Problems

Ted Eisenberg, Section Editor

****

This section of the Journal offers readers an opportunity to exchange interesting mathematical problems and solutions. Please send them to Ted Eisenberg, Department of Mathematics, Ben-Gurion University, Beer-Sheva, Israel or fax to: 972-86-477-648. Questions concerning proposals and/or solutions can be sent e-mail to <eisenbt@013.net>. Solutions to previously stated problems can be seen at <http://www.ssma.org/publications>.

Solutions to the problems stated in this issue should be posted before March 15, 2021

- **5619:** *Proposed by Kenneth Korbin, New York, NY*

  If x, y and z are positive integers such that

  $$x^2 + xy + y^2 = z^2$$

  then there are two different Pythagorean triangles with area $K = xyz(x+y)$.

  Find the sides of the triangles if $z = 61$.

- **5620:** *Proposed by Daniel Sitaru, National Economic College "Theodor Costescu" Drobeta Turnu-Severin, Mehedinti, Romania*

  Prove: If $a, b, \in [0,1]$; $a \leq b$, then

  $$4\sqrt{ab} \leq a\left(\left(\frac{b}{a}\right)^{\sqrt{ab}} + \sqrt{\left(\frac{b}{a}\right)^{a+b}}\right) + b\left(\left(\frac{a}{b}\right)^{\sqrt{ab}} + \sqrt{\left(\frac{a}{b}\right)^{a+b}}\right) \leq 2(a+b).$$

- **5621:** *Proposed by Stanley Rabinowitz, Brooklyn, NY*

  Given non-negative integer n, real numbers a and c with $ac \neq 0$, and the expression $a + cx^2 \geq 0$.

  Express: $\int \left(a + cx^2\right)^{\frac{2n+1}{2}} dx$ as the sum of elementary functions.

- **5622:** *Proposed by Albert Natian Los Angeles Valley College, Valley Glen, CA*

  Suppose f is a real-valued function such that for all real numbers x;

  $$[f(x - 8/15)]^2 + [f(x + 47/30)]^2 + [f(x + 2/75)]^2 =$$

  $$= f(x - 8/15)f(x + 47/30) + f(x + 47/30)f(x + 2/75) + f(x + 2/75)f(x - 8/15).$$

  If $f\left(\frac{49}{5}\right) = \frac{11}{3}$, then find $f\left(\frac{1}{2}f\left(\frac{28}{50}\right) - \frac{2}{25}f\left(-42\right)\right)$.

- **5623:** *Proposed by José Luis Díaz-Barrero, Barcelona Tech, Barcelona, Spain*

  Let P be an interior point to an equilateral triangle of altitude one. If x, y, z are the distances from P to the sides of the triangle, then prove that

  $$x^2 + y^2 + z^2 \geq x^3 + y^3 + z^3 + 6xyz.$$

- **5624:** *Proposed by Seán M. Stewart, Bomaderry, NSW, Australia*

  Evaluate: $\displaystyle\int_0^1 \left(\frac{\tan^{-1} x - x}{x^2}\right)^2 dx.$

Solutions

- **5601:** *Proposed by Kenneth Korbin, New York, NY*

  Solve:

  $$\frac{\sqrt{x(x-1)^2}}{(x+1)^2} = \frac{\sqrt{77}}{36}.$$

Solution 1 by David A. Huckaby, Angelo State University, San Angelo, TX

Squaring both sides of the equation and cross-multiplying yields $36^2 x(x-1)^2 = 77(x+1)^4$. Expanding gives the quartic equation $77x^4 - 988x^3 + 3054x^2 - 988x + 77 = 0$.

Note that the coefficients of the quartic polynomial p on the left side of the equation have a palindrome pattern. So if $r \neq 0$ is a root of p, then $\frac{1}{r}$ is also a root of p. Indeed, for $x \neq 0$ we have $x^4 p\left(\frac{1}{x}\right) = p(x)$. So if $r \neq 0$ is a root of p, then $r^4 p\left(\frac{1}{r}\right) = p(r) = 0$, so that $p\left(\frac{1}{r}\right) = 0$. (Note that all roots of p are nonzero, since $p(0) = 77$.)

Let $q(x) = \frac{1}{77}p(x) = x^4 - \frac{988}{77}x^3 + \frac{3054}{77}x^2 - \frac{988}{77}x + 1$, and let r_1 and r_2 be two roots of p and hence also of q. Then

$$\begin{aligned}
q(x) &= (x - r_1)(x - r_2)\left(x - \frac{1}{r_1}\right)\left(x - \frac{1}{r_2}\right) \\
&= (x - r_1)\left(x - \frac{1}{r_1}\right)(x - r_2)\left(x - \frac{1}{r_2}\right) \\
&= (x^2 - \left(r_1 + \frac{1}{r_1}\right)x + 1)(x^2 - \left(r_2 + \frac{1}{r_2}\right)x + 1) \\
&= x^4 - \left(r_1 + \frac{1}{r_1} + r_2 + \frac{1}{r_2}\right)x^3 + \left[\left(r_1 + \frac{1}{r_1}\right)\left(r_2 + \frac{1}{r_2}\right) + 2\right]x^2 \\
&\qquad - \left(r_1 + \frac{1}{r_1} + r_2 + \frac{1}{r_2}\right)x + 1
\end{aligned}$$

Equating the x^3 (equivalently, the x) coefficients and the x^2 coefficients, we obtain the following system of two equations:

$$r_1 + \frac{1}{r_1} + r_2 + \frac{1}{r_2} = \frac{988}{77}$$
$$\left(r_1 + \frac{1}{r_1}\right)\left(r_2 + \frac{1}{r_2}\right) + 2 = \frac{3054}{77}$$

Solving for $r_2 + \frac{1}{r_2}$ in the first equation and substituting into the second yields the quadratic equation $\left(r_1 + \frac{1}{r_1}\right)^2 - \frac{988}{77}\left(r_1 + \frac{1}{r_1}\right) + \frac{3054}{77} - 2 = 0$. Since the system is symmetric in $r_1 + \frac{1}{r_1}$ and $r_2 + \frac{1}{r_2}$, the quadratic equation is also true with $r_1 + \frac{1}{r_1}$ replaced by $r_2 + \frac{1}{r_2}$. The two solutions of this quadratic equation are $\frac{58}{7}$ and $\frac{50}{11}$, so that $r_1 + \frac{1}{r_1} = \frac{58}{7}$, say, and then $r_2 + \frac{1}{r_2} = \frac{50}{11}$.

So $q(x) = (x^2 - \frac{58}{7}x + 1)(x^2 - \frac{50}{11}x + 1)$. The solutions to the quadratic equation $x^2 - \frac{58}{7}x + 1 = 0$ are $r_1 = \frac{29 + 6\sqrt{22}}{7}$ and $\frac{1}{r_1} = \frac{29 - 6\sqrt{22}}{7}$, and the solutions to the quadratic equation $x^2 - \frac{50}{11}x + 1 = 0$ are $r_2 = \frac{25 + 6\sqrt{14}}{11}$ and $\frac{1}{r_2} = \frac{25 - 6\sqrt{14}}{11}$. None of the four is an extraneous solution to the original equation, so these are its four solutions.

Solution 2 by Bruno Salgueiro Fanego, Viveiro, Spain

$$\frac{\sqrt{x(x-1)^2}}{(x+1)^2} = \frac{\sqrt{77}}{36} \Rightarrow \left(\frac{\sqrt{x(x-1)^2}}{(x+1)^2}\right)^2 = \left(\frac{\sqrt{77}}{36}\right)^2 \Rightarrow \frac{x(x-1)^2}{(x+1)^4} = \frac{77}{1296}$$

$\Rightarrow 1296x(x^2-2+1) = 77(x^4+4x^3+6x^2+4+1) \Rightarrow 0 = 77x^4-988x^3+3054x^23-988x+77 \Rightarrow$

$\Rightarrow (7x^2 - 58x + 7)(11x^2 - 50x + 11) = 0 \Rightarrow 7x^2 - 58x + 7 = 0$ or $11x^2 - 50x + 11 = 0 \Rightarrow$

$$\Rightarrow x = \frac{58 \pm \sqrt{(-58)^2 - 4\cdot 7\cdot 7}}{2\cdot 7} \text{ or } x = \frac{50 \pm \sqrt{50^2 - 4\cdot 11\cdot 11}}{2\cdot 11} \Rightarrow$$
$$\Rightarrow x = \frac{29}{7} \pm \frac{6\sqrt{27}}{7} \text{ or } x = \frac{25}{11} \pm \frac{6\sqrt{14}}{11}.$$

These four numbers are the roots to the given equation.

Solution 3 by Anthony J. Bevelacqua, University of North Dakota, Grand Forks, ND

We have

$$36\sqrt{x(x-1)^2} = \sqrt{77}(x+1)^2$$

and so squaring gives

$$1296x(x-1)^2 = 77(x+1)^4$$

which yields

$$1296(x^3 - 2x^2 + x) = 77(x^4 + 4x^3 + 6x^2 + 4x + 1)$$

and so finally

$$77x^4 - 988x^3 + 3054x^2 - 988x + 77 = 0.$$

Now divide this equation by x^2 to find

$$77x^2 - 988x + 3054 - 988\frac{1}{x} + 77\frac{1}{x^2} = 0$$

and so

$$77\left(x^2 + \frac{1}{x^2}\right) - 988\left(x + \frac{1}{x}\right) + 3054 = 0.$$

Since $x^2 + \frac{1}{x^2} = \left(x + \frac{1}{x}\right)^2 - 2$ we have

$$77\left(x + \frac{1}{x}\right)^2 - 988\left(x + \frac{1}{x}\right) + 2900 = 0.$$

Now, by the quadratic formula, we have

$$x + \frac{1}{x} = \frac{58}{7} \text{ or } \frac{50}{11}$$

which gives

$$x^2 - \frac{58}{7}x + 1 = 0 \text{ with roots } x = \frac{29 \pm 6\sqrt{22}}{7} \approx 8.16,\ 0.12$$

or

$$x^2 - \frac{50}{11}x + 1 = 0 \text{ with roots } x = \frac{25 \pm 6\sqrt{14}}{11} \approx 4.31,\ 0.23$$

Since each of these roots are positive, our original equation has the four solutions

$$\frac{29 \pm 6\sqrt{22}}{7}, \quad \frac{25 \pm 6\sqrt{14}}{11}$$

Solution 4 by Peter Fulop, Gyomro, Hungary

$$\frac{\sqrt{x(x-1)^2}}{(x+1)^2} = \frac{\sqrt{77}}{36} \tag{1}$$

Starting with realign (1) in the following way:

$$18\sqrt{4x(x^2 - 2x + 1)} = \sqrt{77}((x^2 - 2x + 1) + 4x) \tag{2}$$

Let $a = 4x$ and $b = x^2 - 2x + 1$

So we can write that

$$18\sqrt{ab} = \sqrt{77}(a + b) \tag{3}$$

Divided (3) by $\sqrt{ab}$, we get a quadratic equation in $\sqrt{\frac{a}{b}}$

$$\sqrt{77}\frac{a}{b} - 18\sqrt{\frac{a}{b}} + \sqrt{77} = 0 \tag{4}$$

$$\frac{a}{b} = \begin{cases} \dfrac{7}{11} \\ \\ \dfrac{11}{7} \end{cases} \tag{5}$$

On the other hand

$$\frac{a}{b} = \frac{4x}{x^2 - 2x + 1} \tag{6}$$

Finally from (5) and (6) we have the four roots:

$$\boxed{x_{1,2} = 1 + \frac{22 \pm 6\sqrt{22}}{7}}$$

$$\boxed{x_{3,4} = 1 + \frac{14 \pm 6\sqrt{14}}{11}}$$

Solution 5 by Albert Natian, Los Angeles Valley College, Valley Glen, California.

