\cos z=1$. Thus,

$$3^{\cos^2 x-\cos y-\cos z}+3^{\cos^2 y-\cos x-\cos z}+3^{\cos^2 z-\cos x-\cos y}=1$$

if and only if $\cos x=\cos y=\cos z=1$. The solutions to

$$3^{\cos x+\cos y+\cos z}=3^{\cos^2 x+\cos x}+3^{\cos^2 y+\cos y}+3^{\cos^2 z+\cos z}$$

are therefore the ordered triples $(x, y, z) = (2\pi j, 2\pi k, 2\pi \ell)$ for any integers j, k, and ℓ.

Solution 2 by Albert Stadler, Herrliberg, Switzerland

By the AM-GM inequality

$$3^{\cos^2 x+\cos x} + 3^{\cos^2 y+\cos y} + 3^{\cos^2 z+\cos z} \geqslant 3 \cdot 3^{\frac{\cos^2 x+\cos x+\cos^2 y+\cos y+\cos^2 z+\cos z}{3}}.$$

Hence,

$$3^{\cos x+\cos y+\cos z-1-\frac{\cos^2 x+\cos x+\cos^2 y+\cos y+\cos^2 z+\cos z}{3}} =$$
$$= 3^{-\frac{(\cos x-1)^2+(\cos y-1)^2+(\cos z-1)^2}{3}} \geqslant 1,$$

which implies $(\cos x-1)^2+(\cos y-1)^2+(\cos z-1)^2 \leqslant 0$, $\cos x = \cos y = \cos z = 1$, and finally, x$\equiv y \equiv z \equiv 0 (\text{mod } 2\pi)$.

Solution 3 by Pratik Donga, India

Let $\cos x = a, \cos y = b$, and $\cos z = c$. This implies:

$$\begin{aligned} 3^{a+b+c} &= 3^{a^2+a} + 3^{b^2+b} + 3^{c^2+c} \\ &\Rightarrow 3^a \cdot 3^b \cdot 3^c = 3^{a^2} \cdot 3^{b^2} + 3^b \cdot 3^{c^2} \cdot 3 \\ &\Rightarrow 1 = 3^{a^2-b-c} + 3^{b^2-a-c} + 3^{c^2-a-b} \qquad (1) \end{aligned}$$

In Eq (1) if $a^2-b-c = b^2-a-c = c^2-a-b = -1$, then

$$1 = 3^{-1}+3^{-1}+3^{-1} = \frac{1}{3}+\frac{1}{3}+\frac{1}{3} = \frac{1}{3}(3) = 1.$$

This implies, to make Eq (1) true, that the LHS=RHS, that is:

$$a^2-b-c = b^2-a-c = c^2-a-b = -1, \qquad (A)$$

Now,

$$a^2-b-c = b^2-a-c \Rightarrow a^2+a = b^2+b \Rightarrow a^2-b^2 = b-a \qquad (2)$$

Similarly,

$$b^2-a-c = c^2-a-b \Rightarrow b^2+b = c^2+b = c \Rightarrow b^2-c^2 = c-b \qquad (3)$$

In Eq(2) $a^2 - b^2 = b - a \Rightarrow a + b = -1 \Rightarrow c^2 = -2$, and
in Eq(3) $b^2 - c^2 = c - b \Rightarrow b + c = -1 \Rightarrow a^2 = -2$, which are not possible for any real numbers. Therefore, we cannot divide Eq(2) by $a - b$ nor Eq(3) by $b - c$.

So $a - b = b - c = 0 \Rightarrow a = b = c$ and also

$$a^2 + a = b^2 + b = c^2 + c \Rightarrow a = b = c \tag{4}$$

Put Eq(4) into Eq (A). So

$$a^2 = b + c - 1 = a + a - 1 \Rightarrow \left(a - 1\right)^2 = 0 \Rightarrow a = 1.$$

But since $a = b = c$ we have $a = b = c = 1$ so $\cos x = \cos y = \cos z = 1$ and since

$$x = \cos^{-1} 1,\ y = \cos^{-1} 1,\ z = \cos^{-1} 1 \rightarrow x = y = z = \{2k\pi | k \in Z\}$$

x=2kπ, $y = 2l\pi$, and $z = 2m\pi$, with k, l and m being integers.

Also solved by Hatef I. Arshagi, Guilford Technical Community College, Jamestown, NC; Michel Bataille, Rouen, France; David E. Manes, Oneonta NY; Kee-Wai Lau, Hong Kong, China; Titu Zvonaru, Comănesti, Romania, and the proposer.

5592: *Proposed by Michel Bataille, Rouen, France*

Let n be a positive integer. Evaluate $\sum_{k=1}^{n} a_k$ where

$$a_k = \left(\prod_{i=1}^{k}(2i-1)\right) \cdot \left(\prod_{j=k+1}^{n}(k+j)\right).$$

[The second factor is 1 if $k = n$.]

Solution 1 by Seán M. Stewart, Bomaderry, NSW, Australia

We will show the desired sum is equal to $n!(2^n - 1)$.

We begin by first proving the following binomial identity

$$\sum_{k=0}^{n} \binom{n+k}{k} \frac{1}{2^k} = 2^n. \tag{1}$$

Let

$$f(n) = \sum_{k=0}^{n} \binom{n+k}{k} \frac{1}{2^k}.$$

Then

$$f(n+1) = \sum_{k=0}^{n+1} \binom{n+k+1}{k} \frac{1}{2^k},$$

which, on application of Pascal's identity

$$\binom{n+1}{k} = \binom{n}{k} + \binom{n}{k-1},$$

can be rewritten as

$$f(n+1) = \sum_{k=0}^{n+1} \left[\binom{n+k}{k} + \binom{n+k}{k-1} \right] \frac{1}{2^k} = \sum_{k=1}^{n+1} \binom{n+k}{k-1} \frac{1}{2^k} + \sum_{k=0}^{n+1} \binom{n+k}{k} \frac{1}{2^k}.$$

Reindexing the first sum by $k \mapsto k+1$ we have

$$\begin{aligned} f(n+1) &= \frac{1}{2} \sum_{k=0}^{n} \binom{n+k+1}{k} \frac{1}{2^k} + \sum_{k=0}^{n+1} \binom{n+k}{k} \frac{1}{2^k} \\ &= \frac{1}{2} \left[\sum_{k=0}^{n+1} \binom{n+k+1}{k} \frac{1}{2^k} - \binom{2n+2}{n+1} \frac{1}{2^{n+1}} \right] \\ &\quad + \sum_{k=0}^{n} \binom{n+k}{k} \frac{1}{2^k} + \binom{2n+1}{n+1} \frac{1}{2^{n+1}} \\ &= \frac{1}{2} f(n+1) + f(n) + \frac{1}{2^{n+2}} \left[\binom{2n+1}{n+1} - \binom{2n+2}{n+1} \right]. \end{aligned} \tag{2}$$

Noting that

$$2\binom{2n+1}{n+1} - \binom{2n+2}{n+1} = 0,$$

the expression for (3), after rearranging, reduces to

$$f(n+1) = 2f(n).$$

Observing that

$$\begin{aligned} n = 0 &: f(1) = 2f(0) \\ n = 1 &: f(2) = 2f(1) = 2^2 f(0) \\ n = 2 &: f(3) = 2f(2) = 2^3 f(0). \end{aligned}$$

As $f(0) = 1$, from induction on n we see that $f(n) = 2^n f(0) = 2^n$, as required to prove.

Now for the sum of interest. Since

$$\prod_{i=1}^{k} (2i-1) = 1 \cdot 3 \cdot 5 \cdots (2k-1) = \frac{(2k)!}{2^k k!},$$

and

$$\prod_{j=k+1}^{n} (k+j) = (2k+1)(2k+2) \cdots (k+n) = \frac{(k+n)!}{(2k)!},$$

for the term a_k we have

$$\begin{aligned}
a_k &= \left(\prod_{i=1}^{k}(2i-1)\right)\cdot\left(\prod_{j=k+1}^{n}(k+j)\right)\\
&= \frac{(2k)!}{2^k k!}\cdot\frac{(k+n)!}{(2k)!}\\
&= \frac{(k+n)!}{2^k k!}\\
&= \binom{k+n}{k}\frac{n!}{2^k}.
\end{aligned}$$

So for the sum we have

$$\begin{aligned}
\sum_{k=1}^{n} a_k &= n!\sum_{k=1}^{n}\binom{k+n}{k}\frac{1}{2^k}\\
&= n!\left[\sum_{k=0}^{n}\binom{k+n}{k}\frac{1}{2^k}-1\right]\\
&= n!(2^n-1),
\end{aligned}$$

where we have made use of the binomial identity given in (1), as announced.

