# Problems

Ted Eisenberg, Section Editor

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This section of the Journal offers readers an opportunity to exchange interesting mathematical problems and solutions. Please send them to Ted Eisenberg, Department of Mathematics, Ben-Gurion University, Beer-Sheva, Israel or fax to: 972-86-477-648. Questions concerning proposals and/or solutions can be sent e-mail to <eisenbt@013.net>. Solutions to previously stated problems can be seen at <http://www.ssma.org/publications>.

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*Solutions to the problems stated in this issue should be posted before February 15, 2021*

- **5613:** *Proposed by Kenneth Korbin, New York, NY*

  Given the equations:

$$\begin{cases} \sqrt{3x^2+6x+1} = y + \sqrt{3y^2-3} \\ \text{and} \\ ax^2+by^2+cxy+dx+ey+f=0. \end{cases}$$

  Find integers $(a, b, c, d, e, f)$ so that infinitely many pairs of positive integers $(x, y)$ satisfy both equations.

- **5614:** *Proposed by Michael Brozinsky, Central Islip, NY*

  Solve:

$$\cos^2\theta + 6\cos(\theta)\cos\left(\frac{\theta}{3}\right) + 9\cos^2\left(\frac{\theta}{3}\right) = \sin^2\theta - 6\sin\theta\sin\left(\frac{\theta}{3}\right) + 9\sin^2\left(\frac{\theta}{3}\right).$$

- **5615:** *Proposed by Pedro Henrique Oliveira Pantoja, University of Campina Grande, Brazil*

  Solve in $\Re \times \Re$ the system:

$$\begin{cases} \sqrt[3]{2x+2} + \sqrt[3]{4-x} + \sqrt[3]{2-x} = 2 \\ \\ \sqrt[5]{20-2y} + \sqrt[5]{7-y} + \sqrt[5]{3y+5} = 2 \end{cases}$$

- **5616:** *Proposed by D.M. Bătinetu-Giurgiu "Matei Basarb" National College, Bucharest and Neculai Stanciu, "George Emil Palade" Secondary School Buzău, Romania*

  Prove that in all tetrahedrons $[ABCD]$ the following inequality holds:

$$\frac{1}{h_a}\sqrt[3]{\frac{h_bh_c}{h_a^2}} + \frac{1}{h_b}\sqrt[3]{\frac{h_ch_d}{h_b^2}} + \frac{1}{h_c}\sqrt[3]{\frac{h_dh_a}{h_c^2}} + \frac{1}{h_d}\sqrt[3]{\frac{h_ah_b}{h_d^2}} \geq \frac{1}{r},$$

where $r$ is the radius of the insphere of the tetrahedron.

- **5617:** *Proposed by José Luis Díaz-Barrero, Barcelona Tech, Barcelona, Spain*

  Let $a, b, c$ be the roots of the equation $x^3 + rx + s = 0$. Without the aid of a computer, calculate

$$\det \begin{vmatrix} 2bc - a^2 & c^2 & b^2 \\ c^2 & 2ca - b^2 & a^2 \\ b^2 & a^2 & 2ab - c^2 \end{vmatrix}$$

- **5618:** *Proposed by Ovidiu Furdui and Alina Sîntămărian, Technical University of Cluj-Napoca, Cluj-Napoca, Romania*

  Let $k > 0$ be a real number. Calculate

$$\lim_{n\to\infty} n^2 \left( \frac{1}{n^3} + \frac{1}{(n+k)^3} + \frac{1}{(n+2k)^3} + \cdots \right).$$

*Solutions*

- **5595:** *Proposed by Kenneth Korbin, New York, NY*

  Trapezoid $ABCD$ with integer length sides is inscribed in a circle with diameter $23^3$. Side $\overline{AB} = 4439$. Find the other three sides.

  **Solution by Kee-Wai Lau, Hong Kong, China**

  We show that the other three sides are 2717, 4439, and 10051.

  Let $O$ be the center and $r = \dfrac{12167}{2}$ be the radius of the circle.
  **Case I:** $AB = 4439$ and $CD = 4439$ are the legs of the trapezoid.
  Let $BC = s, AD = t, \angle AOB = \angle DOC = \theta$ and $\angle BPC = \varphi$.

  **a)** $O$ lies outside the trapezoid.

  It suffices to consider the case $t \ge s$. Then $\cos\theta = \dfrac{205343}{279841}$
  $\sin\theta = \dfrac{29336\sqrt{42}}{279841}$, $\cos\dfrac{\varphi}{2} = \dfrac{\sqrt{148035889 - s^2}}{12167}$ and $\sin\dfrac{\varphi}{2} = \dfrac{s}{12167}$, so that

$$\text{t=2rsin}\left(\theta + \frac{\varphi}{2}\right) = 2r\left(\sin\theta\frac{\varphi}{2} + \cos\theta\sin\frac{\varphi}{2}\right) = \frac{29336\sqrt{42\,(148035889 - s^2)} + 205343s}{279841}.$$

  Since $s \le t \le 2r = 12167$, we have $1 \le s \le 8927$. By a computer search, we find that $s = 2717, t = 10051$ is the only solution in integers.

  **b)** $O$ lies inside the trapezoid.

We still have $t = \dfrac{29336\sqrt{42(148035889 - s^2)} + 205343s}{279841}$, but here we have $8928 \le 12167$.

A computer search finds no integral solutions for $t$.

**Case II:** $AB = 4439$ is one of the parallel sides of the trapezoid.

Let $AD = BC = x$ and $CD = y$.

**a)** $O$ lies outside the trapezoid and $1 \le y < 4439$

Here we have $y = 4439 - \dfrac{x\left(386x + 152\sqrt{42(148035889 - x^2)}\right)}{6436343}$, where $1 \le x \le 2258$.

**b)** $O$ lies outside the trapezoid and $4439 < y < 12167$. Here we have

$y = 4439 - \dfrac{x\left(386x - 152\sqrt{42(148035889 - x^2)}\right)}{6436343}$, where $1 \le x \le 6856$.

**c)** $O$ lies inside or on the trapezoid so that $1 \le y \le 12167$. We still have

$y = 4439 - \dfrac{x\left(386x - 152\sqrt{42(148035889 - x^2)}\right)}{6436343}$, but here we have $6857 \le x \le 11955$.

A computer search for cases IIa), IIb) and IIc) finds no integral solutions for $y$.

This completes the solution.

**Observations made by Ken Korbin, proposer of the problem:**

Diameter = (23)(23)(23).

Sides of inscribed trapezoid are $(4439, 2717, 4439, 10051)$.

4439 =(2)(19)(19)(23)-(23)(23)(23)

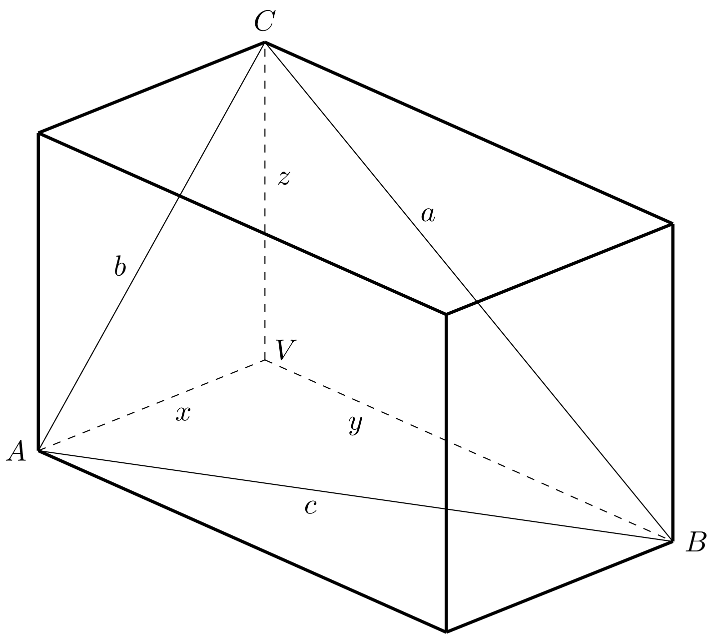
10051 = (19)(23)(23)

2717 = (3)(19)(23)(23) − (4)(19)(19)(19).

If the 19 is replaced by an 18, the inscribed trapezoid will have sides $(2737,\ 5238,\ 2737,\ 9522)$.

**Also solved by Ioannis D. Sfikas, National Technical University of Athens, Greece, and the proposer.**

• **5596:** *Proposed by Albert Natian, Los Angeles Valley College, Valley Glen, CA*




Let $V$ be a vertex of a rectangular box. Let $\overline{VA}$, $\overline{VB}$ and $\overline{VC}$ be the three edges meeting at vertex $V$. Suppose the area of the triangle ABC is $6\sqrt{26}$. The volume of the box is 144. And the sum of the edges of the box is 76. Find the total surface area of the box.

**Solution 1 by Titu Zvonaru, Comănesti, Romania**

We have $x+y+z=19$. Since $16[ABC]=2a^2b+2b^2c^2+2c^2a^2-a^4-b^4-c^4$, we obtain

$$16{\cdot}36{\cdot}26=2(x^2{+}y^2)(y^2{+}z^2){+}2(y^2{+}z^2)(z^2{+}x^2){+}2(z^2{+}x^2)(x^2{+}y^2){-}(x^2{+}y^2)^2{-}(y^2{+}z^2)^2{-}(z^2{+}x^2)^2$$

$$4(x^2y^2+y^2z^2+z^2x^2)=16\cdot 36\cdot 26$$

$$x^2y^2+y^2z^2+z^2x^2=144\cdot 26$$

$$(xy+yz+zx)^2-2xyz(x+y+z)=144\cdot 26$$

$$(xy+yz+zx)^2=2\cdot 144\cdot 19+144\cdot 26,$$

hence $xy+yz+zx=96$ and the total surface area of the box is $2\cdot 96=192$.

