Problems

Ted Eisenberg, Section Editor

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This section of the Journal offers readers an opportunity to exchange interesting mathematical problems and solutions. Please send them to Ted Eisenberg, Department of Mathematics, Ben-Gurion University, Beer-Sheva, Israel or fax to: 972-86-477-648. Questions concerning proposals and/or solutions can be sent e-mail to <eisenbt@013.net>. Solutions to previously stated problems can be seen at <http://www.ssma.org/publications>.

Solutions to the problems stated in this issue should be posted before
May 15, 2021

- **5631:** *Proposed by Kenneth Korbin, New York, NY*

  Trapezoid $ABCD$ with integer length sides is inscribed in a circle with diameter P^3 where P is a prime number great than 14. Base $\overline{AB}$ =7P^2. Express the the lengths of the other three sides in terms P.

- **5632:** *Proposed by Toyesh Prakash Sharma (Student), St. C.F. Andrews School, Agra, India*

  Show that

$$\frac{1-\sqrt{2}\sin(89/2)^\circ}{1+\sqrt{2}\sin(89/2)^\circ} < \tan^2\left(\frac{1}{2}\right)^\circ \cdot \tan^2\left(\frac{2}{2}\right)^\circ \cdot \tan^2\left(\frac{3}{2}\right)^\circ \cdots \tan^2\left(\frac{89}{2}\right)^\circ.$$

- **5633:** *Proposed by Goran Conar, Varaž din, Croatia*

  Calculate:

$$\lim_{n\to+\infty} \sqrt[n]{\sinh n + \tanh n}.$$

- **5634:** *Proposed by Daniel Sitaru, National Economic College, "Theodor Costescu" Drobeta Turna-Severin, Romania*

  a) Find all real numbers x such that $\tan 3x = \tan 2x + \tan x$.

  b) Find:

$$\Omega = \int \tan\left(x+\frac{\pi}{3}\right)\tan 3x \tan\left(2x-\frac{\pi}{3}\right)dx.$$

- **5635:** *Proposed by José Luis Díaz-Barrero, Barcelona Tech, Barcelona, Spain*

  Assuming that $\left(1-3x+3x^2\right)^8\left(1-x-x^2+x^3+3x^4+3x^5\right)^{17} =$

$a_0 + a_1x + a_2x^2 + \ldots + a_{100}x^{100} + a_{101}x^{101}$ calculate the value of $a_1 + a_2 + a_3 + \ldots + a_{49} + a_{50}$ and $a_0 + a_4 + a_8 + \ldots + a_{100}$.

- **5636:** *Proposed by Ovidiu Furdui Technical University of Cluj-Napoca, Cluj-Napoca, Romania*

Prove that
$$\sum_{n=0}^{\infty} n!\left(e - 1 - \frac{1}{1!} - \frac{1}{2!} - \cdots - \frac{1}{n!}\right)^2 = e\sum_{n=1}^{\infty} \frac{1}{n \cdot n!}.$$

Solutions

- **5613:** *Proposed by Kenneth Korbin, New York, NY*

Given the equations:
$$\begin{cases} \sqrt{3x^2+6x+1} = y + \sqrt{3y^2-3} \\ \text{and} \\ ax^2 + by^2 + cxy + dx + ey + f = 0. \end{cases}$$

Find integers (a, b, c, d, e, f) so that infinitely many pairs of positive integers (x, y) satisfy both equations.

Solution 1 by Albert Stadler, Herrliberg, Switzerland

We have
$$\sqrt{3x^2+6x+1} - \sqrt{3y^2-3} = y$$
and obtain
$$3x^2 + 6x + 1 + 3y^2 - 3 - y^2 = 2\sqrt{3x^2+6x+1}\sqrt{3y^2-3}.$$
We square again and obtain:
$$0 = (3x^2 + 6x + 2y^2 - 2)^2 - 4(3x^2 + 6x + 1)(3y^2 - 3) =$$
$$= (3x^2 + 2y^2 - 6xy + 6x - 6y + 4)(3x^2 + 2y^2 + 6xy + 6x + 6y + 4).$$
So either
$$(i)\ \ 3x^2 + 2y^2 - 6xy + 6x - 6y + 4 = 0$$
or
$$(ii)\ \ 3x^2 + 2y^2 + 6xy + 6x + 6y + 4 = 0.$$

Obviously (ii) has no solutions in positive integers x and y. So we must have $(a, b, c, d, e, f) = (3, 2, -6, 6, -6, 4)$, and it remains to prove that there are infinitely many pairs of positive integers (x, y) that satisfy (i). Case (i) is equivalent to
$$y^2 - 3(x - y + 1)^2 = 1.$$

The Pell equation $u^2 - 3v^2 = 1$ has infinitely many solutions in positive integers (u, v). They are given by $(u, v) = (u_j, v_j), j \geq 2$ where

$$u_j = \frac{(2+\sqrt{3})^j + (2-\sqrt{3})^j}{2}, \quad v_j = \frac{(2+\sqrt{3})^j - (2-\sqrt{3})^j}{2\sqrt{3}}.$$

If $x_j = u_j + v_j - 1, y = u_j$, then $(x, y) = (x_j, y_j)$ is a solution of $y^2 - 3(x - y + 1)^2 = 1$. Finally,

$$\sqrt{3x^2+6x+1} - y - \sqrt{3y^2-3} = \sqrt{3(u_j+v_y-1)^2 + 6(u_j+v_j-1)+1} - u_j - \sqrt{3u_j^2-3} =$$

$$\sqrt{3u_j^2+3v_j^2+6u_jv_j-2} - u_j - 3v_j = \sqrt{u_j^2+9v_j^2+6u_jv_j} - u_j - 3v_j =$$
$$= \sqrt{(u_j+3v_j)^2} - u_j - 3v_j = 0.$$

Solution 2 by Trey Smith, Angelo State University, San Angelo, TX

Noting that $3x^2 + 6x + 1 = 3(x+1)^2 - 2$, and letting $z = x + 1$ and $q = \sqrt{3z^2 - 2}$, we have the associated equation

$$q^2 - 3z^2 = -2.$$

Letting $p = \sqrt{3y^2 - 3}$ yields the equation

$$p^2 - 3y^2 = -3.$$

We make the following claim: If z_1, y_1, p_1, q_1 satisfy the following:

1. $q^2 - 3z^2 = -2$
2. $p^2 - 3y^2 = -3$
3. $p = 3z - 3y$
4. $q = 3z - 2y$
5. $q = p + y$

then so will z_2, y_2, p_2, and q_2 where

$$\begin{aligned} z_2 &= 2z_1 + q_1 \\ y_2 &= 2y_1 + p_1 \\ p_2 &= 3y_1 + 2p_1 \\ q_2 &= 3z_1 + 2q_1. \end{aligned}$$

The proof of the claim follows:

1.

$$\begin{aligned} (q_2)^2 - 3(z_2)^2 &= (3z_1 + 2q_1)^2 - 3(2z_1 + q_1)^2 \\ &= 9(z_1)^2 + 12(z_1)(q_1) + 4(q_1)^2 - 12(z_1)^2 - 12(z_1)(q_1) - 3(q_1)^2 \\ &= (q_1)^2 - 3(z_1)^2 \\ &= -2. \end{aligned}$$

2.

$$
\begin{aligned}
(p_2)^2 - 3(y_2)^2 &= (3y_1 + 2p_1)^2 - 3(2y_1 + p_1)^2 \\
&= 9(y_1)^2 + 12(y_1)(p_1) + 4(p_1)^2 - 12(y_1)^2 - 12(y_1)(p_1) - 3(p_1)^2 \\
&= (p_1)^2 - 3(y_1)^2 \\
&= -3.
\end{aligned}
$$

3.

$$
\begin{aligned}
p_2 &= 3y_1 + 2p_1 \\
&= 3(q_1 - p_1) + 2(3z_1 - 3y_1) \\
&= 3q_1 - 3p_1 + 6z_1 - 6y_1 \\
&= 3(2z_1 + q_1) - 3(2y_1 + p_1) \\
&= 3z_2 - 3y_2.
\end{aligned}
$$

4.

$$
\begin{aligned}
q_2 &= 3z_1 + 2q_1 \\
&= 3z_1 + q_1 + q_1 \\
&= 3z_1 + (3z_1 - 2y_1) + q_1 \\
&= 6z_1 - 2y_1 + q_1 \\
&= 6z_1 - 4y_1 + q_1 + 2y_1 \\
&= 6z_1 - 4y_1 + q_1 + 2(q_1 - p_1) \\
&= 6z_1 + 3q_1 - 4y_1 - 2p_1 \\
&= 3(2z_1 + q_1) - 2(2y_1 + p_1) \\
&= 3z_2 - 2y_2.
\end{aligned}
$$

5.

$$
\begin{aligned}
q_2 &= 3z_1 + 2q_1 \\
&= p_1 + 3y_1 + 2p_1 + 2y_1 \\
&= 3y_1 + 2p_1 + 2y_1 + p_1 \\
&= p_2 + y_2.
\end{aligned}
$$

Starting with $z_1 = 1,\ y_1 = 1,\ p_1 = 0$, and $q_1 = 1$ – which do satisfy all five equations – we have an infinite number of positive integer solutions for the five equations. Then for any such solution (z, y) (with $z > 1$) we know that, for $x = z - 1$, (x, y) will be a positive integer solution to the equation

$$\sqrt{3x^2+6x+1} = y + \sqrt{3y^2-3}.$$

We obtain our coefficient values using the fact that if (z, y, p, q) is obtained using the above recursion, then,

$$\begin{aligned}
& q = p + y \\
\Longrightarrow\quad & q^2 = (p+y)^2 \\
\Longrightarrow\quad & q^2 = p^2 + 2py + y^2 \\
\Longrightarrow\quad & 3z^2 - 2 = 3y^2 - 3 + 2(3z - 3y)y + y^2 \\
\Longrightarrow\quad & 3z^2 - 2 = -2y^2 + 6zy - 3 \\
\Longrightarrow\quad & 3(x+1)^2 + 2y^2 - 6(x+1)y + 1 = 0 \\
\Longrightarrow\quad & 3x^2 + 2y^2 - 6xy + 6x - 6y + 4 = 0.
\end{aligned}$$

Letting n be any real number, the coefficients are

$$[a, b, c, d, e, f] \quad = \quad [3n, 2n, -6n, 6n, -6n, 4n].$$

Obviously, n can be chosen to insure integer coefficients.

Note: the actual recursive formulas for x and y that will work with the two equations are

$$\begin{aligned}
x_2 &= 5x_1 - 2y_1 + 4 \\
y_2 &= 3x_1 - y_1 + 3,
\end{aligned}$$

where the initial pair is $x_1 = 0$ and $y_1 = 1$. These are easily obtained using the four recursive formulas and the claim.

Solution 3 by Albert Natian, Los Angeles Valley College, Valley Glen, California

Answer: $(a = 3,\ b = 2,\ c = -6,\ d = 6,\ e = -6,\ f = 4)$.

Broadly $\quad (a = 3\mu,\ b = 2\mu,\ c = -6\mu,\ d = 6\mu,\ e = -6\mu,\ f = 4\mu)\quad$ with $\mu \in Z$.