Editorial Comment by Solver :

The binomial identity given in (1) seems to be reasonably well known. It appears as Eq. (5.20) on page 167 of *Concrete Mathematics* (2nd ed.) by R. L. Graham, D. E. Knuth, and O. Patashnik (Addison-Wesley: Massachusetts, 1994) and as Identity 60 on page 40 of *The Art of Proving Binomial Identities* by M. Z. Spivey (CRC Press: Boca Raton, 2019).]

Solution 2 by Brian Bradie, Christopher Newport University, Newport News, VA

First,

$$\prod_{i=1}^{k}(2i-1)=\frac{(2k)!}{2^k k!}\quad\text{and}\quad\prod_{j=k+1}^{n}(k+j)=\frac{(n+k)!}{(2k)!},$$

so

$$a_k=\frac{(n+k)!}{2^k k!}.$$

Now, consider the sum

$$\sum_{k=0}^{n}\frac{1}{2^k}\binom{n+k}{k}.$$

We have

$$\begin{aligned}
\sum_{k=0}^{n}\frac{1}{2^k}\binom{n+k}{k} &= \sum_{k=0}^{n-1}\frac{1}{2^k}\binom{n-1+k}{k}+\sum_{k=0}^{n-1}\frac{1}{2^{k+1}}\binom{n+k}{k}+\frac{1}{2^n}\binom{2n-1}{n}\\
&= \sum_{k=0}^{n-1}\frac{1}{2^k}\binom{n-1+k}{k}+\frac{1}{2}\sum_{k=0}^{n}\frac{1}{2^{k+1}}\binom{n+k}{k}-\frac{1}{2^{n+1}}\binom{2n}{n}+\frac{1}{2^n}\binom{2n-1}{n}.
\end{aligned}$$

Because

$$-\frac{1}{2^{n+1}}\binom{2n}{n}+\frac{1}{2^n}\binom{2n-1}{n}=\frac{1}{2^n}\binom{2n-1}{n}\left(1-\frac{2n}{2n}\right)=0,$$

it follows that

$$\sum_{k=0}^{n}\frac{1}{2^k}\binom{n+k}{k}=2\sum_{k=0}^{n-1}\frac{1}{2^k}\binom{n-1+k}{k}=4\sum_{k=0}^{n-2}\frac{1}{2^k}\binom{n-2+k}{k}=\cdots=2^n\sum_{k=0}^{0}\frac{1}{2^k}\binom{k}{k}=2^n.$$

Finally,

$$\sum_{k=1}^{n}a_k=\sum_{k=1}^{n}\frac{(n+k)!}{2^k k!}=n!\sum_{k=1}^{n}\frac{1}{2^k}\binom{n+k}{k}=n!\left(\sum_{k=0}^{n}\frac{1}{2^k}\binom{n+k}{k}-1\right)=n!(2^n-1).$$

Solution 3 by David E. Manes, Oneonta, NY

If n is a positive integer, then the sum $a_1+a_2+\cdots+a_n=n!(2^n-1)$.

We will use the following identity: $\displaystyle\sum_{k=1}^{n}\frac{1}{2^k}\binom{n+k}{k}=2^{n-1}$, a combinatorial proof of which occurs in A. Engel's book, *Problem-Solving Strategies*, Springer-Verlag, New York, 1998, pp 96-97(E18).

Observe that if n is a positive integer, then the terms $a_1,a_2,\ldots,a_n$ are given by

$$\begin{aligned}
a_1 &= \left(\prod_{i=1}^{1}(2i-1)\right)\cdot\left(\prod_{j=2}^{n}(1+j)\right)=(1)\cdot(3\cdot 4\ldots(n+1)),\\
a_2 &= \left(\prod_{i=1}^{2}(2i-1)\right)\cdot\left(\prod_{j=3}^{n}(2+j)\right)=(1\cdot 3)\cdot(5\cdot 6\ldots(n+2)),\\
&\vdots\\
a_{n-1} &= \left(\prod_{i=1}^{n-1}(2i-1)\right)\cdot\left(\prod_{j=n}^{n}(n-1+j)\right)=(1\cdot 3\ldots(2n-3))\cdot(2n-1),\\
a_n &= \left(\prod_{i=1}^{n}(2i-1)\right)\cdot\left(\prod_{j=n+1}^{n}(n+j)\right)=(1\cdot 3\ldots(2n-3)\cdot(2n-1))\cdot(1).
\end{aligned}$$

Therefore,

$$\begin{aligned}
\sum_{k=1}^{n} a_k &= \sum_{k=1}^{n}\left(\prod_{i=1}^{k}(2i-1)\right)\cdot\left(\prod_{j=k+1}^{n}(k+j)\right)\\
&= \sum_{k=1}^{n}(1\cdot 3\cdots(2k-1))\cdot\left(\frac{(n+k)!}{(2k)!}\right)\\
&= \sum_{k=1}^{n}(1\cdot 3\cdots(2k-1))\cdot\left(\frac{(n+k)!}{(1\cdot 3\cdots(2k-1))(2\cdot 4\ldots(2k))}\right)\\
&= \sum_{k=1}^{n}\frac{(n+k)!}{2^k\cdot k!} = (n!)\sum_{k=1}^{n}\frac{1}{2^k}\left(\frac{(n+k)!}{n!\cdot k!}\right)\\
&= (n!)\sum_{k=1}^{n}\frac{1}{2^k}\cdot\binom{n+k}{k}\\
&= n!(2^n-1),
\end{aligned}$$

by the identity stated at the start of the solution. This completes the solution.

Solution 4 by Kee-Wai Lau, Hong Kong, China

We show that

$$\sum_{k=1}^{n} a_k = (2^n-1)n!. \tag{1}$$

It is easy to check that $a_k = 2^{-k}\binom{n+k}{k}n!$, and so (1) will follow from

$$\sum_{k=0}^{n} 2^{-k}\binom{n+k}{k} = 2^n. \tag{2}$$

Denote the left side of (2) by S_n. By the well-known relation

$\binom{m+1}{j} = \binom{m}{j} + \binom{m}{j-1}$ for positive integers satisfying $1 \leqslant j \leqslant m$, we have

$$\begin{aligned}
S_{n+1} &= 1 + \sum_{k=1}^{n+1} 2^{-k}\left(\binom{n+k}{k} + \binom{n+k}{k-1}\right)\\
&= \sum_{k=0}^{n+1} 2^{-k}\binom{n+k}{k} + \sum_{k=0}^{n} 2^{-(k+1)}\binom{n+1+k}{k}
\end{aligned}$$

$$= \sum_{k=0}^{n} 2^{-k}\binom{n+k}{k} + \sum_{k=0}^{n+1} 2^{-(k+1)}\binom{n+1+k}{k} = S_n + \frac{S_{n+1}}{2}.$$

Hence $S_{n+1} = 2S_n$, and by induction we obtain (2). This completes the solution.