Answer. The solution set is $\left\{ \frac{6+\sqrt{14}}{6-\sqrt{14}},\ \frac{6-\sqrt{14}}{6+\sqrt{14}},\ \frac{6+\sqrt{22}}{6-\sqrt{22}},\ \frac{6-\sqrt{22}}{6+\sqrt{22}} \right\}$.

We will first find real solutions and then argue that there can be no other solutions, not even non-real solutions.

The above equation can be written as

$$\frac{|x-1|\sqrt{x}}{(x+1)^2} = \frac{\sqrt{77}}{36}.$$

Case One. $x > 1$.

We have

$$\frac{(x-1)\sqrt{x}}{(x+1)^2} = \frac{\sqrt{77}}{36},$$

$$\left[\frac{\sqrt{x}}{x+1}\right] / \left[\frac{x+1}{x-1}\right] = \left[\sqrt{77}\right] / [36]$$

which suggests there exists a positive real number m such that

$$\frac{\sqrt{x}}{x+1} = m\sqrt{77} \qquad \text{and} \qquad \frac{x+1}{x-1} = 36m,$$

$$x/(x+1)^2 = 77m^2 \qquad \text{and} \qquad x = \frac{36m+1}{36m-1} \qquad \text{and} \qquad x+1 = \frac{72m}{36m-1},$$

$$\left(\frac{36m+1}{36m-1}\right) / \left(\frac{72m}{36m-1}\right)^2 = 77m^2,$$

$$\frac{(36m-1)(36m+1)}{(72m)^2} = 77m^2,$$

$$36^2m^2 - 1 = 77 \cdot 72^2 m^4,$$

$$36\left(36m^2\right) - 1 = 77 \cdot 4\left(36m^2\right)^2$$

which, upon the substitution $u = 36m^2$, becomes

$$36u - 1 = 308u^2 \qquad \text{or} \qquad 308u^2 - 36u + 1 = 0 \qquad \text{or} \qquad (14u-1)(22u-1) = 0$$

whose solutions are

$$36m^2 = u = \frac{1}{14} \qquad \text{or} \qquad 36m^2 = u = \frac{1}{22},$$

$$m = \frac{1}{6\sqrt{14}} \qquad \text{or} \qquad m = \frac{1}{6\sqrt{22}}$$

which, upon insertion into $x = \frac{36m+1}{36m-1}$, gives

$$x = \frac{6+\sqrt{14}}{6-\sqrt{14}} \qquad \text{or} \qquad x = \frac{6+\sqrt{22}}{6-\sqrt{22}}$$

Case Two. $0 < x \leq 1$.

We have

$$\frac{(1-x)\sqrt{x}}{(x+1)^2} = \frac{\sqrt{77}}{36},$$

which, in a manner as in the above,

$$\left[\frac{\sqrt{x}}{x+1}\right] / \left[\frac{x+1}{1-x}\right] = \left[\sqrt{77}\right] / [36]$$

which suggests there exists a positive real number m such that

$$\frac{\sqrt{x}}{x+1} = m\sqrt{77} \qquad \text{and} \qquad \frac{x+1}{1-x} = 36m,$$

$$x/(x+1)^2 = 77m^2 \qquad \text{and} \qquad x = \frac{36m-1}{36m+1} \qquad \text{and} \qquad x+1 = \frac{72m}{36m+1},$$

$$\left(\frac{36m-1}{36m+1}\right) / \left(\frac{72m}{36m+1}\right)^2 = 77m^2,$$

$$\frac{(36m-1)(36m+1)}{(72m)^2} = 77m^2,$$

$$36^2m^2 - 1 = 77 \cdot 72^2 m^4,$$

$$36\left(36m^2\right)-1=77\cdot 4\left(36m^2\right)^2$$

and so, as before, we get

$$m=\frac{1}{6\sqrt{14}} \qquad \text{or} \qquad m=\frac{1}{6\sqrt{22}}$$

which, upon insertion into $x=\frac{36m-1}{36m+1}$, gives

$$x=\frac{6-\sqrt{14}}{6+\sqrt{14}} \qquad \text{or} \qquad x=\frac{6-\sqrt{22}}{6+\sqrt{22}}.$$

The solution set of the given radical equation contains at least four different numbers and is a subset of the solution set of a 4-th degree polynomial equation that can be derived from the given equation (by squaring both sides of the equation). Since a 4-th degree polynomial equation has at most 4 (different) solutions, then we can be certain that there are no other solutions of the given equation and that the solution set of the given equation is

$$\left\{\frac{6+\sqrt{14}}{6-\sqrt{14}},\ \frac{6-\sqrt{14}}{6+\sqrt{14}},\ \frac{6+\sqrt{22}}{6-\sqrt{22}},\ \frac{6-\sqrt{22}}{6+\sqrt{22}}\right\}.$$

Comments by other solvers :

David Stone and John Hawkins of Georgia Southern University stated that this problem is a classic "where does a line intersect a hyperbola?" They went on to say that on their graphing calculator, the graphs of $Y_1=\dfrac{\sqrt{x(x-1)^2}}{(x+1)^2}$ and $Y_2=\dfrac{\sqrt{77}}{36}$ do not even appear to intersect four times until some significant zooming is done. The curve Y_1 starts at the origin and rises quickly to a maximum of 0.25, which is barely above the horizontal line Y_2, then drops quickly to its x-intercept at $x=1$. Then it rises again to the same maximum height of 0.25, barely creeping above Y_2 once again, before descending asymptotically toward the x-axis. (The maximum points of Y_1 occur at $x=3\pm 2$.)

It is amazing to find such nice solutions. The use of the quadratic formula to find (y,z) produced rational solutions only because of the numbers chosen in the problem as posed. How did the poser see all of this?

Comment by Ken Korbin, the proposer:

In problem 5583, the four radii have lengths 16, 49, 9, and 121. And $\sin A=\dfrac{3696}{4225.}$

In problem 5601, if the fraction to the right of the equal sign is replaced by $\dfrac{\sin A}{4}$, then, the four roots of the equation will be $\dfrac{16}{49}, \dfrac{49}{16}, \dfrac{19}{21}$, and $\dfrac{121}{9}$. *Note* : $\dfrac{\sin A}{4}=\dfrac{924}{4225}$.

Also solved by Brian D. Beasley, Presbyterian College, Clinton, SC; Brian Bradie, Christopher Newport University, Newport News, VA; Pat Costello, Eastern Kentucky University, Richmond, KY; Pratik Donga, Junagadh, India; Eagle Problem Solvers, Georgia Southern University, Statesboro, GA and Savannah, GA; Farid Huseynov (student; communicated by his instructor Yagub Aliyev), ADA University, Baku, Azerbaijan; Kee-Wai Lau, Hong Kong, China; David E. Manes, Oneonta, NY; Ronald Martins, Brazil; Albert Stadler, Herrliberg, Switzerland; Seán M. Stewart, Bomaderry, NSW, Australia; David

Stone and John Hawkins, Georgia Southern University, Statesboro, GA, and the proposer.

- **5602:** *Proposed by Pedro Henrique Oliveira Pantoja. University of Campina Grande, Brazil*

Prove that:

$$\det\begin{vmatrix} 1 & \cos\frac{\pi}{7} & \sin\frac{3\pi}{7} \\ \sin\frac{3\pi}{7} & \sin\frac{2\pi}{7} & \sin^2\frac{\pi}{7} \\ 0 & \tan\frac{\pi}{7} & 2\sin^2\frac{\pi}{7} \end{vmatrix} = \frac{\sqrt{7}}{8}.$$

Solution 1 by Albert Stadler, Herrliberg, Switzerland

Let $x = \sin\frac{\pi}{7}, y = \cos\frac{\pi}{7} = \sqrt{1-x^2}$. Then $\sin\frac{2\pi}{7} = 2xy, \sin\frac{3\pi}{7} = 3x - 4x^3, \tan\frac{\pi}{7} = \frac{x}{y}$. Let d be the value of the determinate. We expand the determinant along the first column and get

$$d = \det\begin{vmatrix} \sin\frac{2\pi}{7} & \sin^2\frac{\pi}{7} \\ \tan\frac{\pi}{7} & 2\sin^2\frac{\pi}{7} \end{vmatrix} - \sin\frac{3\pi}{7}\det\begin{vmatrix} \cos\frac{\pi}{7} & \sin\frac{3\pi}{7} \\ \tan\frac{\pi}{7} & 2\sin^2\frac{\pi}{7} \end{vmatrix} =$$

$$= 2\sin^2\frac{\pi}{7}\sin\frac{2\pi}{7} - \tan\frac{\pi}{7}\sin^2\frac{2\pi}{7} - \sin\frac{3\pi}{7}\left(2\sin^2\frac{\pi}{7}\cos\frac{\pi}{7} - \tan\frac{\pi}{7}\sin\frac{3\pi}{7}\right) =$$

$$= 4x^3y - \frac{x^3}{y} - (3x-4x^3)\left(2x^2y - \frac{3x^2-4x^4}{y}\right) = \frac{2x^3(4-12x^2+8x^4-y^2+4x^2y^2)}{y} =$$

$$= \frac{2x^3(4-12x^2+8x^4-(1-x^2)+4x^2(1-x^2))}{y} = \frac{2(x-1)x^3(x+1)(4x^2-3)}{y} =$$

$$= \frac{2y^2x^2(3x-4x^3)}{y} = \sin\frac{\pi}{7}\sin\frac{2\pi}{7}\sin\frac{3\pi}{7} = \sqrt{\sin\frac{\pi}{7}\sin\frac{2\pi}{7}\sin\frac{3\pi}{7}\sin\frac{4\pi}{7}\sin\frac{5\pi}{7}\sin\frac{6\pi}{7}} = \sqrt{\frac{7}{64}}$$

since for all integers $n \geq 2$

$$\prod_{k=1}^{n-1} 2\sin\left(\frac{k\pi}{n}\right) = \prod_{k=1}^{n-1}(-i)\left(e^{\frac{\pi ik}{n}} - e^{\frac{-\pi ik}{n}}\right) = (-i)^{n-1}e^{\sum_{k=1}^{n-1}\frac{\pi ik}{n}}\prod_{k=1}^{n-1}\left(1 - e^{-\frac{2\pi ik}{n}}\right) =$$

$$= (-i)^{n-1}e^{\frac{\pi i(n-1)}{2}}\prod_{k=1}^{n-1}\left(1 - e^{\frac{2\pi ik}{n}}\right) = \lim_{x\to 1}\frac{x^n-1}{x-1} = n, \text{(see problem 5497, April 2018)}.$$