Derivation:

By way of Brahmagupta's identity

$$(ax + Hby)^2 - H\,(ay + bx)^2 = \left(a^2 - Hb^2\right)\left(x^2 - Hy^2\right),$$

we will develop formulas that generate infinitely many positive integers x and y for which $\sqrt{3x^2+6x+1}$ and $\sqrt{3y^2-3}$ are integers. First set $u = x+1$ so that $3x^2+6x+1 = 3u^2-2$. We would like the quantity $3u^2 - 2$ to be a perfect square so that $\sqrt{3x^2+6x+1}$ or $\sqrt{3u^2-2}$ is an integer. That is, for some integer v, we want

$$\sqrt{3u^2-2} = v, \quad 3u^2 - 2 = v^2, \quad v^2 - 3u^2 = -2.$$

So, we ask: For what integer values of u and v does the preceding Diophantine equation hold? A general form of the above Diophantine equation is

$$x^2 - Hy^2 = k$$

where all the variables are integers, with $H > 0$. Suppose we know of two triplets (x_1, y_1, k_1) and (x_2, y_2, k_2) of integers that satisfy the equations

$$x_1^2 - Hy_1^2 = k_1 \qquad \text{and} \qquad x_2^2 - Hy_2^2 = k_2.$$

Then, by Brahmagupta's identity

$$(x_1x_2 + Hy_1y_2)^2 - H(x_1y_2 + x_2y_1)^2 = \left(x_1^2 - Hy_1^2\right)\left(x_2^2 - Hy_2^2\right) = k_1k_2,$$

the triplet

$$(x_1x_2 + Hy_1y_2,\ x_1y_2 + x_2y_1,\ k_1k_2)$$

satisfies the equation

$$X^2 - HY^2 = k_1k_2.$$

So, in particular, if the triplets $(x_0, y_0, 1)$ and (x_n, y_n, k) of integers satisfy the equations

$$x_0^2 - Hy_0^2 = 1 \qquad \text{and} \qquad x_n^2 - Hy_n^2 = k,$$

then the triplet

$$(x_{n+1}, y_{n+1}, k) := (x_0x_n + Hy_0y_n,\ x_0y_n + x_ny_0,\ k)$$

of integers satisfies the equation

$$x_{n+1}^2 - Hy_{n+1}^2 = k.$$

This leads to a recursion:

$$\left\{ \begin{array}{r} x_{n+1} = x_0x_n + Hy_0y_n \\ y_{n+1} = y_0x_n + x_0y_n \end{array} \right\},$$

$$\begin{pmatrix} x_{n+1} \\ y_{n+1} \end{pmatrix} = \begin{pmatrix} x_0 & Hy_0 \\ y_0 & x_0 \end{pmatrix} \begin{pmatrix} x_n \\ y_n \end{pmatrix} = \begin{pmatrix} x_0 & Hy_0 \\ y_0 & xt_0 \end{pmatrix}^n \begin{pmatrix} x_1 \\ y_1 \end{pmatrix}$$

where (x_1, y_1) satisfies $x_1^2 - Hy_1^2 = k$.

Via diagonalization (assuming distinct eigenvalues), we have

$$\begin{pmatrix} x_0 & Hy_0 \\ y_0 & x_0 \end{pmatrix}^n =$$

$$\frac{1}{2\sqrt{H}} \begin{pmatrix} \sqrt{H}\left[\left(x_0 + y_0\sqrt{H}\right)^n + \left(x_0 - y_0\sqrt{H}\right)^n\right] & H\left[\left(x_0 + y_0\sqrt{H}\right)^n - \left(x_0 - y_0\sqrt{H}\right)^n\right] \\ \left(x_0 + y_0\sqrt{H}\right)^n - \left(x_0 - y_0\sqrt{H}\right)^n & \sqrt{H}\left[\left(x_0 + y_0\sqrt{H}\right)^n + \left(x_0 - y_0\sqrt{H}\right)^n\right] \end{pmatrix}.$$

Thus

$$\left\{ \begin{array}{l} x_{n+1} = \frac{1}{2}\left[\left(x_1 + y_1\sqrt{H}\right)\left(x_0 + y_0\sqrt{H}\right)^n + \left(x_1 - y_1\sqrt{H}\right)\left(x_0 - y_0\sqrt{H}\right)^n\right] \\ y_{n+1} = \frac{1}{2\sqrt{H}}\left[\left(x_1 + y_1\sqrt{H}\right)\left(x_0 + y_0\sqrt{H}\right)^n - \left(x_1 - y_1\sqrt{H}\right)\left(x_0 - y_0\sqrt{H}\right)^n\right] \end{array} \right\}.$$

Recalling the equation $v^2 - 3u^2 = -2$, we see that $(v_0 = 2, u_0 = 1)$ satisfies $v^2 - 3u^2 = 1$ and $(v_1 = 1, u_1 = 1)$ satisfies $v^2 - 3u^2 = -2$. Thus

$$\left\{ \begin{array}{l} v_{n+1} = \frac{1}{2}\left[\left(1+\sqrt{3}\right)\left(2+\sqrt{3}\right)^n + \left(1-\sqrt{3}\right)\left(2-\sqrt{3}\right)^n\right] \\ \\ u_{n+1} = \frac{1}{2\sqrt{3}}\left[\left(1+\sqrt{3}\right)\left(2+\sqrt{3}\right)^n - \left(1-\sqrt{3}\right)\left(2-\sqrt{3}\right)^n\right] \end{array} \right\}.$$

Since $2 \pm \sqrt{3} = \frac{1}{2}\left(1 \pm \sqrt{3}\right)^2$, then the preceding can be written as

$$\left\{ \begin{array}{l} v_n = \frac{1}{2^n}\left[\left(1+\sqrt{3}\right)^{2n-1} + \left(1-\sqrt{3}\right)^{2n-1}\right] \\ \\ u_n = \frac{1}{2^n\sqrt{3}}\left[\left(1+\sqrt{3}\right)^{2n-1} - \left(1-\sqrt{3}\right)^{2n-1}\right] \end{array} \right\}.$$

Now we turn to the expression $\sqrt{3y^2-3}$, which we would like to be an integer, say, z. Thus

$$\sqrt{3y^2-3} = z, \quad 3y^2 - 3 = z^2, \quad z^2 - 3y^2 = -3.$$

The triplets $(z_0 = 2, y_0 = 1, 1)$ and $(z_1 = 3, y_1 = 2, -3)$ satisfy

$$z_0^2 - 3y_0^2 = 1 \qquad \text{and} \qquad z_1^2 - 3y_1^2 = -3.$$

Thus

$$\left\{ \begin{array}{l} z_{m+1} = \frac{1}{2}\left[\left(3+2\sqrt{3}\right)\left(2+\sqrt{3}\right)^m + \left(3-2\sqrt{3}\right)\left(2-\sqrt{3}\right)^m\right] \\ \\ y_{m+1} = \frac{1}{2\sqrt{3}}\left[\left(3+2\sqrt{3}\right)\left(2+\sqrt{3}\right)^m - \left(3-2\sqrt{3}\right)\left(2-\sqrt{3}\right)^m\right] \end{array} \right\}$$

which can be written as

$$\left\{ \begin{array}{l} z_m = \frac{\sqrt{3}}{2^{m+1}}\left[\left(1+\sqrt{3}\right)^{2m} - \left(1-\sqrt{3}\right)^{2m}\right] \\ \\ y_m = \frac{1}{2^{m+1}}\left[\left(1+\sqrt{3}\right)^{2m} + \left(1-\sqrt{3}\right)^{2m}\right] \end{array} \right\}.$$

Inserting the above results into the original equation

$$\sqrt{3x^2+6x+1} = y + \sqrt{3y^2-3} \qquad \text{or} \qquad \sqrt{3u^2-2} = y + \sqrt{3y^2-3}$$

or

$$v = y + z,$$

we have

$$v_n = y_m + z_m,$$

$$\frac{1}{2^n}\left[\left(1+\sqrt{3}\right)^{2n-1} + \left(1-\sqrt{3}\right)^{2n-1}\right] = \frac{1}{2^{m+1}}\left[\left(1+\sqrt{3}\right)^{2m} + \left(1-\sqrt{3}\right)^{2m}\right] +$$

$$+\frac{\sqrt{3}}{2^{m+1}}\left[\left(1+\sqrt{3}\right)^{2m} - \left(1-\sqrt{3}\right)^{2m}\right],$$

$$\frac{1}{2^n}\left[\left(1+\sqrt{3}\right)^{2n-1}+\left(1-\sqrt{3}\right)^{2n-1}\right]=\frac{1}{2^{m+1}}\left[\left(1+\sqrt{3}\right)^{2m+1}+\left(1-\sqrt{3}\right)^{2m+1}\right],$$

$$\frac{1}{2^n}\left[\left(1+\sqrt{3}\right)^{2n-1}+\left(1-\sqrt{3}\right)^{2n-1}\right]=\frac{1}{2^{m+1}}\left[\left(1+\sqrt{3}\right)^{2(m+1)-1}+\left(1-\sqrt{3}\right)^{2(m+1)-1}\right],$$

which holds if $n=m+1$.

Since $u=x+1$ or $x=u-1$, then the given equation

$$ax^2+by^2+cxy+dx+ey+f=0$$

becomes

$$au^2+by^2+cuy+(d-2a)\,u+(e-c)\,y+(a-d+f)=0.$$

Claim: For $a=3\mu,\ b=2\mu,\ c=-6\mu,\ d=6\mu,\ \ e=-6\mu,\ f=4\mu$ with $\mu\in Z$, there exist infinitely many pairs of positive integers (x,y) that satisfy the system

$$\left\{\begin{array}{c}\sqrt{3x^2+6x+1}=y+\sqrt{3y^2-3}\\ ax^2+by^2+cxy+dx+ey+f=0\end{array}\right\}.$$

Proof: Since $d-2a=0$, $e-c=0$ and $a-d+f=1$, then the equation

$$au^2+by^2+cuy+(d-2a)\,u+(e-c)\,y+(a-d+f)=0$$

becomes

$$3u^2+2y^2-6uy+1=0.$$

Letting $p=1+\sqrt{3}$ and $q=1-\sqrt{3}$, we insert the above values for y_m and u_{m+1} into the latter equation and write

$$3\left(\frac{1}{2^{m+1}\sqrt{3}}\left[p^{2m+1}-q^{2m+1}\right]\right)^2+2\left(\frac{1}{2^{m+1}}\left[p^{2m}+q^{2m}\right]\right)^2$$
$$-6\left(\frac{1}{2^{m+1}\sqrt{3}}\left[p^{2m+1}-q^{2m+1}\right]\right)\left(\frac{1}{2^{m+1}}\left[p^{2m}+q^{2m}\right]\right)+1=0.$$

We will show the latter equation holds for all positive integer values of m. Taking advantage of the fact that $pq=-2$, we simplify the above as

$$\frac{3}{3\cdot 2^{2m+2}}\left(p^{4m+2}+2^{2m+2}+q^{4m+2}\right)+\frac{2}{2^{2m+2}}\left(p^{4m}+2^{2m+1}+q^{4m}\right)$$
$$-\frac{6}{2^{2m+2}\sqrt{3}}\left(p^{4m+1}+2^{2m}p-2^{2m}q-q^{4m+1}\right)+1=0$$

both sides of which we multiply by 2^{2m+2} to get

$$\left(p^{4m+2}+2^{2m+2}+q^{4m+2}\right)+2\left(p^{4m}+2^{2m+1}+q^{4m}\right)$$
$$-2\sqrt{3}\left(p^{4m+1}+2^{2m}p-2^{2m}q-q^{4m+1}\right)+2^{2m+2}=0,$$

$$\left(p^{4m+2}+2^{2m+2}+q^{4m+2}\right)+2\left(p^{4m}+2^{2m+1}+q^{4m}\right)$$
$$-2\sqrt{3}\left(p^{4m+1}+[p-q]\cdot 2^{2m}-q^{4m+1}\right)+2^{2m+2}=0,$$

$$\left(p^{4m+2}+2^{2m+2}+q^{4m+2}\right)+2\left(p^{4m}+2^{2m+1}+q^{4m}\right)$$
$$-2\sqrt{3}\left(p^{4m+1}+\left[2\sqrt{3}\right]\cdot 2^{2m}-q^{4m+1}\right)+2^{2m+2}=0,$$

$$\left[p^{4m+2}+2p^{4m}-2\sqrt{3}p^{4m+1}\right]+\left[2^{2m+2}+2\cdot 2^{2m+1}-2\sqrt{3}\cdot 2\sqrt{3}\cdot 2^{2m}\right]$$
$$+\left[q^{4m+2}+2q^{4m}+2\sqrt{3}q^{4m+1}\right]+2^{2m+2}=0. \qquad (\star)$$

Since
$$p^{4m+2}+2p^{4m}-2\sqrt{3}p^{4m+1}=p^{4m}\left(p^2+2-2\sqrt{3}p\right)=0$$
and
$$q^{4m+2}+2q^{4m}+2\sqrt{3}q^{4m+1}=q^{4m}\left(q^2+2+2\sqrt{3}q\right)=0,$$
then the above equation $(\star)$ is simplified to
$$\left[2^{2m+2}+2\cdot 2^{2m+1}-2\sqrt{3}\cdot 2\sqrt{3}\cdot 2^{2m}\right]+2^{2m+2}=0$$
which obviously holds for all integer values of m. This completes the proof.

Solution 4 by Kee-Wai Lau, Hong Kong, China

We show that we can take $(a,b,c,d,e,f)=(3,2,-6,6,-6,4)$.

