School Science and Mathematics Problem 5613

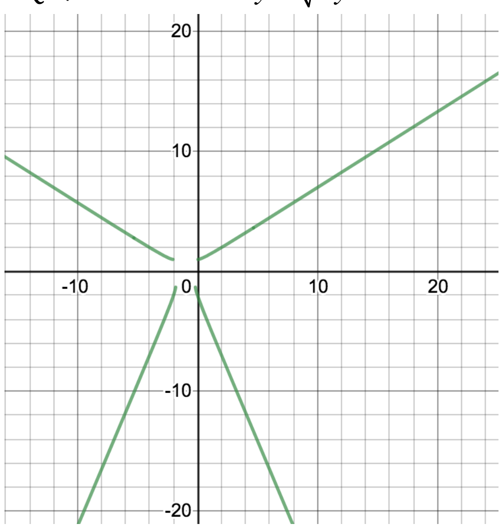
Problem Given the equations:

$$\begin{cases} \sqrt{3x^2+6x+1} = y + \sqrt{3y^2-3} \\ \text{and} \\ ax^2+by^2+cxy+dx+ey+f=0. \end{cases}$$

Find integers {*a, b, c, d, e, f*} so that infinitely many pairs of positive integers (*x, y*) satisfy both equations.
(Proposed by Kenneth Korbin, New York, NY; December 2020)

Solution (David Stone and John Hawkins, Georgia Southern University (retired) Statesboro, Georgia)
We shall show that the hyperbola $3x^2+2y^2-6xy+6x-6y+4=0$ satisfies the given conditions. In fact, *all* of the first quadrant points (x, y) satisfying the first equation also satisfy this generalized quadratic equation.
For convenience, we assign a label, EQ, to the equation $\sqrt{3x^2+6x+1} = y + \sqrt{3y^2-3}$.
From Desmos.com, we have the graph of EQ:

EQ $\sqrt{3x^2+6x+1} = y + \sqrt{3y^2-3}$.



The excluded region in the middle is the rectangle

$$\left\{(x,y) \mid -1-\sqrt{\frac{2}{3}} < x < -1+\sqrt{\frac{2}{3}}, -1 < y < 1\right\}.$$

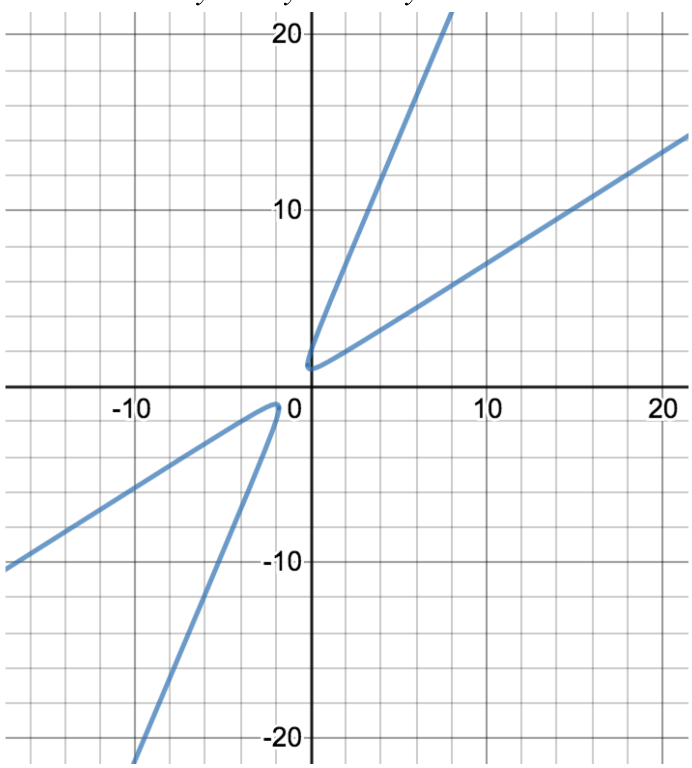
The graph of EQ consists of four separate pieces, which look straight but cannot be line segments because EQ is not linear. We label them S1, S2, S3, S4, where Si lies in Quadrant i. (Actually, S4 creeps into the third quadrant.)
The problem as posed asks us to find a conic which contains infinitely many points (x, y) of S1 for which x and y are both positive integers.

In fact, the conic given above has *all* points of S1 *and* S3 lying on it.

In order that (x, y) have both coordinates positive integers, it must be the case that each quantity under a radical in EQ is a perfect square. Imposing that condition leads us to two Pell's Equation problems, which we solve using standard techniques in Appendix 1. This produces infinitely many such points on S1. Here are the first five of them: (0, 1), (2, 2), (10, 7), (40, 26) (152, 97).
(All other points can be obtained recursively or via Binet-like formulas.)

These five points determine a unique conic; we derive its equation (given above) in Appendix 2. Here is its graph – thanks to Desmos.com -- which we label as CONIC.

CONIC $3x^2+2y^2-6xy+6x-6y+4=0$



If we were to overlay the pictures, it certainly appears that S1 and S3 form the lower halves of the two branches of the hyperbola.
To verify this, we solve each equation for x, focusing on quadrant 1.

From EQ: $x=-1\pm\sqrt{\dfrac{4y^2-1+2y\sqrt{3y^2-3}}{3}}$.

Using the + sign with a positive y produces the points on S1.

From our conic equation: $x=-1+y\pm\sqrt{\dfrac{3y^2-3}{9}}$.

