Problems

Ted Eisenberg, Section Editor

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This section of the Journal offers readers an opportunity to exchange interesting mathematical problems and solutions. Please send them to Ted Eisenberg, Department of Mathematics, Ben-Gurion University, Beer-Sheva, Israel or fax to: 972-86-477-648. Questions concerning proposals and/or solutions can be sent e-mail to <eisenbt@013.net>. Solutions to previously stated problems can be seen at <http://www.ssma.org/publications>.

Solutions to the problems stated in this issue should be posted before June 15, 2021

- **5625:** *Proposed by Kenneth Korbin, New York, NY*

  Trapezoid $ABCD$ with integer length sides is inscribed in a circle with diameter y^4.

  $$\begin{aligned}\text{Side } \overline{AB} &= \overline{CD} = 3xy^3 - 4x^3y \\ \text{Side } \overline{BC} &= 2x^2y^2 - y^4.\end{aligned}$$

  Express the length of side $\overline{AD}$ in terms of positive integers x and y.

- **5626:** *Roger Izard, Dallas,TX*

  In triangle ABC, cevians AF, BE, and CD are drawn so that they intersect at point O. Prove or disprove that

  $$\overline{AC} \cdot \overline{EO} \cdot \overline{OC} \cdot \overline{DB} \cdot \overline{FB} \cdot = \overline{AB} \cdot \overline{OD} \cdot \overline{OB} \cdot \overline{CE} \cdot \overline{CF}.$$

- **5627:** *Proposed by Albert Natian, Los Angeles Valley College, Valley Glen, CA*

  Solve $ax + 7 = bx + a$, given that $a, b \in \Re$ and

  $$a = \begin{cases} 2 & \text{if } b \text{ is less than } 3 \\ x-6 & \text{if } \ 3 \leq b \end{cases} \Bigg\} \text{ and } \ b = \begin{cases} 3x & \text{if } x \text{ is less than } a \\ 7 & \text{if } \ a \leq x \end{cases} \Bigg\}$$

- **5628:** *Daniel Sitaru, National Economic College,"Theodor Costescu" Drobeta Turnu-Severin, Mehedinti, Romania*

  Prove that if $x, y, z, u, v, w \in (0, \infty)$, then

  $$\frac{x^2}{u}e^{\frac{u}{x}} + \frac{y^2}{v}e^{\frac{v}{y}} + \frac{z^2}{w}e^{\frac{w}{z}} \geq \frac{(x+y+z)^2}{u+v+w}e^{\frac{u+v+w}{x+y+z}}.$$

- **5629:** *Proposed by José Luis Díaz-Barrero, Barcelona Tech, Barcelona, Spain*

  A square is divided into a finite number of blue and red rectangles, each with sides parallel to the sides of the square. Within each blue rectangle the ratio between its width and its height is written. Within each red rectangle the ratio between its height and its width is written. Finally, the sum S of these numbers is computed. If the total area of the blue rectangles equals twice the total area of the red rectangles, what is the smallest possible value of S?

- **5630:** *Proposed by Arkady Alt, San Jose, CA*

  Find the integer part of the minimal value of $k + \dfrac{n}{k}, k \in N$.

Solutions

- **5607:** *Proposed by Kenneth Korbin, New York, NY*

  Given $\triangle ABC$ with integer area and with altitude $\overline{CD}$. Find the sides if $\overline{BC} = \overline{AC} + 1 = \overline{CD} + 100$.

  Solution 1 by Albert Stadler, Herrliberg, Switzerland

  Let $a := \overline{BC}, y := \overline{AC}, c := \overline{AB}, h := \overline{CD}$. Let Δ be the area of the triangle.

  By assumption $a = b + 1 = h + 100$. Let $x = a + b \geq c$.Then $a = \dfrac{x+1}{2}, b = \dfrac{x-1}{2}$, and by Heron's formula,

$$4\Delta = \sqrt{(a+b+c)((-a+b+c)(a-b+c)(a+b-c)} = \sqrt{(x^2-c^2)(c^2-1)}.$$

  We have also

$$4\Delta = 2ch = 2c(a-100) = 2c\left(\frac{x+1}{2} - 100\right) = c(x-199).$$

  We eliminate x from these two equations and get

$$x^2 = \frac{16\Delta^2}{c^2-1} + c^2 = \left(\frac{4\Delta}{c} + 199\right)^2.$$

  We clear denominators and find

$$16\Delta^2 - 1592c(c^2-1)\Delta + c^2(c^2-1)(c^2-199^2) = 0 \qquad (*)$$

$$x = \frac{4\Delta}{c} + 199 \qquad (**)$$

So given Δ we can calculate c from $(*)$ and then x from $(**)$. Finally $a = \frac{x+1}{2}, b = \frac{x-1}{2}$. It's clear that Δ integral does not imply c integral. For instance , if $\Delta = 1 (*)$ has two positive (real) roots which are approximately equal to 1.00019 and 199.02.

The challenge starts if we mandate c to be an integer. $(*)$ is a quadratic equation in Δ whose roots are

$$\Delta = \frac{199}{4}c(c^2-1) \pm 15\sqrt{11}c^2\sqrt{c^2-1}.$$

By assumption Δ is integral. Therefore, if c is an integer, there is an integer k such that

$$11c^2 - 11 = k^2. \qquad (***)$$

This is a Pellian equation whose roots are given by

$$(c,k) \in \left\{ \left(\frac{\left(10-3\sqrt{11}\right)^n + \left(10+3\sqrt{11}\right)^n}{2}, \frac{\sqrt{11}\left(-(10-3\sqrt{11})^n + (10+3\sqrt{11})^n\right)}{2} \right) \middle| n \geq 0 \right\}.$$

Let (c_n, k_n) be the parametric solutions to $(* * *)$. We find

$$c_n = \frac{(10-3\sqrt{11})^n + (10+3\sqrt{11})^n}{2} = \sum_{j=0}^{\left[\frac{n}{2}\right]} \binom{n}{2j} 99^j 10^{n-2j}$$

and

$$k_n = \sqrt{11} \cdot \frac{-(10-3\sqrt{11})^n + (10+3\sqrt{11})^n}{2} = 33 \sum_{j=0}^{\left[\frac{n-1}{2}\right]} \binom{n}{2j+1} 99^j 10^{n-2j-1}.$$

These two sums show that c_n has the opposite parity of n, while k_n has the same parity as n.

Furthermore

$$\Delta_n = \frac{1}{4}\left(199c_n(c_n^2-1) \pm 60\sqrt{11}c_n^2\sqrt{c_n^2-1}\right) = \frac{c_n k_n}{4}\left(199 \cdot \frac{k_n}{11} \pm 60c_n\right),$$

$$x_n = \frac{4\Delta}{c_n} + 199 = k_n\left(\frac{199}{11}k_n \pm 60c_n\right) + 199.$$

We see that x_n has the opposite parity of n. There a_n and b_n are half integers for odd n and are integers of even n. Clearly $\Delta_n \geq 0$. Therefore the minus sign in the formula for Δ_n applies only if $199c_n\left(c_n^2-1\right) > 60\sqrt{11}c_n^2\sqrt{c_n^2-1}$ which means that $c_n > 199$,

We take the + sign in the above formula for Δ_n. We find for the first few values of n.

n	c_n	k_n	Δ_n	x_n	a_n	b_n
1	10	33	197505/2	39700	39701/2	39699/2
2	199	660	784099800	15760999	7880500	7880499
3	3970	13167	12451504627485/2	6272798500	6272798501/2	6272798499/2

We take the $-$ sign in the above formula for Δ_n. We find for the first few values of n.

$$\begin{pmatrix} n & c_n & k_n & \Delta_n & x_n & a_n & b_n \\ & & & & & & \\ 3 & 3970 & 13167 & 78409485/2 & 39700 & 39701/2 & 39699/2 \\ & & & & & & \\ 4 & 79201 & 262680 & 312067780200 & 15760999 & 7880500 & 7880499 \\ & & & & & & \end{pmatrix}$$

Solution 2 by David Stone and John Hawkins, Georgia Southern University, Statesboro,GA

We'll show there are infinitely many solutions with integer area, integer sides and integer altitude satisfying the conditions of the problem. (That is, Heronian triangle solutions.) One solution:

$$\begin{aligned} \overline{AC} &= 7,880,499 \\ \overline{BC} &= 7,880,500 \\ \overline{CD} &= 7,880,400 \\ \overline{AB} &= 79,201 \\ Area &= 312,067,780,200 \end{aligned}$$

Consider the triangle with vertices A, B, C and height $h = \overline{CD}$. Therefore, we can express the sides in terms of h: $\overline{BC} = h + 100$ and $\overline{AC} = h + 99$.