For positive integers k let $x_k=\dfrac{(1+\sqrt{3})(2+\sqrt{3})^k-(1-\sqrt{3})(2-\sqrt{3})^k}{2\sqrt{3}}-1$ and $y_k=\dfrac{(2+\sqrt{3})^k+(2-\sqrt{3})^k}{2}$. It is easy to see that both x_k and y_k are positive integers.

It can be checked readily that
$$\sqrt{3x_k^2+6x_k+1}+=y_k+\sqrt{3y_k^2-3}=\frac{(1+\sqrt{3})(2+\sqrt{3})^k+(1-\sqrt{3})(2-\sqrt{3})^k}{2}$$
and
$$3x_k^2+2y_k^2-6x_ky_k+6x_k-6y_k+4=0.$$
Hence our claim.

Solution 5 by Charles Burnette, Xavier University of Louisiana (New Orleans, LA)

Let (x,y) be a pair of positive integer solutions to the first equation. If we square both sides of the first equation and isolate the radical term remaining its right side, we get
$$3x^2-4y^2+6x+4=2y\sqrt{3y^2-3}. \qquad (1)$$

Squaring (1) yields

$$9x^4 + 16y^4 - 24x^2y^2 + 36x^3 - 48xy^2 + 60x^2 - 32y^2 + 48x + 16 = 12y^4 - 12y^2. \quad (2)$$

This simplifies to

$$9x^4 + 4y^4 - 24x^2y^2 + 36x^3 - 48xy^2 + 60x^2 + 12y^2 + 48x + 16 = 0. \quad (3)$$

The left side of (3) can be reorganized to beget a difference of two squares:

$$(9x^4 + 4y^4 + 12x^2y^2 + 36x^3 + 24xy^2 + 60x^2 + 48y^2 + 48x + 16) - (36x^2y^2 + 72xy^2 + 36y^2)$$
$$= (3x^2 + 2y^2 + 6x + 4)^2 - (6xy + 6y)^2.$$

We therefore end up with the equation

$$(3x^2 + 2y^2 + 6xy + 6x + 6y + 4)(3x^2 + 2y^2 - 6xy + 6x - 6y + 4) = 0. \quad (4)$$

In assuming that x and y are positive, we must then have

$$3x^2 + 2y^2 - 6xy + 6x - 6y + 4 = 0. \quad (5)$$

Because the above steps are reversible provided that

$$1 \leq |y| \leq \frac{1}{2}\sqrt{3x^2 + 6x + 4},$$

it suffices to show that (5) has infinitely many positive integer solutions in this range. A Legendre transformation changes (5) into a Pell-type diophantine equation. Indeed,

$$u^2 - 3v^2 = -2, \quad (6)$$

with $u = 3x - 2y + 3$ and $v = x + 1$, is equivalent to doubling (5). Equation (6) has a particular solution of $(1, 1)$, and since 3 is not a perfect square, the corresponding Pell resolvent $u^2 - 3v^2 = 1$ and, consequently, (6) have infinitely many integer solutions in each quadrant. Furthermore, if (u, v) is an integer solution to (6), then u and v must share the same parity. As a result, x and

$$y = \frac{3x + 3 - u}{2} = \frac{3v - u}{2}$$

provide integer solutions of (5) whenever (u, v) is an integer solution of (6). If we also specifically restrict our attention to positive solutions for u and v so that $x \geq 1$ and

$$u = \sqrt{3v^2 - 2} = \sqrt{3x^2 + 6x + 1},$$

then

$$y = \frac{3x + 3 - \sqrt{3x^2 + 6x + 1}}{2} \geq \frac{3x + 3 - \sqrt{9x^2 + 6x + 1}}{2} = \frac{3x + 3 - (3x + 1)}{2} = 1,$$

and

$$\begin{aligned} y &= \frac{3x + 3 - \sqrt{3x^2 + 6x + 1}}{2} \\ &= \frac{\sqrt{9x^2 + 18x + 9} - \sqrt{3x^2 + 6x + 1}}{2} \\ &\leq \frac{\sqrt{9x^2 + 18x + 3x^2 + 6x + 4} - \sqrt{3x^2 + 6x + 1}}{2} \\ &= \frac{\sqrt{12x^2 + 24x + 4} - \sqrt{3x^2 + 6x + 1}}{2} \\ &= \frac{\sqrt{3x^2 + 6x + 1}}{2} \leq \frac{\sqrt{3x^2 + 6x + 4}}{2}, \end{aligned}$$

as required. We can now conclude that

$$a = 3,\ b = 2,\ c = -6,\ d = 6,\ e = -6,\ f = 4.$$

Solution 6 by David Stone and John Hawkins, Georgia Southern University, Statesboro, GA

Editor′s Comment: The solution by the above authors contains some nice graphics and interesting comments to ponder, but I received it too close to the deadline for submitting papers for me to recopy it into the format of the other solutions in this issue. However, I believe their solution to be very instructive and so I have included it as an appendix to this March 2021 issue of the column. It listed as a separate pdf file.

Comments about 5613 from the author, Kenneth Korbin:

The roots of the first equation are:

$$\begin{aligned}
(x, y) &= (10, 7),\ (152, 97),\ (2130, 1351),\ \cdots,\\
x &= (10, 152, 2130, \cdots) \text{ with}\\
x_{N+1} &= 14x_N - x_{N-1} + 12.\\
\\
y &= (7, 97, 1351, \cdots) \text{ with}\\
y_{N+1} &= 14y_N - y_{N-1}.\\
\\
x - y &= (3, 55, 779, 1063, \cdots),
\end{aligned}$$

The second equation can be written as: $3x^2 + 2y^2 - 6xy + 6x - 6y + 4 = 0$

Also solved by Peter Fulop, Gyomro, Hungary, and the proposer.

- **5614:** *Proposed by Michael Brozinsky, Central Islip, NY*

Solve:

$$\cos^2\theta + 6\cos(\theta)\cos\left(\frac{\theta}{3}\right) + 9\cos^2\left(\frac{\theta}{3}\right) = \sin^2\theta - 6\sin\theta\sin\left(\frac{\theta}{3}\right) + 9\sin^2\left(\frac{\theta}{3}\right).$$

Solution 1 by Dionne Bailey, Elsie Campbell, and Charles Diminnie, Angelo State University, San Angelo, TX

We will make repeated use of the following basic trig identities:
a. $\cos(2x) = \cos^2 x - \sin^2 x$
b. $\cos(x \pm y) = \cos x \cos y \mp \sin x \sin y$
c. $\sin(x \pm y) = \sin x \cos y \pm \cos x \sin y$
d. $\cos(3x) = 4\cos^3 x - 3\cos x$
e. $\sin(3x) = 3\sin x - 4\sin^3 x$.
To begin, the given equation can be re-written in the form

$$\left(\cos^2\theta - \sin^2\theta\right) + 6\left(\cos\theta\cos\left(\frac{\theta}{3}\right) + \sin\theta\sin\left(\frac{\theta}{3}\right)\right) + 9\left(\cos^2\left(\frac{\theta}{3}\right) - \sin^2\left(\frac{\theta}{3}\right)\right) = 0.$$

Using identities a and b above, this equation becomes

$$\cos(2\theta) + 6\cos\left(\theta - \frac{\theta}{3}\right) + 9\cos\left(\frac{2\theta}{3}\right) = 0,$$

i.e.,

$$\cos(2\theta) + 15\cos\left(\frac{2\theta}{3}\right) = 0.$$

Since $2\theta = 3\left(\frac{2\theta}{3}\right)$, identity d implies that

$$\cos(2\theta) = \cos\left[3\left(\frac{2\theta}{3}\right)\right] = 4\cos^3\left(\frac{2\theta}{3}\right) - 3\cos\left(\frac{2\theta}{3}\right)$$

and our equation reduces to

$$\begin{aligned} 4\cos^3\left(\frac{2\theta}{3}\right) - 3\cos\left(\frac{2\theta}{3}\right) + 15\cos\left(\frac{2\theta}{3}\right) &= 0 \\ 4\cos^3\left(\frac{2\theta}{3}\right) + 12\cos\left(\frac{2\theta}{3}\right) &= 0 \\ \cos\left(\frac{2\theta}{3}\right)\left[\cos^2\left(\frac{2\theta}{3}\right) + 3\right] &= 0. \end{aligned}$$

Because $\cos^2\left(\frac{2\theta}{3}\right) + 3 > 0$, we are left with

$$\cos\left(\frac{2\theta}{3}\right) = 0$$

whose solutions are

$$\frac{2\theta}{3} = \frac{\pi}{2} + n\pi,$$

i.e.,

$$\theta = \frac{3\pi}{4} + n\left(\frac{3\pi}{2}\right) \tag{1}$$

for all $n \in Z$.

To complete our solution, we must check whether (1) contains any extraneous solutions. To aid in this task, we note first that identities d and e yield

$$\begin{aligned} \cos^2\theta + 6\cos\theta\cos\left(\frac{\theta}{3}\right) + 9\cos^2\left(\frac{\theta}{3}\right) &= \left[\cos\theta + 3\cos\left(\frac{\theta}{3}\right)\right]^2 \\ &= \left[4\cos^3\left(\frac{\theta}{3}\right) - 3\cos\left(\frac{\theta}{3}\right) + 3\cos\left(\frac{\theta}{3}\right)\right]^2 \\ &= 16\cos^6\left(\frac{\theta}{3}\right) \end{aligned} \tag{2}$$

and

$$\begin{aligned} \sin^2\theta - 6\sin\theta\sin\left(\frac{\theta}{3}\right) + 9\sin^2\left(\frac{\theta}{3}\right) &= \left[\sin\theta - 3\sin\left(\frac{\theta}{3}\right)\right]^2 \\ &= \left[3\sin\left(\frac{\theta}{3}\right) - 4\sin^3\left(\frac{\theta}{3}\right) - 3\sin\left(\frac{\theta}{3}\right)\right]^2 \\ &= 16\sin^6\left(\frac{\theta}{3}\right). \end{aligned} \tag{3}$$

If $\theta = \frac{3\pi}{4} + n\left(\frac{3\pi}{2}\right)$, then $\frac{\theta}{3} = \frac{\pi}{4} + n\left(\frac{\pi}{2}\right)$ and identity b implies that

$$\begin{aligned}
\cos\left(\frac{\theta}{3}\right) &= \cos\left[\frac{\pi}{4} + n\left(\frac{\pi}{2}\right)\right] \\
&= \cos\left(\frac{\pi}{4}\right)\cos\left(n\frac{\pi}{2}\right) - \sin\left(\frac{\pi}{4}\right)\sin\left(n\frac{\pi}{2}\right) \\
&= \frac{\sqrt{2}}{2}\left[\cos\left(n\frac{\pi}{2}\right) - \sin\left(n\frac{\pi}{2}\right)\right].
\end{aligned}$$

If $n = 2k$, we get

$$\begin{aligned}
\cos\left(\frac{\theta}{3}\right) &= \frac{\sqrt{2}}{2}\left[\cos(k\pi) - \sin(k\pi)\right] \\
&= \frac{\sqrt{2}}{2}\left[(-1)^k - 0\right] \\
&= (-1)^k\,\frac{\sqrt{2}}{2} \qquad (4)
\end{aligned}$$

while $n = 2k + 1$ yields

$$\begin{aligned}
\cos\left(\frac{\theta}{3}\right) &= \frac{\sqrt{2}}{2}\left[\cos(2k+1)\frac{\pi}{2} - \sin(2k+1)\frac{\pi}{2}\right] \\
&= \frac{\sqrt{2}}{2}\left[0 - \sin\left(k\pi + \frac{\pi}{2}\right)\right] \\
&= -\frac{\sqrt{2}}{2}\left[\sin(k\pi)\cos\left(\frac{\pi}{2}\right) + \cos(k\pi)\sin\left(\frac{\pi}{2}\right)\right] \\
&= -\frac{\sqrt{2}}{2}\left[0 + \cos(k\pi)\right] \\
&= -\frac{\sqrt{2}}{2}(-1)^k \\
&= (-1)^{k+1}\,\frac{\sqrt{2}}{2} \qquad (5)
\end{aligned}$$