**Solution 2 by Bruno Salgueiro Fanego, Viveiro, Spain**

Let, as it appears in the figure, $x=\overline{VA}, y=\overline{VB}$ and $z=\overline{VC}$. Since the box is rectangular, triangles $\triangle VAB, \triangle VBC$, and $\triangle VAC$ are situated in such a way that forces $V$ a right angle. So the Pythagorean theorem, applied to the triangles mentioned above, implies that $x^2+y^2=c^2, y^2+z^2=a^2$ and $x^2+z^2=b^2$.

From the hypothesis in the problem, the area of the $\triangle ABC$ is, (from Herons formula), $\sqrt{\dfrac{a+b+c}{2}\cdot\dfrac{-a+b+c}{2}\cdot\dfrac{a-b+c}{2}\cdot\dfrac{a+b-c}{2}}=6\sqrt{26}$, the volume of the box is $xyz=144$, and the sum of the edges of the box is $4x+4y+4z=76$.

Since

$$(a+b+c)(-a+b+c)(a-b+c)(a+b-c) = \left((b+c)^2-a^2\right)\left(a^2-(b-c)^2\right)$$

$$\begin{aligned} &= \left(2bc+b^2+c^2-a^2\right)\left(2bc-b^2-c^2+a^2\right) = (2bc)^2-(b^2+c^2-a^2) \\ &= 4\left(x^2+z^2\right)\left(x^2+y^2\right)-4x^4 = x^2y^2+y^2z^2+x^2z^2, \end{aligned}$$

the givens in the statement of the problem are equivalent to:

$$\begin{cases} x+y+z=26 \\ x^2y^2+y^2z^2+x^2z^2=3744 \\ xyz=144. \end{cases}$$

This implies that $(x,y,z)=(3,4,12)$ and hence the total surface area of the box is $2(xy+yz+zx)=192$.

**Solution 3 by Pratik Donga, Junagadh, India.**

The volume of the box is $xyz=144$ and the area of $\triangle ABC = 6\sqrt{26}$. The sum of the edges is $4(x+y+z)=76 \Rightarrow x+y+z=19$.

Total surface area $S.A. = 2(xy+yz+zx)$.

Volume of the tetrahedron $VABC = \dfrac{Volume\ of\ the\ cuboid}{6} = \dfrac{144}{6} = 24$. Volume of the tetrahedron

$$\begin{aligned} VABC &= \frac{1}{3}\times height \times area\ of\ the\ base \\ &= \frac{1}{3}\times h \times area\ of\ \triangle ABC = 24 \\ &\Rightarrow\ h = \frac{3\times 24}{6\sqrt{26}} = \frac{12}{\sqrt{26}} \\ &\Rightarrow\ h^2 = \frac{144}{26}. \end{aligned}$$

For the tetrahedron

$$\frac{1}{h^2} = \frac{1}{x^2}+\frac{1}{y^2}+\frac{1}{z^2} \Rightarrow h^2 = \frac{x^2y^2z^2}{x^2y^2+y^2z^2+z^2x^2} \Rightarrow (xy)^2+(yz)^2+(zx)^2 = \frac{(144)^2\times 26}{144} = 3744.$$

Now, $(S.A.)^2 = 4(xy+yz+zx)^2 = 4\left((xy)^2+(yz)^2+(zx)^2+2(xyz)(x+y+z)\right)$
so, $(S.A.)^2 = 4\left(3744+2(144)\times 19)\right) = 4(3744+5472) = 4\times 9216$, and hence the *total surface area of the box is* $\sqrt{4\times 9216} = 192$.

**Solution 4 Brian D. Beasley, Presbyterian College, Clinton, SC**

We have $xyz=144$ and $x+y+z=19$. Then $xy(19-x-y)=144$, or

$$x^2y-19xy+xy^2+144=0.$$

Next, using Heron's formula, we obtain

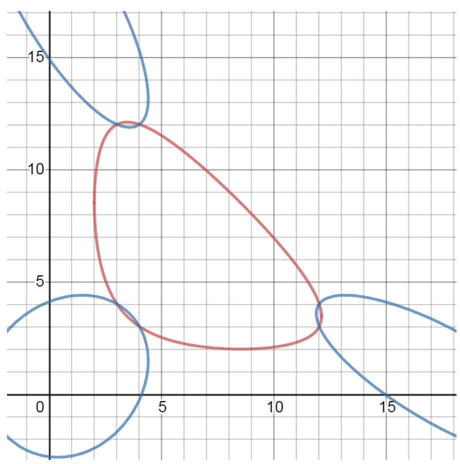
$$(a+b+c)(a+b-c)(a-b+c)(-a+b+c) = 14976.$$

Since $a^2 = y^2 + z^2$, $b^2 = x^2 + z^2$, and $c^2 = x^2 + y^2$, this yields $d_1 d_2 d_3 d_4 = 14976$, where

$$d_1 = a+b+c = \sqrt{y^2 + (19-x-y)^2} + \sqrt{x^2 + (19-x-y)^2} + \sqrt{x^2+y^2};$$
$$d_2 = a+b-c = \sqrt{y^2 + (19-x-y)^2} + \sqrt{x^2 + (19-x-y)^2} - \sqrt{x^2+y^2};$$
$$d_3 = a-b+c = \sqrt{y^2 + (19-x-y)^2} - \sqrt{x^2 + (19-x-y)^2} + \sqrt{x^2+y^2};$$
$$d_4 = -a+b+c = -\sqrt{y^2 + (19-x-y)^2} + \sqrt{x^2 + (19-x-y)^2} + \sqrt{x^2+y^2}.$$

The intersection of these graphs in the first quadrant (see below) consists of six points, corresponding to $\{x, y, z\} = \{3, 4, 12\}$. Hence the total surface area of the box is $2(xy + yz + zx) = 192$.

*Addenda.* (1) The diagram below shows the graphs in the first quadrant of the equations $x^2y - 19xy + xy^2 + 144 = 0$ (in red) and $d_1d_2d_3d_4 = 14976$ (in blue).




(2) We note that $\{a, b, c\} = \{5, 3\sqrt{17}, 4\sqrt{10}\}$.

*Editor's comment*: Shortly after receiving the above solution Brian sent a note expanding on the above question. His note;

Motivated by how the graphs of the two equations barely intersect in the first quadrant (in our solution for the original problem), we pose the following questions:

1. Given a triangle area of $6\sqrt{26}$ and a box edge sum of 76, what is the maximum possible

box volume?

2. Given a box volume of 144 and a box edge sum of 76, what is the minimum possible triangle area?

3. Given a box volume of 144 and a triangle area of $6\sqrt{26}$, what is the maximum possible box edge sum?

*Solution.* In each case, due to symmetry, we assume $x = y$ in order to produce three points (instead of six) where the two graphs intersect.

1. Given $x+y+z = 19$ and $(a+b+c)(a+b-c)(a-b+c)(-a+b+c) = 14976$ with $x = y$, we have $a = b = \sqrt{x^2+z^2}$ and $c = \sqrt{2}x$. Then using $z = 19-2x$ yields

$$2x^2(18x^2 - 304x + 1444) = 14976,$$

or $9x^4 - 152x^3 + 722x^2 - 3744 = 0$. Thus $x \approx 3.56611$, so the maximum volume is $xyz \approx 150.92443$ (not too much larger than the originally given 144).

2. Given $x+y+z = 19$ and $xyz = 144$ with $x = y$, we have $x^2(19-2x) = 144$, or $2x^3 - 19x^2 + 144 = 0$. Then $x \approx 3.44966$, so the minimum triangle area is approximately 30.11067 (not too much smaller than the originally given $6\sqrt{26} \approx 30.59412$).

3. Given $xyz = 144$ and $(a+b+c)(a+b-c)(a-b+c)(-a+b+c) = 14976$ with $x = y$, we let $S = x+y+z$. Then $2x+z = S$, so $a = b = \sqrt{x^2+(S-2x)^2}$ and $c = \sqrt{2}x$. Thus

$$36x^4 - 32Sx^3 + 8S^2x^2 = 14976.$$

Using $z = S - 2x = 144/x^2$, we have $S = (2x^3+144)/x^2$ and hence

$$36x^4 - 32x(2x^3+144) + \frac{8(2x^3+144)^2}{x^2} = 14976.$$

Then $x \approx 3.38838$, so the maximum box edge sum is $4S \approx 77.27628$ (not too much larger than the originally given 76).

In summary, we salute the proposer of the original problem for finding constraints that not only produce integer solutions for the box dimensions but also cannot be tweaked too much more without losing any chance of a solution.