Solution 2 by Brian Bradie, Christopher Newport University, Newport News,VA

Subtracting $\sin\frac{3\pi}{7}$ times the first row from the second row yields

$$\det\begin{vmatrix} 1 & \cos\frac{\pi}{7} & \sin\frac{3\pi}{7} \\ \sin\frac{3\pi}{7} & \sin\frac{2\pi}{7} & \sin^2\frac{\pi}{7} \\ 0 & \tan\frac{\pi}{7} & 2\sin^2\frac{\pi}{7} \end{vmatrix} = \det\begin{vmatrix} 1 & \cos\frac{\pi}{7} & \sin\frac{3\pi}{7} \\ 0 & \sin\frac{2\pi}{7} - \cos\frac{\pi}{7}\sin\frac{3\pi}{7} & \sin^2\frac{\pi}{7} - \sin^2\frac{3\pi}{7} \\ 0 & \tan\frac{\pi}{7} & 2\sin^2\frac{\pi}{7} \end{vmatrix}.$$

Now,

$$\begin{aligned} \sin\frac{2\pi}{7} - \cos\frac{\pi}{7}\sin\frac{3\pi}{7} &= \sin\left(\frac{3\pi}{7} - \frac{\pi}{7}\right) - \cos\frac{\pi}{7}\sin\frac{3\pi}{7} \\ &= \sin\frac{3\pi}{7}\cos\frac{\pi}{7} - \cos\frac{3\pi}{7}\sin\frac{\pi}{7} - \cos\frac{\pi}{7}\sin\frac{3\pi}{7} = -\cos\frac{3\pi}{7}\sin\frac{\pi}{7} \end{aligned}$$

and

$$\begin{aligned} \sin^2\frac{\pi}{7} - \sin^2\frac{3\pi}{7} &= \left(\sin\frac{\pi}{7} - \sin\frac{3\pi}{7}\right)\left(\sin\frac{\pi}{7} + \sin\frac{3\pi}{7}\right) \\ &= \left(-2\sin\frac{\pi}{7}\cos\frac{2\pi}{7}\right)\left(2\sin\frac{2\pi}{7}\cos\frac{\pi}{7}\right) = -\sin\frac{2\pi}{7}\sin\frac{4\pi}{7}, \end{aligned}$$

so

$$\begin{aligned} \det\begin{vmatrix} 1 & \cos\frac{\pi}{7} & \sin\frac{3\pi}{7} \\ \sin\frac{3\pi}{7} & \sin\frac{2\pi}{7} & \sin^2\frac{\pi}{7} \\ 0 & \tan\frac{\pi}{7} & 2\sin^2\frac{\pi}{7} \end{vmatrix} &= -2\sin^3\frac{\pi}{7}\cos\frac{3\pi}{7} + \tan\frac{\pi}{7}\sin\frac{2\pi}{7}\sin\frac{4\pi}{7} \\ &= \frac{\sin\frac{\pi}{7}}{\cos\frac{\pi}{7}}\left(-2\sin^2\frac{\pi}{7}\cos\frac{\pi}{7}\cos\frac{3\pi}{7} + \sin\frac{2\pi}{7}\sin\frac{4\pi}{7}\right) \\ &= \frac{\sin\frac{\pi}{7}\sin\frac{2\pi}{7}}{\cos\frac{\pi}{7}}\left(-\sin\frac{\pi}{7}\cos\frac{3\pi}{7} + \sin\frac{4\pi}{7}\right) \\ &= \frac{\sin\frac{\pi}{7}\sin\frac{2\pi}{7}}{\cos\frac{\pi}{7}}\left(-\sin\frac{\pi}{7}\cos\frac{3\pi}{7} + \sin\frac{3\pi}{7}\cos\frac{\pi}{7} + \cos\frac{3\pi}{7}\sin\frac{\pi}{7}\right) \\ &= \sin\frac{\pi}{7}\sin\frac{2\pi}{7}\sin\frac{3\pi}{7}. \end{aligned}$$

To show

$$\sin\frac{\pi}{7}\sin\frac{2\pi}{7}\sin\frac{3\pi}{7} = \frac{\sqrt{7}}{8},$$

let $n \geq 2$ be an integer. The roots of $z^n - 1$ are $\omega_k = e^{2ik\pi/n}$, for $k = 0, 1, 2, \ldots, n-1$. Then

$$z^n - 1 = (z-1)\sum_{k=0}^{n-1} z^k = (z-1)\prod_{k=1}^{n-1}(z - \omega_k),$$

or

$$\sum_{k=0}^{n-1} z^k = \prod_{k=1}^{n-1}(z - \omega_k).$$

Substituting $z = 1$ yields

$$n = \prod_{k=1}^{n-1}(1 - \omega_k).$$

Next,

$$|1-\omega_k| = \left|1-\cos\frac{2k\pi}{n} - i\sin\frac{2k\pi}{n}\right| = \sqrt{2-2\cos\frac{2k\pi}{n}} = 2\sin\frac{k\pi}{n},$$

so

$$n = |n| = \prod_{k=1}^{n-1}|1-\omega_k| = 2^{n-1}\prod_{k=1}^{n-1}\sin\frac{k\pi}{n}.$$

Take $n=7$ and note

$$\sin\frac{4\pi}{7} = \sin\frac{3\pi}{7}, \quad \sin\frac{5\pi}{7} = \sin\frac{2\pi}{7}, \quad \text{and} \quad \sin\frac{6\pi}{7} = \sin\frac{\pi}{7}.$$

It follows that

$$7 = 2^6\sin^2\frac{\pi}{7}\sin^2\frac{2\pi}{7}\sin^2\frac{3\pi}{7} \quad \text{or} \quad \sin\frac{\pi}{7}\sin\frac{2\pi}{7}\sin\frac{3\pi}{7} = \frac{\sqrt{7}}{8}.$$

Editor's comment : The solution submitted by **Seán M. Stewart of Bomaderry, Australia** started off by proving three identities:

1. $C = \cos\frac{\pi}{7} - \cos\frac{2\pi}{7} + \cos\frac{3\pi}{7} = \frac{1}{2}$
2. $S = -\sin\frac{\pi}{7} + \sin\frac{2\pi}{7} + \sin\frac{3\pi}{7} = \frac{\sqrt{7}}{2}$
3. $T = \dfrac{\sin\frac{\pi}{7} + 2\sin\frac{3\pi}{7}}{\cos\frac{\pi}{7}} = \sqrt{7}.$

The proof of C was straight forward, in proving S he first showed that the square of the LHS equals the square of the RHS, and he then chose the positive square roots; in proving T he showed that $T = \dfrac{S}{1-C} = \dfrac{\frac{\sqrt{7}}{2}}{1-\frac{1}{2}} = \sqrt{7}.$

He then expanded the Determinate D down the third column and showed that $D = \dfrac{T}{8} = \sqrt{7}8$. Lots of algebra, but it worked.

Solution 3 by Kee-Wai Lau, Hong Kong, China

Let $\alpha = \dfrac{\pi}{7}$. The given determinate, denoted by D equals

$$2\sin 2\alpha\sin^2\alpha + \sin^2 3\alpha\alpha - \tan\alpha\sin^2\alpha - 2\sin^2\alpha\cos\alpha\sin 3\alpha$$

$$= \frac{4\sin^3\alpha\cos^2\alpha + \sin^2 3\alpha\sin\alpha - \sin^3\alpha - 2\sin^2\alpha\cos^2\alpha\sin 3\alpha}{\cos\alpha}.$$

Let $k=\sin\alpha$. By using the relations $\cos^2\alpha = 1-k^2$ and $\sin 3\alpha = 3k-4k^2$, we see that the numerator of D equals

$$8k^7 - 14k^5 + 6k^3 = 2k^3(1+k)(1-k)(3-4k^2).$$

Since $0 < k < \sin\dfrac{\pi}{6} = \dfrac{1}{2}$, so $D > 0$. Hence to prove that $D = \dfrac{\sqrt{7}}{8}$, it suffices to show that $D^2 = \dfrac{7}{64}$ or $64(8k^7 - 14k^5 + 6k^3)^2 - 7(1-k^2) = 0$, or

$$(k+1)(k-1)(64k^5 - 48k^4 - 8k^2 - 1)(64k^6 - 112k^6 + 56k^2 - 7) = 0. \qquad (1)$$

It is well known that $\sin 7\theta = -\sin\theta(64\sin^6\theta - 112\sin^4\theta + 56\sin^2\theta - 7)$

for any real number θ. Hence, $64k^6 - 112k^4 + 56k^2 - 7 = \dfrac{-\sin 7\alpha}{k} = 0.$

Thus (1) holds and this completes the solution.

Editor's comment : **David Stone and John Hawkins of Georgia Southern University** used a statement in their solution that was proved by P dilip k Stefan V., that $\sin\left(\dfrac{\pi}{7}\right)\sin\left(\frac{2\pi}{7}\right)\sin\left(\dfrac{3\pi}{7}\right) = \dfrac{\sqrt{7}}{8}$ (see: (https://socratic.org/questions/how-do-you-evaluate $\sin\dfrac{\pi}{7}\sin\left(\dfrac{2\pi}{7}\right)\sin\left(\dfrac{3\pi}{7}\right)$). They concluded their solution by stating "in this problem, the surprise is that the given determinate equals $\sin\pi 7\sin\left(\dfrac{2\pi}{7}\right)\sin\left(\dfrac{3\pi}{7}\right)$)."

Also solved by Brian D. Beasley, Presbyterian College, Clinton, SC; Bruno Salgueiro Fanego, Viveiro, Spain; Peter Fulop, Gyomro, Hungary; Seán M. Stewart, Bomaderry, NSW, Australia; David Stone and John Hawkins, Georgia Southern University, Statesboro, GA, and the proposer.