The base $\overline{AB}$ can be computed by applying the Pythagorean Theorem in each of the right triangles $\triangle ADC$ and $\triangle BDC$:

$$\begin{aligned} AB &= \overline{AD} + \overline{DB} = \sqrt{\overline{AC}^2 - h^2} + \sqrt{\overline{BC}^2 - h^2} \\ &= \sqrt{(h+99)^2 - h^2} + \sqrt{(h+100)^2 - h^2} \\ &= \sqrt{198h + 9801} + \sqrt{200h + 10000}. \quad \text{Therefore,} \\ Area &= \frac{1}{2} h \left(\sqrt{(198h + 9801} + \sqrt{200h + 10000} \right). \end{aligned}$$

We shall see how to choose the altitude h to be an even integer such that each expression under its radical is a perfect square; thus the area will be an integer.

(1) $198h + 9801 = 9 \cdot 11(2h + 99)$.

To be a square, we must have $2h + 99$ divisible by 11; say $h = 11k$.
Thus $198h + 9801 = 9 \cdot 11(22k + 99) = 33^2(2k + 9)$.

So we choose k such that $2k + 9$ is a square; that is let $k = \dfrac{m^2 - 9}{2}$.

To make k an integer, we must choose m odd. Say $m = 2z + 1$.
Thus

$$k = \frac{m^2 - 9}{2} = \frac{(2z+1)^2 - 9}{2} = \frac{4z^2 + 4z + 1 - 9}{2} = \frac{4z^2 + 4z - 8}{2} = 2z^2 + 2z - 4 = 2\left(z^2 + z - 2\right).$$

Thus $h = 11k = 22\left(z^2 + z - 2\right)$.
We verify that this choice of h does indeed serve its role:

$198h+9801 = 99(2h+99) = 9{\cdot}11\left(2\left[22\left(z^2+z-2\right)\right]+99\right) = 9{\cdot}11^2\left(4z^2+4z-8+9\right) = 33^2\left(2z+1\right)^2.$

(2) Similarly, we find that $200h + 10000$ is a perfect square if $h = 2\left(v^2 - 25\right)$.
For then $200h+10000 = 2{\cdot}10^2(h+50) = 2{\cdot}10^2\left(2\left(v^2-25\right)+50\right) = 2^2{\cdot}10^2\left(v^2-25+25\right) = (20v)^2$.
However, we need both radicals to be integers. We can make both (1) and (2) happen if we can find z and w such that $h = 11k = 22\left(z^2+z-2\right) = 2\left(v^2-25\right)$.
That is, we need $11\left(z^2+z-2\right) = v^2 - 25$
$11\left(2z+1\right)^2 = (2v)^2 - 1$.
Letting $x = 2v, y = 2z+1$, we have the Pell's equation $x^2 - 11y^2 = 1$.
This equation has infinitely many solutions
A fundamental solution is $x_1 = 10, y_1 = 3$. All other solutions are given by the recurrence relations $\begin{cases} x_{k+1} &= 10x_k + 33y_k \\ y_{k+1} &= 3x_k + 10y_k. \end{cases}$
This is easily visualized using matrix multiplication:

$$\begin{pmatrix} x_{k+1} \\ y_{k+1} \end{pmatrix} = \begin{pmatrix} 10 & 33 \\ 2 & 10 \end{pmatrix} \begin{pmatrix} x_k \\ y_k \end{pmatrix}.$$

For our problem, the parity needs of x and y require that we use every second solution, achieved by applying the square of the matrix and rescaling the indexing:

$$\begin{pmatrix} x_{k+1} \\ y_{k+1} \end{pmatrix} = \begin{pmatrix} 199 & 660 \\ 60 & 199 \end{pmatrix} \begin{pmatrix} x_k \\ y_k \end{pmatrix} \ \mathit{with} \ \begin{pmatrix} x_1 \\ y_1 \end{pmatrix} = \begin{pmatrix} 10 \\ 3 \end{pmatrix}.$$

The next solution is $x = 3970, y = 1197$, which produces the triangle given initially:
$v = \frac{x}{2} = \frac{3970}{2} = 1985$ and $z = \frac{y-1}{2} = \frac{1197-1}{2} = 598$.
Then, as expected, both of these produce the same value for h :
$h = 22\left(z^2+z-2\right) = 22\left(598^2+598-2\right) = 22{\cdot}358200 = 7,880,400$ and $h = 2\left(v^2-25\right) = 2\left(1985^2-25\right) = 7,880,400$.
Then

$$\begin{aligned} Area &= \frac{1}{2}(7880400)\left(\sqrt{198\cdot 7880400+9801}+\sqrt{200\cdot 7880400+10000}\right) = \\ &= 3940200\left(\sqrt{1560329001}+\sqrt{1576090000}\right) = 3940200\cdot 79201 = 312,067,780,200. \end{aligned}$$

These numbers are already pretty large, but well we exhibit one more solution: $x = 1,580,050,\ y = 476,403 \Rightarrow h = 1,248,279,001,200$;
Sides $\overline{AC} = 1,248,279,001,299$; $\overline{BC} = 1,248,279,001,300$
Base $\overline{AB} = 31,521,799$ and $Area = 19,673,999,885,873,579,400$.

A final note. For any of our solutions (x, y) of the Pell equation (with x even and y odd), we find that
$h = \dfrac{x^2-100}{2}$ and $Area = \dfrac{\left(x^2-100\right)\left(10x+33y\right)}{4}$.

Comment: Consider the function $Area = \frac{1}{2}h\left(\sqrt{198h+9801}+\sqrt{200h+10000}\right)$ giving the triangle's area in terms of the altitude.

When $h = 0$, Area $= 0$. Then the Area slowly grows monotonically to infinity. That is, it takes on all values for the Area, including all integer values.

However, given a specific integer N for the area, it seems impossible to solve

$$N = \frac{1}{2}h\left(\sqrt{198h+9801}+\sqrt{200h+10000}\right).$$

By algebra, we convert this equation to the sixth degree polynomial equation
$4h^6 + 796h^5 + 39601h^4 - 3184N^2h^3 - 158408N^2h^2 + 16N^4 = 0$.
This equation does not seem any more tractable.

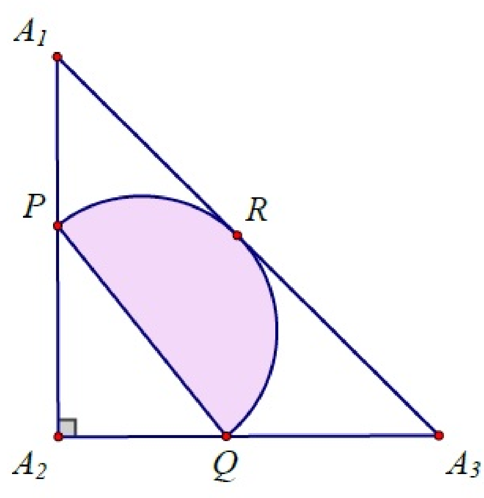
Using a TI-83 Emulator and the Wolfram Alpha Equation Solver, we obtained approximate values of h for a few proscribed values of N. Nothing nice showed up.

$N(area)$	$h = \overline{CD}$ $(altitude)$	$\overline{BC}$	$\overline{AC}$	$Base = \overline{AB}$
1	0.010049	100.01004	99.010049	199.020818
10	0.100401	100.100401	99.100401	199.200707
100	0.995122	100.995122	99.995122	200.980382
1000	9.230380	109.230380	108.230380	216.675803
2000	17.312640	117.312640	116.312640	231.045063
3000	24.654455	124.654455	123.654455	243, 363725

Also solved by the proposer.

5608: *Proposed by Arsalan Wares, Valdosta State University, Valdosta, Georgia*

Triangle $A_1A_2A_3$ is a right isosceles triangle with $\angle A_1A_2A_3 = 90°$. Point P is on side A_1A_2 such that $\frac{A_1P}{PA_2} = \frac{4}{5}$. Point Q is on side A_2A_3 and arc PQR, touching side A_1A_3 at point R, is semicircular. If $A_1A_2 = 3$, find the exact length A_1R.



Solution 1 by Bruno Salgueiro Fanego, Viveiro, Spain

Since R belongs to the semicircle with diameter PQ, then angle $\angle PRQ$ is inscribed in the circle C with diameter PQ so $\angle PRQ = 90°$. And since angle

$\angle QA_2P = \angle A_3A_2A_1 = 90°$, the vertex A_2 of triangle $A_1A_2A_3$ belongs to circle C. Thus, R and A_2 belong to circle C.

Since $3 = A_1A_2 = A_1P + PA_2 = A_1P + \frac{5}{4}A_1P = \frac{9}{4}A_1P \Rightarrow A_1P = \frac{4}{3}$, we conclude that the length A_1R is the square root of power of the vertex A_1 with respect to circle C, which is also equal to $\sqrt{A_1P \cdot A_1A_2} = \sqrt{\frac{4}{3} \cdot 3} = 2$.