**Also solved by Michel Bataille, Rouen, France; Kee-Wai Lau, Hong Kong, China; Annabel Ma, (student), New Trier High School, Winnetka, Il; Ioannis D. Sfikas National Technical University of Athens, Greece; Daniel Văcaru, Pitesti Romania; David Stone and John Hawkins, Georgia Southern University, Statesboro, GA, and the proposer.**

- **5597:** *Proposed by Daniel Sitaru, National Economic College "Theodor Costescu," Mehedinti, Romania*

If $x, y, z > 0; xyz = 1$ then:

$$\left(x+y-\frac{1}{\sqrt{z}}\right)^2 + \left(y+z-\frac{1}{\sqrt{x}}\right)^2 + \left(z+x-\frac{1}{\sqrt{y}}\right)^2 \geq 3$$

**Solutions 1, 2, and 3 by Henry Ricardo, Westchester Area Math Circle, NY**

**Solution 1:** First we note that Maclaurin's inequality gives us $\sqrt{(xy+yz+zx)/3} \geq \sqrt[3]{xyz} = 1$, or $xy+yz+zx \geq 3$. Then the AGM inequality yields

$$\begin{aligned}\sum_{cyclic}\left(x+y-\frac{1}{\sqrt{z}}\right)^2 &= \sum_{cyclic}(x+y-\sqrt{xy})^2 \\ &\geq \sum_{cyclic}(\sqrt{xy})^2 \;=\; xy+yz+zx \;\geq\; 3.\end{aligned}$$

Equality holds if and only if $x = y = z = 1$.

**Solution 2:** First we note that the AGM inequality gives us $(x+y+z)/3 \geq \sqrt[3]{xyz} = 1$, or $x+y+z \geq 3$. Then the AGM inequality and Radon's inequality yield

$$\begin{aligned}\sum_{cyclic}\left(x+y-\frac{1}{\sqrt{z}}\right)^2 &= \sum_{cyclic}(x+y-\sqrt{xy})^2 \\ &\geq \sum_{cyclic}\left(\frac{x+y}{2}\right)^2 \;\geq\; \frac{[2(x+y+z)]^2}{12} \;\geq\; \frac{6^2}{12} \;=\; 3.\end{aligned}$$

Equality holds if and only if $x = y = z = 1$.

**Solution 3:** Using the AGM inequality twice, we see that

$$\begin{aligned}\sum_{cyclic}\left(x+y-\frac{1}{\sqrt{z}}\right)^2 &= \sum_{cyclic}(x+y-\sqrt{xy})^2 \\ &\geq \sum_{cyclic}(\sqrt{xy})^2 \\ &\geq 3\sqrt[3]{(xyz)^2} \;=\; 3.\end{aligned}$$

**Solution 4 by Kee-Wai Lau, Hong Kong, China**

By the AM-GM inequality, we have

$$\begin{aligned}&2(x+y+z) - (\sqrt{xy}+\sqrt{yz}+\sqrt{zx}) \\ &= (x+y+z) + \frac{(\sqrt{x}-\sqrt{y})^2+(\sqrt{y}-\sqrt{z})^2+(\sqrt{z}-\sqrt{x})^2}{2} \\ &\geq x+y+z \\ &\geq 3\sqrt[3]{xyz} \\ &= 3.\end{aligned}$$

Hence by the Cauchy-Schwarz inequality, we have

$$\begin{aligned}
& \left(x+y-\frac{1}{\sqrt{z}}\right)^2+\left(y+z-\frac{1}{\sqrt{x}}\right)^2+\left(z+x-\frac{1}{\sqrt{y}}\right)^2 \\
\geq\ & \frac{1}{3}\left(\left(x+y-\frac{1}{\sqrt{z}}\right)+\left(y+z-\frac{1}{\sqrt{x}}\right)+\left(z+x-\frac{1}{\sqrt{y}}\right)\right)^2 \\
=\ & \frac{1}{3}\left((x+y-\sqrt{xy})+(y+z-\sqrt{yz})+\left(z+x-\sqrt{zx}\right)\right)^2 \\
=\ & \frac{1}{3}\left(2\left(x+y+z\right)-\left(\sqrt{xy}+\sqrt{yz}+\sqrt{zx}\right)\right)^2 \\
\geq\ & 3, \text{ as required.}
\end{aligned}$$

**Comments from the solution of David Stone and John Hawkins of Georgia Southern University, Statesboro, GA:** Like others they commented that $A(x)$ is the Lagrange Interpolating Polynomial passing through $n+1$ given points $\left(i, 3^i\right), i=0.\dots n$. They then went on to say that "the value 3 in the problem is not necessary. That is if $a>1$ is any real number $a$ we require that $A(x)$ be a polynomial of degree $n$ such that $A(i)=a^i$ for $0\leq i\leq n$, the exact same proof shows that $A(n+1)=a^{n+1}-(a-1)^{n+1}$."

They continued: "This example demonstrates the problem with using the polynomial $A(x)$ to approximate the exponential function $3^x$. Even when $A(x)$ passes through the $n+1$ 'nice' points, $\left(i, 3^i\right), i=0,1,\dots n$, it misses the 'next' value, $3^{n+1}$, by a long way. Of course, it is beautiful that the amount of the miss is known to be $2^{n+1}$."

**Also solved by Hatef I. Arshagi, Guilford Technical Community College, Jamestown, NC; Dionne Bailey, Elsie Campbell, and Charles Diminnie, Angelo State University, San Angelo, TX; Michel Bataille, Rouen, France; Brian Bradie, Christopher Newport University, Newport News, VA; Marin Chirciu, National College "Zinca Golescu," Pitesti, Romania; Bruno Salgueiro Fanego, Viveiro, Spain; Pratik Donga, India; Oleh Faynshteyn, Leipzig, Germany; Igbal Z. Hasanli (student, mentored by Yagub Aliyev), ADA University, Baku, Azerbaijan; Moti Levy, Rehovot, Israel; Ángel Plaza, Universidad de Las Palmas de Gran Canaria, Spain; Ioannis D. Sfikas National Technical University of Athens, Greece; Albert Stadler, Herrliberg, Switzerland; Daniel Văcaru, Pitesti, Romania; Titu Zvonaru, Comănesti, Romania, and the proposer.**

- **5598:** *Proposed by José Luis Díaz-Barrero, Barcelona Tech, Barcelona, Spain*

Let $A(x)$ be a polynomial of degree $n$ such that $A(i)=3^i$ for $0\leq i\leq n$. Find the value of $A(n+1)$.

**Solution 1 by Ulrich Abel, Technische Hochschule Mittelhessen, Germany**

We consider the following generalization: Let $a\in\mathbb{R}$ and suppose that $A\left(x\right)$ is a polynomial of degree (at most) $n$ such that $A(i) = a^i$, for $0 \le i \le n$. Then

$$A(n+1) = \sum_{k=0}^{n} a^k L_{n,k}(n+1)$$

with the Lagrange polynomials

$$L_{n,k}(n+1) = \sum_{\substack{j=0 \\ j\neq k}}^{n} \frac{n+1-j}{k-j} = \frac{(n+1)!/(n+1-k)}{k!(-1)^{n-k}(n-k)!} = (-1)^{n-k}\binom{n+1}{k}.$$

Hence,

$$A(n+1) = -\sum_{k=0}^{n} (-1)^{n+1-k}\binom{n+1}{k} a^k = a^{n+1} - (a-1)^{n+1}.$$

**Solution 2 by Albert Stadler, Herrliberg, Switzerland** By Lagrange's interpolation formula (see for instance
https://en.wikipedia.org./wiki/Langrangepolynomial),

$$A(n+1) = \sum_{i=0}^{n} 3^i \prod_{\substack{j=0 \\ j\neq t}}^{n} \frac{n+1-j}{i-j} = \sum_{i=0}^{n} 3^i \frac{(-1)^{n-1}(n+1)!}{i!(n+1-i)!} =$$

$$= (-1)^n \sum_{i=0}^{n} (-1)^i 3^i \binom{n+1}{i} = (-1)^n(1-3)^{n+1} + 3^{n+1} = 3^{n+1} - 2^{n+1}.$$

**Solution 3 by Ángel Plaza, Universidad de Las Palmas de Gran Canaria, Spain**

It may be shown by induction that

$$A(x) = 1 + 2x + \frac{2^2}{2}x(x-1) + \frac{2^3}{3!}x(x-1)(x-2) + \cdots + \frac{2^n}{n!}x(x-1)\cdots(x-n+1) = \sum_{k=0}^{n}\binom{x}{k}2^k,$$

and therefore $A(n+1) = \sum_{k=0}^{n}\binom{n+1}{k}2^k = 3^{n+1} - 2^{n+1}$.

**Solution 4 by Brian Bradie, Christopher Newport University, Newport News, VA**

Using divided differences, the Newton form for the polynomial $A(x)$ is

$$A(x) = 1 + \sum_{k=1}^{n} \frac{2^k}{k!} \prod_{j=0}^{k-1}(x-j).$$

Thus,

$$\begin{aligned}
A(n+1) &= 1+\sum_{k=1}^{n}\frac{2^k}{k!}\prod_{j=0}^{k-1}(n+1-j)\\
&= 1+\sum_{k=1}^{n}2^k\binom{n+1}{k}\\
&= \sum_{k=0}^{n+1}2^k\binom{n+1}{k}-2^{n+1}\\
&= 3^{n+1}-2^{n+1}.
\end{aligned}$$

**Solution 5 by Michel Bataille, Rouen, France**

$A(x)$ is the Lagrange polynomial associated to the values $3^i$ taken for $x=i$ $(0\le i\le n)$. This polynomial is

$$A(x)=\sum_{i=0}^{n}3^i\cdot\frac{x(x-1)\cdots(x-i+1)\widehat{(x-i)}(x-i-1)\cdots(x-n)}{i!(-1)^{n-i}(n-i)!}=\sum_{i=0}^{n}3^i\cdot\frac{1}{P'(i)}\cdot\frac{P(x)}{x-i}$$

where the hat indicates the omitted factor and $P(x)=\prod_{i=0}^{n}(x-i)$.