- **5603:** *Proposed by Michael Brozinsky, Central Islip, NY*

In an election 50 votes were cast for candidate A and 50 for candidate B. The candidates decide to end the tie as follows; by tallying the votes at random and if A is ever in the lead by 3 votes, then Candidate A will be declared the winner. Otherwise Candidate B wins. What is the probability that A wins?

Solution 1 by Albert Stadler, Herriliberg, Switzerland

Consider the 2-dimensional integer lattice Z^2 in the Euclidean space R^2 whose lattice points are 2-tuples of integers. Every random tallying of votes can be mapped bijectively to a path from $(0,0)$ to $(50,50)$ under the rule that if starting at $(0,0)$ and if a vote for A is picked a step from (x,y) to $(x+1,y)$ is done and if a vote for B is picked a step from (x,y) to $(x,y+1)$ is done. In total there are $\dbinom{100}{50}$ paths from $(0,0)$ to $(50,50)$. By the reflection principle, the number of paths from $(0,0)$ to $(50,50)$ where A will be in the lead by 3 votes at some point is equal to the number of paths from $(0,0)$ to $(47,53)$ which equals $\dbinom{100}{47}$. Therefore the probability that A wins equals

$$\frac{\dbinom{100}{47}}{\dbinom{100}{50}} = \frac{100!50!50!}{47!53!100!} = \frac{48\cdot 49\cdot 50}{51\cdot 52\cdot 53} = \frac{9800}{11713} \approx 83.7\%.$$

Solution 2 by Albert Natian, Los Angeles College, Valley Glen, CA

Answer: $\frac{48\cdot 49\cdot 50}{51\cdot 52\cdot 53} \approx 0.836677.$

First we generalize the problem as follows:

In an election m votes are cast for candidate A and n for candidate B. The candidates agree to break the tie as follows: the votes will be tallied uniformly at random and if A is ever in the lead (for the first time) by L votes, then A is declared the winner; otherwise B wins. In this generalization, unlike the original statement of the problem, an additional advantage/disadvantage of i votes is initially accorded candidate A. That is, if, say, $i = 1$ and $L = 3$, and the first two tallies are for A, then A wins. But if the first tally is for B, then A's 1-point advantage is lost (and so i becomes 0) and now A needs to lead B by 3 in order to win, and if the following tally is again for B, then now A's advantage/disadvantage i becomes -1. So i measures A's lead over B by the latest tally. It's clear that i can be negative, zero or positive. So when (if ever) i becomes L, then A wins. What is the probability that A wins?

We let $P(m, n, i, L)$ denote the probability that A wins. It's immediate that P satisfies the following conditions

$$P(m, n, i, L) = 1 \quad \text{if} \quad L = i \quad \text{or} \quad i = n + L - m$$

$$P(m, n, i, L) = 0 \quad \text{if} \quad m < L - i$$

$$P(L - i, n, i, L) = \binom{L - i + n}{n}^{-1}.$$

The first tally is a vote for A with probability $m/(m+n)$ and a vote for B with probability $n/(m+n)$. If the first tally is a vote for A, then there will now be $(m-1)$ votes remaining for A and n votes for B. Also if the first tally is a vote for A, then i is incremented by 1. However, if the first tally is a vote for B, then there will now be $(n-1)$ votes remaining for B and m votes for A. Also if the first tally is a vote for B, then i is decremented by 1. Se we assert

$$P(m, n, i, L) = \frac{m}{m+n} \cdot P(m-1, n, i+1, L) + \frac{n}{m+n} \cdot P(m, n-1, i-1, L).$$

In order to solve the above recursion with the given conditions, we define f as

$$f(m, n, i, L) = \binom{m+n}{m} P(m, n, i, L).$$

So

$$P(m, n, i, L) = \binom{m+n}{m}^{-1} f(m, n, i, L).$$

Now

$$f(m,n,i,L) = \binom{m+n}{m} P(m,n,i,L) = \binom{m+n}{n} P(m,n,i,L)$$

$$= \binom{m+n}{m}\left(\frac{m}{m+n}\cdot P(m-1,n,i+1,L) + \frac{n}{m+n}\cdot P(m,n-1,i-1,L)\right)$$

$$= \binom{m+n}{m}\frac{m}{m+n}\cdot P(m-1,n,i+1,L) + \binom{m+n}{n}\frac{n}{m+n}\cdot P(m,n-1,i-1,L)$$

$$= \binom{m-1+n}{m-1}\cdot P(m-1,n,i+1,L) + \binom{m+n-1}{n-1}\cdot P(m,n-1,i-1,L)$$

$$= f(m-1,n,i+1,L) + f(m,n-1,i-1,L).$$

The conditions for f are

$$f(m,n,i,L) = \binom{m+n}{m} \quad \text{if} \quad L = i \quad \text{or} \quad i = n + L - m$$

$$f(m,n,i,L) = 0 \quad \text{if} \quad m < L - i$$

$$P(L-i,n,i,L) = 1.$$

The Solution for the recursion

$$f(m,n,i,L) = f(m-1,n,i+1,L) + f(m,n-1,i-1,L)$$

satisfying the aforementioned conditions is

$$f(m,n,i,L) = \binom{m+n}{n+L-i} = \binom{m+n}{m-L+i}.$$

Thus

$$P(m,n,i,L) = \binom{m+n}{m}^{-1}\binom{m+n}{m-L+i} = \frac{m!\ n!}{(m-L+i)!\ (n+L-i)!}.$$

To answer the original question, we let $m = 50$, $n = 50$, $i = 0$, $L = 3$. Then

$$P(50,50,0,3) = \frac{50!\ 50!}{47!\ 53!} = \frac{48\cdot 49\cdot 50}{51\cdot 52\cdot 53} \approx 0.836677.$$

Solution 3 by Moti Levy, Rehovot, Israel

The process of counting $2N$ votes can be modeled as a random walk from $(0,0)$ to $(2N,0)$ with up-steps and down-steps of one unit each. The paths are called grand-Dyck path (or free-Dyck path).

The number of all grand-Dyck paths from $(0,0)$ to $(2N,0)$ is equal to $\binom{2N}{N}$. Clearly all paths have equal probability.

Let us denote by $\mathcal{N}_n$ the number of "losing paths", i.e., grand-Dyck paths from $(0,0)$ to $(n,0)$ that never exceed the height 2.

Then the probability that candidate A will be declared the winner is

$$P\left(\text{A} \quad \text{wins}\right) = 1 - \frac{\mathcal{N}_{2N}}{\binom{2N}{N}}.$$

Let $\Psi\left(z\right)$ be the generating function of the sequence of numbers of grand-Dyck paths from $(0,0)$ to $(n,0)$ that never exceed the height 2.

$$\Psi\left(z\right) = \sum_{n=0}^{\infty} \mathcal{N}_n z^n$$

The article by Panny and Prodinger [1] , states that (see theorem 4.1, page 130) the generating function is

$$\Psi\left(z\right) = \frac{1+v^2}{1-v^2}\left(1-v^6\right), \qquad z = \frac{v}{1+v^2},$$

or

$$\Psi\left(z\right) = 1 + 2v^2 + 2v^4 + v^6, \quad v = \frac{1}{2z}\left(1 - \sqrt{1-4z^2}\right).$$

Now we find several generating functions on our way to evaluate $\Psi\left(z\right)$.

By definition of the binomial coefficient $\binom{\frac{1}{2}}{n} = \frac{\frac{1}{2}\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)\cdots\left(-\frac{2n-3}{2}\right)}{n!}$, hence

$$\binom{\frac{1}{2}}{n} = \begin{cases} 1 & n = 0, \\ \frac{1}{2} & n = 1, \\ (-1)^{n-1}\,\frac{1}{n-1}\,\frac{\binom{2n-2}{n}}{2^{2n-1}} & n > 1. \end{cases} \tag{1}$$

By the binomial theorem and (1),

$$\left(1-4z^2\right)^{\frac{1}{2}} = 1 + \sum_{m=1}^{\infty} (-1)^m\, 2^{2m} \binom{\frac{1}{2}}{m} z^{2m}, \tag{2}$$

$$\begin{aligned}\left(1-4z^2\right)^{\frac{3}{2}} &= 1 + \sum_{m=1}^{\infty} (-1)^m\, 2^{2m} \binom{\frac{3}{2}}{m} z^{2m} \\ &= 1 - 4z^2 + \sum_{m=1}^{\infty} (-1)^m\, 2^{2m} \binom{\frac{1}{2}}{m} z^{2m} - 4z^2 \sum_{m=1}^{\infty} (-1)^m\, 2^{2m} \binom{\frac{1}{2}}{m} z^{2m}\end{aligned} \tag{3}$$

$$1+2v^2+2v^4+v^6 = -\frac{1}{16z^6}\left(8z^2\left(1-4z^2\right)^{\frac{3}{2}} + \left(40z^4 - 4z^2 + 3\right)\sqrt{1-4z^2} + 32z^2 - 24z^4 - 8\right) \tag{4}$$

Substituting (2) and (3) in (4) and after tedious simplifications we get,

$$\begin{aligned}\Psi\left(z\right) &= 1 + \sum_{m=1}^{\infty} (-1)^m\, 2^{2m+1}\left(\binom{\frac{1}{2}}{m+1} + 8\binom{\frac{1}{2}}{m+2} + 16\binom{\frac{1}{2}}{m+3}\right) z^{2m} \\ &= \sum_{m=0}^{\infty} \binom{2m}{m} \frac{9m^2+9m+6}{m^3+6m^2+11m+6} z^{2m}.\end{aligned}$$

It follows that

$$\mathcal{N}_{2N} = \binom{2N}{N} \frac{9N^2+9N+6}{N^3+6N^2+11N+6}, \tag{5}$$

and that

$$P\,(\text{A} \quad \text{wins}) = 1 - \frac{9N^2+9N+6}{N^3+6N^2+11N+6}.$$

For the case $N = 50$, we have $\mathrm{P(A} \quad \mathrm{wins)} = 1 - \frac{9*50^2+9*50+6}{50^3+6*50^2+11*50+6} \cong 0.836\,68.$

Remarks:

1) Equation (5) can be verified for small values of N by manual counting of the paths:

N	$\mathcal{N}_{2N}$	$\binom{2N}{N}$	$P\,(\text{A} \quad \text{wins})$
1	2	2	0
2	6	6	0
3	19	20	$\frac{1}{20} = 0.05$
4	62	70	$\frac{8}{70} \cong 0.114\,29$

2) For large N

$$P\,(\text{A} \quad \text{wins}) = 1 - \frac{9}{N}\left(1 + O\left(\frac{1}{N}\right)\right)$$