Solution 2 by David A. Huckaby, Angelo State University, San Angelo, TX

Since $A_1A_2 = 3$ and $\frac{A_1P}{PA_2} = \frac{4}{5}$, we have $PA_2 = \frac{5}{3}$. Apply a Cartesian coordinate system to triangle $A_1A_2A_3$, with A_2 at the origin, P with coordinates $(0, \frac{5}{3})$, and Q with coordinates $(a, 0)$. Since triangle $A_1A_2A_3$ is isosceles, side A_1A_3 has the equation $y = 3 - x$, so we can let R have coordinates $(b, 3 - b)$.

Semicircle PQR is half of a circle that has its center C at the midpoint of PQ, namely at $C(\frac{a}{2}, \frac{5}{6})$. The radius of the circle is the distance from P to C, which is $\sqrt{\left(\frac{a}{2} - 0\right)^2 + \left(\frac{5}{6} - \frac{5}{3}\right)^2} = \sqrt{\frac{a^2}{4} + \frac{25}{36}}$. So the equation of the circle is

$$\left(x - \frac{a}{2}\right)^2 + \left(y - \frac{5}{6}\right)^2 = \frac{a^2}{4} + \frac{25}{36} \tag{2}$$

Since the point $R(b, 3 - b)$ lies on the circle, we have $\left(b - \frac{a}{2}\right)^2 + \left(3 - b - \frac{5}{6}\right)^2 = \frac{a^2}{4} + \frac{25}{36}$, which simplifies to

$$2b^2 - \left(a + \frac{13}{3}\right)b + \frac{144}{36} = 0. \tag{1}$$

Taking the derivative of the equation of the circle (2) with respect to x gives $2\left(x - \frac{a}{2}\right) + 2\left(y - \frac{5}{6}\right)\frac{dy}{dx} = 0$. Since the circle is tangent to the line $y = 3 - x$ at point $R(b, 3 - b)$, $\frac{dy}{dx} = -1$ there, and so $2\left(b - \frac{a}{2}\right) + 2\left(3 - b - \frac{5}{6}\right)(-1) = 0$, which simplifies to

$$b = \frac{1}{4}a + \frac{13}{12}. \tag{2}$$

Inserting this expression for b into equation (1) gives $2\left(\frac{a}{4} + \frac{13}{12}\right)^2 - \left(a + \frac{13}{3}\right)\left(\frac{a}{4} + \frac{13}{12}\right) + \frac{144}{36} = 0$, which simplifies to the quadratic equation $a^2 + \frac{26}{3}a - \frac{119}{9}$, which has the solutions $a = -\frac{13}{3} \pm 4\sqrt{2}$. Discarding the negative solution, we have $a = -\frac{13}{3} + 4\sqrt{2}$.

Inserting this expression for a into equation (2) gives $b = \frac{1}{4}\left(-\frac{13}{3} + 4\sqrt{2}\right) + \frac{13}{12} = \sqrt{2}$. Therefore point R has coordinates $(b, 3 - b) = (\sqrt{2}, 3 - \sqrt{2})$. Since point A_1 has coordinates $(0,3)$, the distance formula gives the exact length of A_1R to be $\sqrt{(\sqrt{2}-0)^2+(3-\sqrt{2}-3)^2}=\sqrt{2+2}=2.$

Editor's comment: Most of the solutions received to 5608 followed the paths in the above featured solutions. They either coordinitized the circle and constructed the necessary equations, or they used the secant tangent theorem of a given point P to a circle C. This theorem says that if P is outside the circle C, and if a line through P intersects the circle circle in points M and N, and if another line through P is tangent to the circle at point T then $\overline{PM}\cdot\overline{PN}=\overline{PT}^2$. This is also called the "power of point P to circle C".

David Stone and John Hawkins of Georgia Southern University took the problem a bit further by asking themselves: "How are the other sides of the triangle cut by *extreme points* of the inscribed semicircle?"

Specifically, they coordinitized the triangle assigning A_1 the coordinates $(0,3)$, $A_2(0,0)$ and $A_3(3,0)$. Calling the midpoint of the arc of the semicircle R, they assigned to it the coordinates (u,v). They then drew RP and RQ. Note that $\triangle PRQ$ is a right triangle with $\angle PRQ=90°$. Letting M be the midpoint of the chord PQ they then found the coordinates of P and Q, and computed the ratios $\dfrac{A_2Q}{QA_3}$ and $\dfrac{A_1R}{RA_3}$.

For the coordinates they assigned, they found that

$$\frac{A_2Q}{QA_3}=\frac{1+54\sqrt{2}}{98}\approx 0.78946$$

$$\frac{A_1R}{RA_3}=\frac{2+3\sqrt{2}}{7}\approx 0.89181,$$

and concluded that "the three sides are cut in different ratios by P,Q and R, but the legs suffer nearly equal effects."

Their solution ended with the following comment:

" Centuries ago, mankind investigated and answered the questions of inscribing a circle in a triangle and inscribing a triangle in a circle. Sometimes people have considered a semicircle inscribed in a triangle in which the diameter lies along one side of the triangle. Here, the problems poser asks more generally about inscribing a semicircle in a (very nice) triangle, given only the ratio of the parts of one of the separated sides. It turns out there is a solution and it is unique. Would that be true if the given ratio were any real number (between 0 and 1) instead of 4/5? A question probably worth investigating."

Also solved by Michel Bataille, Rounen, France; Brian D. Beasley, Presbyterian College, Clinton, SC; Michael Brozinsky (3 solutions), Central Islip, New York; Charles Burnette, Xavier University of Louisiana, New Orleans, LA; Kenneth Korbin, New York, NY; Kee-Wai Lau, Hong Kong, China; David E. Manes, Oneonta, NY; Albert Natian, Los Angeles Valley College, Valley Glen, CA; Albert Stadler, Herrliberg, Switzerland; Seán M. Stewart, Bomaderry, NSW, Australia; David Stone and John Hawkins, Georgia Southern University, Statesboro, GA, and the proposer.

5609: *Proposed by Ángel Plaza, Universidad de Las Palmas de Gran Canaria, Spain*

Evaluate the following limit without using derivatives:

$$\lim_{x\to 0}\frac{(1+a\ln(1+bx^3))^{1/x}-1}{x^2}.$$

Solution 1 by Ulrich Abel, Technische Hochschule Mittelhessen, Germany

Let $a, b \in R$. Taking advantage of the relations

$$\ln(1+t) = t + O\left(t^2\right) \qquad \text{and} \qquad e^t = 1 + t + O\left(t^2\right) \qquad (t \to 0)$$

we obtain

$$\begin{aligned}\left(1+a\ln\left(1+bx^3\right)\right)^{1/x} &= \exp\left(\frac{1}{x}\ln\left(1+a\ln\left(1+bx^3\right)\right)\right)\\ &= \exp\left(\frac{1}{x}\ln\left(1+abx^3+O\left(x^6\right)\right)\right)\\ &= \exp\left(abx^2+O\left(x^5\right)\right) = 1+abx^2+O\left(x^4\right)\end{aligned}$$

as $x \to 0$. Hence,

$$\lim_{x\to 0}\frac{\left(1+a\ln\left(1+bx^3\right)\right)^{1/x}-1}{x^2} = ab.$$

Solution 2 by Pratik Donga, Junagadh, India

We want to calculate

$$\lim_{x\to 0}\frac{\left(1+a\ln(1+bx^3)\right)^{\frac{1}{x}}-1}{x^2}. \tag{1}$$

Replace x by $y = \dfrac{1}{x}$ implies as $x \to 0, y = \dfrac{1}{x} \to \infty$, implies $y^3 \to \infty$.

So (1) can be written as

$$\lim_{y\to\infty} y^2\left\{\left(1+a\ln\left(1+\frac{b}{y^3}\right)\right)^y-1\right\}$$

$$\Rightarrow \lim_{y\to\infty} y^2\left\{1^y + ya\ln\left(1+\frac{b}{y}\right)+\binom{y}{2}\left(a\ln\left(1+\frac{b}{y^3}\right)\right)^2+\ldots+\left(a\ln\left(1+\frac{b}{y^3}\right)\right)^y-1\right\}$$

$$\Rightarrow \left\{y^3 a\ln\left(1+\frac{b}{y^3}\right)+\frac{y^2(y-1)}{2!}a^2\left(\ln\left(1+\frac{b}{y^3}\right)\right)^2\right.$$

$$\left.+\frac{y^3(y-1)(y-2)}{3!}a^3\left(\ln\left(1+\frac{b}{y^3}\right)\right)^3+\ldots+\frac{y^3}{y}a^y\left(\ln\left(1+\frac{b}{y^3}\right)\right)^y\right\}$$

$$\Rightarrow \lim_{y\to\infty}\left\{a\ln\left(1+\frac{b}{y^3}\right)^{y^3}+\frac{(y-1)^2}{2!}\left(\ln\left(1+\frac{b}{y^3}\right)^{y^3}\right)\left(\ln\left(1+\frac{b}{y^3}\right)\right)+\right.$$

$$\frac{(y-1)(y-2)}{3!}a^2\left(\ln\left(1+\frac{b}{y^3}\right)^{y^3}\right)\left(\ln\left(1+\frac{b}{y^3}\right)\right)^2+\ldots+\frac{a^y}{y}\left(\ln\left(1+\frac{b}{y^3}\right)^{y^3}\right)\left(\ln\left(1+\frac{b}{y^3}\right)\right)^{y-1}\right\}$$