Solution 5 by Albert Natian, Los Angeles Valley College, Valley Glen, CA

Answer. $n!\,(2^n - 1)$**.**

Computation. Since a_k depends on both k and n, then we write $a(n,k) := a_k$. We have

$$a(n,k) = \left(\prod_{i=1}^{k}(2i-1)\right)\cdot\left(\prod_{j=k+1}^{n}(k+j)\right) = \frac{(2k)!}{k!\cdot 2^k}\cdot\frac{(n+k)!}{(2k)!} = n!\cdot\binom{n+k}{n}\frac{1}{2^k}.$$

Define

$$Q_n := \sum_{k=0}^{n}\binom{n+k}{n}\frac{1}{2^k}.$$

Then

$$\sum_{k=1}^{n+1}\binom{n+k}{n+1}\frac{1}{2^k} = \sum_{j=0}^{n}\binom{n+1+j}{n+1}\frac{1}{2^{j+1}} = -\binom{2n+2}{n+1}\frac{1}{2^{n+2}} + \frac{1}{2}Q_{n+1}.$$

Now

$$Q_{n+1} = \sum_{k=0}^{n+1}\binom{n+1+k}{n+1}\frac{1}{2^k} = 1 + \sum_{k=1}^{n+1}\left[\binom{n+k}{n} + \binom{n+k}{n+1}\right]\frac{1}{2^k},$$

$$Q_{n+1} = 1 + \sum_{k=1}^{n+1}\binom{n+k}{n}\frac{1}{2^k} + \sum_{k=1}^{n+1}\binom{n+k}{n+1}\frac{1}{2^k},$$

$$Q_{n+1} = \sum_{k=0}^{n}\binom{n+k}{n}\frac{1}{2^k} + \binom{2n+1}{n}\frac{1}{2^{n+1}} - \binom{2n+2}{n+1}\frac{1}{2^{n+2}} + \frac{1}{2}Q_{n+1}.$$

Since

$$\binom{2n+1}{n}\frac{1}{2^{n+1}} - \binom{2n+2}{n+1}\frac{1}{2^{n+2}} = 0,$$

then

$$Q_{n+1} = Q_n + \frac{1}{2}Q_{n+1},$$

$$Q_{n+1} = 2Q_n.$$

Since $Q_0 = 1, Q_1 = 2$, thenA

$$Q_n = 2^n.$$

Thus

$$\sum_{k=1}^{n} a_k = \sum_{k=1}^{n} a(n,k) = n!\sum_{k=1}^{n}\binom{n+k}{n}\frac{1}{2^k} = n!\,(-1+Q_n) = n!\,(2^n-1)\,.$$

Solution 6 by Moti Levy, Rehovot, Israel

$$\prod_{i=1}^{k}(2i-1)=\frac{(2k)!}{2^k k!}, \qquad \prod_{j=k+1}^{n}(k+j)=\frac{(n+k)!}{(2k)!}.$$

$$a_k=\frac{(2k)!}{2^k k!}\frac{(n+k)!}{(2k)!}=\frac{(n+k)!}{k!}\left(\frac{1}{2}\right)^k$$

The required sum is

$$S_n := \sum_{k=1}^{n} a_k = n! \sum_{k=1}^{n}\binom{n+k}{n}\left(\frac{1}{2}\right)^k.$$

Now we show by mathematical induction that

$$\sum_{k=1}^{n}\binom{n+k}{n}\left(\frac{1}{2}\right)^k = 2^{n+1}-2^n-1.$$

For $n=1$, $\sum_{k=1}^{1}\binom{1+k}{n}\left(\frac{1}{2}\right)^k = 1 = 2^2-2-1.$

Suppose $\sum_{k=1}^{n}\binom{n+k}{n}\left(\frac{1}{2}\right)^k = 2^{n+1}-2^n-1$ is true, then we have to show that $\sum_{k=1}^{n+1}\binom{n+1+k}{n+1}\left(\frac{1}{2}\right)^k = 2^{n+2}-2^{n+1}-1.$

We apply the binomial identity,

$$\binom{n+1+k}{n+1}=\binom{n+k}{n+1}+\binom{n+k}{n}.$$

$$\begin{aligned}
&\sum_{k=1}^{n+1}\binom{n+1+k}{n+1}\left(\frac{1}{2}\right)^k \\
&=\sum_{k=1}^{n+1}\binom{n+k}{n+1}\left(\frac{1}{2}\right)^k+\sum_{k=1}^{n+1}\binom{n+k}{n}\left(\frac{1}{2}\right)^k \\
&=\frac{1}{2}\sum_{k=0}^{n}\binom{n+1+k}{n+1}\left(\frac{1}{2}\right)^k+\sum_{k=1}^{n}\binom{n+k}{n}\left(\frac{1}{2}\right)^k+\binom{2n+1}{n}\left(\frac{1}{2}\right)^{n+1} \\
&=\frac{1}{2}+\frac{1}{2}\sum_{k=1}^{n}\binom{n+1+k}{n+1}\left(\frac{1}{2}\right)^k+\sum_{k=1}^{n}\binom{n+k}{n}\left(\frac{1}{2}\right)^k+\binom{2n+1}{n}\left(\frac{1}{2}\right)^{n+1} \\
&=\frac{1}{2}+\frac{1}{2}\sum_{k=1}^{n+1}\binom{n+1+k}{n+1}\left(\frac{1}{2}\right)^k-\frac{1}{2}\binom{2n+2}{n+1}\left(\frac{1}{2}\right)^{n+1}+\sum_{k=1}^{n}\binom{n+k}{n}\left(\frac{1}{2}\right)^k+\binom{2n+1}{n}\left(\frac{1}{2}\right)^{n+1}.
\end{aligned}$$

Since $\frac{1}{2}\binom{2n+2}{n+1}\left(\frac{1}{2}\right)^{n+1}=\binom{2n+1}{n}\left(\frac{1}{2}\right)^{n+1}$, we get

$$\sum_{k=1}^{n+1}\binom{n+1+k}{n+1}\left(\frac{1}{2}\right)^k=\frac{1}{2}+\frac{1}{2}\sum_{k=1}^{n+1}\binom{n+1+k}{n+1}\left(\frac{1}{2}\right)^k+\sum_{k=1}^{n}\binom{n+k}{n}\left(\frac{1}{2}\right)^k,$$

or

$$\sum_{k=1}^{n+1} \binom{n+1+k}{n+1} \left(\frac{1}{2}\right)^k = 1 + 2\left(\sum_{k=1}^{n} \binom{n+k}{n} \left(\frac{1}{2}\right)^k\right) = 2^{n+2} - 2^{n+1} - 1.$$

We conclude that

$$\sum_{k=1}^{n} a_k = n!\left(2^{n+1} - 2^n - 1\right).$$

Also solved by Albert Stadler, Herrliberg, Switzerland, and the proposer.

5593: *Proposed by José Luis Díaz-Barrero, Barcelona Tech, Barcelona, Spain*

Let A be the set of quadruples of positive integers (i, j, k, l) such that $i + j + k + l = 23$. Compute the following sum

$$\sum_{(i,j,k,l)\in A} ijkl.$$

Solution 1 by Seán M. Stewart, Bomaderry, NSW, Australia

We show the desired sum is equal to $\binom{26}{7} = 657800$.

Let

$$S_n = \sum_{i+j+k+l=n} ijkl,$$

where $n \geqslant 4$ is an integer. The sum we require is S_{23}.

Recalling

$$\frac{1}{1-x} = \sum_{n\geqslant 0} x^n,$$

differentiating with respect to x gives

$$\frac{1}{(1-x)^2} = \sum_{n\geqslant 0} nx^{n-1},$$

or

$$\frac{x}{(1-x)^2} = \sum_{n\geqslant 0} nx^n, \tag{3}$$

after multiplying both sides of the expression by x. We now find a generating function for the sequence $\{S_n\}_{n\geqslant 0}$. Now

$$\begin{aligned}\sum_{n\geqslant 0} S_n x^n &= \sum_{n\geqslant 0}\left(\sum_{i+j+k+l=n} ijkl\right)x^n\\ &= \left(\sum_{n\geqslant 0} nx^n\right)^4\\ &= \left(\frac{x}{(1-x)^2}\right)^4 = \frac{x^4}{(1-x)^8},\end{aligned}$$

where we have made use of the result given in (3). As S_n is the coefficient of x^n in the generating function we have

$$\begin{aligned} S_n &= [x^n]\left(\frac{x^4}{(1-x)^8}\right) \\ &= \frac{1}{7!}(n+3)(n+2)\cdots(n-2)(n-3) \\ &= \frac{(n+3)!}{7!(n-4)!} \\ &= \binom{n+3}{7}, \quad n \geqslant 4. \end{aligned}$$

Thus

$$S_{23} = \binom{26}{7} = 657\,800,$$

as announced.

Solution 2 by David E. Manes, Oneonta, NY

Let N denote the set of natural numbers and let $S = \sum_{(i,j,k,l)\in A} ijkl$. If $A = \{(i,j,k,l) \mid i+j+k+l = 23\}$, then $S = 657,800$.