3) We give here the essential steps in Panny and Prodinger derivation (see [2]).
We consider grand-Dyck paths form $(0,0)$ to $(n,0)$. We allow the path to touch $-h$ and k but not $-h-1$ and $k+1$. Let $\mathcal{N}_{n,h}$ be the number of paths which do not touch $-h-1$ and $k+1$ and lead to level i. Let Ψ_i be the generating function of the sequence $(\mathcal{N}_{n,h})_{n=0}^{\infty}$,

$$\Psi_i = \sum_{n=0}^{\infty} \mathcal{N}_{n,h} z^n.$$

Looking at the last step of the paths we write the following recurrences:

$$\begin{aligned}
\mathcal{N}_{n,-h} &= \mathcal{N}_{n-1,-h} \\
\mathcal{N}_{n,-h+1} &= \mathcal{N}_{n-1,-h} + \mathcal{N}_{n-1,-h+2} \\
&\vdots \\
\mathcal{N}_{n,-1} &= \mathcal{N}_{n-1,-2} + \mathcal{N}_{n-1,0} \\
\mathcal{N}_{n,0} &= \begin{cases} \mathcal{N}_{n-1,-1} + \mathcal{N}_{n-1,1} & if \quad n > 0 \\ 1 & if \quad n = 0 \end{cases} \\
\mathcal{N}_{n,1} &= \mathcal{N}_{n-1,0} + \mathcal{N}_{n-1,2} \\
&\vdots \\
\mathcal{N}_{n,k-1} &= \mathcal{N}_{n-1,k-2} + \mathcal{N}_{n-1,k} \\
\mathcal{N}_{n,k} &= \mathcal{N}_{n-1,k}
\end{aligned}$$

which implies (in terms of the generating functions)

$$\begin{bmatrix} 1 & -z & 0 & \cdots & \cdots & 0 \\ -z & 1 & -z & 0 & \cdots & 0 \\ 0 & -z & 1 & -z & \cdots & 0 \\ & \cdots & \cdots & & & \\ 0 & \cdots & 0 & -z & 1 & -z \\ 0 & \cdots & \cdots & 0 & -z & 1 \end{bmatrix} \begin{bmatrix} \Psi_{-h} \\ \vdots \\ \vdots \\ \Psi_0 \\ \vdots \\ \Psi_k \end{bmatrix} = \begin{bmatrix} 0 \\ \vdots \\ \vdots \\ 1 \\ \vdots \\ 0 \end{bmatrix} \tag{6}$$

Of course, we are interested in Ψ_0.
The matrix is tridiagonal, hence its determinant f_n of $n \times n$ matrix can be found by solving the recurrence relation (see [3])

$$f_n = f_{n-1} - z^2 f_{n-2}.$$

Setting $z = \frac{v}{(1+v)^2}$, for convenience, the solution is $f_n = a\left(\frac{1}{v^2+1}\right)^2 + b\left(\frac{v^2}{v^2+1}\right)^2$. Applying the initial conditions $f_1 = 1$ and $f_2 = 1 - z^2$, we find that

$$f_n = \frac{1}{1-v^2} \frac{1-v^{2n+2}}{(v^2+1)^n}.$$

Applying Cramer's rule, to solve (6), we get

$$\Psi_0 = \frac{f_h f_k}{f_{h+k+1}} = \frac{1+v^2}{1-v^2}\left(1 - v^{2k+2}\right)\frac{v^{2h+2}-1}{v^{2h+2k+4}-1}$$

Now we send h to -∞, to get

$$\Psi_0(z) = \frac{1+v^2}{1-v^2}\left(1 - v^{2k+2}\right).$$

References:
[**1**] Wolfgang Panny, Helmut Prodinger, "*The expected height of paths for several notions of height*", Studia Scientiarum Mathematicarum Hungarica 20 (1985), 119-132.
[**2**] Helmut Prodinger, "*The number of restricted lattice paths revisited*", Filomat 26:6, 1133-1134, published by Faculty of Sciences and Mathematics, University of Niŝ, Serbia.
[3] Wikipedia, "*Tridiagonal Matrix*" entry.

Solution 4 by Kee-Wai Lau, Hong Kong, China

We show that A wins with probability $\dfrac{9800}{11713}$.
Suppose, in general, that A obtained a votes and B obtained b votes with $a \geq b$.
Denote by $P_0(a,b) =$ the probability that there will ever be a tie in a full tallying. For $k = 1, 2, 3$, let $P_k(a,b)$ be the probability that B will ever lead by k votes in a full tallying. It is known (1), pp. 6, 37, 38, Problem 22: The ballot box that $P_0(a,b) = \dfrac{2b}{a+b}$.
We use this result to find $P_1(a,b), P_2(a,b)$, and $P_3(a,b)$
successively. The probability that A gets the first vote equals $\dfrac{a}{a+b}$ and the probability that B gets the first vote equals $\dfrac{b}{a+b}$. Hence by conditioning on the first

vote, we obtain $\frac{2b}{a+b} = P_0(a,b) = \frac{a}{a+b}P_1(a-1,b) + \frac{b}{a+b}/$. Thus $P_1(a-1,b) = \frac{b}{a}$ and $P_1(a,b) = \frac{b}{a+1}$. By conditioning again, we have

$\frac{b}{a+1} = P_1(a,b)P_2(a-1,b) + \frac{b}{a+b}$, giving $P_2(a-1,b) = \frac{b(b-1)}{a(a+1)}$ and

$P_2(a,b) = \frac{b(b-1)}{(a+1)(a+2)}$. Finally, we have

$\frac{b(b-1)}{(a+1)(a+2)} = P_2(a,b) = \frac{a}{a+b}P_3(a-1,b) +$

This gives $P_3(a-1,b) = \frac{b(b-1)(b-2)}{a(a+1)(a_2)}$ and $P_3(a,b)\frac{b(b-1)(b-2)}{(a+1)(a+2)(a+3)}$,

If $a = b$, then $P(A$ is ever in the lead by 3 votes$)= P(B$ is ever in the lead by 3 votes$)$

$$\begin{aligned} &= P_3(a,a), \\ &= \frac{a(a-1)(a-2)}{(a+1)(a+2)(a+3)}. \end{aligned}$$

By putting $a = 50$, we obtain the result stated at the beginning.

Reference: 1. F. Mosteller: Fifty Challenging Problems in Probability with Solutions, Dover Publications, Inc., 1987.

Solution 5 by the Eagle Problem Solvers, Georgia Southern University, Statesboro, GA and Savannah, GA

Let S be the set of all sequences of $50A's$ and $50B's$, and let A be the subset of S consisting of the sequences in which there is an initial subsequence with three more $A's$ than $B's$. We claim that there is a one-to-one correspondence between A and the set B of all sequences of $47A's$ and 53 $B's$.

Let $(x_1, x_2, \cdots, x_{100})$ be a sequence in A, and let k be the smallest integer for which the initial sequence $(x_1, x_2 \cdots,\ x_k)$ has exactly three more $A's$ than $B's$. Then the initial sequence $(x_1, x_2, \cdots,\ x_k)$ has b $B's$ and $b+3$ $A's$ for some integer b, where $0 \le b \le 47$, and the remaining sequence $(x_{k+1}, x_{k+2}, \cdots x_{100})$ will have $50 - bB's$ and $47 - bA's$. Now transform the sequence $(x_1, x_2, \cdots, x_{100})$ by changing the $A's$ to $B's$ and the $B's$ to $A's$ in the initial k terms, and leaving the remaining $100 - k$ terms unchanged. This new sequence will have $b + (47 - b) = 47A's$ and $(b+3) + (50 - b) = 53B's$, and thus will be a sequence in B.

In the other direction, if we begin with a sequence $(y_1, y_2, \cdots, y_{100})$ of $47A's$ and $53B's$, then let k be the smallest integer for which the initial sequence $(y_1, y_2, \cdots y_k)$ has exactly three more $B's$ than $A's$. Change the $A's$ to $B's$ and $B's$ to $A's$ in this initial sequence and leave the tail end of the sequence unchanged. The resulting sequence will have $50A's$ and $50B's$, and the initial sequence of k terms will have exactly three more $A's$ than $B's$.

Thus, the sets A and B have the same cardinality, $|A| = |B| = \binom{100}{47}$ and the probability that Candidate A will win is

$$\begin{aligned}\frac{|A|}{|B|} &= \frac{\binom{100}{47}}{\binom{100}{50}} \\ &= \frac{100!}{47!53!}\cdot\frac{50!50!}{100!} \\ &= \frac{50\cdot 49\cdot 48}{51\cdot 52\cdot 53} \\ &= \frac{9800}{11,713} \\ &\approx 0.836677.\end{aligned}$$

More generally, suppose A is declared the winner if A is ever in the lead by k votes, where k is a positive integer less than or equal to 50. Let A be the subset of S containing sequences in which there is an initial subsequence containing exactly k more $A's$ than B'. As before, there is a one−to−one correspondence between A and the set B of sequences with $50-kA's$ and $50+kB's$. Thus, the cardinality of A is $\binom{100}{50-k}$ and the probability that A is declared the winner is

$$\frac{\binom{100}{50-k}}{\binom{100}{50}} = \frac{(50!)^2}{(50-k)!(50+k)!} = \frac{50\cdot 49\cdots(51-k)}{51\cdot 52\cdots(50+k)}.$$

The value of k for which this probability comes closest to $\frac{1}{2}$ is $k=6$, for which the probability is

$$\frac{189,175}{386,529}\approx 0.48942.$$

Also solved by the proposer.

- **5604:** *Proposed by Albert Natian, Los Angeles Valley College, Valley Glen, CA*

Prove:

$$\binom{N}{r} = \lim_{n\to\infty}\frac{1}{n}\sum_{\mu=0}^{n-1} e^{-ir\mu\frac{2\pi}{n}}\left(1+e^{i\mu\frac{2\pi}{n}}\right)^N$$

where $N, r \in N$ and $i^2=-1$.