$$\Rightarrow \lim_{y\to\infty}\left\{ln\left(1+\frac{b}{y^3}\right)^{y^3}\right\}\left\{a+\frac{(y-1)}{2!}a^3\left(1+\frac{b}{y^3}\right.+\frac{(y-1)(y-2)}{3!}a^2\left(\ln\left(1+\frac{b}{y^3}\right)\right)+\right.$$

$$\left.\ldots+\frac{a^y}{y}\left(\ln\left(1+\frac{b}{y^3}\right)\right)^{y-1}\right\}$$

$$\Rightarrow\left\{\ln\lim_{y^3\to\infty}\left(1+\frac{b}{y^3}\right)^{y^3}\right\}\lim_{y\to\infty}\left\{a+\frac{(y-1)}{2!}a^2\left(\ln\left(1+\frac{b}{y^3}\right)\right)+\frac{(y-1)(y-2)}{3!}a^3\right.$$

$$\left.\left(\ln\left(1+\tfrac{b}{y^3}\right)\right)^2+\ldots+\tfrac{a^y}{y}\left(\ln\left(1+\tfrac{b}{y^3}\right)\right)^{y-1}\right\}$$

$\Rightarrow$

$$\{\ln e^b\}\left\{a+\lim_{y\to\infty}\frac{(y-1)}{2!}a^2(0)+\lim_{y\to\infty}\frac{(y-1)(y-2)}{3!}a^3(0)+\ldots+\lim_{y\to\infty}\frac{a^y}{y}(0)\right\}\Rightarrow ba=ab$$

Hence, $\displaystyle\lim_{x\to 0}\frac{\left(1+a\cdot\ln(1+bx^3)\right)^{\frac{1}{x}}-1}{x^2}=ab$

Solution 3 by Seán M. Stewart, Bomaderry, NSW, Australia

Denote the limit to be found by ℓ. We show, where its evaluation is to be achieved without using derivatives, that $\ell=ab$. To achieve this we will make use of the following two asymptotic expansions as $z\to 0$

$$\log(1+z)=z+\mathcal{O}(z^2)\quad\text{and}\quad\exp(z)=1+z+\mathcal{O}(z^2).$$

These we freely use.
For $x\to 0$ we have $\log(1+bx^3)=bx^3+\mathcal{O}(x^6)$. Now let

$$y=\left(1+a\log(1+bx^3)\right)^{\frac{1}{x}}.$$

Thus

$$\begin{aligned}\log y&=\frac{1}{x}\log\left(1+a\log(1+bx^3)\right)=\frac{1}{x}\log\left(1+abx^3+\mathcal{O}(x^6)\right)\\&=\frac{1}{x}\left(abx^3+\mathcal{O}(x^6)\right)=abx^2+\mathcal{O}(x^5),\end{aligned}$$

or

$$y=\exp\left(abx^2+\mathcal{O}(x^5)\right)=1+abx^2+\mathcal{O}(x^4).$$

So for the limit ℓ we have

$$\ell=\lim_{x\to 0}\frac{\left(1+abx^2+\mathcal{O}(x^4)\right)-1}{x^2}=\lim_{x\to 0}\left(ab+\mathcal{O}(x^2)\right)=ab,$$

as announced.

Solution 4 by Brian Bradie, Christopher Newport University, Newport News, VA

For $x \to 0$,

$$\begin{aligned}
\ln(1+bx^3) &= bx^3 - \frac{b^2x^6}{2} + O(x^9), \\
a\ln(1+bx^3) &= abx^3 - \frac{ab^2x^6}{2} + O(x^9), \\
\ln(1+a\ln(1+bx^3)) &= abx^3 - \frac{b^2x^6}{2}(a+a^2) + O(x^9), \\
\frac{1}{x}\ln(1+a\ln(1+bx^3)) &= abx^2 - \frac{b^2x^5}{2}(a+a^2) + O(x^8),
\end{aligned}$$

and

$$(1+a\ln(1+bx^3))^{1/x} = e^{abx^2+O(x^5)} = 1 + abx^2 + O(x^4).$$

Thus,

$$\lim_{x\to 0} \frac{(1+a\ln(1+bx^3))^{1/x} - 1}{x^2} = ab.$$

Solution 5 by Michel Bataille, Rouen, France

The required limit is ab.
We start from the following well-known result: for all $x > 0$, we have $\ln(x) \le x - 1$. Applying this to $\frac{1}{x}$ instead of x, we obtain $\ln(x) \ge \frac{x-1}{x}$, hence

$$\frac{x-1}{x} \le \ln(x) \le x - 1. \tag{1}$$

It follows that for $x > -1$

$$\frac{x}{x+1} \le \ln(1+x) \le x. \tag{2}$$

We also deduce from (1) that $\frac{e^x-1}{e^x} \le x \le e^x - 1$ so that $x \le e^x - 1 \le \frac{x}{1-x}$ when $x < 1$. Therefore $e^x - 1 \ \sim \ x$ as $x \to 0$.
First suppose that $a > 0$. Then, for x close to 0, (2) yields

$$1 + \frac{abx^3}{1+bx^3} \le 1 + a\ln(1+bx^3) \le 1 + abx^3,$$

hence

$$\ln\left(1 + \frac{abx^3}{1+bx^3}\right) \le \ln(1+a\ln(1+bx^3)) \le \ln(1+abx^3).$$

Using (2) again, we deduce

$$\frac{abx^3}{1+b(a+1)x^3} \le \ln(1+a\ln(1+bx^3)) \le abx^3$$

and so $\ln(1+a\ln(1+bx^3)) \ \sim \ abx^3$ as $x \to 0$.
If $a < 0$, then similarly $abx^3 \le a\ln(1+bx^3) \le \frac{abx^3}{1+bx^3}$ leads to

$$\frac{abx^3}{1+abx^3} \le \ln(1+a\ln(1+bx^3)) \le \frac{abx^3}{1+bx^3}$$

and $\ln(1+a\ln(1+bx^3)) \sim abx^3$ as $x \to 0$ still holds.
Now, let $f(x) = \frac{(1+a\ln(1+bx^3))^{1/x}-1}{x^2} = \frac{e^{u(x)}-1}{x^2}$ where $u(x) = \frac{\ln(1+a\ln(1+bx^3))}{x}$. Then, from above, for $a \neq 0$ we have $\lim\limits_{x\to 0} u(x) = 0$ with $u(x) \sim abx^2$ as $x \to 0$. Since $e^{u(x)} - 1 \sim u(x)$, we obtain $f(x) \sim \frac{abx^2}{x^2} = ab$ and so $\lim\limits_{x\to 0} f(x) = ab$. This result also holds if $a = 0$ (since then $f(x) = 0$)

Editor's Note: **Albert Natian of Los Angeles Valley College, Valley Glen, California** mentioned the following in his solution.

Although the proscription in the statement of the problem precludes differentiation (in fact, L'Hôpital), there's a neat way to find the limit without the use of L'Hôpital that nonetheless uses differentiation:

Define $f : R^2 \to R$ by setting

$$f(a,b) = \lim_{x\to 0} \frac{\left(1+a\ln\left(1+bx^3\right)\right)^{1/x}-1}{x^2} \qquad \forall (a,b) \in R^2.$$

Then if we can show that $\partial f/\partial a = b$ and $\partial f/\partial b = a$, then (in light of the fact that $f(0,b) = 0$ and $f(a,0) = 0$) it will follow that $f(a,b) = ab$. Bypassing fine points over the validity of sending the derivative operator through the limit operator, we have

$$\frac{\partial f}{\partial a} = \lim_{x\to 0} \frac{1}{x^3}\left(1+a\ln\left(1+bx^3\right)\right)^{1/x} / \left(1+a\ln\left(1+bx^3\right)\right) = \cdots = b,$$

$$\frac{\partial f}{\partial b} = \lim_{x\to 0} \frac{a\left(1+a\ln\left(1+bx^3\right)\right)^{1/x}}{(1+bx^3)\left(1+a\ln\left(1+bx^3\right)\right)} = \cdots = a.$$

Also solved by Charles Burnette, Xavier University of Louisiana, New Orleans, LA; G. C. Greubel of Newport News, VA; Moti Levy, Rehovot, Israel; Albert Natian, Los Angeles Valley College, Valley Glen, CA; Albert Stadler, Herrliberg, Switzerland, and the proposer.