If $n = 2k$, conditions (2) and (4) imply that

$$\begin{aligned}
\cos^2\theta + 6\cos\theta\cos\left(\frac{\theta}{3}\right) + 9\cos^2\left(\frac{\theta}{3}\right) &= 16\cos^6\left(\frac{\theta}{3}\right) \\
&= 16\left[(-1)^k\,\frac{\sqrt{2}}{2}\right]^6 \\
&= (16)\left(\frac{1}{8}\right) \\
&= 2.
\end{aligned}$$

If $n = 2k+1$, conditions (2) and (5) give

$$\begin{aligned}
\cos^2\theta + 6\cos\theta\cos\left(\frac{\theta}{3}\right) + 9\cos^2\left(\frac{\theta}{3}\right) &= 16\cos^6\left(\frac{\theta}{3}\right) \\
&= 16\left[(-1)^{k+1}\,\frac{\sqrt{2}}{2}\right]^6 \\
&= 16\left(\frac{1}{8}\right) \\
&= 2.
\end{aligned}$$

Hence, for all $n \in Z$, $\theta = \dfrac{3\pi}{4} + n\left(\dfrac{3\pi}{2}\right)$ forces

$$\cos^2\theta + 6\cos\theta\cos\left(\frac{\theta}{3}\right) + 9\cos^2\left(\frac{\theta}{3}\right) = 2.$$

As before, if $\theta = \dfrac{3\pi}{4} + n\left(\dfrac{3\pi}{2}\right)$, then $\dfrac{\theta}{3} = \dfrac{\pi}{4} + n\left(\dfrac{\pi}{2}\right)$ and identity c gives

$$\begin{aligned}
\sin\left(\frac{\theta}{3}\right) &= \sin\left[\frac{\pi}{4} + n\left(\frac{\pi}{2}\right)\right] \\
&= \sin\left(\frac{\pi}{4}\right)\cos n\left(\frac{\pi}{2}\right) + \cos\left(\frac{\pi}{4}\right)\sin n\left(\frac{\pi}{2}\right) \\
&= \frac{\sqrt{2}}{2}\left[\cos\left(n\frac{\pi}{2}\right) + \sin\left(n\frac{\pi}{2}\right)\right].
\end{aligned}$$

If $n = 2k$, then

$$\begin{aligned}
\sin\left(\frac{\theta}{3}\right) &= \frac{\sqrt{2}}{2}\left[\cos(k\pi) + \sin(k\pi)\right] \\
&= \frac{\sqrt{2}}{2}\left[(-1)^k + 0\right] \\
&= (-1)^k\,\frac{\sqrt{2}}{2},
\end{aligned}$$

while $n = 2k+1$ leads to $\dfrac{\theta}{3} = \dfrac{\pi}{4} + (2k+1)\left(\dfrac{\pi}{2}\right)$ and identity c gives

$$\begin{aligned}
\sin\left(\frac{\theta}{3}\right) &= \frac{\sqrt{2}}{2}\left[\cos\left((2k+1)\frac{\pi}{2}\right) + \sin\left((2k+1)\frac{\pi}{2}\right)\right] \\
&= \frac{\sqrt{2}}{2}\left[0 + \sin\left(k\pi + \frac{\pi}{2}\right)\right] \\
&= \frac{\sqrt{2}}{2}\left[\sin(k\pi)\cos\left(\frac{\pi}{2}\right) + \cos(k\pi)\sin\left(\frac{\pi}{2}\right)\right] \\
&= \frac{\sqrt{2}}{2}\left[0 + (-1)^k\right] \\
&= (-1)^k\,\frac{\sqrt{2}}{2}.
\end{aligned}$$

In both situations, condition (3) yields

$$\begin{aligned}\sin^2\theta - 6\sin\theta\sin\left(\frac{\theta}{3}\right) + 9\sin^2\left(\frac{\theta}{3}\right) &= 16\sin^6\left(\frac{\theta}{3}\right)\\ &= 16\left[(-1)^k\frac{\sqrt{2}}{2}\right]^6\\ &= 16\left(\frac{1}{8}\right)\\ &= 2.\end{aligned}$$

Thus, for all $n \in Z$, $\theta = \dfrac{3\pi}{4} + n\left(\dfrac{3\pi}{2}\right)$ makes

$$\begin{aligned}\cos^2\theta + 6\cos\theta\cos\left(\frac{\theta}{3}\right) + 9\cos^2\left(\frac{\theta}{3}\right) &= 16\cos^6\left(\frac{\theta}{3}\right)\\ &= 2\\ &= 16\sin^6\left(\frac{\theta}{3}\right)\\ &= \sin^2\theta - 6\sin\theta\sin\left(\frac{\theta}{3}\right) + 9\sin^2\left(\frac{\theta}{3}\right)\end{aligned}$$

and (1) contains no extraneous solutions. This completes our solution.

Solution 2 by David A. Huckaby, Angelo State University, San Angelo, TX

We have

$$\cos^2\theta - \sin^2\theta + 6\cos\theta\cos\left(\frac{\theta}{3}\right) + 6\sin\theta\sin\left(\frac{\theta}{3}\right) + 9\cos^2\left(\frac{\theta}{3}\right) - 9\sin^2\left(\frac{\theta}{3}\right) = 0$$

$$\cos 2\theta + 6\cos\left(\frac{2\theta}{3}\right) + 9\cos\left(\frac{2\theta}{3}\right) = 0$$

$$\cos 2\theta + 15\cos\left(\frac{2\theta}{3}\right) = 0$$

$$4\cos^3\left(\frac{2\theta}{3}\right) - 3\cos\left(\frac{2\theta}{3}\right) + 15\cos\left(\frac{2\theta}{3}\right) = 0$$

$$4\cos^3\left(\frac{2\theta}{3}\right) + 12\cos\left(\frac{2\theta}{3}\right) = 0$$

$$4\cos\left(\frac{2\theta}{3}\right)\left[\cos^2\left(\frac{2\theta}{3}\right) + 3\right] = 0$$

So $\cos\left(\frac{2\theta}{3}\right) = 0$, whence $\frac{2\theta}{3} = \frac{\pi}{2} + \pi n$ for any integer n, so that $\theta = \frac{3\pi}{4} + \frac{3\pi}{2}n$ for any integer n.

Solution 3 by Titu Zvonaru, Comănesti, Romania,

We have

$$\cos\theta = \cos\frac{\theta}{3}\left(4\cos^3\frac{\theta}{3} - 3\right)$$

$$\Rightarrow \cos^2\theta + 6\cos\theta\cos\frac{\theta}{3} + 9\cos^2\frac{\theta}{3} = \cos^2\frac{\theta}{3}\left(16\cos^4\frac{\theta}{3} + 9 - 24\cos^2\frac{\theta}{3}\right) + 6\cos^2\frac{\theta}{3}\left(4\cos^2\frac{\theta}{3} - 3\right) -$$

$$\cos^2\frac{\theta}{3}\left(16\cos^4\frac{\theta}{3} + 9 - 24\cos^2\frac{\theta}{3} + 24\cos^2\frac{\theta}{3} - 18 + 9\right)$$

$$= 16\cos^6\frac{\theta}{3}$$

$$\begin{aligned}
\sin\theta &= \sin\frac{\theta}{3}\left(3 - 4\sin^2\frac{\theta}{3}\right) = \\
&= \sin^2\theta - 6\sin\theta\sin\frac{\theta}{3} + 9\sin^2\frac{\theta}{3} = \sin^2\frac{\theta}{3}\left(16\sin^4\frac{\theta}{3} + 9 - 24^2\frac{\theta}{3}\right) - 6\sin^2\frac{\theta}{3}\left(3 - 4\sin^2\frac{\theta}{3}\right) \\
&+ 9\sin^2\frac{\theta}{3} = \\
&= \sin^2\frac{\theta}{3}\left(16\sin^4\frac{\theta}{3} + 9 - 24\sin^2\frac{\theta}{3} + 24\sin^2\frac{\theta}{3} - 18 + 9\right) = 16\sin^6\frac{\theta}{3}.
\end{aligned}$$

We obtain the equation,

$$\tan^6\frac{\theta}{3} = 1 \Rightarrow \tan^3\frac{\theta}{3} = 1, \tan^3\frac{\theta}{3} = -1$$

hence, $\theta = \frac{3\pi}{4} + 3k\pi,\ \theta = -\frac{3\pi}{4} + 3k\pi$ where k is an integer.

Solution by 4 by Michel Bataille, Rouen, France

From the formulas $\cos 2x = \cos^2 x - \sin^2 x,\ \cos a\cos b + \sin a\sin b = \cos(a-b)$, we deduce that the given equation is equivalent to

$$\cos 2\theta + 6\cos\frac{2\theta}{3} + 9\cos\frac{2\theta}{3} = 0. \qquad (1)$$

Using $\cos 3x = 4\cos^3 x - 3\cos x$, (1) can be written as $4\cos^3\frac{2\theta}{3} - 3\cos\frac{2\theta}{3} + 15\cos\frac{2\theta}{3} = 0$, that is,

$$4\cos\frac{2\theta}{3}\left(3 + \cos^2\frac{2\theta}{3}\right) = 0.$$

Since $3 + \cos^2\frac{2\theta}{3} > 0$ for all real θ, the latter reduces to $\cos\frac{2\theta}{3} = 0$. Thus, the solutions are the numbers $\dfrac{3\pi}{4} + \dfrac{3k\pi}{2},\ k \in Z.$

Solution 5 by David E. Manes, Oneonta, NY

The solution of the equation is $\theta = \dfrac{3\pi}{4} + \dfrac{3n\pi}{2}$, where n is any integer.

The given equation is equivalent to

$$(\cos^2\theta - \sin^2\theta) + 6\left(\cos(\theta)\cos\left(\frac{\theta}{3}\right) + \sin(\theta)\sin\left(\frac{\theta}{3}\right)\right) + 9\left(\cos^2\left(\frac{\theta}{3}\right) - \sin^2\left(\frac{\theta}{3}\right)\right) = 0.$$

The addition formula for cosine and the double-angle formula ($\cos 2a = \cos^2 a - \sin^2 a$) imply

$$\cos(2\theta) + 6\cos\left(\frac{2\theta}{3}\right) + 9\cos\left(\frac{2\theta}{3}\right) = \left[\cos(2\theta) + \cos\left(\frac{2\theta}{3}\right)\right] + 14\cos\left(\frac{2\theta}{3}\right) = 0.$$

By the sum-to-product formula, $\cos(2\theta) + \cos(2\theta/3) = 2\cos(4\theta/3)\cdot\cos(2\theta/3)$. Therefore, the equation reduces to

$$2\cos\left(\frac{4\theta}{3}\right)\cdot\cos\left(\frac{2\theta}{3}\right) + 14\cos\left(\frac{2\theta}{3}\right) = 2\cos\left(\frac{2\theta}{3}\right)\left(\cos\left(\frac{4\theta}{3}\right) + 7\right) = 0.$$

Hence, $\cos\left(\dfrac{2\theta}{3}\right) = 0$ since $\cos\left(\dfrac{4\theta}{3}\right) + 7 \neq 0$ for any value of θ. Therefore,

$$\frac{2\theta}{3} = \frac{\pi}{2} + n\pi \text{ or } \theta = \frac{3\pi}{4} + \frac{3n\pi}{2},$$

for any integer n. This completes the solution.

Solution 6 by Albert Stadler Herrliberg, Switzerland

Let $z = \cos\frac{\theta}{3} + i\sin\frac{\theta}{3} = e^{i\frac{\theta}{3}}$. Then $z^3 = e^{i\theta} = \cos\theta + i\sin\theta$. The above equation is equivalent to each of the following lines:

$$\left(\cos\theta + 3\cos\frac{\theta}{3}\right)^2 = \left(\sin\theta - 3\sin\frac{\theta}{3}\right)^2,$$

$$\left(\frac{z^3 + \frac{1}{z^3}}{2} + 3\frac{z + \frac{1}{z}}{2}\right)^2 = \left(\frac{z^3 - \frac{1}{z^3}}{2i} - 3\frac{z - \frac{1}{z}}{2i}\right)^2,$$

$$\left(z^3 + \frac{1}{z^3} + 3z + \frac{3}{z}\right)^2 + \left(z^3 - \frac{1}{z^3} - 3z + \frac{3}{z}\right)^2 = 0,$$

$$z^6 + \frac{1}{z^6} + 15z^2 + \frac{15}{z^2} = 0,$$

$$\left(z^2 + \frac{1}{z^2}\right)^3 + 12\left(z^2 + \frac{1}{z}^2\right) = 0.$$

The equation $u^3 + 12u = 0$ has the roots $u = 0, u = 2i\sqrt{3}, u = -2i\sqrt{3}$.
$z^2 + \frac{1}{z^2} = 0$, has the four roots $z = e^{\frac{\pi i}{4}} i^j, j = 0, 1, 2, 3$.
$z^2 + \frac{1}{z^2} = 2i\sqrt{3}$ has the four roots $z = (1+i)\left(\frac{1+\sqrt{3}}{2}\right), (-1-i)\left(\frac{1+\sqrt{3}}{2}\right)$,
(1-i)$\left(\frac{-1+\sqrt{3}}{2}\right), (-1+i)\left(\frac{-1+\sqrt{3}}{2}\right)$, none of which has modulus 1.