Solution 1 by Albert Stadler, Herrliberg, Switzerland

The sum on the right hand side is a Riemann sum that converges to

$$\int_0^1 e^{-2\pi irx}(1+e^{2\pi ix})^N dx = \sum_{n=0}^{N}\binom{N}{n}\int_0^1 e^{-2\pi irx+2\pi irnx}dx = \binom{N}{r},$$

as claimed. We have used the binomial theorem and the fact that for an integer k

$$\int_0^1 e^{2\pi ikx}dx = \begin{cases} 1, & k=0 \\ 0, & k\neq 0. \end{cases}$$

Solution 2 by Brian Bradie, Christopher Newport University, Newport News,VA

Let $\theta = \frac{2\pi}{n}$. By the binomial theorem,

$$\left(1+e^{i\mu\theta}\right)^N = \sum_{k=0}^{N}\binom{N}{k}e^{i\mu k\theta},$$

so

$$\sum_{\mu=0}^{n-1} e^{-ir\mu\theta}\left(1+e^{i\mu\theta}\right)^N = \sum_{\mu=0}^{n-1}\sum_{k=0}^{N}\binom{N}{k}e^{i\mu\theta(k-r)} = \sum_{k=0}^{N}\binom{N}{k}\sum_{\mu=0}^{n-1}e^{i\mu\theta(k-r)}.$$

Now, for $r > N$,

$$\binom{N}{r} = 0,$$

and, for each $n > r$,

$$\sum_{\mu=0}^{n-1} e^{i\mu\theta(k-r)} = \frac{1-e^{in\theta(k-r)}}{1-e^{i\theta(k-r)}} = \frac{1-e^{2\pi i(k-r)}}{1-e^{i\theta(k-r)}} = 0,$$

so

$$\lim_{n\to\infty}\frac{1}{n}\sum_{\mu=0}^{n-1} e^{-ir\mu\frac{2\pi}{n}}\left(1+e^{i\mu\frac{2\pi}{n}}\right)^N = 0 = \binom{N}{r}.$$

Finally, for $r \le N$ and for each $n > N$

$$\begin{aligned} \sum_{k=0}^{N}\binom{N}{k}\sum_{\mu=0}^{n-1}e^{i\mu\theta(k-r)} &= \binom{N}{r}\sum_{\mu=0}^{n-1}1 + \sum_{k=0,k\neq r}^{N}\binom{N}{k}\sum_{\mu=0}^{n-1}e^{i\mu\theta(k-r)} \\ &= n\binom{N}{r} + \sum_{k=0,k\neq r}^{N}\binom{N}{k}\frac{1-e^{in\theta(k-r)}}{1-e^{i\theta(k-r)}} \\ &= n\binom{N}{r}, \end{aligned}$$

so

$$\lim_{n\to\infty}\frac{1}{n}\sum_{\mu=0}^{n-1} e^{-ir\mu\frac{2\pi}{n}}\left(1+e^{i\mu\frac{2\pi}{n}}\right)^N = \binom{N}{r}.$$

Solution 3 by Seán M. Stewart, Bomaderry, NSW, Australia

Let

$$S(r,N) = \sum_{\mu=0}^{n-1} e^{-ir\mu\frac{2\pi}{n}}\left(1+e^{i\mu\frac{2\pi}{n}}\right)^N,$$

where $r, N \in N$. To prove the result given we also need to make the additional assumption that $r \leq N$. From the binomial theorem we can write the term appearing in the brackets inside the sum as

$$\left(1 + e^{i\mu\frac{2\pi}{n}}\right)^N = \sum_{k=0}^{N} \binom{N}{k} e^{\frac{2\pi i\mu k}{n}}.$$

Thus

$$S(r, N) = \sum_{k=0}^{N} \binom{N}{k} \sum_{\mu=0}^{n-1} e^{\mu\theta_k},$$

after the order of the summations have been interchanged. Here $\theta_k = \frac{2\pi i}{n}(k - r)$. As

$$\sum_{\mu=0}^{n-1} e^{\mu\theta_k} = \frac{e^{n\theta_k} - 1}{e^{\theta_k} - 1},$$

we see that

$$f_k = \sum_{\mu=0}^{n-1} e^{\mu\theta_k} = \frac{e^{2\pi i(k-r)} - 1}{e^{\frac{2\pi i}{n}(k-r)} - 1} = 0,$$

provided $k \neq r$ since $e^{2\pi i(k-r)} = 1$ as k is a non-negative integer and r a positive integer. When $k = rN$ we have

$$f_k = \sum_{\mu=0}^{n-1} e^{\mu\theta_k} = \sum_{\mu=0}^{n-1} 1 = n.$$

So for $r \leq N$ where $r, N \in N$ we have

$$f_k = \begin{cases} 0, & k \neq r, \\ n, & k = r. \end{cases}$$

Thus

$$S(r, N) = \sum_{k=0}^{N} \binom{N}{k} f_k = n\binom{N}{r}.$$

So for the desired limit we have

$$\lim_{n\to\infty} \frac{1}{n} \sum_{\mu=0}^{n-1} e^{-ir\mu\frac{2\pi}{n}} \left(1 + e^{i\mu\frac{2\pi}{n}}\right)^N = \lim_{n\to\infty} \frac{1}{n} \cdot n\binom{N}{r} = \binom{N}{r},$$

as required to prove.

Solution 4 by Ulrich Abel, Technische Hochschule Mittelhessen, Germany

Let $N, r \in N$. Applying the binomial formula we obtain

$$\sum_{\mu=0}^{n-1} e^{-ir\mu\frac{2\pi}{n}} \left(1 + e^{i\mu\frac{2\pi}{n}}\right)^N = \sum_{k=0}^{N} \binom{N}{k} \sum_{\mu=0}^{n-1} e^{i\mu\frac{2\pi}{n}(k-r)}.$$

If $k = r$, the inner sum has value equal to 1. Otherwise, the formula for the geometric sum tells us that its value is equal to

$$\frac{1 - e^{2\pi i(k-r)}}{1 - e^{2\pi i(k-r)/n}} = 0,$$

provided that $n > |k - r|$. Hence,

$$\lim_{n\to\infty} \frac{1}{n} \sum_{\mu=0}^{n-1} e^{-ir\mu\frac{2\pi}{n}} \left(1 + e^{i\mu\frac{2\pi}{n}}\right)^N = \binom{N}{r}.$$

Solution 5 by Moti Levy, Rehovot, Israel

$$\begin{aligned}
\lim_{n\to\infty} \frac{1}{n} \sum_{\mu=0}^{n-1} e^{-ir\mu\frac{2\pi}{n}\left(1+e^{i\mu\frac{2\pi}{n}}\right)} &= \int_0^1 e^{-ir2\pi x} \left(1 + e^{i2\pi x}\right)^N dx \\
&= \int_0^1 e^{-ir2\pi x} \sum_{k=0}^{N} \binom{N}{k} e^{i2\pi kx} dx \\
&= \sum_{k=0}^{N} \binom{N}{k} \int_0^1 e^{-i2\pi rx} e^{i2\pi kx} dx
\end{aligned}$$

$$\int_0^1 e^{-i2\pi rx} e^{i2\pi kx} dx = \begin{cases} 1 & if \quad k = r \\ 0 & if \quad k \neq r \end{cases}$$

and the result follows.

Solution 6 by Peter Fulop, Gyomro, Hungary

$$\boxed{\textbf{Prove } S = \lim_{n\to\infty} \frac{1}{n} \sum_{\mu=0}^{n-1} e^{-ir\mu\frac{2\pi}{n}} \left(1 + e^{i\mu\frac{2\pi}{n}}\right)^N = \tbinom{N}{r}}$$

where $N, r \in N$

Riemann sum

We know that Riemann sum gives us the following formula for a function $f \in C^1$:

$$\lim_{n\to\infty} \frac{1}{n} \sum_{\mu=0}^{n} f\left(\frac{\mu}{n}\right) = \int_0^1 f(x)dx \tag{8}$$

In our case:

$$\lim_{n\to\infty} \frac{1}{n} \sum_{\mu=0}^{n} e^{-ir\mu\frac{2\pi}{n}} \left(1 + e^{i\mu\frac{2\pi}{n}}\right)^N - \underbrace{\lim_{n\to\infty} \frac{2^N}{n}}_{\to 0} = \int_0^1 e^{-2\pi ixr} \left(1 + e^{2\pi ix}\right)^N dx \tag{9}$$

Contour integral

Applying the $z = e^{2\pi ix}$ substitution in the integral of (2) we get the following contour integral:

$$S = \frac{1}{2\pi i} \oint_{|z|=1} \frac{(1+z)^N}{z^{r+1}} dx \tag{10}$$

Using the Cauchy's differentiation formula and realized that $g(z) = \frac{(1+z)^N}{z^{r+1}}$ function has $r+1$ poles at $z=0$ which are inside the unit circle $|z|=1$, so we get:

$$S = \frac{1}{2\pi i} \oint\limits_{|z|=1} \frac{(1+z)^N}{z^{r+1}} dx = \frac{1}{r!} \frac{d^r}{dz^r} (1+z)^N \Big|_{z=0} \tag{11}$$

Performing the derivates ($(1+z)^N$ can be differentiated r times):

$$S = \frac{1}{r!} \frac{d^r}{dz^r} (1+z)^N \Big|_{z=0} = \frac{1}{r!} \Big(N(N-1)(N-2)....(N-r+1) \Big) (1+z)^{N-r} \Big|_{z=0} \tag{12}$$

Easy to realize that $N(N-1)(N-2)....(N-r+1) = \frac{N!}{(N-r)!}$
Finally substitute back to (5) we give the result:

$$S = \frac{1}{2\pi i} \oint\limits_{|z|=1} \frac{(1+z)^N}{z^{r+1}} dx = \binom{N}{r} \tag{13}$$

So the statement is proved.

Also solved by Pratik Donga, Junagadh, India; Kee-Wai Lau, Hong Kong, China, and the proposer.

- **5605:** *Proposed by José Luis Díaz-Barrero, Barcelona Tech, Barcelona, Spain*

Let b and c be distinct coprime numbers. Find the smallest positive integer a for which

$$\gcd(a^b - 1, a^c - 1) = 100.$$

Solution 1 by Albert Stadler, Herrliberg, Switzerland

Let n, t, s be natural numbers. We claim that

$$\gcd(n^r - 1, n^s - 1) = n^{\gcd(r,s)} - 1. \tag{*}$$

For the proof we proceed by induction on $\max(r, s)$. The statement is trivial for $\max(r, s) = 1$ and $r = s$. Suppose the statement holds true for all r, s with $\max(r, s) \leq m$. Suppose that $r = m + 1 > s$. Then

$$\begin{aligned} \gcd(n^r - 1, n^s - 1) &= \gcd(n^r - 1 - (n^s - 1), n^s - 1) = \gcd(n^s(n^{r-s} - 1), n^s - 1) = \\ &= \gcd(n^{r-s} - 1, n^s - 1) = n^{\gcd(r-s,s)} - 1 = n^{\gcd(r,s)} - 1, \end{aligned}$$

which concludes the proof $(*)$.

Clearly, $b \geq 0,\ c \geq 0$. By $(*)$ 100=$\gcd(a^b-1, a^c-1) = a^{\gcd(b,c)} - 1 = a - 1$. Hence $a = 101$.