Since $\frac{P(n+1)}{n+1-i}=\frac{(n+1)!(n-i)!}{(n+1-i)!}$, we obtain

$$\begin{aligned}
A(n+1) &= \sum_{i=0}^{n}3^i\cdot\frac{1}{i!(-1)^{n-i}(n-i)!}\cdot\frac{(n+1)!(n-i)!}{(n+1-i)!}\\
&= \sum_{i=0}^{n}(-1)^{n-i}3^i\cdot\frac{(n+1)!}{i!(n+1-i)!}\\
&= (-1)^n\sum_{i=0}^{n}\binom{n+1}{i}(-3)^i\\
&= (-1)^n\left(\sum_{i=0}^{n+1}\binom{n+1}{i}(-3)^i-(-3)^{n+1}\right)\\
&= (-1)^n\left((1-3)^{n+1}-(-1)^{n+1}3^{n+1}\right)
\end{aligned}$$

and finally $A(n+1)=3^{n+1}-2^{n+1}$.

**Solution 6 by Dionne Bailey, Elsie Campbell, and Charles Diminnie, Angelo State University, San Angelo, TX**

Our solution makes use of the Binomial Theorem: For $m\ge 1$ and $a,b>0$,

$$(a+b)^m=\binom{m}{0}a^m+\binom{m}{1}a^{m-1}b+\ldots+\binom{m}{m-1}ab^{m-1}+\binom{m}{m}b^m,$$

where $\binom{m}{k} = \dfrac{m!}{k!\,(m-k)!}$ for $k = 0, \ldots, m$.
We begin by introducing the new function

$$f(x) = 1 + \sum_{k=1}^{n} \frac{2^k}{k!} x(x-1)\cdots(x-k+1)$$

for $n \geq 1$. Note first of all that $f(x)$ is a polynomial of degree $n$ and $f(0) = 1 = 3^0$. Next, when $0 < i < n$, we have

$$i(i-1)\cdots(i-k+1) = 0$$

for $k \geq i+1$. Hence,

$$\begin{aligned}
f(i) &= 1 + \sum_{k=1}^{n} \frac{2^k}{k!} i(i-1)\cdots(i-k+1) \\
&= 1 + \sum_{k=1}^{i} \frac{2^k}{k!} i(i-1)\cdots(i-k+1) \\
&= 1 + \frac{2}{1!} i + \frac{2^2}{2!} i(i-1) + \ldots + \frac{2^i}{i!} i(i-1)\cdots(1) \\
&= 1 + \binom{i}{1} 2 + \binom{i}{2} 2^2 + \ldots + \binom{i}{i} 2^i \\
&= \binom{i}{0} 1^i + \binom{i}{1} 1^{i-1} 2 + \binom{i}{2} 1^{i-2} 2^2 + \ldots + \binom{i}{i} 2^i \\
&= (1+2)^i \\
&= 3^i.
\end{aligned}$$

Finally,

$$\begin{aligned}
f(n) &= 1 + \frac{2}{1!} n + \frac{2^2}{2!} n(n-1) + \ldots + \frac{2^n}{n!} n(n-1)\cdots(1) \\
&= \binom{n}{0} 1^n + \binom{n}{1} 1^{n-1} 2 + \binom{n}{2} 1^{n-2} 2^2 + \ldots + \binom{n}{n} 2^n \\
&= (1+2)^n \\
&= 3^n.
\end{aligned}$$

Since $f(x)$ and $A(x)$ are both polynomials of degree $n$ and
$f(i) = 3^i = A(i)$ for $0 \leq i \leq n$, it follows that $f(x) = A(x)$ for all real $x$. As a result, for

$n \geq 1$,

$$\begin{aligned}
A(n+1) &= f(n+1) \\
&= 1 + \sum_{k=1}^{n} \frac{2^k}{k!} (n+1)(n) \cdots (n+2-k) \\
&= 1 + \frac{2^1}{1!}(n+1) + \frac{2^2}{2!}(n+1)(n) + \ldots + \frac{2^n}{n!}(n+1)(n)\cdots(2) \\
&= 1 + \binom{n+1}{1} 2^1 + \binom{n+1}{2} 2^2 + \ldots + \binom{n+1}{n} 2^n \\
&= 1 + \binom{n+1}{1} 2^1 + \binom{n+1}{2} 2^2 + \ldots + \binom{n+1}{n} 2^n + \binom{n+1}{n+1} 2^{n+1} \\
&- \binom{n+1}{n+1} 2^{n+1} \\
&= \binom{n+1}{0} 1^{n+1} + \binom{n+1}{1} 1^n 2^1 + \binom{n+1}{2} 1^{n-1} 2^2 + \ldots + \binom{n+1}{n} 1^1 2^n \\
&+ \binom{n+1}{n+1} 2^{n+1} - 2^{n+1} \\
&= (1+2)^{n+1} - 2^{n+1} \\
&= 3^{n+1} - 2^{n+1}. \qquad (1)
\end{aligned}$$

To complete our claim, we note that when $n = 0$, $A(x)$ is a polynomial of degree 0 (i.e., $A(x)$ is a constant polynomial) with $A(0) = 3^0 = 1$. It follows that $A(x) = 1$ for all real $x$ and we have

$$A(0+1) = A(1) = 1 = 3 - 2 = 3^{0+1} - 2^{0+1}.$$

This also demonstrates that (1) is true when $n = 0$.

**Solution 7 by Ioannis D. Sfikas, National Technical University of Athens, Greece**

Using [1] we have : $a_i = 3^i$, so:

$$A(n+1) = \sum_{i=1}^{n+1} (-1)^{i-1} \binom{n+1}{i} 3^{n+1-i} = 3^{n+1} - 2^{n+1}$$

Since:

$$\sum_{i=1}^{n+1} (-1)^{i-1} \binom{n+1}{i} 3^{n+1-i} = \sum_{i=0}^{n} (-1)^i \binom{n+1}{i+1} 3^{n-i}$$

$$= \sum_{i=0}^{n-1} (-1)^i \left[ \binom{n}{i+1} + \binom{n}{i} \right] 3^{n-i} + (-1)^n$$

$$= \sum_{i=0}^{n-1} (-1)^i \binom{n}{i+1} 3^{n-i} + \sum_{i=0}^{n-1} (i-1)^i \binom{n}{i} 3^{n-i} + (-1)^{n-1}$$

$$= \sum_{i=1}^{n-1} (-1)^{i-1} \binom{n}{i} 3^{n-i+1} - \sum_{i=0}^{n} (-1)^{i-1} \binom{n}{i} 3^{n-i} + (-1)^n + (-1)^{n-1}$$

$$= \sum_{i=1}^{n} (-1)^{i-1} \binom{n}{i} 3^{n-i+1} - \sum_{i=0}^{n} (-1)^{i-1} \binom{n}{i} 3^{n-i}$$

$$= 3^{n+1} - 3 \cdot 2^n - 2^n$$

$$= 3^{n+1} - 2^{n+1}.$$


[1] Alt, Arkady M. (2019). Numerical sequences and polynomials. *Arhimede Mathematical Journal,* **6**(2) : 114−120. *http : //amj−math.com/wp−content/uploads/*2020/02/*AMJ*2019−*vol*6*iss*2*.pdf*

[2] *http://math.stackexchnge.com/questions/2161052/showing−3n− sum−k−0n−1k−binomnk4n-k*


**Solution 9 by Albert Natian, Los Angeles Valley College, Valley Glen, California**

**Answer.** $A(n+1) = 3^{n+1} - 2^{n+1}$.

**Justification.** We begin with the following lemma whose proof is provided toward the end of this solution.

**Lemma.**

$$\sum_{i=0}^{n} (-1)^{n+i} \binom{n+1}{i} i^k = (n+1)^k \quad \text{for} \quad k = 0, 1, 2, \cdots, n.$$

Now let's suppose

$$A(i) = b^i \quad \text{for} \quad 0 \le i \le n$$

where $b$ is a fixed number (e.g., $b = 3$) and $A(x) = \sum_{k=0}^{n} a_k x^k$.