$z^2+\frac{1}{z^2}=-2i\sqrt{3}$has the four roots $z=(1+i)\left(\frac{-1+\sqrt{3}}{2}\right),(-1-i)\left(\frac{1+\sqrt{3}}{2}\right)$,
$(1-i)\left(\frac{1+\sqrt{3}}{2}\right),(-1+i)\left(\frac{1+\sqrt{3}}{2}\right)$, none of which has modulus 1.

Therefore the roots of the given equation satisfy $e^{i\frac{\theta}{3}=e^{\frac{\pi i}{4}}i^j, j=0,1,2,3}$ and thus the given equation has the roots $\theta=\frac{3\pi}{4}(2j+1)$, where j is an arbitrary integer.

Comment: the Mathematica command
TrigFactor[Cos[x]^2+6Cos[x]Cos[x/3]+9Cos[x/3]^2-Sin[x]^2-6Sin[x]Sin[x/3]+9Sin[x/3]^2)]
produces the output

$$\text{-8Sin}\left[\frac{\pi}{4}-\frac{x}{3}\right]\text{Sin}\left[\frac{\pi}{4}+\frac{x}{3}\right]\left(-2+\text{Sin}\left[\frac{2x}{3}\right]\right)\left(2+\text{Sin}\left[\frac{2x}{3}\right]\right).$$

It's clear that $\pm2+\sin((2x/3)=0$ has no realist,since $|\sin(y)|\le 1$ for all real y. Therefore the roots of the given equation are the roots of $\sin(\pi/4-x/3)\sin(\pi/4+x/3)=0$, who are given by $\theta=\dfrac{3\pi}{4}(2j+1)$ where j is an arbitrary integer.

Solution 7 by Daniel Văcaru, Pitesti, Romania

Equivalently:
$\cos^2\vartheta+6\cos\vartheta\cos\frac{\vartheta}{3}+9\cos^2\frac{\vartheta}{3}=\sin^2\vartheta-6\sin\vartheta\sin\frac{\vartheta}{3}+9\sin^2\frac{\vartheta}{3}\Leftrightarrow\cos^2\vartheta-\sin^2\vartheta+6\cos\vartheta\cos\frac{\vartheta}{3}+6\sin\vartheta\sin\frac{\vartheta}{3}+9\cos^2\frac{\vartheta}{3}-9\sin^2\frac{\vartheta}{3}=0$,
which is

$$\cos 2\vartheta+6\cos\left(\vartheta-\frac{\vartheta}{3}\right)+9\cos\frac{2\vartheta}{3}=0\Leftrightarrow\cos 2\vartheta+6\cos\frac{2\vartheta}{3}+9\cos\frac{2\vartheta}{3}=0\Leftrightarrow\cos 2\vartheta+15\cos\frac{2\vartheta}{3}=0\,(1)\,.$$

We use $\cos 3x=4\cos^3 x-3\cos x$, and (1) becomes $4\cos^3\frac{2\vartheta}{3}+12\cos\frac{2\vartheta}{3}=0\Leftrightarrow\cos\frac{2\vartheta}{3}\left(4\cos^2\frac{2\vartheta}{3}+12\right)=0\,(2)$.
We obtain $\cos\frac{2\vartheta}{3}=0$, $\frac{2\vartheta}{3}\in\{\pm\arccos 0+2k\pi k\in Z\}\Leftrightarrow\vartheta\in\left\{\frac{3}{2}\left(\pm\frac{\pi}{2}+2k\pi\right)k\in Z\right\}=\pm\frac{3\pi}{4}+3Z\pi$.

Solution 8 by HyunBin Yoo, South Korea

Move all of the terms to the left-hand side.

$$(\cos^2\theta-\sin^2\theta)+9\left(\cos^2\left(\frac{\theta}{3}\right)-\sin^2\left(\frac{\theta}{3}\right)\right)+6\left(\cos(\theta)\cos\left(\frac{\theta}{3}\right)+\sin(\theta)\sin\left(\frac{\theta}{3}\right)\right)=0$$

Use the angle sum identities to further simplify the equation.

$$\cos(\theta+\theta)+9\cos\left(\frac{\theta}{3}+\frac{\theta}{3}\right)+6\cos\left(\theta-\frac{\theta}{3}\right)=\cos(2\theta)+9\cos\left(\frac{2}{3}\theta\right)+6\cos\left(\frac{2}{3}\theta\right)=\cos(2\theta)+15\cos(2\theta)=0$$

Notice that 2θ is the triple of $\dfrac{2}{3}\theta$. So we can express $\cos(2\theta)$ in terms of $\cos\left(\dfrac{2}{3}\theta\right)$ using the angle sum identities.

$$\cos(2\theta)=\cos\left(\frac{2}{3}\theta+\frac{4}{3}\theta\right)=\cos\left(\frac{2}{3}\theta\right)\cos\left(\frac{4}{3}\theta\right)-\sin\left(\frac{2}{3}\theta\right)\cos\left(\frac{4}{3}\theta\right)=\cos\left(\frac{2}{3}\theta\right)\left(2\cos^2\left(\frac{2}{3}\theta\right)-1\right)-$$

$\sin\left(\frac{2}{3}\theta\right)\left(2\sin\left(\frac{2}{3}\theta\right)\cos\left(\frac{2}{3}\theta\right)\right) = 2\cos^3\left(\frac{2}{3}\theta\right) - \cos\left(\frac{2}{3}\theta\right) - 2\sin^2\left(\frac{2}{3}\theta\right)\cos\left(\frac{2}{3}\theta\right) =$
$2\cos^3\left(\frac{2}{3}\theta\right) - \left(1 + 2\sin^2\left(\frac{2}{3}\theta\right)\right)\cos\left(\frac{2}{3}\theta\right) = 2\cos^3\left(\frac{2}{3}\theta\right) - \left(3 - 2\cos^2\left(\frac{2}{3}\theta\right)\right)\cos\left(\frac{2}{3}\theta\right) =$
$4\cos^3\left(\frac{2}{3}\theta\right) - 3\cos\left(\frac{2}{3}\theta\right)$

Substituting the $\cos(2\theta)$ above, we get $4\cos^3\left(\frac{2}{3}\theta\right) + 12\cos\left(\frac{2}{3}\theta\right) = 0$. Factoring it results in $4\cos^3\left(\frac{2}{3}\theta\right)\left(\cos^2\left(\frac{2}{3}\theta\right) + 3\right) = 0$. Since $\cos^2\left(\frac{2}{3}\theta\right) + 3$ can never be zero, the only solution is $\cos\left(\frac{2}{3}\theta\right) = 0$.

$\frac{2}{3}\theta = 2n\pi \pm \frac{\pi}{2}(n \in Z)$
$\theta = 3n\pi \pm \frac{3}{4}\pi(n \in Z)$

Also solved by Brian D. Beasley, Presbyterian College, Clinton, SC; Anthony Bevelacqua, University of North Dakota, Grand Falls, ND; Brian Bradie, Christopher Newport University, Newport News,VA; Charles Burnette, Xavier University of Louisiana (New Orleans, LA); Bruno Salgueiro Fanego, Viveiro Spain; Peter Fulop, Gyomro, Hungary; Paul M. Harms, North Newton, KS; Kee-Wai Lau, Hong Kong, China; Albert Natian, Los Angeles Valley College,Valley Glen, CA; SQ Mathematical Problem Solving Group,Yogyakarta, Indonesia; Seán M. Stewart, Bomaderry, NSW, Australia; David Stone and John Hawkins, Georgia Southern University, Statesboro, GA, and the proposer.

- **5615:** *Proposed by Pedro Henrique Oliveira Pantoja, University of Campina Grande, Brazil*

  Solve in $\Re \times \Re$ the system:

$$\begin{cases} \sqrt[3]{2x+2} + \sqrt[3]{4-x} + \sqrt[3]{2-x} = 2 \\ \\ \sqrt[5]{20-2y} + \sqrt[5]{7-y} + \sqrt[5]{3y+5} = 2 \end{cases}$$

Solution 1 by Michel Bataille, Rouen, France

As a lemma, we first show the following: Let a, b, c be real numbers such that $a^3+b^3+c^3 = (a+b+c)^3 = 8$ (resp. $a^5+b^5+c^5 = (a+b+c)^5 = 32$). Then, one of a, b, c is equal to 2.
Proof. We introduce the polynomial $P(X) = (X-a)(X-b)(X-c) = X^3 - 2X^2 + mX - p$ where $p = abc, m = ab+bc+ca$. From the identity

$$a^3+b^3+c^3-3abc = (a+b+c)((a+b+c)^2 - 3(ab+bc+ca)),$$

we deduce that $8 = 3abc + 2(4 - 3(ab + bc + ca))$, that is, $p = 2m$. As a result,

$$P(X) = X^3 - 2X^2 + mX - 2m = (X^2 + m)(X - 2),$$

hence one of the roots a, b, c of P is 2.
In the case when $a^5 + b^5 + c^5 = (a + b + c)^5 = 32$, Newton's formulas successively give $a^2 + b^2 + c^2 = 4 - 2m$, $a^3 + b^3 + c^3 = 8 + 3p - 6m$, $a^4 + b^4 + c^4 = 16 + 2m^2 + 8p - 16m$ and $a^5 + b^5 + c^5 = 32 + 20p - 40m + 10m^2 - 5pm$ and the latter yields $(p - 2m)(4 - m) = 0$. However, we have $m \neq 4$ since otherwise $P'(X) = 3X^2 - 4X + 4$ is positive and P, as a strictly monotone function, could not have three real roots. Thus, we again have $p = 2m$ and we conclude as above.
Turning to the problem, if x is a solution, let us set $a = \sqrt[3]{2x + 2}, b = \sqrt[3]{4 - x}, c = \sqrt[3]{2 - x}$. Then we see that $a + b + c = 2$ and $a^3 + b^3 + c^3 = 8$. From the lemma, we obtain $a = 2$ or $b = 2$ or $c = 2$, hence $x = 3$ or $x = -4$ or $x = -6$. Conversely, it is readily checked that $3, -4, -6$ are indeed solutions. Thus, the solutions to the first equation are $3, -4, -6$.
Similarly, if y is a solution to the second equation, this time setting $a = \sqrt[5]{20 - 2y}, b = \sqrt[5]{7 - y}, c = \sqrt[5]{3y + 5}$, we obtain $a + b + c = 2$ and $a^5 + b^5 + c^5 = 32$. The lemma gives that a or b or c is equal to 2, which leads to $y = -6$ or $y = -25$ or $y = 9$. Conversely each of these three numbers is a solution. Thus, the solutions to the second equation are $-6, -25$ and 9.

Solution 2 by David E. Manes, Oneonta, NY

By inspection, the solutions (x, y) of the system are: $(3, -6)$, $(3, 9)$, $(3, -25)$, $(-4, -6)$, $(-4, 9)$, $(-4, -25)$, $(-6, -6)$, $(-6, 9)$ and $(-6, -25)$. One verifies that each of these nine ordered pairs is a solution of the system. They are the only solutions.