5610: *Proposed by Albert Natian, Los Angeles Valley College, Valley Glen, CA*

Find the smallest positive number x such that the following three quantities a, b and c are all integers;

$$a = a(x) = \sqrt[4]{1+\sqrt{19+\sqrt{\frac{x}{15}}} \;+\; \sqrt[4]{1+\sqrt[3]{\frac{x}{100}}}},$$

$$b = b(x) = \sqrt{\frac{5x}{48}} + \sqrt[3]{26+\frac{x}{20000}},$$

$$c = c(x) = \sqrt[3]{\frac{2x}{25}}.$$

Solution 1 by Albert Stadler, Herrliberg, Switzerland

The last equation implies $x = \dfrac{25}{2}c^3$. We express a and b in terms c and get

$$a = \sqrt[4]{1+\sqrt{19+\sqrt{\frac{5}{16}c^3}} \; + \; \sqrt[4]{1+\frac{c}{2}}},$$

$$b = \frac{5}{4}\sqrt{\frac{5}{6}c^3} + \sqrt[3]{26+\frac{1}{1600}c^3} = \frac{5}{24}\sqrt{30c^3} + \frac{1}{20}\sqrt[3]{26\cdot 8000+5c^3}$$

For $c = 30$ we find $a = 2$ and $b = 191$, and neither a nor b is integral for $1 \le c \le 29$. Therefore the smallest positive number of x with the required property is

$$x = \frac{25}{2}30^3 = 337500.$$

Solution 2 by Brian D. Beasley, Presbyterian College, Clinton, SC

We show that $x = (25/2)30^3 = 337500$.

To make $c(x)$ an integer with $x > 0$, we need x in the form $x_n = (25/2)n^3$, where n is a positive integer. Then $c(x_n) = n$,

$$b(x_n) = \frac{5n}{4}\sqrt{\frac{5n}{6}} + \frac{1}{4}\sqrt[3]{\frac{41600+n^3}{25}},$$

and

$$a(x_n) = \sqrt[4]{1+\sqrt{19+n\sqrt{\frac{5n}{6}}}+\sqrt[4]{1+\frac{n}{2}}}.$$

This allows us to verify that neither $b(x_n)$ nor $a(x_n)$ is an integer for $1 \le n \le 29$. However, $b(x_{30}) = 191$ and $a(x_{30}) = 2$, so the smallest positive value for x is $x_{30} = 337500$ as claimed.

Solution 3 by Dionne Bailey, Elsie Campbell, and Charles Diminnie, Angelo State University, San Angelo, TX

To begin, we note that for $x > 0$, we have

$$\begin{aligned} a\left(x\right) &> \sqrt[4]{1+\sqrt{19}+1} \\ &> 1 \end{aligned} \tag{1}$$

and also, if $0 < x_1 < x_2$, then

$$\begin{aligned} a\left(x_1\right) &= \sqrt[4]{1+\sqrt{19+\sqrt{\frac{x_1}{15}}}+\sqrt[4]{1+\sqrt[3]{\frac{x_1}{100}}}} \\ &< \sqrt[4]{1+\sqrt{19+\sqrt{\frac{x_2}{15}}}+\sqrt[4]{1+\sqrt[3]{\frac{x_2}{100}}}} \\ &= a\left(x_2\right). \end{aligned} \tag{2}$$

If we set $x = 15^3 \times 100$, then,

$$\begin{aligned}
a\left(15^3 \times 100\right) &= \sqrt[4]{1+\sqrt{19+\sqrt{\frac{15^3 \times 100}{15}}}+\sqrt[4]{1+\sqrt[3]{\frac{15^3 \times 100}{100}}}} \\
&= \sqrt[4]{1+\sqrt{19+150}+\sqrt[4]{1+15}} \\
&= \sqrt[4]{1+\sqrt{169}+\sqrt[4]{16}} \\
&= \sqrt[4]{1+13+2} \\
&= 2. \qquad (3)
\end{aligned}$$

By (1), (2), and (3), we conclude that if $0 < x < 15^3 \times 100$, then

$$1 < a(x) < a\left(15^3 \times 100\right) = 2$$

and hence, $a(x)$ is not an integer.

Further, for $x = 15^3 \times 100$, we also obtain

$$\begin{aligned}
b\left(15^3 \times 100\right) &= \sqrt{\frac{5\left(15^3 \times 100\right)}{48}} + \sqrt[3]{26+\frac{15^3 \times 100}{20000}} \\
&= \sqrt{\frac{5^6 \cdot 9}{4}} + \sqrt[3]{26+\frac{135}{8}} \\
&= \frac{5^3 \cdot 3}{2} + \sqrt[3]{\frac{343}{8}} \\
&= \frac{375}{2} + \frac{7}{2} \\
&= 191
\end{aligned}$$

and

$$c\left(15^3 \times 100\right) = \sqrt[3]{\frac{15^3 \times 200}{25}} = \sqrt[3]{8 \cdot 15^3} = 30.$$

It follows that for $x = 15^3 \times 100$, a, b, and c are integers, while for $0 < x < 15^3 \times 100$, $a(x)$ is not an integer. Therefore, $x = 15^3 \times 100$ is the smallest positive number for which a, b, and c are all integers.

Also solved by Kee-Wai Lau, Hong Kong, China, and the proposer.

5611: *Proposed by José Luis Díaz-Barrero, Barcelona Tech, Barcelona, Spain*

Let $n \geq 1$ be an integer number. Prove that the following inequality holds:

$$\sum_{k=1}^{n} \sqrt{\frac{k \tan^{-1}(k)}{\tan^{-1}(k) + \tan^{-1}(n-k)}} \leq \frac{(n+1)\sqrt{n}}{2}.$$

(Here, $\tan^{-1}(x)$ represents $\arctan(x)$ for all real x.)

Solution 1 by Moti Levy, Rehovot, Israel

Lemma: Let $(a_k)_{k\geq 0}^n$ be a sequence of numbers such that $a_k + a_{n-k} \neq 0$ and $a_0 = 0$, then

$$S_n := \sum_{k=1}^{n} \frac{a_k}{a_k + a_{n-k}} = \frac{n+1}{2}.$$

Proof:
Suppose n is even. Then $n = 2m$ and

$$S_{2m} = 1 + \left(\sum_{k=1}^{m-1} \frac{a_k}{a_k + a_{n-k}}\right) + \left(\sum_{k=m+1}^{2m} \frac{a_k}{a_k + a_{n-k}}\right) + \frac{a_m}{a_m + a_{2m-m}} = 1 + (m-1) + \frac{1}{2} = m + 1.$$

Suppose n is odd. Then $n = 2m - 1$ and

$$S_{2m-1} = 1 + \left(\sum_{k=1}^{m-1} \frac{a_k}{a_k + a_{n-k}}\right) + \left(\sum_{k=m}^{2m-1} \frac{a_k}{a_k + a_{n-k}}\right) = 1 + (m-1) = m.$$

By Cauchy-Schwarz inequality,

$$\sum_{k=1}^{n} \sqrt{k} \sqrt{\frac{a_k}{a_k + a_{n-k}}} \leq \left(\sum_{k=1}^{n} k\right)^{\frac{1}{2}} \left(\sum_{k=1}^{n} \frac{a_k}{a_k + a_{n-k}}\right)^{\frac{1}{2}}.$$

Applying the lemma,

$$\sum_{k=1}^{n} \sqrt{k} \sqrt{\frac{a_k}{a_k + a_{n-k}}} \leq \sqrt{\frac{n(n+1)}{2}} \sqrt{\frac{n+1}{2}} = \frac{\sqrt{n}\,(n+1)}{2}.$$

Now set $a_k = \arctan(k)$ and the problem is solved.

Remark: The inequality has nothing to do with the properties of the arctan function (except for $\arctan(0) = 0$). The use of the arctan function here is tricky and is meant to send the potential solver off-track to the realms of inverse trigonometric functions. Another example of this trick is to compose an inequality which is valid for any convex function but to state it for the Gamma function $\Gamma(x)$ (which is convex function for $x > 0$).

Solution 2 by Albert Stadler, Herrliberg, Switzerland

We have

$$\sum_{k=1}^{n} \sqrt{\frac{k \arctan k}{\arctan k + \arctan(n-k)}} = \sum_{k=0}^{n} \sqrt{\frac{k \arctan k}{\arctan k + \arctan(n-k)}} =$$

$$= \frac{1}{2} \sum_{k=0}^{n} \sqrt{\frac{k \arctan k}{\arctan k + \arctan(n-k)}} + \frac{1}{2} \sum_{k=0}^{n} \sqrt{\frac{(n-k) \arctan(n-k)}{\arctan k + \arctan(n-k)}} \leq$$

$$\leq \frac{n+1}{2} \max_{0\leq k\leq n} \left(\sqrt{\frac{k \arctan k}{\arctan k + \arctan(n-k)}} + \sqrt{\frac{(n-k) \arctan(n-k)}{\arctan k + \arctan(n-k)}} \right. =\leq$$

$$\leq n + 12 \max_{0\leq k\leq n, 0\leq a\leq 1} \left(\sqrt{ak} + \sqrt{(1-a)(n-k)}\right).$$

It remains to prove that

$$\sqrt{ak}+\sqrt{(1-a)(n-k)}\le\sqrt{n}$$

if $0\le k\le n,\ 0\le 1)$. Consider the left-hand side as a function of a, The function gets maximal either at $a=0$ or at $a=1$ or at the stationary point:$\frac{1}{2\sqrt{a}}\sqrt{k}-\frac{1}{2}\sqrt{1-a}\sqrt{n-k)}=0$. So $a\in\{0,1,k/n\}$. For the stationary point have $\sqrt{ak}+\sqrt{(1-a)(n-k)}\le\frac{1}{\sqrt{n}}+\frac{n-k}{\sqrt{n}}=\sqrt{n}$. as claimed.