We will use the following result: the number of partitions of the integer 23 consisting of exactly 4 parts is 94. The number of those partitions such that no two of the positive integers i, j, k, l are equal is 39. They start with the partition $(17, 3, 2, 1)$ with product 102 and end with $(8, 6, 5, 4)$ with product 960. Note that each of these 39 partitions contributes a total of 24 quadruples in the set A by permuting the entries of the quadruple. The number of partitions in which exactly two of the integers i, j, k and l are equal is 48. Examples of such partitions are $(19, 2, 1, 1)$ with product 38 and $(7, 6, 5, 5)$ with product 1050. Note that each of these 48 partitions contributes a total of 12 quadruples in A. Finally, there are 7 partitions of 23 in which three of the four integers i, j, k, l are equal. They are $(20, 1, 1, 1)$, $(17, 2, 2, 2)$, $(14, 3, 3, 3)$, $(11, 4, 4, 4)$, $(8, 5, 5, 5)$, $(7, 7, 7, 2)$ and $(6, 6, 6, 5)$. Each of these seven partitions contributes 4 quadruples in the set A and the sum of the products for the seven partitions is $4,004$.

The partitions of 23 such that no two of i, j, k, l are equal are:

(17, 3, 2, 1) (16, 4, 2, 1) (15, 5, 2, 1) (15, 4, 3, 1) (14, 6, 2, 1) (14, 5, 3, 1) (14, 4, 3, 2)
(13, 7, 2, 1) (13, 6, 3, 1) (13, 5, 4, 1) (13, 5, 3, 2) (12, 8, 2, 1) (12, 7, 3, 1) (12, 6, 4, 1)
(12, 6, 3, 2) (12, 5, 4, 2) (11, 9, 2, 1) (11, 8, 3, 1) (11, 7, 4, 1) (11, 7, 3, 2) (11, 6, 5, 1)
(11, 6, 4, 2) (11, 5, 4, 3) (10, 9, 3, 1) (10, 8, 4, 1) (10, 8, 3, 2) (10, 7, 5, 1) (10, 7, 4, 2)
(10, 6, 5, 2) (10, 6, 4, 3) (9, 8, 5, 1) (9, 8, 4, 2) (9, 7, 6, 1) (9, 7, 5, 2) (9, 7, 4, 3)
(9, 6, 5, 3) (8, 7, 6, 2) (8, 7, 5, 3) (8, 6, 5, 4)

Note that the sum of the products of these 39 partitions is $16,016$.

The partitions of 23 such that exactly two of the integers i, j, k, l are equal are:

(19, 2, 1, 1) (18, 3, 1, 1) (18, 2, 2, 1) (17, 4, 1, 1) (16, 5, 1, 1) (16, 3, 2, 2) (16, 3, 3, 1)
(15, 6, 1, 1) (15, 4, 2, 2) (15, 3, 3, 2) (14, 7, 1, 1) (14, 5, 2, 2) (14, 4, 4, 1) (13, 8, 1, 1)
(13, 6, 2, 2) (13, 4, 3, 3) (13, 4, 4, 2) (12, 9, 1, 1) (12, 7, 2, 2) (12, 5, 5, 1) (12, 5, 3, 3)
(12, 4, 4, 3) (11, 10, 1, 1) (11, 8, 2, 2) (11, 6, 3, 3) (11, 5, 5, 2) (10, 10, 2, 1) (10, 9, 2, 2)
(10, 7, 3, 3) (10, 6, 6, 1) (10, 5, 5, 3) (10, 5, 4, 4) (9, 9, 4, 1) (9, 9, 3, 2) (9, 8, 3, 3)
(9, 6, 6, 2) (9, 6, 4, 4) (9, 5, 5, 4) (8, 8, 6, 1) (8, 8, 5, 2) (8, 8, 4, 3) (8, 7, 7, 1)
(8, 7, 4, 4) (8, 6, 6, 3) (7, 7, 6, 3) (7, 7, 5, 4) (7, 6, 6, 4) (7, 6, 5, 5)

The sum of the products of these 48 entries is $21,450$. Hence, the sum S of the products enumerated by set A is

$$S = \sum_{(i,j,k,l)\in A} ijkl = 24(16,016) + 12(21,450) + 4(4004) = 657,800.$$

Solution 3 by Albert Stadler, Herrliberg, Switzerland

We note that

$$\sum_{i+j=n, i\geqslant 1, j\geqslant 1} ij = \sum_{i=1}^{n-1} i(n-i) = \sum_{ik=1}^{n-1} i\left\{\binom{n-i+1}{2} - \binom{n-i}{2}\right\} =$$

$$= \sum_{i=1}^{n-1} i\binom{n-i+1}{2} - \sum_{i=2}^{n}(i-1)\binom{n-i+1}{2} = \sum_{i=1}^{n-1}\binom{n-i+1}{2} =$$

$$= \sum_{i=1}^{n-1}\left\{\binom{n-i+2}{3} - \binom{n-i+1}{3}\right\} = \binom{n+1}{3}.$$

$$\sum_{i+j+k=n, i\geqslant 1, j\geqslant 1, k\geqslant 1} ijk = \sum_{i=1}^{n-2} i \sum_{j+k=n-i, j\geqslant 1, k\geqslant 1} jk = \sum_{i=1}^{n-2} i\binom{n-i+1}{3} =$$

$$= \sum_{i=1}^{n-2} i\left\{\binom{n-i+2}{4} - \binom{n-i+1}{4}\right\} = \sum_{i=1}^{n-2} i\binom{n-i+2}{4} - \sum_{i=2}^{n-1}\{i-1\binom{n-i+2}{4} =$$

$$= \sum_{i=1}^{n-2}\binom{n-i+2}{4} = \sum_{i=1}^{n-2}\left\{\binom{n-i+3}{5} - \binom{n-i+2}{5}\right\} = \binom{n+2}{5},$$

and finally (using the same line of arguments),

$$\sum_{i+j+k+l=n, i\geqslant 1, j\geqslant 1, k\geqslant 1, l\geqslant 1} ijkl = \binom{n+3}{7}.$$

In particular, if $n = 23$ then $\binom{n+3}{7} = \binom{26}{7} = 657800.$

Solution 4 by Moti Levy, Rehovot, Israel

Let $S := \sum\limits_{(i,j,k,l)\in A} ijkl$. Let $(n)_{n=1}^{\infty}$ be the sequence $1, 2, 3, \ldots$.

The generating function of the sequence $(n)_{n=1}^{\infty}$ is

$$N(z) := \sum_{n=1}^{\infty} nz^n = \frac{z}{(z-1)^2}.$$

The sum S is the 23-rd coefficient of four consecutive convolutions of the sequence $(n)_{n=1}^{\infty}$ with itself. The generating function of the four convolutions is $(N(z))^4$.

Then

$$\begin{aligned} S &= \left[z^{23}\right]\left(\frac{z}{(z-1)^2}\right)^4 \\ &= \left[z^{23}\right]\frac{z^4}{(z-1)^8} = \left[z^{19}\right]\frac{1}{(z-1)^8} = (-1)^{19}\binom{-8}{19} \\ &= \binom{19+8-1}{19} = 657800. \end{aligned}$$

Solution 5 by Albert Natian, Los Angeles Valley College, Valley Glen, Ca

Answer: $n!\,(2^n-1)$.