For the first equation in the system, consider the equivalent equation $(2x + 2)^{1/3} + (4 - x)^{1/3} = 2 - (2 - x)^{1/3}$. Raising each side of the equation to the third power and simplifying, one obtains the following

$$3(2x + 2)^{1/3}(4 - x)^{1/3}\left[2 - (2 - x)^{1/3}\right] = -3 \cdot 2(2 - x)^{1/3}\left[2 - (2 - x)^{1/3}\right].$$

Noting that $(2x + 2)^{1/3} + (4 - x)^{1/3} = 2 - (2 - x)^{1/3}$. Therefore,

$$(2x + 2)^{1/3}(4 - x)^{1/3} = -2(2 - x)^{1/3}$$

provided $2 - (2 - x)^{1/3} \neq 0$. If $2 - (2 - x)^{1/3} = 0$, then $(2 - x)^{1/3} = 2$ and $x = -6$, a solution for the first equation in the system. Cubing each side of the above displayed equation, one obtains $(2x + 2)(4 - x) = -8(2 - x)$. The resulting quadratic equation $x^2 + x - 12 = 0$ has roots $x = -4$ and $x = 3$, both of which are solutions to the first equation in the system.

For the second equation in the system, consider the equation

$$\left[(20 - 2y)^{1/5} + (7 - y)^{1/5}\right]^5 = \left[2 - (3y + 5)^{1/5}\right]^5.$$

Then expanding and simplifying, one obtains

$$\left[2 - (3y + 5)^{1/5}\right] \cdot \left[(uv)^{1/5}\left(u^{2/5} + (uv)^{1/5} + v^{2/5}\right) + 2\left(w^{1/5}\right)\left(2^2 - 2w^{1/5} + w^{2/5}\right)\right] = 0,$$

where $u = 20 - 2y$, $v = 7 - y$ and $w = 3y + 5$. Therefore, $2 - (3y + 5)^{1/5} = 0$ implies $y = 9$. Similarly, $[(7 - y)^{1/5} + (3y + 5)^{1/5}]^5 = [2 - (20 - 2y)^{1/5}]^5$ implies

$$\left[2 - (20 - 2y)^{1/5}\right] \cdot \left[(uv)^{1/5}\left(u^{2/5} + (uv)^{1/5} + v^{2/5}\right) + 2w^{1/5}\left(2^2 - 2w^{1/5} + w^{2/5}\right)\right] = 0,$$

where $u = 7 - y$, $v = 3y + 5$ and $w = 20 - 2y$. Therefore, $2 - (20 - 2y)^{1/5} = 0$ implies $y = -6$. Finally, if $[(20 - 2y)^{1/5} + (3y + 5)^{1/5}]^5 = [2 - (7 - y)^{1/5}]^5$, then

$$[2 - (7 - y)^{1/5}] \cdot \left[(uv)^{1/5}\left(u^{2/5} + (uv)^{1/5} + v^{2/5}\right) + 2w^{1/5}\left(2^2 - 2w^{1/5} + w^{2/5}\right)\right] = 0,$$

where $u = 20 - 2y$, $v = 3y + 5$ and $w = 7 - y$. Therefore, $2 - (7 - y)^{1/5} = 0$ implies $y = -25$. This completes the solution.

Solution 3 by Kee-Wai Lau, Hong Kong, China

We show that the solutions are $x = -6, -4, 3$ and $y = -25, -6, 9$.

For real numbers a and b, let $s = \sqrt[3]{a} + \sqrt[3]{b}$ and $t = \sqrt[5]{a} + \sqrt[5]{b}$.

By direct expansion, we can prove readily that

$$s^9 - 3(a + b)s^6 + 3(a^2 - 7ab + b^2)s^3 - (a + b)^3 = 0, \tag{1}$$

and

$$\begin{aligned} &t^{25} - 5(a + b)t^{20} + 5(2a^2 - 121ab + 2b^2)t^{15} - 5(a + b)(2a^2 + 379ab + 2b^2)t^{10} + \\ + \; &5(a^4 - 121a^3b + 381a^2b^2 - 121ab^3 + b^4)t^5 - (a + b)^5 = 0. \end{aligned} \tag{2}$$

By the substitution $x = 2 + (s - 2)^3$, we see that the first equation of the problem can be written as $\sqrt[3]{6 + 2(s - 2)^3} + \sqrt[3]{2 - (s - 2)^3} = s$. We put $a = 6 + 2(s - 2^3)$ and $b = 2 - (s - 2)^3$ into (1) and after simplification and factorization we obtain

$$s^3((s - 2)^3 + 6)(s - 3)(s^2 - 3s + 3) = 0.$$

Since s is real, so $s = 0, 2 - \sqrt[3]{6}, 3$, giving $x = -6, -4, 3$.

By the substitution $y = 7 + (t - 2)^5$, we see that the second equation of the problem can be written as $\sqrt[5]{6 - 2(t - 2)^5} + \sqrt[5]{26 + 3(t - 2)^5} = t$. We put $a = 6 - 2(t - 2)^5$ and $b = 26 + 3(t - 2)^5$ into (2) and after simplification and factorization we obtain

$$t^5((t - 2)^5 + 13)((t - 2)^5 - 2)f(t) = 0,$$

where

$$f(t) = 7t^{10} - 130t^9 + 1140t^8 - 6000t^7 + 20880t^6 - 49982t^5 + 83180t^4 - 94640t^3 + 70240t^2 - 30560t + 5924.$$

By using the software *Mathematica,* we find that all the roots of $f(t) = 0$ are not real. Hence, $t = 0,\ 2 - \sqrt[5]{13},\ 2 + \sqrt[5]{2}$ giving $y = -25, -6, 9$.

Solution 4 by Peter Fulop, Gyomro, Hungary

$$\sqrt[3]{2x + 2} + \sqrt[3]{4 - x} + \sqrt[3]{2 - x} = 2 \tag{1}$$

$$\sqrt[5]{20 - 2y} + \sqrt[5]{7 - y} + \sqrt[5]{3y + 5} = 2 \tag{2}$$

Solving equation (1):

Rearrange the equation (1): $\sqrt[3]{4-x}+\sqrt[3]{2-x}=2-\sqrt[3]{2x+2}$

Let's raise to the third power:

$2\sqrt[3]{2x+2}-2+\sqrt[3]{2x+2}=\sqrt[3]{2-x}\sqrt[3]{4-x}\underbrace{\left(\sqrt[3]{2-x}+\sqrt[3]{4-x}\right)}_{2-\sqrt[3]{2x+2}}$

After the cancellations:

$$\left(2-\sqrt[3]{2x+2}\right)\left(2\sqrt[3]{2x+2}+\sqrt[3]{2-x}\sqrt[3]{4-x}\right)=0 \tag{3}$$

Equation (3) gives two equations both are equal to zero.

Solving the $2-\sqrt[3]{2x+2}=0$ we get $\underline{\underline{x_1=3}}$

and the other $2\sqrt[3]{2x+2}=-\sqrt[3]{2-x}\sqrt[3]{4-x}$ and square it: $x^2+10x+24=0$

Resulting two other roots: $\underline{\underline{x_2=-4}}$ and $\underline{\underline{x_3=-6}}$

Solving equation (2):

Let $a=\sqrt[5]{20-2y}$; $b=\sqrt[5]{7-y}$; $c=\sqrt[5]{3y+5}$ and $d=2$.

Based on (2) we have: $a+b=c-d$ raise it on fifth power and let's realize that $a^5+b^5=d^5-c^5$ we get:

$$5ab(a^3+b^3)+10a^2b^2(a+b)=5cd(c^3-d^3)-10c^2d^2(c-d) \tag{4}$$

Using the appropriate identity on both sides of (4):

$$x^3\pm y^3=(x\pm y)(x^2\mp xy+y^2)$$

We get:

$$ab(a+b)(a^2-ab+b^2)+2a^2b^2(a+b)=-cd(a+b)(a^2+ab+b^2)+2c^2d^2(a+b) \tag{5}$$

It can be seen that $a+b=0$ provides first root.

$$\sqrt[5]{20-2y}=-\sqrt[5]{7-y} \text{ namely: } 20-2y=y-7$$

$$\underline{\underline{y_1=9}}$$

Rearranging (5): $ab\Big[(a+b)^2-ab\Big]=-cd\Big[(c-d)^2+cd\Big]$

and using $(c-d)^2=(a+b)^2$ we get:

$$(a+b)^2(ab+cd)=(ab+cd)(ab-cd) \tag{6}$$

$(ab+cd)=0$ provides the second two roots:

Namely $\sqrt[5]{20-2y}\sqrt[5]{7-y}=-2\sqrt[5]{3y+5}$ gives us the following quadratic equation:

$$y^2+31y+150=0. \tag{7}$$

We get further two roots:

$$\underline{\underline{y_2 = -6}} \qquad \text{and} \qquad \underline{\underline{y_3 = -25}}$$

The remaining part of (6):

$$(a+b)^2 = ab - cd \tag{8}$$

$$\begin{aligned} b^2 + (a-1)b - a^2 - a + 2 &= 0 \\ b_{1,2} &= \frac{1}{2}\left(1 - a \pm \sqrt{-3a^2 + 2a - 7}\right) \end{aligned}$$

b should be real number but the discriminant less than zero so (8) will not give real a, b.

On the other hand: $a^2 + ab + b^2 = -4a + 2a + 2b$ and it is known that $\dfrac{a^3 - b^3}{a - b} = a^2 + ab + b^2$; hence we obtain:

$$a(a^2 - 2a + 4) = b(b^2 - 2b + 4)$$

$a = b$ does not provide a solution.

Finally the roots are:

$$\boxed{x_1 = 3\ , \quad x_2 = -4, \quad x_3 = -6}$$

$$\boxed{y_1 = 9\ , \quad y_2 = -6, \quad y_3 = -25}$$

Also solved by Charles Burnette, Xavier University of Louisiana (New Orleans, LA); SQ Mathematical Problem Solving Group, Yogyakarta, Indonesia; Albert Stadler, Herrliberg, Switzerland; David Stone and John Hawkins, Georgia Southern University, Statesboro, GA; Daniel Văcaru, Pitesti, Romania, and the proposer.

- **5616:** *Proposed by D.M. Bătinetu-Giurgiu "Matei Basarb" National College, Bucharest and Neculai Stanciu, "George Emil Palade" Secondary School Buzău, Romania*

Prove that in all tetrahedrons $[ABCD]$ the following inequality holds:

$$\frac{1}{h_a}\sqrt[3]{\frac{h_b h_c}{h_a^2}} + \frac{1}{h_b}\sqrt[3]{\frac{h_c h_d}{h_b^2}} + \frac{1}{h_c}\sqrt[3]{\frac{h_d h_a}{h_c^2}} + \frac{1}{h_d}\sqrt[3]{\frac{h_a h_b}{h_d^2}} \geq \frac{1}{r},$$

where r is the radius of the insphere of the tetrahedron.

Solution 1 by Moti Levy, Rehovot, Israel

Let us substitute h_a, h_b, h_c and h_d as follows:

$$x := \frac{1}{h_a},\ y := \frac{1}{h_b},\ z := \frac{1}{h_c},\ t := \frac{1}{h_d}.$$

It is known that $\sum_{cyc} \frac{1}{h_a} = \frac{1}{r}$, hence

$$x + y + z + t = \frac{1}{r}.$$

Therefore, the original inequality is equivalent to

$$\frac{3x^2}{x+y+z} + \frac{3y^2}{y+z+t} + \frac{3z^2}{z+t+x} + \frac{3t^2}{t+x+y} \geq x + y + z + t.$$

The inequality is homogenous, hence we may assume $x+y+z+t = 1$. With this constraint the inequality becomes :

$$\frac{x^2}{1-t} + \frac{y^2}{1-x} + \frac{z^2}{1-y} + \frac{t^2}{1-z} \geq \frac{1}{3}, \qquad x + y + z + t = 1.$$

Now we apply Radon's inequality

$$\frac{x^2}{1-t} + \frac{y^2}{1-x} + \frac{z^2}{1-y} + \frac{t^2}{1-z}$$
$$\geq \frac{(x+y+z+t)^2}{(1-t+1-x+1-y+1-z)} = \frac{1}{3}.$$

Remark: Radon's inequality states:
If $x_k, a_k > 0,\ k \in \{1, 2\ldots, n\},\ p > 0,$ *then*

$$\frac{x_1^{p+1}}{a_1^p} + \frac{x_2^{p+1}}{a_2^p} + \cdots + \frac{x_2^{n+1}}{a_2^n} \geq \frac{(x_1 + x_2 + \cdots + x_n)^{p+1}}{(a_1 + a_2 + \cdots + a_n)^p}.$$