We have

$$\begin{aligned}
A(n+1) &= \sum_{k=0}^{n} a_k (n+1)^k = \sum_{k=0}^{n} a_k \sum_{i=0}^{n} (-1)^{n+i} \binom{n+1}{i} i^k \\
&= \sum_{i=0}^{n} (-1)^{n+i} \binom{n+1}{i} \sum_{k=0}^{n} a_k i^k = \sum_{i=0}^{n} (-1)^{n+i} \binom{n+1}{i} A(i) \\
&= \sum_{i=0}^{n} (-1)^{n+i} \binom{n+1}{i} b^i = (-1)^n \sum_{i=0}^{n} \binom{n+1}{i} (-b)^i \\
&= (-1)^n \left[ -(-b)^{n+1} + \sum_{i=0}^{n+1} \binom{n+1}{i} (-b)^i \right] \\
&= (-1)^n \left[ (-1)^n b^n + (1-b)^{n+1} \right] \\
&= b^{n+1} - (b-1)^{n+1}.
\end{aligned}$$

**Generalization.** Suppose $A(x)$ is a polynomial of degree $n$ such that

$$A(i) = \sum_{t=1}^{m} \beta_t b_t^i \quad \text{for} \quad 0 \le i \le n$$

where $\beta_t$ and $b_t$ are indexed fixed numbers. Then

$$A(n+1) = \sum_{t=1}^{m} \beta_t b_t^{n+1} - \sum_{t=1}^{m} \beta_t (b_t - 1)^{n+1}.$$

**Corollary.** Suppose $A(x)$ is a polynomial of degree $n$ such that

$$A(i) = \sum_{t=1}^{m} t^i \quad \text{for} \quad 0 \le i \le n.$$

Then

$$A(n+1) = m^{n+1}.$$

**Lemma.**

$$\sum_{i=0}^{n} (-1)^{n+i} \binom{n+1}{i} i^k = (n+1)^k \quad \text{for} \quad k = 0, 1, 2, \cdots, n.$$

**Proof.** Clearly the above result holds for $k = 0$ and all $n$. The remainder of the proof is by induction on $n$ with $k \ge 1$. For $n = 0, 1, 2$, the above statement (clearly) holds. Suppose the statement of the lemma above holds for $n$. Then

$$\begin{aligned}
\sum_{i=0}^{n+1}(-1)^{n+1+i}\binom{n+1+1}{i}i^k &= \sum_{i=1}^{n+1}(-1)^{n+1+i}\binom{n+1+1}{i}i^k \\
&= \sum_{i=1}^{n+1}(-1)^{n+1+i}\left[\binom{n+1}{i}+\binom{n+1}{i-1}\right]i^k \\
&= \left[\sum_{i=1}^{n+1}(-1)^{n+1+i}\binom{n+1}{i}i^k\right]+\left[\sum_{i=1}^{n+1}(-1)^{n+1+i}\binom{n+1}{i-1}i^k\right] \\
&= \left[(n+1)^k-\sum_{i=1}^{n}(-1)^{n+i}\binom{n+1}{i}i^k\right]+ \\
&\qquad\qquad +\left[\sum_{i=1}^{n+1}(-1)^{n+1+i}\binom{n+1}{i-1}i^k\right] \\
&= \left[(n+1)^k-(n+1)^k\right]+\left[\sum_{j=0}^{n}(-1)^{n+j}\binom{n+1}{j}(j+1)^k\right] \\
&= \sum_{j=0}^{n}(-1)^{n+j}\binom{n+1}{j}\sum_{p=0}^{k}\binom{k}{p}j^p \\
&= \sum_{p=0}^{k}\binom{k}{p}\sum_{j=0}^{n}(-1)^{n+j}\binom{n+1}{j}j^p \\
&= \sum_{p=0}^{k}\binom{k}{p}(n+1)^p=(n+1+1)^k\,.
\end{aligned}$$

**Also solved by Bruno Salgueiro Fanego (two solutions), Viveiro, Spain; Kee-Wai Lau, Hong Kong, China; Moti Levy, Rehovot, Israel, and the proposer.**

- **5599:** *Proposed by Ovidiu Furdui and Alina Sîntămărian, Technical University of Cluj-Napoca, Cluj-Napoca, Romania*

Let $n \geq 2$ be an integer. Calculate

$$\int_0^{\frac{\pi}{2}}\frac{\sin x+\cos x}{\sin^{2n-1}x+\cos^{2n-1}x}\mathrm{d}x.$$

**Solution 1 by Seán M. Stewart, Bomaderry, NSW, Australia**

Denote the integral to be found by $I_n$ where $n$ is a positive integer such that $n \geqslant 2$. Splitting the integral as follows

$$I_n=\int_0^{\frac{\pi}{2}}\frac{\sin x}{\sin^{2n-1}x+\cos^{2n-1}x}\,dx+\int_0^{\frac{\pi}{2}}\frac{\cos x}{\sin^{2n-1}x+\cos^{2n-1}x}\,dx,$$

if in the first of these integrals we enforce a substitution of $x \mapsto \dfrac{\pi}{2} - x$ one finds

$$I_n = 2\int_0^{\frac{\pi}{2}} \frac{\cos x}{\sin^{2n-1} x + \cos^{2n-1} x}\, dx.$$

Rearranging the integrand we have

$$\begin{aligned}
I_n &= 2\int_0^{\frac{\pi}{2}} \frac{\cos x}{\cos^{2n-1} x(1 + \tan^{2n-1} x)}\, dx \\
&= 2\int_0^{\frac{\pi}{2}} \frac{\sec^{2n-2} x}{1 + \tan^{2n-1} x}\, dx \\
&= 2\int_0^{\frac{\pi}{2}} \frac{\sec^{2n-4} x \sec^2 x}{1 + \tan^{2n-1} x}\, dx.
\end{aligned}$$

Letting $u = \tan x$, as $\sec^{2n-4} x = (1+u^2)^{n-2}$ one has

$$I_n = 2\int_0^{\infty} \frac{(1+u^2)^{n-2}}{1+u^{2n-1}}\, du.$$

From the binomial theorem, since

$$(1+u^2)^{n-2} = \sum_{k=0}^{n-2} \binom{n-2}{k} u^{2k},$$

the integral for $I_n$ can be rewritten as

$$I_n = 2\sum_{k=0}^{n-2} \binom{n-2}{k} \int_0^{\infty} \frac{u^{2k}}{1+u^{2n-1}}\, du.$$

Identification of the integral that remains as a beta function is now made. Setting $t = u^{2n-1}$ in the integral we find

$$\begin{aligned}
I_n &= \frac{2}{2n-1} \sum_{k=0}^{n-2} \binom{n-2}{k} \int_0^{\infty} \frac{t^{\frac{2k+2-2n}{2n-1}}}{1+t}\, dt \\
&= \frac{2}{2n-1} \sum_{k=0}^{n-2} \binom{n-2}{k} \int_0^{\infty} \frac{t^{\left(\frac{2k+1}{2n-1}\right)-1}}{(1+t)^{\left(\frac{2k+1}{2n-1}\right)+\left(\frac{2n-2k-2}{2n-1}\right)}}\, dt \\
&= \frac{2}{2n-1} \sum_{k=0}^{n-2} \binom{n-2}{k} \mathrm{B}\left(\frac{2k+1}{2n-1}, \frac{2n-2k-2}{2n-1}\right) && (0) \\
&= \frac{2}{2n-1} \sum_{k=0}^{n-2} \binom{n-2}{k} \Gamma\left(\frac{2k+1}{2n-1}\right) \Gamma\left(\frac{2n-2k-2}{2n-1}\right) \\
&= \frac{2}{2n-1} \sum_{k=0}^{n-2} \binom{n-2}{k} \Gamma\left(\frac{2k+1}{2n-1}\right) \Gamma\left(1 - \frac{2k+1}{2n-1}\right) && (1) \\
&= \frac{2\pi}{2n-1} \sum_{k=0}^{n-2} \binom{n-2}{k} \operatorname{cosec}\left(\frac{2k+1}{2n-1}\right)\pi. && (2)
\end{aligned}$$

Explanation for the changes made are as follows:

(1) Identification of the integral as a beta function, namely $\mathrm{B}(x,y)=\int_0^\infty \frac{t^{x-1}}{(1+t)^{x+y}}\,dt$, for $x,y>0$.

(2) Relationship between the beta function and gamma function of $\mathrm{B}(x,y)=\Gamma(x)\Gamma(y)/\Gamma(x+y)$ has been used.

(3) Euler's reflexion formula for the gamma function $\Gamma(x)\Gamma(1-x)=\pi/\sin(\pi x)$ has been used.

So in conclusion the value for the integral is given by

$$I_n=\frac{2\pi}{2n-1}\sum_{k=0}^{n-2}\binom{n-2}{k}\operatorname{cosec}\left(\frac{2k+1}{2n-1}\right)\pi,\quad n=2,3,4,\ldots.$$

**Solution 2 by Albert Stadler, Herrliberg, Switzerland**

We perform a change of variables: $y=\tan x, 1+y^2=\frac{1}{\cos^2 x}, dy=\frac{dx}{\cos^2 x}=(1+y^2)dx$ that results in

$$\int_0^{\frac{\pi}{2}}\frac{\sin x+\cos x}{\sin^{2n-1}x}dx=2\int_0^{\frac{\pi}{4}}\frac{\sin x+\cos x}{\sin^{2n-1}x+\cos^{2n-1}x}dx=$$

$$=\quad 2\int_0^{\frac{\pi}{4}}\frac{1+\tan x}{\cos^{2n-2}x(1+\tan^{2n-1}x)}dx=2\int_0^1\frac{(1+y^2)^{n-2}(1+y)}{1+y^{2n-1}}dy.$$