Computation: Since a_k depends on both k and n, then we write $a(n,k) := a_k$. We have

$$a(n,k) = \left(\prod_{i=1}^{k}(2i-1)\right)\cdot\left(\prod_{j=k+1}^{n}(k+j)\right) = \frac{(2k)!}{k!\cdot 2^k}\cdot\frac{(n+k)!}{(2k)!} = n!\cdot\binom{n+k}{n}\frac{1}{2^k}.$$

Define

$$Q_n := \sum_{k=0}^{n}\binom{n+k}{n}\frac{1}{2^k}.$$

Then

$$\sum_{k=1}^{n+1}\binom{n+k}{n+1}\frac{1}{2^k} = \sum_{j=0}^{n}\binom{n+1+j}{n+1}\frac{1}{2^{j+1}} = -\binom{2n+2}{n+1}\frac{1}{2^{n+2}} + \frac{1}{2}Q_{n+1}.$$

Now

$$Q_{n+1} = \sum_{k=0}^{n+1}\binom{n+1+k}{n+1}\frac{1}{2^k} = 1 + \sum_{k=1}^{n+1}\left[\binom{n+k}{n} + \binom{n+k}{n+1}\right]\frac{1}{2^k},$$

$$Q_{n+1} = 1 + \sum_{k=1}^{n+1}\binom{n+k}{n}\frac{1}{2^k} + \sum_{k=1}^{n+1}\binom{n+k}{n+1}\frac{1}{2^k},$$

$$Q_{n+1} = \sum_{k=0}^{n}\binom{n+k}{n}\frac{1}{2^k} + \binom{2n+1}{n}\frac{1}{2^{n+1}} - \binom{2n+2}{n+1}\frac{1}{2^{n+2}} + \frac{1}{2}Q_{n+1}.$$

Since

$$\binom{2n+1}{n}\frac{1}{2^{n+1}} - \binom{2n+2}{n+1}\frac{1}{2^{n+2}} = 0,$$

then

$$Q_{n+1} = Q_n + \frac{1}{2}Q_{n+1},$$
$$Q_{n+1} = 2Q_n.$$

Since $Q_0 = 1, Q_1 = 2$, then

$$Q_n = 2^n.$$

Thus

$$\sum_{k=1}^{n} a_k = \sum_{k=1}^{n} a(n,k) = n! \sum_{k=1}^{n} \binom{n+k}{n}\frac{1}{2^k} = n!\,(-1+Q_n) = n!\,(2^n - 1)\,.$$

Also solved by Kee-Wai Lau, Hong Kong, China, and the proposer.

5594: *Proposed by Ovidiu Furdui and Alina Sîntămărian, Technical University of Cluj-Napoca, Cluj-Napoca, Romania*

Let $k > 1$. Calculate:

$\bullet[(a)]$ $L = \lim_{n\to\infty} \int_0^1 \left(\frac{k}{\sqrt[n]{x} + k - 1}\right)^n \mathrm{d}x$

$\bullet[(b)]$ $\lim_{n\to\infty} n \left[L - \int_0^1 \left(\frac{k}{\sqrt[n]{x} + k - 1}\right)^n \mathrm{d}x\right].$

Solution 1 by Brian Bradie, Christopher Newport University, Newport News, VA

With

$$\begin{aligned}
\frac{k}{\sqrt[n]{x} + k - 1} &= \frac{1}{1 - \frac{1}{k}(1 - e^{\ln x/n})} \\
&= 1 - \frac{\ln x}{kn} - \frac{(k-2)\ln^2 x}{2k^2n^2} - \sum_{j=3}^{\infty} f_j(k)\frac{\ln^j x}{n^j},
\end{aligned}$$

it follows that

$$n \ln\left(\frac{k}{\sqrt[n]{x} + k - 1}\right) = -\frac{\ln x}{k} - \frac{(k-1)\ln^2 x}{2k^2 n} - \sum_{j=3}^{\infty} \tilde{f}_j(k)\frac{\ln^j x}{n^{j-1}},$$

and

$$\left(\frac{k}{\sqrt[n]{x} + k - 1}\right)^n = x^{-1/k}\left(1 - \frac{(k-1)\ln^2 x}{2k^2 n} - \sum_{j=3}^{\infty} \hat{f}_j(k)\frac{\ln^j x}{n^{j-1}}\right),$$

where each function f_j, $\tilde{f}_j$, and $\hat{f}_j$ is bounded for $k > 1$. Because

$$\int_0^1 x^{-1/k} \ln^j x \, dx$$

is finite for each j,

$$\begin{aligned}\int_0^1 \left(\frac{k}{\sqrt[n]{x}+k-1}\right)^n dx &= \int_0^1 x^{-1/k}\,dx - \frac{k-1}{2k^2 n}\int_0^1 x^{-1/k}\ln^2 x\,dx + O\left(\frac{1}{n^2}\right)\\ &= \frac{k}{k-1} - \frac{k}{(k-1)^2 n} + O\left(\frac{1}{n^2}\right).\end{aligned}$$

Hence,

$$L = \lim_{n\to\infty}\int_0^1 \left(\frac{k}{\sqrt[n]{x}+k-1}\right)^n dx = \frac{k}{k-1}$$

and

$$\lim_{n\to\infty} n\left[L - \int_0^1 \left(\frac{k}{\sqrt[n]{x}+k-1}\right)^n dx\right] = \frac{k}{(k-1)^2}.$$

Solution 2 by Seán M. Stewart, Bomaderry, NSW, Australia

If $k>1$ we will show that

(a) $L = \lim_{n\to\infty}\int_0^1 \left(\frac{k}{\sqrt[n]{x}+k-1}\right)^n dx = \frac{k}{k-1}$, and

(b) $\lim_{n\to\infty} n\left[L - \int_0^1 \left(\frac{k}{\sqrt[n]{x}+k-1}\right)^n dx\right] = \frac{k}{(k-1)^2}$.

For large n the asymptotic expansion for the integrand of the integral found in the limit L is

$$\left(\frac{k}{\sqrt[n]{x}+k-1}\right)^n = x^{-\frac{1}{k}} - \frac{k-1}{2nk^2}x^{-\frac{1}{k}}\log^2(x) + \mathcal{O}\left(\frac{1}{n^2}\right).$$

Let

$$I_n = \int_0^1 \left(\frac{k}{\sqrt[n]{x}+k-1}\right)^n dx,$$

then

$$I_n = \int_0^1 x^{-\frac{1}{k}}\,dx - \frac{k-1}{2nk^2}\int_0^1 x^{-\frac{1}{k}}\log^2(x)\,dx + \mathcal{O}\left(\frac{1}{n^2}\right). \qquad (1)$$

The first of the integrals to the right of the equality is elementary. The result is

$$\int_0^1 x^{-\frac{1}{k}}\,dx = \frac{k}{k-1}.$$

For the second integral to the right of the equality, enforcing a substitution of $x \mapsto x^k$ gives

$$\int_0^1 x^{-\frac{1}{k}}\log^2(x)\,dx = k^3\int_0^1 x^{k-2}\log^2(x)\,dx.$$

Integrating by parts twice leads to

$$\int_0^1 x^{-\frac{1}{k}}\log^2(x)\,dx = \frac{2k^3}{(k-1)^3}.$$

Thus (1) becomes

$$I_n = \frac{k}{k-1} - \frac{k}{(k-1)^2 n} + \mathcal{O}\left(\frac{1}{n^2}\right). \tag{2}$$

Using the result given in (2) we are now in a position to answer the questions given in (a) and (b). For (a)

$$L = \lim_{n\to\infty} I_n = \lim_{n\to\infty}\left[\frac{k}{k-1} - \frac{k}{(k-1)^2 n} + \mathcal{O}\left(\frac{1}{n^2}\right)\right] = \frac{k}{k-1},$$

as announced. And for (b)

$$\begin{aligned}
\lim_{n\to\infty} n(L - I_n) &= \lim_{n\to\infty} n\left[\frac{k}{k-1} - \left\{\frac{k}{k-1} - \frac{k}{(k-1)^2 n} + \mathcal{O}\left(\frac{1}{n^2}\right)\right\}\right] \\
&= \lim_{n\to\infty}\left[\frac{k}{(k-1)^2} + \mathcal{O}\left(\frac{1}{n}\right)\right] \\
&= \frac{k}{(k-1)^2},
\end{aligned}$$

as announced.