Solution 2 by Pratik Donga, Junagadh, India

We can use the following formula to find a: $\gcd\left(n^b-1, n^c-1\right) = n^{\gcd(b,c)} - 1$. Here we let $n = a$ and so we obtain $\gcd\left(a^b-1, a^c-1\right) = a^{\gcd(b,c)-1=100}$. But $\gcd(b,c) = 1$ since b and c are coprime we have $a - 1 = 100 \to a = 101$.

Editor's Comment : Kee-Wai Lau of Hong Kong, China noted that the proof of this formula can be found in T. Andreescu, D. Andrica, and Feng, Z. *104 Number Theory Problems, From the Training of the USA IMO Team, Birkhauser, 2007, on page 112 as part of the solution to problem 38 on page 79.*

Solution 3 by David E. Manes, Oneonta, NY

More generally, if $a,\ b$ and c are positive integers,then

$$\gcd(a^b-1, a^c-1) = a^{\gcd(b,c)} - 1 \tag{1}$$

Therefore, if $\gcd(b,c) = 1, (b \neq c)$, then

$$\gcd(a^b-1, a^c-1) = a^{\gcd(b,c)} - 1 = a - 1 = 100$$

so that the only positive integer satisfying the given equation is $a = 101$.

To prove equation (1), we will us the following result: Let m be a positive integer and a and b are integers relatively prime to m. If x and y are integers such that $a^x \equiv b^x \pmod{m}$ and $a^y \equiv b^y \pmod{m}$, then $a^{\gcd(x,y)} \equiv b^{\gcd(x,y)} \pmod{m}$

Since $\gcd(b,c)$ divides both b and c, the polynomial $x^{\gcd(b,c)} - 1$ divides both $x^b - 1$ and $x^c - 1$. Hence, $a^{\gcd(b,c)-1}$ divides both $a^b - 1$ and $a^c - 1$ so that $a^{\gcd(b,c)} - 1$ is a divisor of $\gcd(a^b-1, a^c-1)$. On the other hand, if m divides both $a^b - 1$ and $a^c - 1$, then $\gcd(a,m) = 1$ and $a^b \equiv 1 \equiv 1^b \pmod{m}$. By the above stated result, it follows that $a^{\gcd(b,c)} \equiv 1 \pmod{m}$; that is to say, m is a divisor of $a^{\gcd(b,c)} - 1$. Therefore, $\gcd(a^b-1, a^c-1)$ is a divisor of $a^{\gcd(b,c)} - 1$. Hence, $a^{\gcd(b,c)} - 1 = \gcd(a^b-1, a^c-1)$.

Solution 4 by David Stone and John Hawkins, Georgia Southern University, Statesboro, GA

We shall show that $a = 101$.

Note that $a^n - 1 = (a-1)\left(a^{n-1} + a^{n-2} + \ldots + a + 1\right)$.

For convenience, let $P_n = \dfrac{a^{m+1}-1}{a-1} = a^m + a^{m-1} + \ldots + a + 1$. Note that P_n has $m+1$ terms.

Thus $\gcd\left(a^b-1, a^c-1\right) = \gcd\left((a-1)P_{b-1}\right), (a-1)P_{c-1} = (a-1)\gcd(P_{b-1}, P_{c-1})$.

In Comment 1 below, we will use the Euclidean Algorithm to show that, for coprime b and c, we have $\gcd(P_{b-1}, P_{c-1}) = 1$. Hence we have $\gcd\left(a^b-1, a^c-1\right) = a - 1$.

That is, $100 = \gcd\left(a^b, a^c\right) = (a-1) \cdot 1 = a - 1$, so $a = 101$.

<u>Comment 1</u>: The Euclidean Algorithm process which is used to determine that $\gcd(b,c) = 1$ can be used as a guide to show that $\gcd(P_{b-1}, P_{c-1}) = 1$.

We demonstrate with an illustrative example.
Let $b = 25$ and $c = 9$.
Apply the Euclidean Algorithm:

$$(1) \qquad \begin{cases} 25 = 2 \cdot 9 + 7 \\ 9 = 1 \cdot 7 + 2 \\ 7 = 3 \cdot 2 + 1 \end{cases}$$

The final remainder is 1, verifying that $\gcd(25, 9) = 1$.

Now we want to demonstrate that $\gcd(P_{24}, P_8) = 1$ by applying the Euclidean Algorithm. The first step in (1) above tells us exactly what to do as a first step here. Divide P_{24} by P_8: P_8 goes 2 times with a remainder of 7. That is, the 25 terms of $P_{24}a$ are grouped into 2 blocks of (9 terms each) and there are 7 terms left over.

$$\begin{aligned}
P_{24} &= a^{24} + a^{23} + a^{22} + \ldots + a + 1 = \\
&= \left(a^{24} + a^{23} + a^{22} + a^{21} + a^{20} + a^{19} + a^{18} + a^{17} + a^{16}\right) + \\
&+ \left(a^{15} + a^{14} + a^{13} + a^{12} + a^{11} + a^{10} + a^{9} + a^{8} + a^{7}\right) + \\
&+ a^{6} + a^{5} + a^{4} + a^{3} + a^{2} + a^{1} + 1 = \\
&= a^{16}\left(a^{8} + a^{7} + a^{6} + a^{5} + a^{4} + a^{3} + a^{2} + a + 1\right) + \\
&+ a^{7}\left(a^{8} + a^{7} + a^{6} + a^{5} + a^{4} + a^{3} + a^{2} + a + 1\right) + \\
&+ a^{6} + a^{5} + a^{4} + a^{3} + a^{2} + a + 1 = \\
&= \left(a^{16} + a^{7}\right) P_8 + P_6.
\end{aligned}$$

For the next step, we divide P_8 by P_6. The second step of (1) above tells us exactly what happens: P_6 goes 1 times with a remainder 2. That is the 9 terms of P_8 are grouped into 1 block (of 7 terms) and there are 2 terms left over.

$$\begin{aligned}
P_8 &= a^{8} + a^{7} + a^{6} + a^{5} + a^{4} + a^{3} + a^{2} + a + 1 = \\
&= \left(a^{8} + a^{7} + a^{6} + a^{5} + a^{4} + a^{3} + a^{2}\right) + (a + 1) = \\
&= a^{2}\left(+a^{6} + a^{5} + a^{4} + a^{3} + a^{2} + a + 1\right) + (a + 1) = \\
&= a^{2} P_6 + P_1.
\end{aligned}$$

For the third step, we divide P_6 by P_1. The third step of (1) above tells exactly what happens: P_1 goes 3 times with a remainder 1. That is, the 7 terms of P_6 are grouped into 3 blocks of 2 terms, and there is 1 term remaining (with must be a 1):

$$P_6 = a^{6} + a^{5} + a^{4} + a^{3} + a^{2} + a + 1$$

$$
\begin{aligned}
&= \left(a^6+a^5\right)+\left(a^4+a^3\right)+\left(a^2+a\right)+1 \\
&= a^5(a+1)+a^3(a+1)+a(a+1)+1 \\
&= \left(a^5+a^3+a\right)(a+1)+1 \\
&= \left(a^5+a^3+a\right)P_1+1.
\end{aligned}
$$

Summarizing this Euclidean Algorithm:

$$
\begin{aligned}
P_{24} &= \left(a^{15}+a^7\right)P_8+P_6 \\
P_8 &= a^2P_6+P_1 \\
P_6 &= (a^5+a^3+a)P_1+1. \text{ Therefore,} \\
\gcd\left(P_{24},P_8\right) &= 1.
\end{aligned}
$$

This process can be formalized. For any coprime b and c, the Euclidean Algorithm can be employed to show that $\gcd\left(P_{b-1},P_{c-1}\right)=1$; the process demonstrated above will work and terminate with 1, by repeatedly applying the following Lemma.

<u>Lemma</u>: For $1 \le n < m$, when P_m is divided by P_n, the remainder is a P_k, with $0 \le k < n$.
Proof: Use the Division Algorithm to divide $m+1$ by $n+1$:
$m+1 = q(n+1)+r$ where $0 \le r \le n+1$. Then it is straight forward to see that

$$P_m = Q \cdot P_n + P_{r-1}, \text{where } Q = \sum_{j=1}^{q} x^{m+1)-j(n+1)}.$$

Note that the operations with the P_n can be considered as with integers or as with polynomials with integer coefficients.

<u>Comment 2:</u> It seems surprising that the answer is independent of the values of b and c. What if b and c are not relatively prime? Suppose that $\gcd(b,c)=d$ with $b=dB$ and $c=dC$ for B and C coprime.

Then

$$\gcd\left(a^b-1,a^c-1\right)=\gcd\left(a^{dB}-1,a^{dC}-1\right)=\gcd\left(\left(a^d\right)^B-1,\left(a^d\right)^C-1\right)=a^d-1,$$

by applying the above result.
So if $d=1$, the effect of b and c disappears.
But $d>1$, asking that $\gcd\left(a^b-1,b^c-1\right)=100$ would be impossible because 101 is not a power.

Also solved by Hatef I. Arshagi, Guilford Technical Community College, Jamestown, NC; Kee-Wai Lau, Hong Kong, China, and the proposer.

- **5606:** *Proposed by Ovidiu Furdui and Alina Sîntămărian, Technical University of Cluj-Napoca, Cluj-Napoca, Romania*

Let $a, b > 0, c \geq 0$ and $4ab - c^2 > 0$. Calculate

$$\int_{-\infty}^{\infty} \frac{x}{ae^x + be^{-x} + c} dx.$$

Solution 1 by Seán M. Stewart, Bomaderry, NSW, Australia

Denote the integral to be evaluated by $I(a, b, c)$ where $a, b > 0$ and $c \geq 0$ such that $4ab - c^2 > 0$. We shall show that

$$I(a, b, c) = \frac{\log(b/a)}{\sqrt{4ab - c^2}} \arctan\left(\frac{\sqrt{4ab - c^2}}{c}\right).$$

Writing the integral as

$$I(a, b, c) = \int_{-\infty}^{\infty} \frac{xe^x}{ae^{2x} + ce^x + b} dx,$$

enforcing a substitution of $x \mapsto \log(x)$ produces

$$I(a, b, c) = \int_0^{\infty} \frac{\log(x)}{ax^2 + cx + b} dx.$$

Since $a, b > 0$, letting $x = t\sqrt{\frac{b}{a}}$ yields

$$\begin{aligned} I(a, b, c) &= \sqrt{\frac{b}{a}} \int_0^{\infty} \frac{\log\left(t\sqrt{\frac{b}{a}}\right)}{bt^2 + ct\sqrt{\frac{b}{a}} + b} dt \\ &= \frac{1}{2}\sqrt{\frac{b}{a}} \log\left(\frac{b}{a}\right) \int_0^{\infty} \frac{dt}{bt^2 + ct\sqrt{\frac{b}{a}} + b} \\ &\quad + \sqrt{\frac{b}{a}} \int_0^{\infty} \frac{\log(t)}{bt^2 + ct\sqrt{\frac{b}{a}} + b} dt. \end{aligned} \tag{1}$$