A partial fraction decomposition applied to the integrand yields

$$\frac{(1+y^2)^{n-2}(1+y)}{1+y^{2n-1}}=\frac{1}{2n-1}\sum_{k=0}^{2n-2}\frac{1}{y-e^{\frac{\pi i(2k+1)}{2n-1}}}\cdot\frac{\left(1+e^{\frac{2\pi i(2k+1)}{2n-1}}\right)^{n-2}\left(1+e^{\frac{\pi i(2k+1)}{2n-1}}\right)}{e^{\frac{\pi i(2k+1)(2n-2)}{2n-1}}}=$$

$$=\frac{2^{n-1}i}{2n-1}\sum_{k=0}^{2n-2}(-1)^k\frac{e^{-\frac{\pi i(2k+1)}{2n-1}}\cos^{n-2}\left(\frac{\pi(n(2k+1)}{2n-1}\right)\cos\left(\frac{\pi(2k+1)}{2(2n-1)}\right)}{1-ye^{\frac{-\pi i(2k+1)}{2n-1}}}.$$

Furthermore,

$$\int_0^1\frac{e^{-\frac{\pi i(2k+1)}{2n-1}}}{1-ye^{\frac{-\pi i(2k+1)}{2n-1}}}dy=-\log\left(1-ye^{\frac{-\pi i(2k+1)}{2n-1}}\right)\Big|_0^1=-\log\left(1-e^{-\frac{\pi i(2k+1)}{2n-1}}\right)=$$

$$=-\log\left(2\sin\left(\frac{\pi(2k+1)}{4n-2}\right)\right)+\frac{\pi i(2k+1)}{4n-2}-\frac{\pi i}{2}.$$

We conclude

$$\int_0^{\frac{\pi}{2}}\frac{\sin x+\cos x}{\sin^{2n-1}x+\cos^{2n-1}x}dx=$$

$$\frac{2^n i}{2n-1}\sum_{k=0}^{2n-2}(-1)^k\cos^{n-2}\left(\frac{\pi(2k+1)}{2n-1}\right)\cos\left(\frac{\pi(2k+1)}{2n-1}\right)\left(-\log\left(2\sin\left(\frac{\pi(2k+1}{4-2}\right)\right)+\frac{\pi i(2k+1)}{4n-2}-\frac{\pi i}{2}\right)=$$

$$=\frac{2^n}{2n-1}\sum_{k=0}^{2n-2}(-1)^k\cos^{n-2}\left(\frac{\pi(2k+1)}{2n-1}\right)\cos\left(\frac{\pi(2k+1)}{2(2n-1)}\right)\left(\frac{\pi}{2}-\frac{\pi(2k+1)}{4\pi-2}\right)=$$

$$=\frac{2^{n+1}\pi}{2n-1}\sum_{k=0}^{2n-2}(-1)^k\cos^{n-2}\left(\frac{\pi(2k+1)}{2n-1}\right)\cos\left(\frac{\pi(2k+1)}{2(2n-1)}\right)\left(\frac{n-1-k}{2n-1}\right).$$

**Solution 3 by Moti Levy, Rehovot, Israel**

$$I_n:=\int_0^{\frac{\pi}{2}}\frac{\sin x+\cos x}{\sin^{2n-1}x+\cos^{2n-1}x}dx=2\int_0^{\frac{\pi}{2}}\frac{\sin x}{\sin^{2n-1}x+\cos^{2n-1}x}dx. \tag{1}$$

After changing the variable $w=\tan^2 x$, we get

$$I_n=\int_0^{\infty}\frac{(1+w)^{n-2}}{1+w^{n-\frac{1}{2}}}dw. \tag{2}$$

The following definite integral from Gradshteyn I , Ryzhik I Table Of Integrals, Series And Products (7Ed , Elsevier, 2007), entry 3.241 is used:

$$\int_0^{\infty}\frac{t^{\mu-1}}{1+t^{\nu}}dw=\frac{1}{\nu}B\left(\frac{\mu}{\nu},1-\frac{\mu}{\nu}\right), \tag{3}$$

where $B(x,y):=\int_0^1 t^{x-1}(1-t)^{y-1}\,dt$ is the Beta function.

Proof of (3):

By definition of the Beta function

$$\frac{1}{\nu}B\left(\frac{\mu}{\nu},1-\frac{\mu}{\nu}\right)=\frac{1}{\nu}\int_0^1 u^{\frac{\mu}{\nu}-1}(1-u)^{-\frac{\mu}{\nu}}\,du$$

After changing the variable $u=\frac{t^{\nu}}{1+t^{\nu}},\qquad \frac{du}{dt}=\frac{\nu t^{\nu-1}}{(t^{\nu}+1)^2}$

$$\frac{1}{\nu}\int_0^1 u^{\frac{\mu}{\nu}-1}(1-u)^{-\frac{\mu}{\nu}}\,du=\frac{1}{\nu}\int_0^{\infty}\left(\frac{t^{\nu}}{1+t^{\nu}}\right)^{\frac{\mu}{\nu}-1}\left(\frac{1}{1+t^{\nu}}\right)-\frac{\mu}{\nu}\frac{\nu t^{\nu-1}}{(t^{\nu}+1)^2}dt$$
$$=\int_0^{\infty}\frac{t^{\mu-1}}{1+t^{\nu}}dx.$$

Expanding $(1+w)^{n-2}$ gives,

$$I_n=\int_0^{\infty}\sum_{m=0}^{n-2}\binom{n-2}{m}\frac{w^m}{1+w^{n-\frac{1}{2}}}dw$$
$$=\sum_{m=0}^{n-2}\binom{n-2}{m}\int_0^{\infty}\frac{w^m}{1+w^{n-\frac{1}{2}}}dw$$
$$=\sum_{m=0}^{n-2}\frac{\binom{n-2}{m}}{n-\frac{1}{2}}B\left(\frac{m+1}{n-\frac{1}{2}},1-\frac{m+1}{n-\frac{1}{2}}\right).$$

Applying the reflection rule

$$B(x, 1-x) = \frac{\pi}{\sin(\pi x)},$$

we conclude that

$$I_n = \frac{2\pi}{2n-1} \sum_{m=0}^{n-2} \frac{\binom{n-2}{m}}{\sin\left(\frac{2m+2}{2n-1}\pi\right)}.$$

**Solution 4 by Ioannis D. Sfikas, National Technical University of Athens, Greece**

We have

$$I_n = \int_0^{\frac{\pi}{2}} \frac{\sin x + \cos x}{\sin^{2n-1} x + \cos^{2n-1} x} dx =$$

$$2\int_0^{\frac{\pi}{2}} \frac{\cos x}{\sin^{2n-1} x + \cos^{2n-1} x} dx = 2\int_0^{\frac{\pi}{2}} \frac{dx}{(1+\tan^{2n-1} x)(\cos^{2n-2} x)}.$$

We may assume: $t = \tan x$. So,

$$\begin{aligned} I_n &= 2\int_0^{+\infty} \frac{(1+t^2)^{n-2}}{1+t^{2n-1}} dt \\ &= 2\int_0^{+\infty} \frac{1}{1+t^{t^{2n-1}}} \sum_{k=0}^{n-2} \binom{n-2}{k} t^{2k} dt = 2\sum_{i=0}^{n-2} \binom{n-2}{k} \int_0^{+\infty} \frac{t^{2k}}{1+t^{2n-1}} dt \end{aligned}$$

Since:

$$J(a,b) = \int_0^{+\infty} \frac{t^a}{1+t^b} dt = \frac{\pi}{b\sin\left[\frac{(a+1)\pi}{b}\right]}, \text{with}$$

a,b $\in \Re$ and $b > a+1 > 0$. With the substitution $\dfrac{1}{1+t^b} = z$, we have:

$$\begin{aligned} J(a,b) = \frac{1}{b}\int_0^1 z^{\frac{a+1}{b}} (1-z)^{\frac{a+1}{b}-1} dz = \frac{1}{b} B\left(\frac{a+1}{b}, 1-\frac{a+1}{b}\right) &= \frac{1}{b}\Gamma\left(\frac{a+1}{b}\right)\Gamma\left(1-\frac{a+1}{b}\right) \\ &= \frac{\pi}{b\sin\left[\frac{(a+1)\pi}{b}\right]}. \end{aligned}$$

So,we have

$$I_n = \frac{2\pi}{2n-1} \sum_{k=0}^{n-2} \frac{\binom{n-2}{k}}{\sin\left[\dfrac{(2k+1)\pi}{2n-1}\right]}.$$

**Also solved by Michel Bataille, Ruen, France; Kee-Wai Lau, Hong Kong, China; and the proposer.**

**5600:** *Proposed by Seán M. Stewart, Bomaderry, NSW, Australia*

Evaluate:

$$\int_0^{\pi} \log\left(1+2a\cos x+a^2\right)\log\left(1+2b\cos x+b^2\right)dx,$$

if $a, b \in \Re$ are such that the product $ab$ with $|a|,|b|<1$ satisfies the equation $a^2b^2+ab=1$.