Solution 2 by Michel Bataille, Rouen, France

Regrettably, this problem is not new. It is part (b) of the *College Mathematics Journal* problem 1018 proposed by the same authors in May 2014. A solution was published in the May 2015 issue (Vol. 46, No 3).
Here is the solution that I sent in 2014.
If F_a, F_b, F_c, F_d are the areas of faces BCD, CDA, DAB, ABC, respectively, of tetrahedron $ABCD$ and V its volume, we have

$$V = \frac{1}{3}h_a \cdot F_a = \frac{1}{3}h_b \cdot F_b = \frac{1}{3}h_c \cdot F_c = \frac{1}{3}h_d \cdot F_d = \frac{1}{3}(r \cdot F_a + r \cdot F_b + r \cdot F_c + r \cdot F_d)$$

and we deduce that

$$\frac{1}{h_a} + \frac{1}{h_b} + \frac{1}{h_c} + \frac{1}{h_d} = \frac{1}{r}.$$

Now, with $\alpha = \frac{1}{h_a}, \beta = \frac{1}{h_b}, \gamma = \frac{1}{h_c}, \delta = \frac{1}{h_d}$, the left-hand side L of the inequality rewrites as

$$\frac{\alpha^2}{\sqrt[3]{\alpha\beta\gamma}} + \frac{\beta^2}{\sqrt[3]{\beta\gamma\delta}} + \frac{\gamma^2}{\sqrt[3]{\gamma\delta\alpha}} + \frac{\delta^2}{\sqrt[3]{\delta\alpha\beta}}.$$

From the arithmetic mean-geometric mean inequality, we deduce

$$L \geq 3\left(\frac{\alpha^2}{\alpha+\beta+\gamma} + \frac{\beta^2}{\beta+\gamma+\delta} + \frac{\gamma^2}{\gamma+\delta+\alpha} + \frac{\delta^2}{\delta+\alpha+\beta}\right). \qquad (1)$$

Using the Cauchy-Schwarz inequality, we obtain

$$\left(\frac{\alpha^2}{\alpha+\beta+\gamma}+\frac{\beta^2}{\beta+\gamma+\delta}+\frac{\gamma^2}{\gamma+\delta+\alpha}+\frac{\delta^2}{\delta+\alpha+\beta}\right)(3(\alpha+\beta+\gamma+\delta)\geq(\alpha+\beta+\gamma+\delta)^2. \tag{2}$$

From (1) and (2) it now follows that

$$L\geq\alpha+\beta+\gamma+\delta=\frac{1}{r},$$

as required.

Editor′s comment: Normally it does not happen that an identical problem appears in two different journals. But in this case it happened and the fault is most likely mine. I have a stack of problems submitted from many individuals from around the world. Sometimes these problems have been submitted to our column several years earlier before being selected for publication. At that time my system did not have a date of submission written on every proposed problem. So it could well be that I had said to the authors of the problem that I liked it and that it would be published in some future issue of the column. But I never specify which issue and sometimes the authors might have to wait years before seeing it in print. That might be what happened here; and if so, I accept the blame for this "fashla." Future editors will have to develop a better protocol for acknowledging submissions.

Solution 3 by Albert Stadler, Herrliberg, Switzerland

Let Δ be the volume of the tetrahedron and let $\Delta_A, \Delta_B, \Delta_C$ be the area of the face opposite the vertex A, B, C, respectively. Then (see for instance http://en.wikipedia.org/wiki/Tetrahedron)

$$\Delta=\frac{1}{3}h_A\Delta_A=\frac{1}{3}h_B\Delta_B=\frac{1}{3}h_C\Delta_C=\frac{1}{3}h_D\Delta_D=\frac{1}{3}\left(\Delta_A+\Delta_B+\Delta_C+\Delta_D\right)r.$$

The inequality is therefore equivalent to each of the following lines:

$$\sum_{cycl}\frac{1}{h_A}\sqrt[3]{\frac{th_Bh_C}{(h_A)^2}}\geq\frac{1}{r},$$

$$\sum_{cycl}\frac{3\Delta}{h_A}\sqrt[3]{\frac{h_Bh_C}{(h_A)^2}}\geq\frac{3\Delta}{r},$$

$$\sum_{cycl}\Delta_A\sqrt[3]{\frac{(\Delta_A)^2}{\Delta_B\Delta_C}}\geq\Delta_A+\Delta_B+\Delta_C+\Delta_D,$$

$$\sum_{cycl}\Delta_A\sqrt[3]{\frac{(\Delta_A)^3}{\Delta_A\Delta_B\Delta_C}}\geq\Delta_A+\Delta_B+\Delta_C+\Delta_D,$$

$$\sum_{cycl}\left(\Delta_A^2\right)\sqrt[3]{\Delta_D)}\geq\left(\Delta_A+\Delta_B+\Delta_C+\Delta_D\right)\sqrt[3]{\Delta_A\Delta_B\Delta_C\Delta_D}.$$

However the last inequality is true, since by the AM-GM inequality,

$$\sum_{cycl} (\Delta_A)^2 \sqrt[3]{\Delta_D} = \sum_{cycl} \left(\frac{119}{185} (\Delta_A)^2 \sqrt[3]{\Delta_D} + \frac{26}{185} (\Delta_B)^2 \sqrt[3]{\Delta_A} + \frac{29}{185} (\Delta_C)^2 \sqrt[3]{\Delta_B} + \frac{11}{185} (\Delta_C)^2 \sqrt[3]{\Delta_C} \right) \geq$$

$$\geq \sum_{cycl} \left((\Delta_A)^{2\cdot\frac{119}{185}} (\Delta_D)^{\frac{1}{3}\cdot\frac{119}{185}} (\Delta_B)^{2\cdot\frac{26}{185}} (\Delta_A)^{\frac{1}{3}\cdot\frac{26}{185}} (\Delta_C)^{2\cdot\frac{29}{185}} (\Delta_B)^{\frac{1}{3}\cdot\frac{29}{185}} (\Delta_D)^{2\cdot\frac{11}{185}} (\Delta_C)^{\frac{1}{3}\cdot\frac{11}{185}} = \right.$$

$$(\Delta_A + \Delta_B + \Delta_C + \Delta_D) \sqrt[3]{\Delta_A \Delta_B \Delta_C \Delta_D}.$$

Solution 4 by Kee-Wai Lau, Hong Kong, China

By the inequality of Cauchy-Schwarz we have

$$\frac{1}{h_a}\sqrt[3]{\frac{h_b h_c}{h_a^2}} + \frac{1}{h_b}\sqrt[3]{\frac{h_c h_d}{h_b^2}} + \frac{1}{h_c}\sqrt[3]{\frac{h_d h_a}{h_c^2}} + \frac{1}{h_d}\sqrt[3]{\frac{h_a h_b}{h_d^2}} =$$

$$= \frac{1}{h_a^2}\sqrt[3]{h_a h_b h_c} + \frac{1}{h_b^2}\sqrt[3]{h_b h_c h_d} + \frac{1}{h_c^2}\sqrt[3]{h_c h_d h_a} + \frac{1}{h_d^2}\sqrt[3]{h_d h_a h_b} \geq$$

$$\geq \frac{\left(\frac{1}{h_a} + \frac{1}{h_b} + \frac{1}{h_c} + \frac{1}{h_d}\right)^2}{\sqrt[3]{\frac{1}{h_a h_b h_c}} + \sqrt[3]{\frac{1}{h_b h_c h_d}} + \sqrt[3]{\frac{1}{h_c h_d h_a}} + \sqrt[3]{\frac{1}{h_d h_a h_b}}}$$

By the AM-GM inequality, we see that the denominator of the last expression does not exceed

$$\frac{1}{3}\left(\left(\frac{1}{h_a} + \frac{1}{h_b} + \frac{1}{h_c}\right) + \left(\frac{1}{h_b} + \frac{1}{h_c} + \frac{1}{h_d}\right) + \left(\frac{1}{h_c} + \frac{1}{h_d} + \frac{1}{h_a}\right) + \left(\frac{1}{h_d} + \frac{1}{h_a} + \frac{1}{h_b}\right)\right)$$

$$= \frac{1}{h_a} + \frac{1}{h_b} + \frac{1}{h_c} + \frac{1}{h_d}.$$

Now the inequality of the problem follows from the well known fact that
$\frac{1}{h_a} + \frac{1}{h_b} + \frac{1}{h_c} + \frac{1}{h_d} = \frac{1}{r}$.

Also solved by the proposers

- **5617:** *Proposed by José Luis Díaz-Barrero, Barcelona Tech, Barcelona, Spain*

Let a, b, c be the roots of the equation $x^3 + rx + s = 0$. Without the aid of a computer, calculate

$$\det\begin{vmatrix} 2bc - a^2 & c^2 & b^2 \\ c^2 & 2ca - b^2 & a^2 \\ b^2 & a^2 & 2ab - c^2 \end{vmatrix}$$

Solution 1 by Titu Zvonaru, Comănesti, Romania

We have $a+b+c=0$. Adding the second column and the third column to the first column, we obtain;

$$\det\begin{vmatrix} 2bc-a^2 & c^2 & b^2 \\ c^2 & 2ca-b^2 & a^2 \\ b^2 & a^2 & 2ab-c^2 \end{vmatrix} = \det\begin{vmatrix} (b+c)^2-a^2 & c^2 & b^2 \\ (c+a)^2-b^2 & 2ca-b^2 & a^2 \\ (a+b)^2-c^2 & a^2 & 2ab-c^2 \end{vmatrix} =$$

$$\det\begin{vmatrix} (a+b+c)(b+c-a) & c^2 & b^2 \\ (c+a+b)(c+a-b) & 2ca-b^2 & a^2 \\ (a+b+c)(a+b-c) & a^2 & 2ab-c^2 \end{vmatrix} = \det\begin{vmatrix} 0 & c^2 & b^2 \\ 0 & 2ca-b^2 & a^2 \\ 0 & a^2 & 2ab-c^2 \end{vmatrix} = 0.$$

Solution 2 by Brian Bradie, Christopher Newport University, Newport News, VA

First,

$$\det\begin{vmatrix} 2bc-a^2 & c^2 & b^2 \\ c^2 & 2ca-b^2 & a^2 \\ b^2 & a^2 & 2ab-c^2 \end{vmatrix}$$

is equal to

$$\begin{aligned}
&(2bc-a^2)(2ca-b^2)(2ab-c^2)+2a^2b^2c^2-(2b^4ca-b^6+2a^4bc-a^6+2c^4ab-c^6)\\
&= 9a^2b^2c^2-4abc(a^3+b^3+c^3)+2(a^3b^3+b^3c^3+c^3a^3)-2abc(a^3+b^3+c^3+a^6+b^6+c^6\\
&= 9a^2b^2c^2-6abc(a^3+b^3+c^3)+(a^3+b^3+c^3)^2.
\end{aligned}$$

Next, because a,b,c are the roots of the equation $x^3+rx+s=0$, it follows that

$$a+b+c=0, \quad ab+bc+ca=r, \quad \text{and} \quad abc=-s.$$

Moreover,

$$a^3+b^3+c^3=-r(a+b+c)-3s=-3s.$$

Thus,

$$\det\begin{vmatrix} 2bc-a^2 & c^2 & b^2 \\ c^2 & 2ca-b^2 & a^2 \\ b^2 & a^2 & 2ab-c^2 \end{vmatrix} = 9s^2-6(-s)(-3s)+9s^2=0.$$

Solution 3 by Bruno Salgueiro Fanego, Viveiro, Spain

Expanding the determinate in the normal way we obtain:

$$\det\begin{vmatrix} 2bc-a^2 & c^2 & b^2 \\ c^2 & 2ca-b^2 & a^2 \\ b^2 & a^2 & 2ab-c^2 \end{vmatrix} =$$

$= (2bc{-}a^2)(2ca{-}b^2)(2ab{-}c^2){+}c^2a^2b^2{+}b^2a^2c^2{-}\big(b^2(2ca-b^2)b^2+c^2c^2(2ab-c^2)+(2bc-a^2)a^2a.$

After simplification we obtain:

$$\big((a+b+c)\big)^3-3(a+b)^2c-3(a+b))c^2-3ab(a+b+c)\big)^2$$

$$(a{+}b{+}c)^2\left((a+b+c)^2-3(a+b)c-3ab\right)^2=(a{+}b{+}c)^2\left((a+b+c)^2-3(ab+bc+ca)\right)^2.$$