Enforcing a substitution of $t \mapsto \frac{1}{t}$ in the second of the integrals after the equality in (1) immediately shows it has a value equal to zero. Thus

$$I(a, b, c) = \frac{1}{2}\sqrt{\frac{b}{a}} \log\left(\frac{b}{a}\right) \int_0^{\infty} \frac{dt}{bt^2 + ct\sqrt{\frac{b}{a}} + b}. \tag{2}$$

Completing the square in the denominator of the integrand given in (2) one has

$$I(a, b, c) = \frac{1}{2\sqrt{ab}} \log\left(\frac{b}{a}\right) \int_0^{\infty} \frac{dt}{\left(t + \frac{c}{2\sqrt{ab}}\right)^2 + \left(\frac{4ab - c^2}{4ab}\right)}. \tag{3}$$

The constant term $\frac{4ab-c^2}{4ab}$ appearing in the denominator of the integrand of (3) is positive since

$4ab - c^2 > 0$ and $a, b > 0$. Performing the integration, which is elementary, we have

$$\begin{aligned} I(a,b,c) &= \frac{\log(b/a)}{\sqrt{4ab-c^2}} \left[\arctan\left(\frac{2t\sqrt{ab}+c}{\sqrt{4ab-c^2}} \right) \right]_0^\infty \\ &= \frac{\log(b/a)}{\sqrt{4ab-c^2}} \left[\frac{\pi}{2} - \arctan\left(\frac{c}{\sqrt{4ab-c^2}} \right) \right] \\ &= \frac{\log(b/a)}{\sqrt{4ab-c^2}} \arctan\left(\frac{\sqrt{4ab-c^2}}{c} \right), \end{aligned}$$

as announced. Note in the last line we have made use of the following well-known identity for the arctangent function of

$$\arctan(x) + \arctan\left(\frac{1}{x}\right) = \frac{\pi}{2}, \quad x > 0,$$

Solution 2 by Ulrich Abel, Technische Hochschule Mittelhessen, Germany

We show that for $a, b > 0$ and $c \geq 0$

$$I := \int_{-\infty}^{\infty} \frac{x}{ae^x + be^{-x} + c} dx = \frac{2\log\left(\frac{b}{a}\right) \arctan\left(\sqrt{\frac{\sqrt{4ab}-c}{\sqrt{4ab}+c}}\right)}{\sqrt{4ab-c^2}}.$$

The change of variable $x = t + \frac{1}{2}\log\frac{b}{a}$ leads to

$$I = \int_{-\infty}^{\infty} \frac{t + \frac{1}{2}\log\frac{b}{a}}{\sqrt{ab}\,(e^t + e^{-t}) + c} dt.$$

Since $\frac{t}{\sqrt{ab}(e^t+e^{-t})+c}$ is an odd integrable function, we obtain

$$\begin{aligned} I &= \frac{1}{2}\log\left(\frac{b}{a}\right) \int_{-\infty}^{\infty} \frac{1}{\sqrt{ab}\,(e^t + e^{-t}) + c} dt \\ &= \frac{1}{4\sqrt{ab}} \log\left(\frac{b}{a}\right) \int_{-\infty}^{\infty} \frac{1}{\cosh t + g} dt, \end{aligned}$$

where $g = \sqrt{c^2/(4ab)} < 1$ by assumption. Since

$$\sqrt{1-g^2} \int \frac{1}{\cosh t + g} dt = 2\arctan\left(\sqrt{\frac{1-g}{1+g}} \tanh\frac{t}{2}\right)$$

we have

$$\sqrt{1-g^2} \int_{-\infty}^{\infty} \frac{1}{\cosh t + g} dt = 4\arctan\left(\sqrt{\frac{1-g}{1+g}}\right),$$

which implies the desired formula.

Remark: In the particular case $c = 0$, we obtain $g = 0$ and

$$\int_{-\infty}^{\infty} \frac{x}{ae^x + be^{-x}} dx = \frac{\pi}{4\sqrt{ab}} \log\left(\frac{b}{a}\right).$$

Solution 3 by Kee-Wai Lau, Hong Kong, China

Denote the given Integral by I. We show that

$$I = \frac{(\ln b - \ln a)\cos^{-1}\left(\dfrac{c}{2\sqrt{ab}}\right)}{\sqrt{4ab - c^2}} \tag{1}$$

By the substitution $x = \ln y$, we obtain $I = \dfrac{1}{a}\displaystyle\int_0^{\infty} \frac{\ln y}{y^2 + \frac{cy}{a} + \frac{b}{a}} dy$. It is known ([1], p.537, entry 4.233(5)) that for $k > 0$ and $0 < t < \pi$, we have

$$\int_0^{\infty} \frac{\ln x}{x^2 + 2xk\cos t + k^2} dx = \frac{t\ln k}{k\sin t}.$$

By putting $k = \sqrt{\dfrac{b}{a}}$ and $t = \cos^{-1}\left(\dfrac{c}{2\sqrt{ab}}\right)$, we obtain (1) readily.

1. I.S. Gradshteyn and I.M. Ryzhik. *Table of Integrals, Series, and Products,* Seventh Edition, Elsevier, Inc. 2007.

Solution 4 by Peter Fulop, Gyomro, Hungary

Let $a, b > 0$ and $4ab - c^2 \geq 0$.

$$\boxed{\textbf{Calculate } I = \int_{-\infty}^{\infty} \frac{x}{ae^x + be^{-x} + c} dx}$$

Partial fraction decomposition

Let's transform the integral into two parts:

$$I = \int_0^{\infty} \frac{x}{ae^x + b/e^x + c} dx - \int_{-\infty}^{0} \frac{x}{ae^x + b/e^x + c} dx \tag{15}$$

At first appling the $x \to -x$ substitution for the second integral of the (1) and then the substitution of $t = \frac{1}{e^x}$ for both integrals of (1):

$$I = \frac{1}{a}\int_0^1 \frac{\ln(t)}{t^2 + \frac{c}{a}t + \frac{b}{a}} dt - \frac{1}{b}\int_0^1 \frac{\ln(t)}{t^2 + \frac{c}{b}t + \frac{a}{b}} dt \tag{16}$$

Find the roots of the quadratic expressions of the (2):

$$t_{1,2} = \frac{-c \pm i\sqrt{4ab - c^2}}{2a} = \sqrt{\frac{b}{a}} e^{\mp i\varphi} \tag{17}$$

$$t_{3,4} = \frac{-c \pm i\sqrt{4ab - c^2}}{2b} = \sqrt{\frac{a}{b}} e^{\mp i\varphi} \tag{18}$$

where $\varphi = \arctan(\frac{d}{c})$, $d = \sqrt{4ab - c^2}$ and $i^2 = -1$ the (2) is becoming the following:

$$I = \frac{1}{a}\Big(\int\limits_0^1 \frac{\ln(t)}{(t-t_1)(t-t_2)}dt\Big) - \frac{1}{b}\Big(\int\limits_0^1 \frac{\ln(t)}{(t-t_3)(t-t_4)}dt\Big) \tag{19}$$

Taking into account that:
$\frac{1}{(t-t_1)(t-t_2)} = \frac{a}{id}\Big[-\frac{1}{(t-t_1)} + \frac{1}{(t-t_2)}\Big]$and
$\frac{1}{(t-t_3)(t-t_4)} = \frac{b}{id}\Big[-\frac{1}{(t-t_3)} + \frac{1}{(t-t_4)}\Big]$ the (5) will be the following:

$$I = \frac{i}{d}\Big(-\int\limits_0^1 \frac{\ln(t)}{(t-t_1)}dt + \int\limits_0^1 \frac{\ln(t)}{(t-t_2)}dt + \int\limits_0^1 \frac{\ln(t)}{(t-t_3)}dt - \int\limits_0^1 \frac{\ln(t)}{(t-t_4)}dt\Big) \tag{20}$$

Spence function and its properties

Based on (6) let $\frac{Id}{i} = -I_1 + -I_2 + I_3 - I_4$ respect to t_k roots (k=1,2,3,4). Let's pull out $-t_k$ from all denominators and performing the substitutions $x = \frac{t}{t_k}$.

$$I_k = \int\limits_0^{\frac{1}{t_k}} \frac{\ln(x) + \ln(t_k)}{(1-x)}dx = \int\limits_0^{\frac{1}{t_k}} \frac{\ln(x)}{1-x} + \frac{\ln(t_k)}{(1-x)}dx \tag{21}$$

$$I_k = \int\limits_0^{\frac{1}{t_k}} \frac{\ln(x)}{1-x}dx + \ln(\frac{1}{t_k})\ln(1-\frac{1}{t_k}) \tag{22}$$

Introducing further substitution ($r = 1 - x$) regarding the integral of (8) we get:

$$I_k = \int\limits_0^1 \frac{\ln(1-r)}{r}dr - \int\limits_0^{1-\frac{1}{t_k}} \frac{\ln(1-r)}{r}dx + \ln(\frac{1}{t_k})\ln(1-\frac{1}{t_k}) \tag{23}$$

Using the definition of the Spence function (Dilogaritm function) we have:

$$I_k = -Li_2(1) + Li_2(1-\frac{1}{t_k}) + \ln(\frac{1}{t_k})\ln(1-\frac{1}{t_k}) \tag{24}$$

Applying the following identity of the Spence function:

$Li_2(z) + Li_2(1-z) = Li_2(1) - \ln(z)\ln(1-z)$, (10) will be the following:

$$I_k = -Li_2(\frac{1}{t_k}) \tag{25}$$

Based on (3),(4) can be seen that $t_4 = \frac{1}{t_1}$ and $t_3 = \frac{1}{t_2}$, go back to (6) we get value of the integral (I):

$$I = Li_2(\frac{1}{t_1}) - Li_2(\frac{1}{t_2}) - Li_2(t_2) + Li_2(t_1) \tag{26}$$

Using the following identity $Li_2(z) + Li_2(\frac{1}{z}) = -Li_2(1) - \frac{1}{2}\ln^2(-z)$ twice we get:

$$I = \frac{i}{2d}\ln(\frac{t_2}{t_1})\ln(t_1 t_2) \tag{27}$$

Finally substitute back t_1 and t_2 from (3) we get the result:

$$\boxed{I = \frac{\ln(\frac{a}{b})}{\sqrt{4ab-c^2}}\arctan(\sqrt{\frac{4ab}{c}-1})} \tag{28}$$

Also solved by Albert Stadler, Herrliberg, Switzerland, and the proposers.