**Solution 1 by Albert Stadler, Herrliberg, Switzerland**

We claim that

$$\int_0^{\pi} \log\left(1+2a\cos x+a^2\right)\log\left(1+2b\cos x+b^2\right)dx = 2\pi Li_2(a,b), \qquad (*)$$

where $Li_2(x)$ is the dilogarithm (see for instance https://en.wikipedia.og/wiki/Spence%27$s$ function).

a$^2b^2+ab=1$ implies that $ab=\frac{1}{2}\left(-1+\sqrt{5}\right)$ or $ab=\frac{1}{2}\left(-1-\sqrt{5}\right)$. However $|a|,|b|<1$, so $ab=\frac{1}{2}\left(-1++\sqrt{5}\right)$ It is known (see above reference) that

$$Li_2\left(\frac{1}{2}(-1+\sqrt{5})\right)=\frac{\pi^2}{10}-\ln^2\left(\frac{1}{2}(1+\sqrt{5})\right).$$

Therefore

$$\int_0^{\pi} \log\left(1+2a\cos x+a^2\right)\log\left(1+2b\cos x+b^2\right)dx=\frac{\pi^3}{5}-2\pi\ln^2\left(\frac{1}{2}(1+\sqrt{5})\right).$$

Proof of $(*)$ :

$$\int_0^{\pi} \log(1+2a\cos x+a^2)\log(1+2b\cos x+b^2)dx=$$

$$=\frac{1}{2}\int_0^{2\pi} \log(1+2a\cos x+a^2)\log(1+2b\cos x+b^2)dx=$$

$$=\frac{1}{2}\int_0^{2\pi} \log\left(1+2a(e^{ix}+e^{-ix})+a^2\right)\log\left(1+b(e^{ix}+e^{-ix})+b^2\right)dx=$$

$$=\frac{1}{2}\int_0^{2\pi} \log\left((1+ae^{ix})(1+ae^{-ix})\right)\log\left((1+be^{ix})(1+be^{-ix})\right)dx=$$

$$+\frac{1}{2}\int_0^{2\pi} \log(1+ae^{ix})\log(1+be^{ix})dx+\frac{1}{2}\int_0^{2\pi} \log(1+ae^{ix})\log(1+be^{-ix})dx+$$

$$+\frac{1}{2}\int_0^{2\pi} \log(1+ae^{-ix})\log(1+be^{ix})dx+\frac{1}{2}\int_0^{2\pi} \log(1+ae^{-ix})\log(1+be^{-ix})dx+$$

$$=\frac{1}{2}\int_0^{2\pi}\left(\sum_{k=1}^{\infty}\frac{(-1)^{k+1}}{k}a^ke^{ikx}\right)\left(\sum_{k=1}^{\infty}\frac{(-1)^{k+1}}{k}b^ke^{ikx}\right)dx+$$

$$+\frac{1}{2}\int_0^{2\pi}\left(\sum_{k=1}^{\infty}\frac{(-1)^{k+1}}{k}a^ke^{ikx}\right)\left(\sum_{k=1}^{\infty}\frac{(-1)^{k+1}}{k}b^ke^{-ikx}\right)dx+$$

$$+\frac{1}{2}\int_0^{2\pi}\left(\sum_{k=1}^{\infty}\frac{(-1)^{k+1}}{k}a^k e^{-ikx}\right)\left(\sum_{k=1}^{\infty}\frac{(-1)^{k+1}}{k}b^k e^{ikx}\right)dx+$$

$$+\frac{1}{2}\int_0^{2\pi}\left(\sum_{k=1}^{\infty}\frac{(-1)^{k+1}}{k}a^k e^{-ikx}\right)\left(\sum_{k=1}^{\infty}\frac{(-1)^{k+1}}{k}b^k e^{-ikx}\right)dx=$$

$$=0+\pi\sum_{k=1}^{\infty}\frac{a^kb^k}{k^2}+\pi\sum_{k=1}^{\infty}\frac{a^kb^k}{k^2}+0=2\pi Li_2(ab).$$

interchange of summation and integration is permitted, since all series converge absolutely and uniformity for $0\le x\le 2\pi$.

**Solution 2 by Michel Bataille, Rouen, France**

Let $I$ be the required integral. We show that $I=2\pi\left(\frac{\pi^2}{10}-\log^2\left(\frac{\sqrt{5}-1}{2}\right)\right)$.

We observe that

$$I=\int_0^{\pi}\left(\int_0^a\frac{2(t+\cos x)}{1+2t\cos x+t^2}\,dt\right)\cdot\left(\int_0^b\frac{2(u+\cos x)}{1+2u\cos x+u^2}\,du\right)\,dx=4\int_0^a\int_0^b F(t,u)\,dt\,du$$

where

$$F(t,u)=\int_0^{\pi}\frac{(t+\cos x)(u+\cos x)}{(1+2t\cos x+t^2)(1+2u\cos x+u^2)}\,dx.$$

To calculate $F(t,u)$, we make use of the change of variables defined by $y=\tan(x/2)$, that is, $x=\arctan(2y),\ dx=\frac{2dy}{1+y^2},\ \cos x=\frac{1-y^2}{1+y^2}$. We obtain:

$$F(t,u)=2\int_0^{\infty}\frac{[y^2(1-t)-(1+t)][y^2(1-u)-(1+u)]}{[y^2(1-t)^2+(1+t)^2][y^2(1-u)^2+(1+u)^2]}\cdot\frac{dy}{y^2+1}=\frac{2}{(1-t)(1-u)}J(r,s)$$

where $r=\frac{1+t}{1-t}>0,\ s=\frac{1+u}{1-u}>0$ (since $|t|\le|a|<1$, $|u|\le|b|<1$) and

$$J(r,s)=\int_0^{\infty}\frac{(y^2-r)(y^2-s)}{(y^2+r^2)(y^2+s^2)(y^2+1)}\,dy.$$

Since $J(r,s)=\frac{\pi}{2(r+s)}$ (see a quick proof at the end), we have

$$F(t,u)=\frac{2}{(1-t)(1-u)}\cdot\frac{\pi}{2\left(\frac{1+t}{1-t}+\frac{1+u}{1-u}\right)}=\frac{\pi}{2(1-tu)}.$$

Noticing that from the hypotheses on $ab$, we must have $ab=\frac{\sqrt{5}-1}{2}$, we obtain

$$I=2\pi\int_0^b du\int_0^a\frac{dt}{1-tu}=2\pi\int_0^b-\frac{\log(|1-au|)}{u}=-2\pi\int_0^{ab}\frac{\log|1-w|}{w}\,dw$$

hence

$$I = 2\pi \int_0^{ab} -\frac{\log(1-w)}{w}\,dw = 2\pi \mathrm{Li}_2(ab)$$

where $\mathrm{Li}_2(x) = -\int_0^x \frac{\ln(1-w)}{w}\,dw = \sum_{n=1}^{\infty} \frac{x^n}{n^2}$ denotes the dilogarithm function. The claimed result now follows from

$$\mathrm{Li}_2\left(\frac{\sqrt{5}-1}{2}\right) = \frac{\pi^2}{10} - \log^2\left(\frac{\sqrt{5}-1}{2}\right)$$

(see for example: D. Zagier, The Remarkable Dilogarithm, *J. Math. and Phys. Sciences*, 22(1988), 131-145).

*Proof of* $J(r,s) = \frac{\pi}{2(r+s)}$.
If $r \neq s$, we have

$$\frac{(y^2-r)(y^2-s)}{(y^2+r^2)(y^2+s^2)(y^2+1)} = \frac{1}{(r-1)(s-1)}\cdot\frac{1}{y^2+1} + \frac{r(r^2+s)}{(r-1)(r^2-s^2)}\cdot\frac{1}{y^2+r^2} + \frac{s(r+s^2)}{(s-1)(s^2-r^2)}\cdot\frac{1}{y^2+s^2}$$

and the result readily follows from $\int_0^\infty \frac{dy}{y^2+m^2} = \left[\frac{1}{m}\arctan(y/m)\right]_0^\infty = \frac{\pi}{2m}$ for positive $m$.
If $r = s, r \neq 1$, the result similarly follows from the decomposition

$$\frac{(y^2-r)^2}{(y^2+1)((y^2+r^2)^2} = \frac{1}{(r-1)^2}\left(\frac{1}{y^2+1} - \frac{r^2(r^2-1)}{(y^2+r^2)^2} + \frac{r^2-2r}{y^2+r^2}\right)$$

and $\int_0^\infty \frac{dy}{(y^2+r^2)^2} = \frac{1}{r^3}\int_0^\infty \frac{dx}{(x^2+1)^2} = \frac{1}{r^3}\int_0^{\pi/2} \cos^2 u\,du = \frac{\pi}{4r^3}$.
If $r = s = 1$, the decomposition

$$\frac{(y^2-1)^2}{(y^2+1)^3} = \frac{4}{(y^2+1)^3} - \frac{4}{(y^2+1)^2} + \frac{1}{y^2+1}$$

and $\int_0^\infty \frac{dy}{(y^2+1)^3} = \int_0^{\pi/2} \cos^4 u\,du = \frac{3\pi}{16}$ readily show that the result is still valid.

**Solution 3 by Kee-Wai Lau, Hong Kong, China**

We show that the given integral, denoted by $I$, equals $2\pi\left(\frac{\pi^2}{10} - \log^2\left(\frac{\sqrt{5}-1}{2}\right)\right)$.