This completes the proof.

Solution 3 by Kee-Wai Lau, Hong Kong, China

By the Cauchy-Schwarz inequality, we have

$$\sum_{k=1}^{n}\sqrt{\frac{k\tan^{-1}(k)}{\tan^{-1}(k)+\tan^{-1}(n-k)}}\le\sqrt{\left(\sum_{k=1}^{n}k\right)\left(\sum_{k=1}^{n}\frac{k\tan^{-1}(k)}{\tan^{-1}(k)+\tan^{-1}(n-k)}\right)}.$$

It is well known that $\sum_{k=1}^{n}k=\frac{n(n+1)}{2}$. By putting $k=n-j$, we have

$$\sum_{k=0}^{n}\frac{\tan^{-1}(k)}{\tan^{-1}(k)+\tan^{-1}(n-k)}=\sum_{j=0}^{n}\frac{\tan^{-1}(n-j)}{\tan^{-1}(n-j)+\tan^{-1}(j)}=\sum_{k=0}^{n}\frac{\tan^{-1}(n-k)}{\tan^{-1}(n-k)+\tan^{-1}(k)}.$$

Hence

$$\begin{aligned}\sum_{k=0}^{n}\frac{\tan^{-1}(k)}{\tan^{-1}(k)+\tan^{-1}(n-k)} &= \frac{1}{2}\left(\sum_{k=0}^{n}\frac{\tan^{-1}(k)}{\tan^{-1}(k)+\tan^{-1}(n-k)}+\sum_{k=0}^{n}\frac{\tan^{-1}(n-k)}{\tan^{-1}(n-k)+\tan^{-1}(k)}\right)\\ &= \frac{1}{2}\sum_{k=0}^{n}1=\frac{n+1}{2}.\end{aligned}$$

The inequality of the problem follows.

Solution 4 by Michel Bataille, Rouen, France

If $n=1$, equality holds, so we suppose that $n\ge 2$ in what follows. We observe that the last term in the left-hand sum is $\sqrt{n}$. Setting $u_m=\tan^{-1}(m)$, this reduces the problem to showing that

$$\sum_{k=1}^{n-1}\sqrt{\frac{ku_k}{u_k+u_{n-k}}}\le\frac{(n-1)\sqrt{n}}{2} \tag{1}$$

Let S_n denote the sum on the left-hand side of (1). The change of index $k\to n-k$ yields

$$S_n=\sum_{k=1}^{n-1}\sqrt{\frac{(n-k)u_{n-k}}{u_{n-k}+u_k}}$$

so that

$$2S_n=\sum_{k=1}^{n-1}\frac{\sqrt{ku_k}+\sqrt{(n-k)u_{n-k}}}{\sqrt{u_k+u_{n-k}}}. \tag{2}$$

Now, for $k \in \{1, 2, \ldots, n-1\}$ and $a, b > 0$, we have $\sqrt{ka} + \sqrt{(n-k)b} \le \sqrt{n(a+b)}$. Indeed, this inequality is equivalent to

$$ka + (n-k)b + 2\sqrt{(ka)(n-k)b} \le na + nb,$$

that is, to $2\sqrt{(kb)(n-k)a} \le (n-k)a + kb$, which clearly holds (since $(\sqrt{x} - \sqrt{y})^2 \ge 0$ for $x, y \ge 0$).

A consequence is that for $k = 1, 2, \ldots, n-1$, we have $\frac{\sqrt{ku_k} + \sqrt{(n-k)u_{n-k}}}{\sqrt{u_k + u_{n-k}}} \le \sqrt{n}$, hence, returning to (2), we see that $2S_n \le (n-1)\sqrt{n}$ and so (1) holds.

Also solved by Toyesh Prakash Sharma (student), St. C. F. Andrews School, Agra, India, and the proposer.

5612: *Proposed by Ovidiu Furdui and Alina Sîntămărian, Technical University of Cluj-Napoca, Cluj-Napoca, Romania*

Let $a, b \in R$ with $a - b = \frac{\gamma}{2}$. Calculate

$$\lim_{n\to\infty} \left(2\mathrm{e}^{1+\frac{1}{2}+\cdots+\frac{1}{n}-a} - \sqrt{n}\mathrm{e}^{1+\frac{1}{3}+\cdots+\frac{1}{2n-1}-b} \right).$$

Solution 1 by Seán M. Stewart, Bomaderry, NSW, Australia

The nth *harmonic number* is defined by

$$H_n = \sum_{k=1}^{n} \frac{1}{k} = 1 + \frac{1}{2} + \cdots + \frac{1}{n}.$$

In terms of harmonic numbers, we also have

$$\begin{aligned}\sum_{k=1}^{n} \frac{1}{2k-1} &= 1 + \frac{1}{3} + \frac{1}{5} + \cdots + \frac{1}{2n-1} \\ &= \left(1 + \frac{1}{2} + \cdots \frac{1}{2n}\right) - \left(\frac{1}{2} + \frac{1}{4} + \cdots + \frac{1}{2n}\right) \\ &= H_{2n} - \frac{1}{2}H_n.\end{aligned}$$

Let

$$L_n = 2e^{1+\frac{1}{2}+\cdots+\frac{1}{n}-a} - \sqrt{n}e^{1+\frac{1}{3}+\cdots+\frac{1}{2n-1}-b} = 2e^{H_n - a} - \sqrt{n}e^{H_{2n}-\frac{1}{2}H_n - b}. \qquad (1)$$

From the well-known asymptotic expansion for the harmonic numbers when n is large

$$H_n = \log(n) + \gamma + \frac{1}{2n} - \frac{1}{12n^2} + \mathcal{O}\left(\frac{1}{n^4}\right).$$

Here γ is the Euler-Mascheroni constant. If in the asymptotic expansion for H_n, n is replaced with $2n$ we have

$$H_{2n} = \log(2n) + \gamma + \frac{1}{4n} - \frac{1}{48n^2} + \mathcal{O}\left(\frac{1}{n^4}\right).$$

Thus

$$H_{2n} - \frac{1}{2}H_n = \log(2\sqrt{n}) + \frac{\gamma}{2} + \frac{1}{48n^2} + \mathcal{O}\left(\frac{1}{n^4}\right).$$

We now find asymptotic expansions for the each of the exponential terms appearing in L_n when n is large. In doing so we will make use of the Maclaurin series expansion for the exponential function. Here, as $x \to 0$ we have

$$e^x = 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \mathcal{O}(x^4).$$

For the first exponential term in (1)

$$\begin{aligned}
e^{H_n - a} &= \exp\left[\log(n) + \gamma - a + \frac{1}{2n} - \frac{1}{12n^2} + \mathcal{O}\left(\frac{1}{n^4}\right)\right] \\
&= ne^{\gamma - a}\exp\left[\frac{1}{2n} - 112n^2 + \mathcal{O}\left(\frac{1}{n^4}\right)\right] \\
&= ne^{\gamma - a}\left[1 + \left(\frac{1}{2n} - \frac{1}{12n^2}\right) + \frac{1}{2}\left(\frac{1}{2n} - \frac{1}{12n^2}\right)^2\right. \\
&\qquad \left. + \frac{1}{6}\left(\frac{1}{2n} - \frac{1}{12n^2}\right)^3 + \mathcal{O}\left(\frac{1}{n^4}\right)\right] \\
&= ne^{\gamma - a}\left[1 + \frac{1}{2n} + \frac{1}{24n^2} - \frac{1}{48n^3} + \mathcal{O}\left(\frac{1}{n^4}\right)\right].
\end{aligned}$$

And for the second exponential term in (1)

$$\begin{aligned}
e^{H_{2n} - \frac{1}{2}H_n - b} &= \exp\left[\log(2\sqrt{n}) + \frac{\gamma}{2} - b + \frac{1}{48n^2} + \mathcal{O}\left(\frac{1}{n^4}\right)\right] \\
&= 2\sqrt{n}e^{\frac{\gamma}{2} - b}\exp\left[\frac{1}{48n^2} + \mathcal{O}\left(\frac{1}{n^4}\right)\right] \\
&= 2\sqrt{n}e^{\frac{\gamma}{2} - b}\left[1 + \frac{1}{48n^2} + \mathcal{O}\left(\frac{1}{n^4}\right)\right]. \qquad (2)
\end{aligned}$$

Given $a - b = \frac{\gamma}{2}$ where $a, b \in R$, we can rewrite (2) as

$$e^{H_{2n} - \frac{1}{2}H_n - b} = 2\sqrt{n}e^{\gamma - a}\left[1 + \frac{1}{48n^2} + \mathcal{O}\left(\frac{1}{n^4}\right)\right].$$