Solution 3 by Albert Stadler, Herrliberg, Switzerland

Let

$$I_{k,n} = \int_0^1 \left(\frac{k}{\sqrt[n]{x}}k - 1\right)^n dx.$$

We perform three variable transforms,

first $x = y^n, dx = ny^{n-1}dy$:

$$I_{k,n} = \int_0^1 \left(\frac{k}{y+k-1}\right) ny^{n-1}dy,$$

second, $y = 1-z, dy = -dz$:

$$I_{k,n} = \int_0^1 \left(\frac{k}{k-z}\right)^n n(1-z)^{n-1}dz = n\int_0^1 \left(1-\frac{z}{k}\right)^{-n}(1-z)^{n=1}dz,$$

third $t = \dfrac{1-z}{1-\frac{z}{k}},\ z = k\dfrac{1-t}{k-t},\ dz = -\dfrac{k(k-1)}{(k-t)^2}$:

$$I_{k,n} = kn\int_0^1 \frac{t^{n-1}}{k-t}dt.$$

This representation allows us to drive the full asymptotic expansion of $I_{k,n}$ in terms of decreasing powers of n by applying repeated integration by parts. Indeed,

$$\begin{aligned} I_{k,n} &= \left.\frac{kt^n}{k-t}\right|_0^1 - k\int_0^1 \frac{t^n}{(k-t)^2}dt = \frac{k}{k-1} - k\int_0^1 \frac{t^n}{(k-t)^2}dt = \\ &= \frac{k}{k-1} - \frac{k}{(k-1)^2(n+1)} + \frac{2k}{n+1}\int_0^1 \frac{t^{n+1}}{(k-t)^3}dt = \\ &= \frac{k}{k-1} - \frac{k}{(k-1)^2(n+1)} + \mathcal{O}\left(\frac{1}{n^2}\right), \end{aligned}$$

Since $0 \leqslant \int_0^1 \frac{t^{n+1}}{(k-t^3)} \leqslant \frac{1}{(k-1)^3}\int_0^1 t^{n+1}dt = \frac{1}{(k-1)^3(n+2)}$.

Therefore, $L = \frac{k}{k-1}$ and

$$\lim_{n\to\infty} n\left(L - \int_0^1 \left(\frac{k}{\sqrt[n]{x}+k-1}\right)^n dx\right) = \frac{k}{(k-1)^2}.$$

Solution 4 by Kee-Wai Lau, Hong Kong, China

We show that the limit L in (a) equals $\frac{k}{k-1}$ and in (b) equals $\frac{k}{(k-1)^2}$.

To prove these results, we need to prove that for positive integers n,

$$\int_0^1 \left(\frac{k}{\sqrt[n]{x}+k-1}\right)^n dx = \frac{k}{k-1} - \frac{k}{(n+1)(k-1)^2} + \frac{2k^n}{n+1}\int_{\frac{k-1}{k}}^1 \frac{(1-t)^{n+1}}{t^3}dt. \qquad (1)$$

Denote the integral on the left side of (1) by I_n. By means of the substitution $x = \left(\frac{(k-1)(1-t)}{t}\right)^n$, we see that $I_n = nk^n\int_{\frac{k-1}{k}}^1 \frac{(1-t)^{n-1}}{t}dt$. Integrating by parts, we obtain $I_n = \frac{k}{k-1} - k^n\int_{\frac{k-1}{k}}^1 \frac{(1-t)^n}{t^2}dt$. Integrating by parts again, we obtain (1).

Denote the integral on the right side of (1) by J_n. Then

$$|J_n| \leqslant \int_{\frac{k-1}{k}}^1 \frac{(1-t)^{n+1}}{\left(\frac{k-1}{k}\right)^3}dt = \frac{1}{(n+2)(k-1)^3k^{n-1}}.$$

It follows that $I_n = \frac{k}{k-1} - \frac{k}{(n+1)(k-1)^2} + O\left(\frac{1}{n^2}\right)$, where the constant implied by O

depends at most on k. Hence, our claimed results for (a) and (b).

Solution 5 by Michel Bataille, Rouen France

Let $I_n = \int_0^1 \left(\frac{k}{\sqrt[n]{x}+k-1}\right)^n dx$. The change of variables $x = \frac{(k-1)^n t^n}{(k-t)^n}$ $(t \in [0,1])$ gives $\sqrt[n]{x} + k - 1 = \frac{k(k-1)}{k-t}$ and $dx = nk(k-1)^n \frac{t^{n-1}}{(k-t)^{n+1}}\, dt$ so that

$$I_n = nk \int_0^1 \frac{t^{n-1}}{k-t}\, dt.$$

Integrating by parts, we obtain

$$I_n = k\left(\left[\frac{t^n}{k-t}\right]_{t=0}^1 - \int_0^1 \frac{t^n}{(k-t)^2}\, dt\right) = \frac{k}{k-1} - kJ_n$$

where $J_n = \int_0^1 \frac{t^n}{(k-t)^2}\, dt$.

From a well-known result [see for example Focus On... No 35, *Crux Mathematicorum*, 45(3), March 2019, p. 138], we have

$$\lim_{n\to\infty} nJ_n = \left[\frac{1}{(k-t)^2}\right]_{t=1} = \frac{1}{(k-1)^2}.$$

First, we deduce that $J_n \sim \frac{1}{(k-1)^2} \cdot \frac{1}{n}$ as $n \to \infty$, hence $\lim_{n\to\infty} J_n = 0$ and therefore

$$L = \lim_{n\to\infty} I_n = \frac{k}{k-1}$$

and, second,

$$\lim_{n\to\infty} n(L - I_n) = k(nJ_n) = \frac{k}{(k-1)^2}.$$

Solution 6 by Albert Natian, Los Angeles Valley College, Valley Glen, CA

Answer:

$$(a)\ L = \frac{k}{k-1}$$

$$(b)\ \lim_{n\to\infty} n\left[L - \int_0^1 \left(\frac{k}{\sqrt[n]{x}+k-1}\right)^n dx\right] = \frac{k}{(k-1)^2}$$

Computation: Set

$$Q_n := \int_0^1 \left(\frac{k}{\sqrt[n]{x}+k-1}\right)^n dx$$

and

$$y = f(x) := \left(\frac{k}{\sqrt[n]{x} + k - 1}\right)^n$$

so that

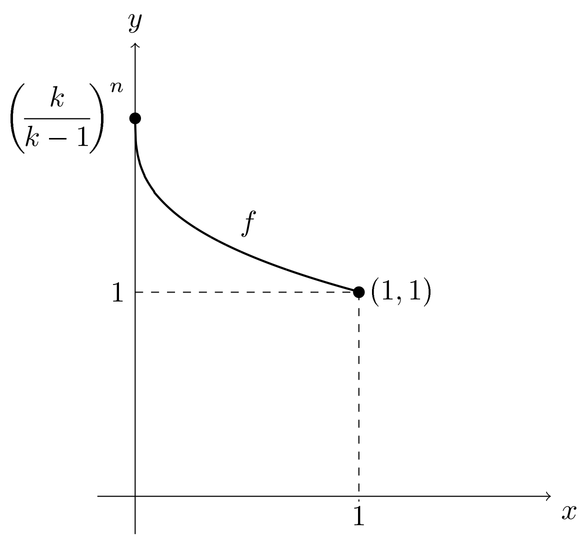
$$x = \frac{\left(k - [k-1]\ \sqrt[n]{y}\right)^n}{y}.$$

From the graph of $y = f(x) = \left(\frac{k}{\sqrt[n]{x} + k - 1}\right)^n$, we see that

$$Q_n = \int_0^1 f(x)\ dx = 1 + \int_1^{\left(\frac{k}{k-1}\right)^n} \frac{\left(k - [k-1]\ \sqrt[n]{y}\right)^n}{y}\ dy.$$

Now set $u = \sqrt[n]{y}$ so that $y = u^n$, $dy = nu^{n-1}\ du$, and

$$Q_n = 1 + n \int_1^{\frac{k}{k-1}} \frac{1}{u} \left(k - [k-1]\, u\right)^n\ du.$$



Now set

$$v = \frac{1}{k} \left(k - [k-1]\, u\right)$$

so that

$$k - [k-1]\, u = kv, \quad u = \frac{k}{k-1} \left(1 - v\right), \quad du = \frac{k}{k-1} \left(-dv\right)$$

and

$$Q_n = 1 + nk^n \int_0^{1/k} \frac{v^n}{1-v}\ dv.$$

Integrating by parts p times, we have

$$\int_0^{1/k} \frac{v^n}{1-v}\ dv = \frac{1}{k^n} \sum_{j=1}^{p} \frac{(-1)^{j-1}\,(j-1)!}{(k-1)^j \prod_{t=1}^{j} (n+t)} + (-1)^p \frac{p!}{\prod_{t=1}^{p} (n+t)} \int_0^{1/k} \frac{v^{n+p}}{(1-v)^{p+1}}\ dv.$$

Then for any positive integer m:

$$1+n\sum_{j=1}^{2m}\frac{(-1)^{j-1}(j-1)!}{(k-1)^j\prod_{t=1}^{j}(n+t)}\leqslant Q_n\leqslant 1+n\sum_{j=1}^{2m-1}\frac{(-1)^{j-1}(j-1)!}{(k-1)^j\prod_{t=1}^{j}(n+t)}.$$