Since a, b, and c are the roots of $x^3+rx+s=0$, we know that $a+b+c=0$. So

$$\det\begin{vmatrix}2bc-a^2 & c^2 & b^2\\ c^2 & 2ca-b^2 & a^2\\ b^2 & a^2 & 2ab-c^2\end{vmatrix}=\Big(\underbrace{a+b+c}_{0}\Big)^2\left((a+b+c)^2-3(ab+bc+ca)\right)^2=0.$$

Solution 4 by Daniel Vácaru, Pitesti, Romania

We add all the lines to the first line, and we obtain

$$\det\begin{vmatrix}b^2+2bc+c^2-a^2 & c^2+2ca+a^2-b^2 & a^2+2ab+b^2-c^2\\ a^2+2ab+b^2-c^2a^2 & & \\ b^2 & a^2 & 2ab-c^2\end{vmatrix}=$$

$$=\det\begin{vmatrix}(b+c)^2-a^2 & (a+c)^2-b^2 & (a+b)^2-c^2\\ c^2 & 2ca-b^2 & a^2\\ b^2 & a^2 & 2ab-c^2\end{vmatrix}=$$

$$=\det\begin{vmatrix}(b+c+a)(b+c-a) & (c+a+b)(c+a-c) & (a+b+c)(a+b-c)\\ c^2 & 2ca-b^2 & a^2\\ b^2 & a^2 & 2ab-c^2\end{vmatrix}$$

By Viéa's formula, $a+b+c=0$, the whole first line is 0, implying that

$$\det\begin{vmatrix}2bc-a^2 & c^2 & b^2\\ c^2 & 2ca-b^2 & a^2\\ b^2 & a^2 & 2ab-c^2\end{vmatrix}=0.$$

Also solved by Dionne Bailey, Elsie Campbell, and Charles Diminnie, Angelo State University, San Angelo, TX; Michel Bataille, Rouen France; Charles Burnette, Xavier University of Louisiana (New Orleans, LA); Pratik Donga, Junagadh, India; Michal N. Fried Ben-Gurion University of the Negev, Beer-Sheva, Israel; Peter Fulop, Gyomro, Hungary; G. C. Greubel, Newport News, VA; Paul M. Harms, North Newton, KS; Kee-Wai Lau, Hong Kong, China; Moti Levy, Rehovot, Israel; David E. Manes, Oneonta, NY; Trey Smith, Angelo State University, San Angelo, TX; SQ Mathematical Problem Solving Group, Yogyakrta, Indonesia; Albert Stadler, Herrliberg, Switzerland; Seán M.Stewart, Bomaderry, NSW, Australia; David Stone and John Hawkins, Georgia Southern University, Statesboro, GA, and the proposer.

5618: *Proposed by Ovidiu Furdui and Alina Sîntămărian, Technical University of Cluj-Napoca, Cluj-Napoca, Romania*

Let $k > 0$ be a real number. Calculate

$$\lim_{n\to\infty} n^2\left(\frac{1}{n^3}+\frac{1}{(n+k)^3}+\frac{1}{(n+2k)^3}+\cdots\right).$$

Solution 1 Brian Bradie, Christopher Newport University, Newport News, VA

First,

$$\frac{1}{n^3}+\frac{1}{(n+k)^3}+\frac{1}{(n+2k)^3}+\cdots=\sum_{j=0}^{\infty}\frac{1}{(n+jk)^3}=\frac{1}{n^3}\sum_{j=0}^{\infty}\frac{1}{(1+j\cdot\frac{k}{n})^3},$$

so

$$n^2\left(\frac{1}{n^3}+\frac{1}{(n+k)^3}+\frac{1}{(n+2k)^3}+\cdots\right)=\frac{1}{n}\sum_{j=0}^{\infty}\frac{1}{(1+j\cdot\frac{k}{n})^3}=\frac{1}{n}\sum_{j=0}^{\infty}\frac{1}{(1+k\cdot\frac{j}{n})^3}.$$

Now, recognize that

$$\frac{1}{n}\sum_{j=0}^{\infty}\frac{1}{(1+k\cdot\frac{j}{n})^3}$$

is a left-endpoint approximation to

$$\int_0^{\infty}\frac{1}{(1+kx)^3}\,dx.$$

Thus,

$$\begin{aligned}\lim_{n\to\infty} n^2\left(\frac{1}{n^3}+\frac{1}{(n+k)^3}+\frac{1}{(n+2k)^3}+\cdots\right) &= \int_0^{\infty}\frac{1}{(1+kx)^3}\,dx\\ &= \left(-\frac{1}{2k(1+kx)^2}\right)\Bigg|_0^{\infty}\\ &= \frac{1}{2k}.\end{aligned}$$

Solution 2 by Michel Bataille, Rouen, France

Let $S_n=\sum\limits_{j=0}^{\infty}\frac{1}{(n+jk)^3}$. We claim that $\lim\limits_{n\to\infty} n^2S_n=\frac{1}{2k}$.

Since the function $x\mapsto\frac{1}{(n+xk)^3}$ is decreasing on $(0,\infty)$, we have

$$\frac{1}{(n+(j+1)k)^3}\le\int_j^{j+1}\frac{dx}{(n+xk)^3}\le\frac{1}{(n+jk)^3}$$

for $j=0,1,2,\ldots$. It follows that for any positive integer J,

$$\int_0^{J+1}\frac{dx}{(n+xk)^3}\le\sum_{j=0}^{J}\frac{1}{(1+jk)^3}\le\frac{1}{n^3}+\int_0^{J}\frac{dx}{(n+xk)^3}. \tag{1}$$

From

$$\int_0^{J}\frac{dx}{(n+xk)^3}\le\int_0^{\infty}\frac{dx}{(n+xk)^3}=\frac{1}{2kn^2}$$

we deduce that

$$\sum_{j=0}^{J} \frac{1}{(1+jk)^3} \leq \frac{1}{n^3} + \frac{1}{2kn^2}.$$

Therefore, the series $\sum\limits_{j=0}^{\infty} \frac{1}{(n+jk)^3}$ is convergent and its sum S_n satisfies $S_n \leq \frac{1}{n^3} + \frac{1}{2kn^2}$. In addition, letting $J \to \infty$ in the left inequality of (1), we obtain

$$\int_0^\infty \frac{dx}{(n+xk)^3} \leq S_n$$

and finally we see that

$$\frac{1}{2kn^2} \leq S_n \leq \frac{1}{n^3} + \frac{1}{2kn^2}.$$

The claimed result directly follows from the Squeeze Theorem.

Solution 3 by Albert Natian, Los Angeles Valley College, Valley Glen, California

Answer. $\dfrac{1}{2k}$.

Computation. Suppose (without loss of generality) that $n > k$. It's immediate that for any integer $j \geq 0$:

$$\int_{n+jk}^{n+(j+1)k} \frac{dx}{x^3} < \frac{k}{(n+jk)^3} < \int_{n+(j-1)k}^{n+jk} \frac{dx}{x^3}.$$

Summing up the latter inequalities, we get

$$\frac{1}{2n^2} = \int_n^\infty \frac{dx}{x^3} = \sum_{j=0}^{\infty} \int_{n+jk}^{n+(j+1)k} \frac{dx}{x^3} < k \sum_{j=0}^{\infty} \frac{1}{(n+jk)^3} < \sum_{j=0}^{\infty} \int_{n+(j-1)k}^{n+jk} \frac{dx}{x^3} = \int_{n-k}^\infty \frac{dx}{x^3} = \frac{1}{2(n-k)^2},$$

$$\frac{1}{2k} < n^2 \sum_{j=0}^{\infty} \frac{1}{(n+jk)^3} < \frac{n^2}{2k(n-k)^2}$$

which (by Squeeze Theorem) implies

$$\lim_{n\to\infty} n^2 \sum_{j=0}^{\infty} \frac{1}{(n+jk)^3} = \frac{1}{2k}$$

since $\lim_{n\to\infty} \frac{1}{2k} = \frac{1}{2k}$ and $\lim_{n\to\infty} \frac{n^2}{2k(n-k)^2} = \frac{1}{2k}$.

A Generalization. Suppose $(\alpha_j)_{j=-1}^{\infty}$ is an increasing sequence of positive real numbers. Then

$$\lim_{n\to\infty} n^\nu \sum_{j=0}^{\infty} \frac{\alpha_j - \alpha_{j-1}}{(n+\alpha_j)^{\nu+1}} = \frac{1}{\nu}.$$

Solution 4 by Moti Levy, Rehovot, Israel

$$\lim_{n\to\infty} n^2 \left(\frac{1}{n^3} + \frac{1}{(n+k)^3} + \frac{1}{(n+2k)^3} + \cdots \right)$$
$$= \lim_{n\to\infty} \frac{1}{n} \sum_{j=0}^{\infty} \frac{1}{\left(1+k\frac{j}{n}\right)^3} = \int_0^\infty \frac{1}{(1+kx)^3} = \frac{1}{2k}, \qquad if \quad k > 0.$$

Solution 5 by Ulrich Abel,Technische Hochschule Mittelhessen, Friedberg, Germany

Let $k > 0$. We have

$$\begin{aligned}
& n^2 \left(\frac{1}{n^3} + \frac{1}{(n+k)^3} + \frac{1}{(n+2k)^3} + \cdots \right) \\
= \; & n^2 \sum_{j=0}^{\infty} \frac{1}{\Gamma(3)} \int_0^{\infty} t^2 e^{-(n+jk)t} dt \\
= \; & \frac{n^2}{2} \int_0^{\infty} e^{-nt} \frac{t^2}{1-e^{-kt}} dt \\
= \; & \frac{n^2}{2} \left(\frac{1}{kn^2} + O\left(n^{-3}\right) \right) \to \frac{1}{2k} \qquad (n \to \infty),
\end{aligned}$$

by Watson's lemma for Laplace integrals, since

$$\frac{t^2}{1-e^{-kt}} = \frac{t}{k} + \frac{t^2}{2} + O\left(t^3\right) \qquad (t \to 0).$$

Solution 6 by G.C. Greubel, Newport News, VA

Consider the series

$$\begin{aligned}
S_k &= \sum_{j=0}^{\infty} \frac{1}{(n+j\,k)^3} \\
&= \frac{1}{2} \sum_{j=0}^{\infty} \int_0^{\infty} e^{-nt-jkt}\, t^2 \, dt \\
&= \frac{1}{2} \int_0^{\infty} \frac{e^{-nt}\, t^2}{1-e^{-kt}} \, dt \\
&= \frac{1}{2\,k^3} \int_0^{\infty} \frac{e^{-(n/k)\,u}\, u^2}{1-e^{-u}} \, du \\
&= \frac{1}{k^3}\, \zeta\left(3, \frac{n}{k}\right).
\end{aligned}$$

The Hurwitz zeta function has the asymptotic expansion, for $a \to \infty$,

$$\zeta(s,a) \approx \frac{a^{1-s}}{s-1} - \frac{1}{2\,a^s} + \sum_{j=1}^{\infty} \frac{B_{2j}\,(s)_{2j-1}}{(2j)!\,a^{2j+s-1}}$$

which leads to

$$\zeta\left(3, \frac{n}{k}\right) \approx \frac{k^2}{2\,n^2} - \frac{k^3}{2\,n^3} + \sum_{j=1}^{\infty} \frac{B_{2j}\,(3)_{2j-1}\,k^{2j+2}}{(2j)!\,n^{2j+2}}$$

and

$$\begin{aligned}
\lim_{n\to\infty} n^2\, S_k &= \lim_{n\to\infty} \frac{n^2}{k^3}\, \zeta\left(3, \frac{n}{k}\right) \\
&= \lim_{n\to\infty} \left(\frac{1}{2\,k} - \frac{1}{2\,n} + \frac{k}{4\,n^2} + \mathcal{O}\left(\frac{1}{n^4}\right) \right) \\
&= \frac{1}{2\,k}.
\end{aligned}$$

Also solved by Charles Burnette, Xavier University of Louisiana (New Orleans, LA); Pratik Donga, Junagedh, India; Albert Stadler, Herrliberg, Switzerland; Seán Stewart, Bomaderry, NSW Australia; HyunBin Yoo, South Korea, and the proposer.

Mea Culpa

COVID-19 has played havoc with the snail-mail in most countries. **Paul M. Harms of North Newton, KS** mailed his solution to **5605** on January 1, 2021; it arrived six weeks later. His solution is correct and he should be credited with having solved 5605.