Returning to (1) we have

$$\begin{aligned}
L_n &= 2ne^{\gamma - a}\left[1 + \frac{1}{2n} + \frac{1}{24n^2} - \frac{1}{48n^3} + \mathcal{O}\left(\frac{1}{n^4}\right)\right] \\
&\quad - 2ne^{\gamma - a}\left[1 + \frac{1}{48n^2} + \mathcal{O}\left(\frac{1}{n^4}\right)\right] \\
&= e^{\gamma - a}\left[1 + \frac{1}{24n} - \frac{1}{24n^2} + \mathcal{O}\left(\frac{1}{n^3}\right)\right].
\end{aligned}$$

So for the limit we are required to find, we have

$$\lim_{n\to\infty} L_n = e^{\gamma-a}\lim_{n\to\infty}\left[1+\frac{1}{24n}-\frac{1}{24n^2}+\mathcal{O}\left(\frac{1}{n^3}\right)\right] = e^{\gamma-a}.$$

Solution 2 by Albert Stadler, Herrliberg, Switzerland

It is well known (https://en.wikipedia.org/wiki/Harmonic number) that the asymptotic expansion of harmonic numbers is given by

$$H_n = \ln n + \gamma + \frac{1}{2n} + O\left(\frac{1}{n^2}\right),$$

as $n \to$. Thus,

$$2e^{1+\frac{1}{2}+\cdots+\frac{1}{n}-a} - \sqrt{n}e^{1+\frac{1}{3}\cdots+\frac{1}{2n-1}-b} = 2e^{H_n-a} - \sqrt{n}e^{H_{2n}-\frac{1}{2}H_n-b} =$$

$$= \quad 2e^{\ln n+\gamma+\frac{1}{2n}+O\left(\frac{1}{n^2}\right)-a} - \sqrt{n}e^{\ln(2n)+\gamma+\frac{1}{4n}+O\left(\frac{1}{n^2}\right)-\frac{1}{2}\ln n-\frac{\gamma}{2}-\frac{1}{4n}+O\left(\frac{1}{n^2}\right)-b} =$$

$$= \quad 2ne^{\gamma+\frac{1}{2n}-a} - 2ne^{\frac{\gamma}{2}-b} + O\left(\frac{1}{n}\right) = 2ne^{\gamma-a} + e^{\gamma-a} - 2ne^{\frac{\gamma}{2}-b} + O\left(\frac{1}{n}\right) =$$

$$= \quad 2ne^{\frac{\gamma}{2}-b}\left(e^{\gamma-a-\frac{\gamma}{2}+b}-1\right) + e^{\gamma-a} + O\left(\frac{1}{n}\right) = e^{\gamma-a} + O\left(\frac{1}{n}\right) \to e^{\gamma-a},$$

as $n \to \infty$.

Solution 3 by Moti Levy, Rehovot, Israel

$$\begin{aligned}&\lim_{n\to\infty}\left(2e^{1+\frac{1}{2}+\cdots+\frac{1}{n}-a} - \sqrt{n}e^{1+\frac{1}{3}+\cdots+\frac{1}{2n-1}-b}\right)\\ &= \lim_{n\to\infty}\left(2e^{H_n-a} - \sqrt{n}e^{H_{2n}-\frac{1}{2}H_n-b}\right).\end{aligned}$$

Let $S_n := 2e^{H_n-a} - \sqrt{n}e^{H_{2n}-\frac{1}{2}H_n-b}$.
Then

$$\begin{aligned}S_n &= 2e^{(H_n-\ln n)+\ln n-a} - \sqrt{n}e^{H_{2n}-\ln 2n+\ln 2n-\frac{1}{2}H_n+\frac{1}{2}\ln(n)-\frac{1}{2}\ln(n)-b} \qquad (4)\\ &= 2e^{(H_n-\ln n)}e^{\ln n-a} - \sqrt{n}e^{H_{2n}-\ln 2n}(2n)\,e^{-\frac{1}{2}(H_n-\ln(n))}e^{-\frac{1}{2}\ln(n)-b}\\ &= 2e^{(H_n-\ln n)}ne^{-a} - e^{H_{2n}-\ln 2n}(2n)\,e^{-\frac{1}{2}(H_n-\ln(n))}e^{-b}\end{aligned}$$

The following two asymptotic expansions are known:

$$H_n - \ln n = \gamma + \frac{1}{2n} + O\left(\frac{1}{n^2}\right). \qquad (1)$$

$$e^{\frac{1}{n}} = 1 + \frac{1}{n} + O\left(\frac{1}{n^2}\right). \qquad (2)$$

(1) and (2) imply

$$e^{(H_{2n}-\ln 2n)} = e^{\gamma+\frac{1}{4n}+O\left(\frac{1}{n^2}\right)} = e^{\gamma}\left(1+\frac{1}{4n}\right)+O\left(\frac{1}{n^2}\right), \tag{3}$$

$$e^{-\frac{1}{2}(H_n-\ln(n))} = e^{-\frac{1}{2}\gamma-\frac{1}{4n}+O\left(\frac{1}{n^2}\right)} = e^{-\frac{1}{2}\gamma}\left(1-\frac{1}{4n}\right)+O\left(\frac{1}{n^2}\right). \tag{5}$$

Plugging (3) and (5) into (4) gives

$$\begin{aligned}
S_n &= 2ne^{\gamma-a}\left(1+\frac{1}{2n}+O\left(\frac{1}{n^2}\right)\right) - e^{\gamma}\left(1+\frac{1}{4n}+O\left(\frac{1}{n^2}\right)\right)(2n)\,e^{-\frac{\gamma}{2}-b}\left(1-\frac{1}{4n}+O\left(\frac{1}{n^2}\right)\right)\\
&= 2ne^{\gamma-a}\left(1+\frac{1}{2n}\right) - e^{\gamma}\left(1+\frac{1}{4n}\right)(2n)\,e^{-\frac{\gamma}{2}-b}\left(1-\frac{1}{4n}\right)+O\left(\frac{1}{n}\right)\\
&= e^{\gamma-a}+2n\left(e^{\gamma-a}-e^{-b+\frac{1}{2}\gamma}\right)+\frac{1}{8n}e^{-b+\frac{1}{2}\gamma}+O\left(\frac{1}{n}\right)\\
&= e^{\gamma-a}+O\left(\frac{1}{n}\right).
\end{aligned}$$

We conclude that

$$\lim_{n\to\infty} S_n = e^{\gamma-a} = e^{a-2b}.$$

Solution 4 by Brian Bradie, Christopher Newport University, Newport News, VA

For $n \to \infty$,

$$1+\frac{1}{2}+\cdots+\frac{1}{n} = H_n = \ln n+\gamma+\frac{1}{2n}+O\left(\frac{1}{n^2}\right)$$

and

$$1+\frac{1}{3}+\cdots+\frac{1}{2n-1} = H_{2n}-\frac{1}{2}H_n = \ln 2+\frac{1}{2}\ln n+\frac{1}{2}\gamma+O\left(\frac{1}{n^2}\right).$$

Thus,

$$2e^{H_n-a} = 2ne^{\gamma-a}e^{1/2n+O(1/n^2)} = 2ne^{\gamma-a}\left(1+\frac{1}{2n}+O\left(\frac{1}{n^2}\right)\right)$$

and

$$\sqrt{n}e^{H_{2n}-\frac{1}{2}H_n-b} = 2ne^{\gamma/2-b}e^{O(1/n^2)} = 2ne^{\gamma/2-b}\left(1+O\left(\frac{1}{n^2}\right)\right).$$

With $a-b=\gamma/2$, it follows that $\gamma-a=\gamma/2-b$, so

$$2e^{H_n-a}-\sqrt{n}e^{H_{2n}-\frac{1}{2}H_n-b} = 2ne^{\gamma-a}\left(\frac{1}{2n}+O\left(\frac{1}{n^2}\right)\right) = e^{\gamma-a}+O\left(\frac{1}{n}\right).$$

Finally,

$$\lim_{n\to\infty}\left(2e^{1+\frac{1}{2}+\cdots+\frac{1}{n}-a}-\sqrt{n}e^{1+\frac{1}{3}+\cdots+\frac{1}{2n-1}-b}\right) = e^{\gamma-a}.$$

Solution 5 by Michel Bataille, Rouen, France

Let $U_n = 2e^{1+\frac{1}{2}+\cdots+\frac{1}{n}-a}-\sqrt{n}e^{1+\frac{1}{3}+\cdots+\frac{1}{2n-1}-b}$. We show that $\lim\limits_{n\to\infty} U_n = e^{\gamma-a}(=e^{\frac{\gamma}{2}-b})$.