For $m=1$:

$$1+n\left[\frac{1}{(k-1)(n+1)}-\frac{1}{(k-1)^2(n+1)(n+2)}\right]\leqslant Q_n\leqslant 1+n\left[\frac{1}{(k-1)(n+1)}\right]$$

which, upon sending $n\to\infty$, collapses to

$$1+\frac{1}{k-1}\leqslant Q_\infty\leqslant 1+\frac{1}{k-1}$$

and so

$$Q_\infty=\frac{k}{k-1}$$

which means

$$L=\lim_{n\to\infty}\int_0^1\left(\frac{k}{\sqrt[n]{x}+k-1}\right)^n\,dx=Q_\infty=\frac{k}{k-1}.$$

For $m=2$:

$$n\left[\frac{k}{k-1}-1-n\sum_{j=1}^{3}\frac{(-1)^{j-1}(j-1)!}{(k-1)^j\prod_{t=1}^{j}(n+t)}\right]$$

$$\leqslant n\left[L-Q_n\right]\leqslant$$

$$n\left[\frac{k}{k-1}-1-n\sum_{j=1}^{4}\frac{(-1)^{j-1}(j-1)!}{(k-1)^j\prod_{t=1}^{j}(n+t)}\right]$$

which, upon sending $n\to\infty$, collapses to

$$\frac{k}{(k-1)^2}\leqslant\lim_{n\to\infty}n\left[L-Q_n\right]\leqslant\frac{k}{(k-1)^2}$$

and so

$$\lim_{n\to\infty}n\left[L-Q_n\right]=\frac{k}{(k-1)^2}$$

which means

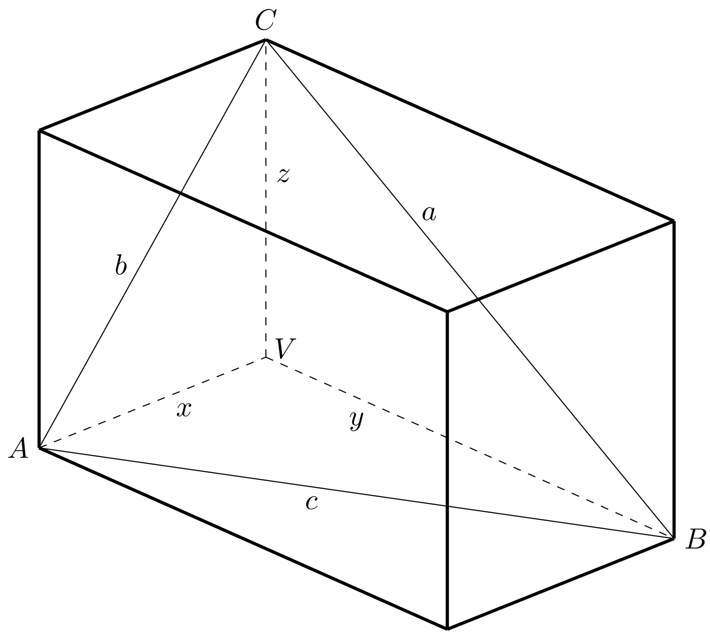
$$\lim_{n\to\infty}n\left[L-\int_0^1\left(\frac{k}{\sqrt[n]{x}+k-1}\right)^n\,dx\right]=\lim_{n\to\infty}n\left[L-Q_n\right]=\frac{k}{(k-1)^2}.$$

Also solved by Moti Levy, Rehovot, Israel, and the proposers.

The following solution uses the 3-dimensional analog of the Pythagorean Theorem.

Solution to 5590 by Albert Natian, Los Angeles Valley College, Valley Glen, CA

Let V be a vertex of a rectangular box. Let $\overline{VA}$, $\overline{VB}$ and $\overline{VC}$ be the three edges meeting at vertex V. We are given that the perimeter of the triangle $\triangle ABC$ is $(28+\sqrt{106})$, the total surface area of the box is 426, and the length of the main diagonal of the box is $5\sqrt{10}$. Find the area of the triangle $\triangle ABC$.



Solution:

Let x, y and z respectively denote the lengths of $\overline{VA}$, $\overline{VB}$ and $\overline{VC}$; i.e., $x = VA$, $y = VB$ and $z = VC$. In $\triangle ABC$, let a, b and c respectively denote the lengths of $\overline{BC}$, $\overline{CA}$ and $\overline{AB}$; i.e., $a = BC$, $b = CA$ and $c = AB$. So $a = \sqrt{y^2+z^2}$, $b = \sqrt{z^2+x^2}$, $c = \sqrt{x^2+y^2}$. The perimeter $p = 28+\sqrt{106}$ can be written as

$$p = c + a + b = \sqrt{x^2+y^2} + \sqrt{y^2+z^2} + \sqrt{z^2+x^2}.$$

For the total surface area $S = 426$ of the box we have: $S = 2\,(xy+yz+zx) = 426$. We let $\sigma = S/2 = 213$. So $S = 2\sigma$ and

$$\sigma = xy + yz + zx.$$

For the length D of the main diagonal of the box we have $D = \sqrt{x^2+y^2+z^2} = 5\sqrt{10}$. We let

$$d = D^2 = x^2 + y^2 + z^2$$

So $d = 250$. For the volume v of the box we have

$$v = xyz.$$

We let the notation $\mathcal{A}(\triangle TRI)$ denote the area of a triangle $\triangle TRI$ and invoke the 3-dimensional Pythagorean Theorem to write:

$$[\mathcal{A}(\triangle AVB)]^2 + [\mathcal{A}(\triangle BVC)]^2 + [\mathcal{A}(\triangle CVA)]^2 = [\mathcal{A}(\triangle ABC)]^2 .$$

Let $\mathcal{A}_* = \mathcal{A}(\triangle ABC)$. Clearly $\mathcal{A}(\triangle AVB) = xy/2$, $\mathcal{A}(\triangle BVC) = yz/2$, $\mathcal{A}(\triangle CVA) = zx/2$.

So

$$\mathcal{A}_*^2 = \left(\frac{xy}{2}\right)^2 + \left(\frac{yz}{2}\right)^2 + \left(\frac{zx}{2}\right)^2 = \left(x^2y^2 + y^2z^2 + z^2x^2\right)/4.$$

We let

$$\alpha = 4\mathcal{A}_*^2 = x^2y^2 + y^2z^2 + z^2x^2.$$

Our task is to find $\mathcal{A}_*$ in terms of (or as a function of) p, D and S. However, we will first find α in terms of p, d and σ where $d = D^2$ and $\sigma = S/2$.

From

$$a = \sqrt{y^2 + z^2},\ b = \sqrt{z^2 + x^2},\ c = \sqrt{x^2 + y^2}$$

we get

$$a^2 + b^2 + c^2 = \left(y^2 + z^2\right) + \left(z^2 + x^2\right) + \left(x^2 + y^2\right) = 2(x^2 + y^2 + z^2) = 2d.$$

In short,

$$a^2 + b^2 + c^2 = 2d. \qquad (\star 1\star).$$

We have

$$x^2 + y^2 + z^2 = d,$$
$$\left(x^2 + y^2 + z^2\right)^2 = d^2,$$
$$x^4 + y^4 + z^4 + 2\left(x^2y^2 + y^2z^2 + z^2x^2\right) = d^2,$$
$$x^4 + y^4 + z^4 + 2\alpha = d^2,$$
$$x^4 + y^4 + z^4 = d^2 - 2\alpha. \qquad (\star 2\star)$$

Since

$$\alpha = x^2y^2 + y^2z^2 + z^2x^2,\ a^2 = y^2 + z^2,\ b^2 = z^2 + x^2,\ c^2 = x^2 + y^2,$$

then

$$c^2a^2 = \left(x^2 + y^2\right)\left(y^2 + z^2\right) = x^2y^2 + y^2z^2 + z^2x^2 + y^4 = \alpha + y^4,$$
$$a^2b^2 = \left(y^2 + z^2\right)\left(z^2 + x^2\right) = x^2y^2 + y^2z^2 + z^2x^2 + z^4 = \alpha + z^4,$$
$$b^2c^2 = \left(z^2 + x^2\right)\left(x^2 + y^2\right) = x^2y^2 + y^2z^2 + z^2x^2 + x^4 = \alpha + x^4.$$