It is known ([2], p.237) that for $|k| < 1$ , we have

$$\log(1+2k\cos x + k^2) = 2\sum_{m=1}^{\infty} \frac{(-1)^{m-1}k^m\cos(mx).}{m}$$

Since

$$\left|\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} \frac{(-1)^{m+n}a^m b^n \cos(mx)\cos(nx)}{mn}\right| \leq \left(\sum_{m=1}^{\infty} \frac{|a|^m}{m}\right)\left(\sum_{n=1}^{\infty} \frac{|b|^n}{n}\right)$$

$$= \log(1-|a|)\log(1-|b|) < \infty,$$

so interchanging the order of integration and summation, we have

$$\begin{aligned} I &= 4\int_0^\pi \left(\sum_{m=1}^\infty \sum_{n=1}^\infty \frac{(-1)^{m+n} a^m b^n \cos(mx)\cos(nx)}{mn}\right) dx \\ &= 4\sum_{m=1}^\infty \sum_{n=1}^\infty \frac{(-1)^{m+n} a^m b^n}{mn} \int_0^\pi \cos(mx)\cos(nx) dx. \end{aligned}$$

It is well known that for positive integers $m$ and $n$, we have

$$\int_0^\pi \cos(mx)\cos(nx)dx = \begin{cases} 0, & m \neq n \\ \frac{\pi}{2}, & m = n. \end{cases}$$

Hence $I = 2\pi \sum_{m=1}^\infty \frac{(ab)^m}{m^2}$; Since $|a|,\ b| < 1$ satisfy the equation

$a^2b^2 + ab = 1$, so $ab = \frac{\sqrt{5}-1}{2}$. According to entry (2.6.12) in theorem 2.63 on p. 105 of [1], we have $\sum_{m=1}^\infty \frac{\left(\frac{\sqrt{5}-1}{2}\right)^m}{m^2} = \frac{\pi^2}{10} - \log^2\left(\frac{\sqrt{5}-1}{2}\right)$. Hence our claim for $I$.

References:


1. G.E. Andrews, R. Askey, R. Roy: *Special Functions,* Cambridge University Press, 1999
2. Paul J. Nahin: *Inside Interesting Integrals,* Springer-Verlag, New York, 2015.


**Solution 4 by Moti Levy, Rehovot, Israel**

Let

$$\begin{aligned} J(a,b) &:= \int_0^\pi \log\left(1 + 2a\cos x + a^2\right) \log\left(1 + 2b\cos x + b^2\right) dx, \\ F(x,a) &:= \frac{1}{1 + 2a\cos x + a^2}. \end{aligned}$$

Then by differentiation under the integral sign,

$$\begin{aligned} &\frac{\partial^2 J}{\partial a \partial b} \\ &= 4\int_0^\pi \frac{(a+\cos x)(b+\cos x)}{(1+2a\cos x + a^2)(1+2b\cos x+b^2)} dx \\ &= \int_0^\pi \frac{1}{ab} dx + \frac{a^2-1}{ab}\int_0^\pi F(x,a)\,dx + \frac{b^2-1}{ab}\int_0^\pi F(x,a)\,dx + \frac{\left(a^2-1\right)b^2-1}{ab}\int_0^\pi F(x,a)\,F(x,b)\,dx \end{aligned} \tag{1}$$

The following definite integrals can be evaluated by substitution $t = \tan \frac{x}{2}$,

$$\int_0^\pi F(x,a)\,dx = \int_0^\infty \frac{2}{(a-1)^2 t^2 + (a+1)^2} dt$$
$$= \frac{2}{1-a^2} \arctan\left(\frac{1-a}{1+a} t\right)\Bigg]_0^\infty = \frac{1}{1-a^2}\pi. \tag{2}$$

$$\int_0^\pi F(x,a)\,F(x,b)\,dx = \int_0^\infty \frac{1}{1+a^2+2a\frac{1-t^2}{1+t^2}} \frac{1}{1+b^2+2b\frac{1-t^2}{1+t^2}} \frac{2}{1+t^2} dt$$
$$2\int_0^\infty \frac{t^2+1}{\left((a-1)^2 t^2 + (a+1)^2\right)\left((b-1)^2 t^2 + (b+1)^2\right)} dt$$
$$= \frac{-2a\left(1-b^2\right)\arctan\left(\frac{1-a}{1+a}t\right) + 2b\left(1-a^2\right)\arctan\left(\frac{1-b}{1+b}t\right)}{(1-a^2)(1-b^2)(b-a+a^2b-ab^2)}\Bigg]_0^\infty$$
$$= \frac{(1+ab)}{(1-a^2)(1-b^2)(1-ab)}\pi. \tag{3}$$

Substitution of (2) and (3) into (1) gives

$$\frac{\partial^2 J}{\partial a \partial b} = \frac{2\pi}{1-ab}.$$

Since

$$\frac{\partial J}{\partial a} = \quad for \quad b = 0,$$
$$J = 0 \quad for \quad a = 0,$$

then

$$J(a,b) = \int_0^b \int_0^a \frac{2\pi}{1-uv} du dv = 2\pi \mathbf{Li}_2(ab),$$

where $\mathbf{Li}_2(x)$ is the Dilogarithm function defined by

$$\mathbf{Li}_2(x) := \sum_{k=1}^\infty \frac{x^k}{k^2}, \quad |x| < 1.$$

The condition $a^2b^2 + ab = 1$ implies that $ab = \frac{1}{\phi}$, where $\phi$ is the golden ratio $\phi = \frac{\sqrt{5}+1}{2}$. The value of the Dilogarithm at $\frac{1}{\phi}$ has been calculated using the properties of the Dilogarithm (see the entry *Spence's function* in Wikipedia),

$$\mathbf{Li}_2\left(\frac{1}{\phi}\right) = \frac{3}{5}\zeta(2) - \ln^2\phi = \frac{\pi^2}{10} - \ln^2\phi.$$

We conclude that the integral is equal to

$$\frac{\pi^3}{5} - 2\pi\ln^2\phi \cong 4.74629$$

**Solution 5 by Ioannis D. Sfikas, National Technical University of Athens, Greece**

The definite integral is the limit of the Riemann sum: The definite integral of a continuos function $f$ over the interval $[a, b]$, denote by $\int_a^b f(x)dx$ is the limit of a Riemann sum as the number of subdivisions approaches infinity. That is:

$$\int_a^b f(x)dx = \lim_{n\to\infty} \sum_{i=1}^{n} \Delta x \cdot f(x_i),$$

where $\Delta x = \dfrac{b-a}{n}$ and $x_i = a + \Delta x \cdot i$. Let:

$$F(a) = \int_0^\pi \log(1 + 2a\cos x + a^2)dx, \qquad F(b) = \int_0^\pi \log(1 + 2b\cos x + b^2)dx,$$

$$F(a, b) = \int_0^\pi \log(1 + 2a\cos x + a^2)\log(1 + 2b\cos x + b^2)dx.$$

First, we have

$$1 + 2a\cos x + a^2 = (a + \cos x)^2 + \sin^2 x = \left|a + e^{ix}\right|,$$
$$1 + 2b\cos x + b^2 = (b + \cos x)^2 + \sin^2 x = |b + e^{ix}|^2.$$

Hence:

$$F(a) = \lim_{n\to\infty} \frac{\pi}{n} \sum_{k=1}^{n} \log\left|a + e^{i\frac{k}{n}}\right|^2 = \pi \lim_{n\to+\infty} \frac{1}{n} \log|a^n + 1|^2 = 0.$$

$$F(b) = \lim_{n\to\infty} \frac{\pi}{n} \sum_{k=1}^{n} \log\left|b + e^{i\frac{k}{n}}\right|^2 = \pi \lim_{n\to+\infty} \frac{1}{n} \log|b^n + 1|^2 = 0$$

for $a, b \in \Re$ are such that the product $ab$ with $|a|,\ |b| < 1$, satisfies the equation $a^2b^2 + ab = 1$. Using the identities:

$$\log(1 - 2a\cos x + a^2) = -2\sum_{n\geq 1} \frac{\cos(nx)}{n} a^n,$$

with $x,\ a \in \Re, |a| < 1$, and $\int_0^\pi \cos mx \cos(nx)dx = \frac{\pi}{2}\delta_{m,n}$ where $\delta$ is the Kroneocker delta, we have:

$$F(a, b) = \lim_{n\to+\infty} \frac{\pi}{n} \sum_{k=1}^{n} \log\left|a + e^{i\frac{k}{n}}\right|^2 \log\left|b + e^{i\frac{k}{n}}\right|^2$$

$$= 4\int_0^n \sum_{n\geq 1} \frac{\cos(nx)}{n}(-a)^n \sum_{m\geq 1} \frac{\cos(mx)}{m}(-b)^m dx$$

$$= \ 4 \sum_{m,n\geq 1} \frac{(-a)^n(-b)^m}{nm} \int_0^\pi \cos(nx)\cos(mx)dx$$

$$= \ 2\pi \sum_{n\geq 1} \frac{(ab)^n}{n^2} = 2\pi Li_2(ab).$$

where $Li_2 x$ is the dilogarithm. Also, we have:

$$ab = \frac{\sqrt{5}-1}{2} = \phi \approx 0.61803,$$

where $\phi$ is the conjugate of the golden ratio. So, we have:

$$Li_2(ab) = Li_2(\phi) = \frac{\pi^2}{10} - \left[\sinh^{-1}\left(\frac{1}{2}\right)\right]^2 \approx 0.75539561953$$

where $\sinh^{-1} x$ the inverse hyperbolic sine. So, we have:

$$F(a,b) = 2\pi Li_2(\phi) = 2\pi\left(\frac{\pi^2}{10} - \left[\sinh^{-1}\left(\frac{1}{2}\right)\right]^2\right) \approx 4.746291.$$

**Also solved by the proposer.**

*Mea Culpa*

**The name of Ioannis D. Sfikas of the National Technical University of Athens, Greece** was inadvertently not listed as having solved problems 5589, 5590, and 5591.

The name of **Albert Stadler of Herrliberg, Switzerland** was also inadvertently omitted from the list of those who solved 5590.