Since

$$1+\frac{1}{3}+\cdots+\frac{1}{2n-1} = H_{2n}-\left(\frac{1}{2}+\frac{1}{4}+\cdots+\frac{1}{2n}\right) = H_{2n}-\frac{H_n}{2}$$

we have $U_n = V_n - W_n$ with $\quad V_n = 2e^{H_n - a}, \quad W_n = \sqrt{n}e^{H_{2n} - \frac{H_n}{2} - b}$.
We shall use the following well-known asymptotic expansion as $n \to \infty$:

$$H_n = \ln(n) + \gamma + \frac{1}{2n} + \frac{\varepsilon_n}{n}$$

where $\lim\limits_{n\to\infty} \varepsilon_n = 0$ [see for example Vito Lampret, *The Euler-Maclaurin and Taylor Formulas: Twin, Elementary Derivation*, Math. Magazine, 74, No 2, April 2001, pp. 109-122, esp. p. 119.]
We obtain $H_n - a = \ln(n) + \gamma - a + \frac{1}{2n} + \frac{\varepsilon_n}{n}$ and so

$$V_n = 2ne^{\gamma - a}e^{1/2n}e^{\varepsilon_n/n}.$$

Similarly, $H_{2n} - \frac{H_n}{2} - b = \ln(2n) + \gamma + \frac{1}{4n} - \frac{\ln(n)}{2} - \frac{\gamma}{2} - b - \frac{1}{4n} + \frac{\varepsilon_{2n} - \varepsilon_n}{2n}$ and so

$$W_n = 2ne^{\gamma/2 - b}e^{\frac{\varepsilon_{2n} - \varepsilon_n}{2n}}.$$

Setting $k = \gamma - a = \frac{\gamma}{2} - b$, we deduce that

$$U_n = V_n - W_n = 2ne^k\left(e^{\frac{1}{2n} + \frac{\varepsilon_n}{n}} - e^{\frac{\varepsilon_{2n} - \varepsilon_n}{2n}}\right).$$

Since $e^x = 1 + x + o(x)$ as $x \to 0$, we obtain

$$U_n = 2ne^k\left(\frac{1}{2n} + \frac{\varepsilon_n}{n} - \frac{\varepsilon_{2n} - \varepsilon_n}{2n} + o(1/n)\right) \sim 2ne^k \cdot \frac{1}{2n} = e^k.$$

The announced result follows.

Solution 6 by Albert Natian, Los Angeles Valley College, Valley Glen, CA

Answer: $e^{\gamma - a}$.
Computation. Set

$$Q_n := 2e^{1 + \frac{1}{2} + \cdots + \frac{1}{n} - a} - \sqrt{n}e^{1 + \frac{1}{3} + \cdots + \frac{1}{2n-1} - b}.$$

We have

$$H_{2n} = \sum_{k=1}^{2n} \frac{1}{k} = \sum_{k=1}^{n} \frac{1}{2k-1} + \frac{1}{2}\sum_{k=1}^{n} \frac{1}{k} = \sum_{k=1}^{n} \frac{1}{2k-1} + \frac{1}{2}H_n,$$

$$\sum_{k=1}^{n} \frac{1}{2k-1} = H_{2n} - \frac{1}{2}H_n,$$

$$Q_n = 2e^{H_n - a} - \sqrt{n}e^{H_{2n} - \frac{1}{2}H_n - b}.$$

Define $\gamma_n = H_n - \ln n$ so that

$$\gamma = \lim_{n\to\infty} \gamma_n,$$

$$H_n = \gamma_n + \ln n,$$

$$H_{2n} - \frac{1}{2}H_n = \gamma_{2n} + \ln(2n) - \frac{1}{2}(\gamma_n + \ln n) = \gamma_{2n} - \frac{1}{2}\gamma_n + \ln\left(2\sqrt{n}\right),$$

$$e^{H_{2n} - \frac{1}{2}H_n - b} = e^{\gamma_{2n} - \frac{1}{2}\gamma_n + \ln\left(2\sqrt{n}\right) - b} = e^{\ln\left(2\sqrt{n}\right)} \cdot e^{\gamma_{2n} - \frac{1}{2}\gamma_n - b} = 2\sqrt{n}e^{\gamma_{2n} - \frac{1}{2}\gamma_n - b},$$

$$\sqrt{n}e^{H_{2n} - \frac{1}{2}H_n - b} = 2ne^{\gamma_{2n} - \frac{1}{2}\gamma_n - b},$$

$$e^{H_n - a} = e^{\gamma_n + \ln n - a} = e^{\ln n} \cdot e^{\gamma_n - a} = n e^{\gamma_n - a},$$

$$Q_n = 2n e^{\gamma_n - a} - 2n e^{\gamma_{2n} - \frac{1}{2}\gamma_n - b},$$

$$Q_n = 2n e^{\gamma_{2n} - \frac{1}{2}\gamma_n - b} \left(e^{\frac{3}{2}\gamma_n - \gamma_{2n} - a + b} - 1 \right),$$

$$Q_n = 2n e^{\gamma_{2n} - \frac{1}{2}\gamma_n - b} \left(e^{\frac{3}{2}\gamma_n - \gamma_{2n} - \frac{1}{2}\gamma} - 1 \right),$$

$$Q_n = 2n e^{\gamma_{2n} - \frac{1}{2}\gamma_n - b} \left(e^{\frac{3}{2}(\gamma_n - \gamma) - (\gamma_{2n} - \gamma)} - 1 \right).$$

By Young's inequality

$$\frac{1}{2(n+1)} < \gamma_n - \gamma < \frac{1}{2n},$$

we have

$$\frac{3}{4(n+1)} < \frac{3}{2}(\gamma_n - \gamma) < \frac{3}{4n},$$

$$-\frac{1}{4n} < -(\gamma_{2n} - \gamma) < -\frac{1}{2(2n+1)},$$

$$\frac{3}{4(n+1)} - \frac{1}{4n} < \frac{3}{2}(\gamma_n - \gamma) - (\gamma_{2n} - \gamma) < \frac{3}{4n} - \frac{1}{2(2n+1)},$$

$$\frac{2n-1}{4n(n+1)} < \frac{3}{2}(\gamma_n - \gamma) - (\gamma_{2n} - \gamma) < \frac{4n+3}{4n(2n+1)},$$

$$e^{\frac{2n-1}{4n(n+1)}} - 1 < e^{\frac{3}{2}(\gamma_n - \gamma) - (\gamma_{2n} - \gamma)} - 1 < e^{\frac{4n+3}{4n(2n+1)}} - 1,$$

$$2n\left(e^{\frac{2n-1}{4n(n+1)}} - 1\right) < 2n\left(e^{\frac{3}{2}(\gamma_n - \gamma) - (\gamma_{2n} - \gamma)} - 1\right) < 2n\left(e^{\frac{4n+3}{4n(2n+1)}} - 1\right).$$

Since

$$\lim_{n\to\infty} 2n\left(e^{\frac{2n-1}{4n(n+1)}} - 1\right) = 1 \qquad \text{and} \qquad \lim_{n\to\infty} 2n\left(e^{\frac{4n+3}{4n(2n+1)}} - 1\right) = 1,$$

then, by Squeeze Theorem,

$$\lim_{n\to\infty} 2n\left(e^{\frac{3}{2}(\gamma_n - \gamma) - (\gamma_{2n} - \gamma)} - 1\right) = 1.$$

Therefore

$$\lim_{n\to\infty} Q_n = \lim_{n\to\infty} 2n e^{\gamma_{2n} - \frac{1}{2}\gamma_n - b} \left(e^{\frac{3}{2}(\gamma_n - \gamma) - (\gamma_{2n} - \gamma)} - 1 \right),$$

$$\lim_{n\to\infty} Q_n = \lim_{n\to\infty} e^{\gamma_{2n} - \frac{1}{2}\gamma_n - b} \cdot \lim_{n\to\infty} 2n \left(e^{\frac{3}{2}(\gamma_n - \gamma) - (\gamma_{2n} - \gamma)} - 1 \right),$$

$$\lim_{n\to\infty} Q_n = e^{\gamma - \gamma/2 - b} = e^{\gamma/2 - b} = e^{\gamma - a}.$$

Also solved by Ulrich Abel, Technische Hochschule Mittelhessen, Germany; Charles Burnette, Xavier University of Louisiana, New Orleans, LA; Kee-Wai Lau, Hong Kong, China; Toyesh Prakash Sharma (student), St. C. F. Andrews School, Agra, India, and the proposer.

Mea Culpa

For various procedural errors made on my part and by others, some solutions and comments were inadvertently omitted from recent issues of this column. Specifically,

Ulrich Abel of Technische Hochschule Mittelhessen, Germany should have been credited with having solved 5575 and 5598.

Albert Natian of Los Angeles Valley College in Valley Green, California should have been credited with having solved 5606.

G. C. Greubel of Newport News, VA and **Paul M. Harms** of North Newton, KS should have been acknowledged for solving 5601.

Michel Bataille of Rouen, France should be credited with having solved 5601, 5602, 5604, 5605, and 5606.

David Stone and John Hawkins, both of Georgia Southern University, in Statesboro should have been credited with having solved 5595.