So, by $(\star 2\star)$,

$$a^2b^2 + b^2c^2 + c^2a^2 = 3\alpha + x^4 + y^4 + z^4 = 3\alpha + d^2 - 2\alpha = \alpha + d^2,$$

$$a^2b^2 + b^2c^2 + c^2a^2 = \alpha + d^2. \qquad (\star 3\star)$$

Let $e = x + y + z$ so that

$$e^2 = (x + y + z)^2 = x^2 + y^2 + z^2 + 2\left(xy + yz + zx\right) = d + 2\sigma.$$

Numerically $e = \sqrt{d + 2\sigma} = \sqrt{250 + 426} = \sqrt{676} = 26.$

Recall $xy + yz + zx = \sigma$. So

$$(xy + yz + zx)^2 = \sigma^2,$$
$$x^2y^2 + y^2z^2 + z^2x^2 + 2\,(xyyz + yzzx + zxxy) = \sigma^2,$$
$$\alpha + 2xyz(x + y + z) = \sigma^2,$$
$$\alpha + 2ve = \sigma^2,$$
$$v = \frac{\sigma^2 - \alpha}{2e},$$
$$v^2 = \frac{\left(\sigma^2 - \alpha\right)^2}{4e^2},$$
$$v^2 = \frac{\left(\sigma^2 - \alpha\right)^2}{4\,(d + 2\sigma)}. \qquad (\star 4\star)$$

We now find an expression for $(abc)^2$. Recall $a = \sqrt{y^2 + z^2}$, $b = \sqrt{z^2 + x^2}$, $c = \sqrt{x^2 + y^2}$. So

$$\begin{aligned}
a^2b^2c^2 &= \left(y^2 + z^2\right)\left(z^2 + x^2\right)\left(x^2 + y^2\right) \\
&= \left(x^2 + y^2 + z^2 - x^2\right)\left(x^2 + y^2 + z^2 - y^2\right)\left(x^2 + y^2 + z^2 - z^2\right) \\
&= \left(d - x^2\right)\left(d - y^2\right)\left(d - z^2\right) \\
&= d^3 - (x^2 + y^2 + z^2)d^2 + \left(x^2y^2 + y^2z^2 + z^2x^2\right)d - x^2y^2z^2 \\
&= d^3 - (d)d^2 + (\alpha)\,d - v^2 \\
&= \alpha d - v^2 \\
&= \alpha d - \frac{\left(\sigma^2 - \alpha\right)^2}{4\,(d + 2\sigma)} \quad \text{by} \quad (\star 4\star) \\
&= \frac{4\,(d + 2\sigma)\,\alpha d - \left(\sigma^2 - \alpha\right)^2}{4\,(d + 2\sigma)} \\
&= \frac{4\,(d + 2\sigma)\,\alpha d - \sigma^4 + 2\sigma^2\alpha - \alpha^2}{4\,(d + 2\sigma)}.
\end{aligned}$$

So

$$(abc)^2 = \frac{-\alpha^2 + \left[4\,(d + 2\sigma)\,d + +2\sigma^2\right]\alpha - \sigma^4}{4\,(d + 2\sigma)}. \qquad (\star 5\star)$$

Recall $a + b + c = p$. So

$$(a + b + c)^2 = p^2,$$
$$a^2 + b^2 + c^2 + 2(a + b + c) = p^2,$$
$$2d + 2(ab + bc + ca) = p^2 \quad \text{by } (\star 1\star),$$
$$2(ab + bc + ca) = p^2 - 2d,$$

$$[2(ab+bc+ca)]^2 = \left(p^2-2d\right)^2,$$

$$4\left[a^2b^2+b^2c^2+c^2a^2+2\left(abbc+bcca+caab\right)\right] = \left(p^2-2d\right)^2,$$

$$4\left[\alpha+d^2+2abc(p)\right] = p^4-4p^2d+4d^2 \qquad \text{by } (\star 3\star).$$

Continuing,

$$4\alpha+4d^2+8abcp = p^4-4p^2d+4d^2,$$

$$8abcp = p^4-4p^2d-4\alpha,$$

$$64\,(abc)^2\,p^2 = (8abcp)^2 = \left[\left(p^4-4p^2d\right)-4\alpha\right]^2.$$

By ($\star 5\star$), the preceding can be re-written as:

$$64\left(\frac{-\alpha^2+\left[4\,(d+2\sigma)\,d+{+2\sigma^2}\right]\alpha-\sigma^4}{4\,(d+2\sigma)}\right)p^2 = \left[p^2\left(p^2-4d\right)-4\alpha\right]^2,$$

$$16\Big(-\alpha^2+\left[4\,(d+2\sigma)\,d+2\sigma^2\right]\alpha-\sigma^4\Big)p^2 = (d+2\sigma)\left[p^2\left(p^2-4d\right)-4\alpha\right]^2.$$

Now we re-express the latter as a quadratic in α. The quadratic re-expression is:

$$16\left[p^2+d+2\sigma\right]\alpha^2-8p^2\left[4\sigma^2+(d+2\sigma)(p^2+4d)\right]\alpha+\left[16\sigma^4p^2+(d+2\sigma)p^4(p^2-4d)^2\right]=0$$

which can be re-written as

$$k\alpha^2-\ell\alpha+m=0$$

where

$k = 16\left[p^2+d+2\sigma\right]$,

$\ell = 8p^2\left[4\sigma^2+(d+2\sigma)(p^2+4d)\right]$,

$m = p^2\left[16\sigma^4+(d+2\sigma)p^2(p^2-4d)^2\right]$.

Recall $d=D^2$ and $2\sigma=S$. So $4\sigma^2=S^2$ and $16\sigma^4=S^4$. So,

$k = 16\left[p^2+D^2+S\right]$,

$\ell = 8p^2\left[S^2+(D^2+S)(p^2+4D^2)\right]$,

$m = p^2\left[S^4+(D^2+S)p^2(p^2-4D^2)^2\right]$.

By quadratic formula, we have:

$$\alpha = \frac{\ell\pm\sqrt{\ell^2-4km}}{2k}.$$

Since $\mathcal{A}_* = \frac{1}{2}\sqrt{\alpha}$, then

$$\mathcal{A}_* = \frac{1}{2}\sqrt{\frac{\ell\pm\sqrt{\ell^2-4km}}{2k}}.$$

At this point of the solution we have essentially come up with the answer to the original question. To produce a numerical answer, all that remains to do is brute computation, which is not shown here. However, the two solutions, as approximations for $\mathcal{A}_*$, are 65.7438 and 392.2613. It's worth mentioning that a box satisfying the above given numerical values for p, S and D is one with dimensions $5 \times 9 \times 12$, which coincides with the answer 65.7438.

Finally, a result worth mentioning is:

$$256\left[p^2 + D^2 + S\right]\mathcal{A}_*^4 - 32p^2\left[S^2 + (D^2 + S)(p^2 + 4D^2)\right]\mathcal{A}_*^2 + p^2\left[S^4 + (D^2 + S)p^2(p^2 - 4D^2)^2\right] = 0.$$

Comments by Albert Natian:

Take a look at the last quadratic-type equation with $\mathcal{A}*$ as the root. It's a beauty! It's a result that I think will intrigue many of your column readers. I'd be curious as to what their take on that equation will be. This problem has actually a very important psychological and logical lesson to teach us all, especially to me, which I've learned well. And it is this, that no amount of work that ends nowhere is no proof that there is no solution. To prove that there is no solution (of a certain kind) one needs to produce positive knowledge that demonstrates there is no such solution – itself often as difficult as the original problem.

I remember all those many days I tried to reach a destination with this problem, but failed every time. And also I remember all those times I was so close to deciding that there is no hope at finding a definitive solution. But somehow I lucked out and, by not giving up, succeeded in finding a solution.

It's notable that there are, by my reckoning, two solutions for the area of triangle ABC; namely, 65.7438 and 392.2613 (as approximations). While I am comfortable with the smaller solution as genuine, I am wondering as to what POSITIVE dimensions (if any) of the box the larger number corresponds.

Thank you,

Albert