Using the + sign with a positive y produces the lower half of the branch of the hyperbola in the first quadrant.

To verify our claim, we need to show that, for x, y positive,

$$\sqrt{\frac{4y^2-1+2y\sqrt{3y^2-3}}{3}} = y+\sqrt{\frac{3y^2-3}{9}}.$$

Simple algebra shows this to be true.

Therefore, S1 equals the lower half of the first quadrant branch of the hyperbola, so the hyperbola satisfies the required conditions.

Similarly, it can be shown that S3 equals the lower half of the third quadrant branch of the hyperbola. So the lower branch of our hyperbola exactly covers S1 and S3.

<u>Comment</u> By the exact same procedure, we find that the lower halves of the branches of the hyperbola $3x^2+2y^2+6xy+6x+6y+4=0$, a rotation of CONIC, exactly cover S2 and S4.

<u>Appendix 1</u> In order that x and y be positive integers, the quadratics under the radicals must each be a perfect square.

First we consider $\sqrt{3x^2+6x+1} = \sqrt{3(x+1)^2-2}$

We need $3(x+1)^2-2=s^2 \Leftrightarrow s^2-3m^2=-2$, where $m = x + 1$.

We have a Pell-like equation with initial solution $s_1=1, m_1=1$.

All other solutions are given by the recurrence relations

$$\begin{cases} s_{k+1}=2s_k+3m_k \\ m_{k+1}=s_k+2m_k \end{cases} \text{expressible as} \begin{bmatrix} s_{k+1} \\ m_{k+1} \end{bmatrix} = \begin{bmatrix} 2 & 3 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} s_k \\ m_k \end{bmatrix}.$$

For each pair (s_k, m_k), we have $x_k = m_k - 1$ and $\sqrt{3x_k^2+6x_k+1} = s_k$.

Next we consider $\sqrt{3y^2-3} = \sqrt{3(y^2-1)}$.

We need $y^2-1=3t^2 \Leftrightarrow y^2-3t^2=1$.

We have a Pell equation with initial solution $y_1=1, t_1=0$.

All other solutions are given by the recurrence relations

$$\begin{cases} y_{k+1}=2y_k+3t_k \\ t_{k+1}=y_k+2t_k \end{cases} \text{expressible as} \begin{bmatrix} y_{k+1} \\ t_{k+1} \end{bmatrix} = \begin{bmatrix} 2 & 3 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} y_k \\ t_k \end{bmatrix}.$$

For each pair (y_k, t_k), we have $\sqrt{3y_k^2-3} = 3t_k$.

Having found the integer values which make each radical expression an integer, we see that they do align to make $\sqrt{3x^2+6x+1} = y+\sqrt{3y^2-3}$.

The first few steps of our process:

k	s	m	x = m-1	$\sqrt{3x^2+6x+1}$ = s	y	t	$y+\sqrt{3y^2-3}$ = y + 3t
1	1	1	0	1	1	0	1
2	5	3	2	5	2	1	5
3	19	11	10	19	7	4	19
4	71	41	40	71	26	15	71
5	265	153	152	265	97	56	265
6	989	571	570	989	362	209	989
7	3691	2131	2130	3691	1351	780	3691

The first few lattice points on S1: (0, 1), (2, 2), (10, 7), (40, 26), (152, 97), (570, 362).
Similarly, we could find all lattice points on S2, S3 and S4.

Appendix 2 To compute the equation of the conic determined by the five points
P(0, 1), Q(2, 2), R(10, 7), S(40, 26), T(152, 97),
we avoid solving a system of equations by using a technique given at an anonymous website:
https://www.qc.edu.hk/math/Advanced%20Level/conic%20through%205%20points.htm
(1) The line through P and Q: x – 2y +2 = 0
The line through R and S: 19 x – 30y +20 = 0
The line through P and S: 5x – 8y +8 = 0
The line through Q and R: 5x – 8y +6 = 0

(2) PQ'RS: (x – 2y +2)(19 x – 30y +20) = 0
PS'QR: (5x – 8y +8) (5x – 8y +6) = 0

(3) A linear combination of these two equations:
(x – 2y +2)(19 x – 30y +20) + k(5x – 8y +8)(5x – 8y +6) = 0

(4) Choose k to make T(152, 97) satisfy the equation:
(152 – 2*97 +2)(19*152 – 30*97 +20)
+ k(5*152 – 8*97 +8)(5*152 – 8*97 +6) = 0.
This forces k = -1.
(5) Final simplification produces the equation $3x^2+2y^2-6xy+6x-6y+4=0$
It is straightforward to verify that our five points satisfy this equation.

FINAL COMMENT – unraveling the mystery.
Where did the original equation EQ come from? How did the problem poser know it would intercept an unknown hyperbola infinitely many times?
If we eliminate the radicals in EQ by squaring (twice), we eventually come to a fourth-degree equation in x and y.

If we compute the product of the two equations representing our two hyperbolas, we obtain the identical fourth-degree equation! The graph of this equation is the union of the two hyperbolas, shown below. Thus, the graph of EQ must be a subset of this union.

We assume that the Ken Korbin started with the two hyperbolas, formed the fourth-degree equation, then strategically manipulated it (e.g. applying quadratic formula) to obtain EQ, then presented it for us to enjoy.

The dual hyperbolas $3x^2 + 2y^2 - 6xy + 6x - 6y + 4 = 0$ and
$3x^2 + 2y^2 + 6xy + 6x + 6y + 4 = 0$